Notes on Banach Algebras and Functional Calculus*

April 23, 2014

1 The Gelfand-Naimark theorem (proved on Feb 7)

Theorem 1. *If $\mathfrak{A}$ is a commutative C^*-algebra and M is the maximal ideal space, of $\mathfrak{A}$ then the Gelfand map Γ is a *-isometric isomorphism of $\mathfrak{A}$ onto $C(M)$.*

Corollary 1. *If $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{A}_1$ is the self-adjoint subalgebra generated by $T \in \mathfrak{A}$, then $\sigma_{\mathfrak{A}}(T) = \sigma_{\mathfrak{A}_1}(T)$.*

Proof. Evidently, $\sigma_{\mathfrak{A}}(T) \subset \sigma_{\mathfrak{A}_1}(T)$. Let $\lambda \in \sigma_{\mathfrak{A}_1}$. Working with $T - \lambda$ instead of T we can assume that $\lambda = 0$. By the Gelfand theorem, T is invertible in $\sigma_{\mathfrak{A}_1}$ iff $\Gamma(T)$ is invertible in $C(M)$. Note that T^*T and TT^* are self-adjoint and if the result holds for self-adjoint elements, then it holds for all elements. Indeed $(T^*T)^{-1}T^*T = I \Rightarrow (T^*T)^{-1}T^*$ is a left inverse of T^* etc. Now $\mathfrak{A}_A$, the C^* sub-algebra generated by $A = T^*T$ is commutative and Theorem 1 applies. Since $\|T\|^2 = \|T^*T| \neq 0$ unless $T = 0$, a trivial case, $\|a\|_\infty = \|T^*T\| > 0$ where $a = \Gamma A$. Let M_A be the maximal ideal space for A. If A is not invertible, by Gelfand's theorem, $a(m) = 0$ for some $m \in M_A$, and for any ε, there is a neighborhood of m s.t. $|a(m)| < \varepsilon/4$. We take $\varepsilon < \|a\|$ and note that the set $S = \{x \in M_A : |a(x) \in (\varepsilon/3, \varepsilon/2)$ is open and disjoint from M_1. There is thus a continuous function g on M which is 1 on S and zero on M_A and zero on $\{m \in M_A : a(m) > \varepsilon\}$. We have $\|g\| = \|G\| = 1$ where $\Gamma G = g$ and clearly, $\|gA\|_{\mathfrak{A}_A} < \varepsilon$. But $\|GA\|_{\mathfrak{A}_A} = \|GA\|_{\mathfrak{A}}$. If A were invertible, then with $\alpha = \|A\|^{-1}$, then $\|G\| = \|GAA^{-1}\| \leqslant \alpha\|GA\| \leqslant \varepsilon\alpha \to 0$ as $\varepsilon \to 0$, a contradiction. □

Exercise* Consider the space $L^1(\mathbb{R}^+)$ with complex conjugation as involution and one sided convolution as product:

$$(f * g)(x) = \int_0^x f(s)g(x-s)ds$$

*Based on material from [1], [5], [4] and original material.

to which we adjoin an identity. That is, let $\mathcal{A} = \{h : h = aI+f, f \in L^1(\mathbb{R}^+),\ a \in \mathbb{C}\}$ with the convention $(\forall g)(I * g = g)$. Show that this is a commutative $C^* - algebra$. Find the multiplicative functionals on $\mathcal{A}$.

2 Operators: Introduction

We start by looking at various simple examples. Some properties carry over to more general settings, and many don't. It is useful to look into this, as it gives us some idea as to what to expect. Some intuition we have on operators comes from linear algebra. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be linear. Then A can be represented by a matrix, which we will also denote by A. Certainly, since A is linear on a finite dimensional space, A is continuous. We use the standard scalar product on $\mathbb{C}^n$,

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y}_i$$

with the usual norm $\|x\|^2 = \langle x, x\rangle$. The operator norm of A is defined as

$$\|A\| = \sup_{x\in\mathbb{C}^n} \frac{\|Ax\|}{\|x\|} = \sup_{x\in\mathbb{C}^n} \left\| A\frac{x}{\|x\|}\right\| = \sup_{u\in\mathbb{C}^n:\|u\|=1} \|Au\| \tag{1}$$

Clearly, since A is continuous, the last sup (on a compact set) is in fact a max, and $\|A\|$ is bounded. Then, we say, A is bounded.

