A RATIONAL WORLD?

Assume we lived in a world where only rational numbers existed, the only ones provided by P1-P12, how would it look like? Let's start first with plane geometry. Lines are defined in analytic geometry by the equation $ax + by + c = 0$ (now everything is in $\mathbb{Q}$: a, b, c, x, y are all rational numbers. As usual two lines coincide if one equation is equivalent to the other. For instance the line $2ax + 2ay + 2c = 0$ is the same as the line $ax + by + c = 0$. The whole plane is now $\mathbb{Q}^2$, the set of all pairs of rationals.

Check that two lines which are not parallel still intersect, that for any two distinct points there is one line and only one joining the two points. More generally, the geometry of lines is not too different from that in "our world" where "real numbers exist". (There are lots of things that would need a serious discussion in the statement "real numbers exist in our world", but for the sake of brevity, we assume we believe in that.)

How about circles? Take first the center at $(0, 0)$ and the radius $R = 1$. Then a circle in the rational world is defined naturally as the set of points $(x, y) \in \mathbb{Q}^2$ s.t.

$$x^2 + y^2 = 1 \tag{1}$$

Does this contain any points. Yes, clearly $(1, 0), (0, 1), (-1, 0), (0, -1)$ are on the circle. But there are many more, in fact infinitely many.

Indeed, consider the following figure:

The coordinates are $N(0, 1)$, $P(r, 0)$ where of course $r \in \mathbb{Q}$.

Exercise 1. *Take any $r \in (0, 1)$. Write the equation of the segment NQ in point-slope form. Intersect it with the circle (1). Show that the (x, y) coordinates of Q are rational numbers.*

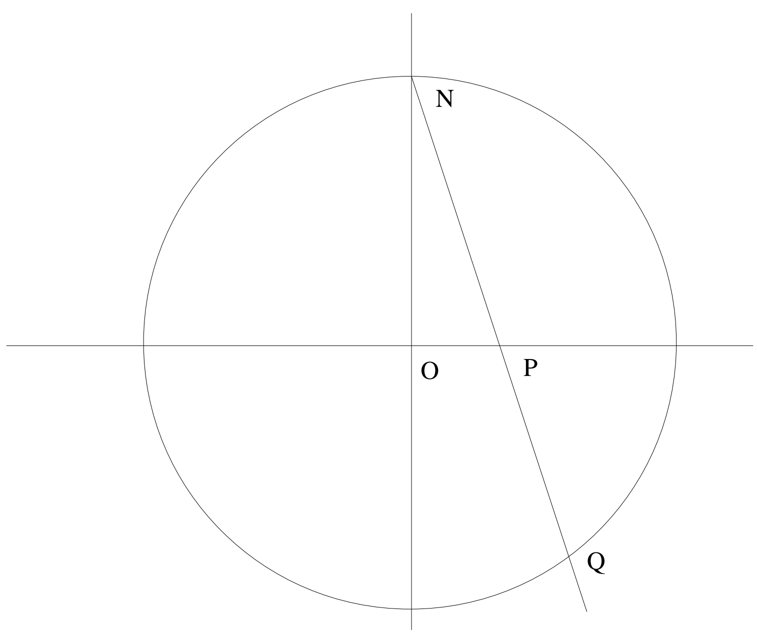

FIGURE 1. A circle and a special line

Thus there are infinitely many points on the circle, and there are points on it arbitrarily close to each other: the points form a "continuum" inasmuch as a "rational" person can define a continuum. This would look like a circle, for sure.

Exercise 2. *Show that a circle with arbitrary center (in Q^2) and arbitrary radius (in Q) also has a continuum of points.*

Exercise 3. *Take now a second circle with center at $(d, 0)$ where $0 < d < 2$ is in $\mathbb{Q}$. In our world, $\mathbb{R}^2$, the circles would intersect. But do they* necessarily *intersect in $\mathbb{Q}^2$?*

Exercise 4. *If $d = 1$, a point of intersection* if it exists *together with the two centers, $(0, 0)$ and $(1, 0)$ would form an equilateral triangle. But does this equilateral triangle exist in $\mathbb{Q}^2$?*

Exercise 5. *Does any equilateral triangle exist in $\mathbb{Q}^2$?*