The resolvent; analyticity

We have already shown that for any point $\lambda \in \rho$ (ρ = resolvent set) $(f-\lambda)^{-1}$ is continuous in λ.

<u>Theorem</u> $(f-\lambda)^{-1}$ is analytic in $\lambda \in \rho \subset \mathbb{C}$

Furthermore, $\lambda \neq \lambda' \Rightarrow (f-\lambda)^{-1}, (f-\lambda')^{-1}$ commute

and $(f-\lambda)^{-1} - (f-\lambda')^{-1} = (\lambda - \lambda')(f-\lambda)^{-1}(f-\lambda')^{-1}$ (1)

Note: (1) is called the <u>first resolvent formula</u>

<u>Proof</u> By the second resolvent formula, we have

$$(f-\lambda)^{-1} - (f-\lambda')^{-1} = (f-\lambda)^{-1}(\lambda-\lambda')(f-\lambda')^{-1} \qquad (2)$$

Interchanging λ and λ' we get

$$(f-\lambda')^{-1} - (f-\lambda)^{-1} = (f-\lambda')^{-1}(\lambda'-\lambda)(f-\lambda)^{-1}$$

Thus

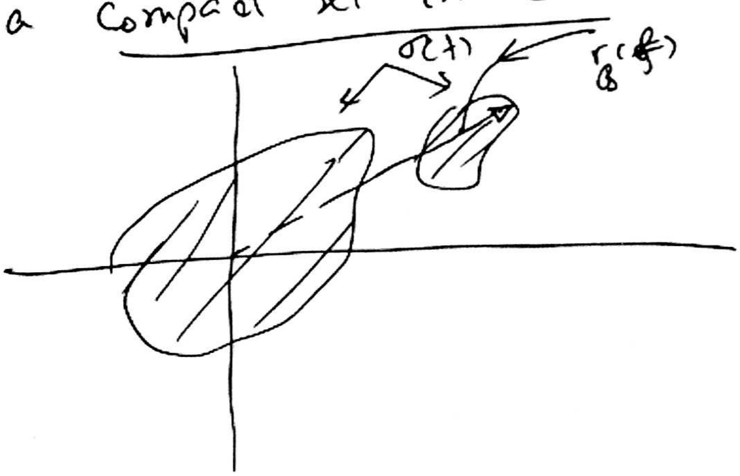
$$(\lambda-\lambda')(f-\lambda)^{-1}(f-\lambda')^{-1} = (\lambda-\lambda')(f-\lambda')^{-1}(f-\lambda)^{-1}$$

implying commutativity and proving (1)

Dividing (1) by $\lambda'-\lambda'$ and using continuity in λ' we get that $[(f-\lambda)^{-1}]'$ exists and equals $(f-\lambda)^{-2}$ as expected.

Spectral radius

We know already that $\sigma(\lambda)$ is closed (since $\rho(\lambda)$ was proved to be open) and bounded since $(f-\lambda)^{-1}$ exist for $|\lambda| > \|f\|$. Thus $\sigma(\lambda)$ is a compact set in $\mathbb{C}$



Definition Let $r_{\mathcal{B}}(f) = \{\sup|\lambda| \mid \lambda \in \sigma(f)\} = \inf\limits_{\substack{\lambda : \lambda' \in \rho(f) \\ \forall |\lambda'| > |\lambda|}} |\lambda|$

Consider the series $\frac{1}{\lambda}\sum_{n=0}^{\infty} \frac{f^n}{\lambda^n}$. This converges

~~iff~~ $|\lambda| > \overline{\lim}\, \|f^n\|^{\frac{1}{n}}$, to $(f-\lambda)^{-1}$

equality may occur

Thus $(f-\lambda)^{-1}$ exists for $|\lambda| > \overline{\lim}\|f^n\|^{\frac{1}{n}} = r_1$

assume $(f-\lambda)^{-1}$ existed for all λ, $|\lambda| = r_1$.

Then, since $(f-\lambda)^{-1}$ is analytic in an open set ~~on~~ for each λ, $|\lambda| = r_1$, by the finite cover theorem it would exist for all $\lambda > r_2$ where $r_2 < r_1$. By standard complex analysis then, $\frac{1}{\lambda}\sum \frac{f^n}{\lambda^n}$ would exist, and converge absolutely for all $\lambda > r_2$, contradiction $\square$

It remains to show that $\lim_{n\to\infty} \|f^n\|^{\frac{1}{n}}$ exists. This has (almost) nothing to do with Banach algebras but it is a direct consequence of the inequality $\|f^n\| \|f^m\| \le \|f^n\| \|f^m\|$ ($\|f^n\|=0$ is trivial). Let $g_n = \ln \|f^n\|$.