The spectrum of A is defined as

$$\sigma(A) = \{\lambda \mid (A - \lambda) \text{ is not invertible}\} \tag{2}$$

This means $\det(A - \lambda) = 0$, which happens iff ker $(A - \lambda) \neq \{0\}$ that is

$$\sigma(A) = \{\lambda \mid (Ax = \lambda x) \text{ has nontrivial solutions}\} \tag{3}$$

For these operators, the spectrum consists exactly of the eigenvalues of A. This however is not generally the case for infinite dimensional operators.

• **Self-adjointness** A is symmetric iff

$$\langle Ax, y\rangle = \langle x, Ay\rangle \tag{4}$$

for all x and y. For matrices, symmetry is the same as self-adjointness, but this is another property that is generally true only in finite dimensional spaces. As an exercise, you can show that this is the case iff $(A)_{ij} = \overline{(A)_{ji}}$.

We can immediately check that all eigenvalues are real, using (4).

We can also check that eigenvectors x_1, x_2 corresponding to distinct eigenvalues λ_1, λ_2 are orthogonal, since

$$\langle Ax_1, x_2\rangle = \lambda_1\langle x_1, x_2\rangle = \langle x_1, Ax_2\rangle = \lambda_2\langle x_1, x_2\rangle \tag{5}$$

More generally, we can choose an orthonormal basis consisting of eigenvectors u_n of A. We write these vectors in matrix form,

$$U = \begin{pmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n2} \\ \cdots & \cdots & \cdots & \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{pmatrix} \tag{6}$$

and note that

$$UU^* = U^*U = I \tag{7}$$

where I is the identity matrix. Equivalently,

$$U^* = U^{-1} \tag{8}$$

We have

$$AU = A \begin{pmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n2} \\ \cdots & \cdots & \cdots & \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} Au_1 & Au_2 & \cdots & Au_n \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{pmatrix} = U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} =: UD \tag{9}$$

where D is a diagonal matrix. In particular

$$U^*AU = D \tag{10}$$

which is a form of the spectral theorem for A. It means the following. If we pass to the basis $\{u_j\}$, that is we write

$$x = \sum_{k=1}^{n} c_k u_k \tag{11}$$

we have

$$c_k = \langle x, u_k \rangle \tag{12}$$

that is, since $\tilde{x} = (c_k)_k$ is the new representation of x, we have

$$\tilde{x} = U^* x \tag{13}$$

We also have

$$Ax = \sum_{k=1}^{n} c_k A u_k = \sum_{k=1}^{n} c_k \lambda_k u_k = D\tilde{x} =: \tilde{A}\tilde{x} \tag{14}$$

another form of (10). This means that after applying U^* to $\mathbb{C}^n$, the new A, $\tilde{A}$ is diagonal (D), and thus is *it acts multiplicatively.*

• **In infinite dimensional spaces:**

- Eq. (1) stays as a definition but the sup may not be attained.
- Definition (2) stays.
- Property (3) will not be generally true anymore.
- (4) will still be valid for bounded operators but generally false for unbounded ones.
- (5) stays.
- (14) properly changed, will be the spectral theorem.

Let us first look at $L^2[0,1]$; here, as we know,

$$\langle f, g\rangle = \int_0^1 f(s)\overline{g(s)}ds$$

We can check that X, defined by

$$(Xf)(x) = xf(x)$$

is symmetric. It is also bounded, with norm $\leqslant 1$ since

$$\int_0^1 s^2|f(s)|^2 ds \leqslant \int_0^1 |f(s)|^2 ds \tag{15}$$

The norm is exactly 1, as it can be seen by choosing f to be the characteristic function of $[1-\epsilon, \epsilon]$ and taking $\epsilon \to 0$. Note that unlike the finite dimensional case, the sup is *not attained: there is no f s.t.* $\|xf\| = 1$.

