FOUR REVIEW PROBLEMS

Exercise 1. True or false? If true, give a proof; if false, provide a counterexample:

- "The sequence $\{s_n\}_{n\in\mathbb{N}}$ converges to $s \in \mathbb{R}$ if and only if for any $\varepsilon \in (0, 1/4)$ there is an n_0 s.t. for all $n \geqslant n_0$ we have $|s_n - s| < \varepsilon$"

Exercise 2. For which values of $a > 0$ is the improper integral

$$\int_0^1 \frac{e^{-t} - 1}{t^a} dt$$

well defined?

Exercise 3. Let g be continuous on $[0, 1]$ and assume that f is continuous on $(0, 1]$. Assume further that for any $\varepsilon \in (0, 1)$ we have

$$\int_\varepsilon^1 f(s)ds = \int_{h(\varepsilon)}^1 g(u)du \tag{1}$$

where $h(\varepsilon) \in (0, 1)$ and $h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(a) Show that the improper integral

$$\int_0^1 f(s)ds$$

is well defined.

(b) Use this result and the substitution $s = u^2$ to show that the integral

$$\int_0^1 \frac{e^s}{\sqrt{s}} ds$$

is well defined.

(c) Find a substitution to show in a similar way that

$$\int_0^1 \frac{e^s}{s^a} ds$$

is well defined for any $a \in (0, 1)$.

(d) Finally, find a substitution showing that

$$\int_1^\infty e^{-x^2} dx$$

is well defined.

Exercise 4. True or false? If true, give a proof; if false, provide a counterexample:

- "Assume that f is continuous on $[0, \infty)$ and that the improper integral $\displaystyle\int_0^\infty f(x)dx$ is well defined. Then $\displaystyle\lim_{x\to\infty} f(x) = 0$."