<u>Lemma</u> Assume g is a sequence with the property

$g_{m+n} \le g_m + g_n \quad \forall\, m, n \in \mathbb{N}$. Then $\lim_{n\to\infty} \frac{g_n}{n}$ exists.

<u>Proof</u> Write $n = mk + \ell$, $\ell < k$ where we first keep k fixed and let $k \to \infty$. We have

$$\frac{g_n}{n} = \frac{g_{mk+\ell}}{mk+\ell} \le \frac{g_{mk}}{mk+\ell} + \frac{g_\ell}{mk+\ell} \le \frac{m g_k}{mk+\ell} + \frac{g_\ell}{mk+\ell}$$

Note that the last term converges to $\frac{g_k}{k}$ as $m \to \infty$.

Thus $\overline{\lim_{n\to\infty}} \frac{g_n}{n} \le \frac{g_k}{k} \;\forall k \Rightarrow \overline{\lim} \frac{g_n}{n} \le \underline{\lim} \frac{g_n}{n}$ $\square$

Therefore $\sigma(f) \subseteq \{ |\lambda| \mid \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} \ge \lambda \}$

We know show that $\sigma(f)$ cannot be empty

<u>Theorem</u> $\sigma_B(f) \ne \emptyset \quad \forall f \in \mathcal{B}$

Proof (by contradiction) Assume $(f-\lambda)^{-1}$ existed for all $\lambda \in \mathbb{C}$. Then $(f-\lambda)^{-1}$ would be entire. Note that $(f-\lambda)^{-1} = -\lambda^{-1}\left(1-\frac{f}{\lambda}\right)^{-1} \to 0$ in norm as $\lambda \to \infty$. By Liouville $(f-\lambda)^{-1} = 0$, cd. $\square$

The following is an important result about <u>division</u> Banach algebras (every nonzero element has an inverse)

<u>Theorem</u> (Gelfand - Mazur) If $\mathcal{B}$ is a division Banach algebra, then there is a unique (up to complex conjugation) isomorphism between $\mathcal{B}$ and $\mathbb{C}$.

<u>Proof</u> Let $f \in \mathcal{B}$ and $\lambda_f \in \sigma(f)$ (nonempty, as we know). By definition $(f-\lambda_f)^{-1}$ does not exist. But $\mathcal{B}$ is a division algebra and thus $f - \lambda_f = 0$ $f = \lambda_f \cdot 1$.
Clearly, if $\lambda \neq \lambda_f$ then $f - \lambda = \lambda_f - \lambda \neq 0$ is invertible. Define the map $\varphi = f \to \lambda_f$.
Since $f = \lambda_f \cdot 1$, clearly φ is an isometric isomorphism (seen already since $\mathcal{B} = \{\lambda_f \cdot 1 \mid \lambda_f \in \mathbb{C}\}$)
Assume φ' is another isomorphism. Then $\varphi^{-1}\varphi' = \varphi$ is an isometric isomorphism between $\mathbb{C}$ and $\mathbb{C}$.
and
$\varphi(z_1+z_2) = \varphi(z_1) + \varphi(z_2) \Rightarrow \varphi(x_1+x_2) = \varphi(x_1) + \varphi(x_2)$
$\qquad y_1 + y_2 \qquad y_1 \qquad y_2$
by continuity $\varphi(x+iy) = ax + iby$ $\quad \varphi(1) = 1 \Rightarrow a = 1$
$(\varphi(i))^2 = \varphi(-1) \Rightarrow b^2 = 1 \quad b = \pm 1$

Stated as "unique" in Douglas glitch pointed out by Joel

Quotient algebras

Let B be a Banach algebra and assume $\mathcal{J}$ is a closed two-sided ideal of $\mathcal{B}$. In particular, $\mathcal{J}$ is a closed <u>subspace</u> of $\mathcal{B}$ and $\mathcal{B}/\mathcal{J}$ is a Banach space. Since $\mathcal{J}$ is a two sided ideal, then $\mathcal{B}/\mathcal{J}$ is a commutative algebra.