What is the spectrum of X? We have to see for which λ $X - \lambda$ is not invertible, that is the equation

$$(x - \lambda)f = g \tag{16}$$

does not have L^2 solutions for all g. This is clearly the case iff $\lambda \in [0,1]$.

But we note that $\sigma(X)$ has no eigenvalues! Indeed,

$$(x-\lambda)f = 0 \Rightarrow f = 0 \forall x \neq \lambda \Rightarrow f = 0 \ a.e, \Rightarrow f = 0 \text{ in the sense of } L^2 \tag{17}$$

Finally, let us look at X on $L^2(\mathbb{R})$. The operator stays symmetric, wherever defined. Note that now X is unbounded, since, with χ the characteristic function,

$$x\chi_{[n,n+1]} \geqslant n\chi_{[n,n+1]} \tag{18}$$

and thus $\|X\| \geqslant n$ for any n. By Hellinger-Toeplitz, proved in the sequel see p. 9, X cannot be everywhere defined. Specifically, $f = (|x|+1)^{-1} \in L^2(\mathbb{R})$ whereas $|x|(|x|+1)^{-1} \to 1$ as $x \to \pm\infty$, and thus Xf is not in L^2. What is the domain of definition of X (domain of X in short)? It consists of all f so that

$$f \in L^2 \ \text{ and } \ xf \in L^2 \tag{19}$$

It is easy to check that (19) is equivalent to

$$f \in L^2(\mathbb{R}^+, (|x|+1)dx) := \{f : (|x|+1)f \in L^2(\mathbb{R}) \tag{20}$$

This is not a closed subspace of L^2. In fact, it is a dense set in L^2 since C_0^∞ is contained in the domain of X and it is dense in L^2. X is said to be densely defined.

3 Bounded and unbounded operators

1. Let X, Y be Banach spaces and $\mathcal{D} \subset X$ a *linear* space, not necessarily closed.

2. A linear operator is a linear map $T : \mathcal{D} \to Y$.

3. $\mathcal{D}$ is the domain of T, sometimes written $\mathrm{dom}\,(T)$, or $\mathcal{D}\,(T)$.

4. The *range* of T, $\mathrm{ran}\,(T)$, is simply $T(\mathcal{D})$.

5. The *graph of* T is

$$\mathbb{G}(T) = \{(x, Tx) | x \in \mathcal{D}\,(T)\}$$

   The graph will play an important role, especially in the theory of unbounded operators.

6. The kernel of T is

$$\ker\,(T) = \{x \in \mathcal{D}\,(T) : Tx = 0\}$$

3.1 Operations

1. If X, Y are Banach spaces, then T is continuous *iff* $\|T\| < \infty$. Indeed, in one direction we have $\|T(x-y)\| \leqslant \|T\|\|x-y\|$, and finite norm implies continuity. It is easy to check that if $\|T\| = \infty$ then there is a sequence $x_n \to 0$ s.t. $\|Tx_n\| \not\to 0$.

2. $aT_1 + bT_2$ is defined on $\mathcal{D}\,(T_1) \cap \mathcal{D}\,(T_2)$.

3. if $T_1 : \mathcal{D}\,(T_1) \subset X \to Y$ and $T_2 : \mathcal{D}\,(T_2) \subset Y \to Z$ then $T_2T_1 : \{x \in \mathcal{D}\,(T_1) : T_1(x) \in \mathcal{D}\,(T_2)\}$.

   In particular, if $\mathcal{D}\,(T)$ and $\mathrm{ran}(T)$ are both in the space X, then, inductively, $\mathcal{D}\,(T^n) = \{x \in \mathcal{D}\,(T^{n-1}) : T(x) \in \mathcal{D}\,(T)\}$. The domain may become trivial.

4. **Inverse**. The inverse is defined iff $\ker\,(T) = \{0\}$. This condition implies T is bijective. Then $T^{-1} : \mathrm{ran}\,(T) \to \mathcal{D}\,(T)$ is defined as the usual function inverse, and is clearly linear. ∂ is not invertible on $C^\infty[0,1]$: $\ker \partial = \mathbb{C}$. How about ∂ on $C_0^\infty((0,1))$ (the set of C^∞ functions with compact support contained in $(0,1)$)?

5. **Spectrum** . Let X be a Banach space. The resolvent set of an operator $T : \mathcal{D} \to X$ is defined as

$$\rho(T) = \{\lambda \in \mathbb{C} : \\ (T - \lambda) \text{ is one-to-one from } \mathcal{D} \text{ to } X \text{ and } (T-\lambda)^{-1} : X \to \mathcal{D} \text{ is bounded}\}$$

See also Corollary 39 below.

The spectrum of T, $\sigma(T) = \mathbb{C} \setminus \rho(T)$. There are various reasons for $(T - \lambda)^{-1}$ not to exist: $(T - \lambda)$ might not be injective, $(T - \lambda)^{-1}$ might be unbounded, or not densely defined. These possibilities correspond to different types of spectra that will be discussed later.

6. **Closable operators**. It is natural to extend functions by continuity, when possible. If $x_n \to x$ and $Tx_n \to y$ we want to see whether we can define $Tx = y$. Clearly, we must have

$$x_n \to 0 \text{ and } Tx_n \to y \Rightarrow y = 0, \tag{21}$$

since $T(0) = 0 = y$. Conversely, (21) implies the extension $Tx := y$ whenever $x_n \to x$ and $Tx_n \to y$ is consistent and defines a linear operator.

An operator satisfying (21) is called *closable*. This condition is the same as requiring

$$\overline{\mathbb{G}(T)} \text{ is the graph of an operator} \tag{22}$$

where the closure is taken in $X \oplus Y$. Indeed, if $(x_n, Tx_n) \to (x, y)$, then, by the definition of the graph of an operator, $Tx = y$, and in particular $x_n \to 0$ implies $Tx_n \to 0$. An operator is closed if its graph is closed. It will turn out that symmetric operators are closable.

7. Clearly, bounded operators are closable, since they are continuous.

8. Common operators are "usually" closable. E.g., ∂ defined on a subset of *continuously differentiable functions* is closable. Assume $f_n \xrightarrow{L^2} 0$ (in the sense of L^2) and $f_n' \xrightarrow{L^2} g$. Since $f_n' \in C^1$ we have

$$f_n(x) - f_n(0) = \int_0^x f_n'(s)ds = \langle f_n', \chi_{[0,x]}\rangle \to \langle g, \chi_{[0,x]}\rangle = \int_0^x g(s)ds \tag{23}$$

Thus

$$(\forall x) \;\; \lim_{n\to\infty} (f_n(x) - f_n(0)) = 0 = \int_0^x g(s)ds \tag{24}$$

implying $g = 0$.

6. As an example of non-closable operator, consider, say $L^2[0,1]$ (or any separable Hilbert space) with an orthonormal basis e_n. Define $Ne_n =$

ne_1, extended by linearity, whenever it makes sense (it is an unbounded operator). Then $x_n = e_n/n \to 0$, while $Nx_n = e_1 \neq 0$. Thus N is not closable.

Every infinite-dimensional normed space admits a nonclosable linear operator. The proof requires the axiom of choice and so it is in general non-constructive.

The closure through the graph of T is called the *canonical closure* of T.

Note: if $\mathcal{D}(T) = X$ and T is closed, then T is continuous, and conversely (see p.9, 5).

7. Non-closable operators have as spectrum the whole of $\mathbb{C}$. Indeed, if T is not closable, you can check that neither is $T - \lambda$, $\lambda \in \mathbb{C}$. We then have to show that T is not invertible. If $x_n \to 0$ and $Tx_n = y_n \to y \neq 0$ then $T^{-1}Tx_n = T^{-1}y_n \to T^{-1}y$ but $T^{-1}Tx_n = x_n \to 0$, thus $T^{-1}y = 0 \in \mathcal{D}(T)$ since $\mathcal{D}(T)$ has to be a linear space and $T(0) = 0$, since T is linear, contradiction.

3.1.1 Adjoints of unbounded operators.

1. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces (we will most often be interested in the case $\mathcal{H} = \mathcal{K}$), with scalar products $\langle , \rangle_{\mathcal{H}}$ and $\langle , \rangle_{\mathcal{K}}$.

2. $T : D(T) \subset \mathcal{H} \to \mathcal{K}$ is **densely defined** if $\overline{D(T)} = \mathcal{H}$.

3. Assume T is densely defined.

4. The adjoint of T is defined as follows. We look for those y for which

$$\exists\ v = v(y) \in \mathcal{H} \text{ s.t.} \forall\ x \in D(T),\ \langle y, Tx \rangle_{\mathcal{K}} = \langle v, x \rangle_{\mathcal{H}} \tag{25}$$

   Since $D(T)$ is dense, such a $v = v(y)$, if it exists, is unique.

5. We define $D(T^*)$ to be the set of y for which $v(x)$ as in (132) exists, and define $T^*(y) = v$. Note that $Tx \in \mathcal{K}$, $y \in \mathcal{K}$, $T^*y \in \mathcal{H}$.

6. Unbounded operators are called self-adjoint if they coincide with their adjoint, *meaning also that they have the same domain.* We will return to this.

Exercise* In $L^2[0,1]$ consider the "extrapolation" operator $\mathcal{E}$ densely defined on polynomials by $\mathcal{E}(P)(x) = P(x+1)$. Is $\mathcal{E}$ bounded? Is it injective? Is it closable? What is the domain of its adjoint? Calculate the spectrum $\sigma(\mathcal{E})$ from the definition of σ. The solution is given in 1, 31.

3.1.2 Absolutely continuous functions

Two (obviously equivalent) definitions of the absolutely continuous functions on $[a,b]$, $AC[a,b]$ are: (I) There exists an L^1 function s.t. $f(x) = f(a) + \int_a^x g(s)ds$ for all $x \in [a,b]$; (II) f is differentiable a.e. on (a,b), $f' \in L^1$ and $f(x) = f(a) + \int_a^x f'(s)ds$ for all $x \in [a,b]$.

Exercise* Consider the operator $T = i\frac{d}{dx}$ defined on

$$H_0^2(\mathbb{R}^+) := \{f \in L^2(\mathbb{R}^+) : f' \in L^2(\mathbb{R}^+);\ f(0) = 0\}$$

Is T symmetric (meaning: $\langle f, Tg\rangle = \langle Tf, g\rangle$ for all $f, g \in \mathcal{D}(T)$)? What is $\sigma(T)$? Can you find a symmetric operator having $\mathbb{C}$ as spectrum?

3.2 A brief review of bounded operators

1. $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y. We write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$.

2. The space $\mathcal{L}(X,Y)$ of *bounded operators* from X to Y is a Banach space too, with the norm $T \to \|T\|$.

3. We see that $T \in \mathcal{L}(X,Y)$ takes bounded sets in X into bounded sets in Y.

4. We have the following topologies on $\mathcal{L}(X,Y)$ in increasing order of weakness:

   (a) The **uniform operator topology** or **norm** topology is the one given by $\|T\| = \sup_{\|u\|=1} \|Tu\|$. Under this norm, $\mathcal{L}(X,Y)$ is a Banach space.

   (b) The **strong operator topology** is the one defined by the convergence condition $T_n \to T \in \mathcal{L}(X,Y)$ iff $T_n x \to Tx$ for all $x \in X$. We note that if $T_n x$ is Cauchy for every x then, in this topology, T_n is convergent *to a* $T \in \mathcal{L}(X,Y)$. Indeed, it follows that $T_n x$ is convergent for every x. Now note that $\|T_n x\| \leqslant C(x)$ for every x because of convergence. Then, $\|T_n\| \leqslant C$ for some C by the uniform boundedness principle. But $\|Tx\| = \|(T - T_{n_0} + T_{n_0})x\| \leqslant \|(T - T_{n_0})x\| + C\|x\| \to C\|x\|$ as $n \to \infty$ thus $\|T\| \leqslant C$.

   We write $T = s - \lim T_n$ in the case of strong convergence.

   Note also that the *strong operator topology* is a pointwise convergence topology while the **uniform operator topology** is the "L^∞" version of it.

   (c) The **weak operator topology** is the one defined by $T_n \to T$ if $\ell(T_n(x)) \to \ell(Tx)$ for all linear functionals on X, i.e. $\forall \ell \in X^*$. In the Hilbert case, $X = Y = \mathcal{H}$, this is the same as requiring that $\langle T_n x, y\rangle \to \langle Tx, y\rangle$ $\forall x, y$ that is the "matrix elements" of T converge. It can be shown that if $\langle T_n x, y\rangle$ converges $\forall x, y$ then there is a T so that $T_n \to^w T$.

5. Reminder: The Riesz Lemma states that if $\varphi \in \mathcal{H}^*$, then there is a unique $y_\varphi \in \mathcal{H}$ so that $\varphi x = \langle x, y_\varphi \rangle$ for all x.

6. Adjoints. Let first $X = Y = \mathcal{H}$ be separable Hilbert spaces and let $T \in \mathcal{L}(\mathcal{H})$. Let's look at $\langle Tx, y \rangle$ as a linear functional $\ell_T(x)$. It is clearly bounded, since by Cauchy-Schwarz we have

$$|\langle Tx, y \rangle| \leqslant (\|y\|\|T\|)\|x\| \tag{26}$$

   Thus $\langle Tx, y \rangle = \langle x, y_T \rangle$ for all x. Define T^* by $\langle x, T^* y \rangle = \langle x, y_T \rangle$. Using (26) we get

$$|\langle x, T^* y \rangle| \leqslant \|y\|\|T\|\|x\| \tag{27}$$

   Furthermore

$$\|T^* x\|^2 = |\langle T^* x, T^* x \rangle| \leqslant \|T^* x\|\|T\|\|x\|$$

   by definition , and thus $\|T^*\| \leqslant \|T\|$. Since $(T^*)^* = T$ we have $\|T\| = \|T^*\|$.

7. In a general Banach space, we mimic the definition above, and write $T'(\ell)(x) =: \ell(T(x))$. It is still true that $\|T\| = \|T'\|$, see [5]. In Hilbert spaces T^* is called the *adjoint*; in Banach spaces it is also sometimes called *transpose*.

1. **Uniform boundedness theorem**. If $\{T_j\}_{j \in \mathbb{N}} \subset \mathcal{L}(X, Y)$ and $\|T_j x\| < C(x) < \infty$ for any x, then for some $C \in \mathbb{R}$, $\|T_j\| < C \ \forall j$.

2. In the following, X and Y are Banach spaces and T is a linear operator.

3. **Open mapping theorem**. Assume $T : X \to Y$ is *onto*. Then $A \subset X$ open implies $T(A) \subset Y$ open.

4. **Inverse mapping theorem**. If $T : X \to Y$ is *one to one*, then T^{-1} is continuous.

   Proof. T is open so $(T^{-1})^{-1} = T$ takes open sets into open sets.

5. **Closed graph theorem** $T \in \mathcal{L}(X, Y)$ (note: T is defined *everywhere*) is bounded *iff* $\mathbb{G}(T))$ is closed.

   Proof. It is easy to see that T bounded implies $\mathbb{G}(T)$ closed.

   Conversely, we first show that $Z = \mathbb{G}(T) \subset X \oplus Y$ is a Banach space, in the norm

$$\|(x, Tx)\|_Z = \|x\|_X + \|Tx\|_Y$$

   It is easy to check that this is a norm, and that $(x_n, Tx_n)_{n \in \mathbb{N}}$ is Cauchy *iff* x_n is Cauchy and Tx_n is Cauchy. Since X, Y are already Banach spaces, then $x_n \to x$ for some x and $Tx_n \to y = Tx$. But then, by the definition of the norm, $(x_n, Tx_n) \to (x, Tx)$, and Z is complete, under this norm, thus it is a Banach space.

Next, consider the *projections* $P_1 : z = (x, Tx) \to x$ and $P_2 : z = (x, Tx) \to Tx$. Since both $\|x\|$ and $\|Tx\|$ are bounded above by $\|z\|$, then P_1 and P_2 are continuous.

Furthermore, P_1 is *one-to one* between Z and X (for any x there is a unique Tx, thus a unique (x, Tx), and $\{(x, Tx) : x \in X\} = \mathbb{G}(T)$, by definition. By the open mapping theorem, thus P_1^{-1} is continuous. But $Tx = P_2 P_1^{-1} x$ and T is also continuous.

6. **Hellinger-Toeplitz theorem**. Assume $T : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. That is, T is everywhere defined. Furthermore, assume T is symmetric, *i.e.* $\langle x, Tx' \rangle = \langle Tx, x' \rangle$ for all x, x'. Then T is bounded.

   Proof. We show that $\mathbb{G}(T)$ is closed. Assume $x_n \to x$ and $Tx_n \to y$. To show that the graph is closed we only need to prove that $Tx = y$. Let x' be arbitrary. Then $\lim_{n\to\infty} \langle x_n, Tx' \rangle = \langle x, Tx' \rangle = \langle Tx, x' \rangle$ by symmetry of T. Also, $\lim_{n\to\infty} \langle x_n, Tx' \rangle = \lim_{n\to\infty} \langle Tx_n, x' \rangle = \langle y, x' \rangle$ by symmetry (and by assumption). Thus, $\langle Tx, x' \rangle = \langle y, x' \rangle \ \forall x' \in \mathcal{H}$ and the graph is closed.

7. Consequence: The differentiation operator $i\partial$, say, with domain is C_0^∞ (and many other *unbounded* symmetric operators in applications), which are symmetric on certain domains *cannot be extended to the whole space.* General L^2 functions are fundamentally nondifferentiable.

   Unbounded symmetric operators come with a nontrivial domain $\mathcal{D}(T) \subsetneq X$, and addition, composition etc are to be done carefully.

Proposition 2. If $\mathcal{H}$ is a Hilbert space, and S, T are in $\mathcal{L}(\mathcal{H})$, then

1. $T^{**} = (T^*)^* = T$
2. $\|T\| = \|T^*\|$
3. $(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*$
4. $(T^*)^{-1} = (T^{-1})^*$ for all invertible T in $\mathcal{L}(\mathcal{H})$.
5. $\|T\|^2 = \|T^* T\|$

Proof. 1. $(\forall f, g), \langle f, T^{**} g \rangle = \langle T^* f, g \rangle = \langle f, Tg \rangle$

2. $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^* Tx, x \rangle \leqslant \|T^* T\| \|x\| \leqslant \|T^*\| \|T\|$ and thus $\|T\| \leqslant \|T^*\|$ and vice-versa.
3. A calculation.
4. $T^*(T^{-1})^* = (T^{-1} T)^* = I$.
5. In one direction we have $\|T^* T\| \leqslant \|T^*\| \|T\| = \|T\|^2$. In the opposite direction, with u unit vectors, we have

$$\|T\|^2 = \sup_u \|Tu\|^2 = \sup_u \langle Tu, Tu \rangle = \sup_u \langle T^* Tu, u \rangle \leqslant \|T^* T\|$$

□

Corollary 3. *The algebra generated by T, T^* is a C^*-algebra.*

Definition 4. If $T \in \mathcal{L}(\mathcal{H})$, then the kernel of T, ker T is the closed subspace $\{x \in \mathcal{H} : Tx = 0\}$. The range of T, ran T is the subspace $\{Tx : x \in \mathcal{H}\}$.

Note that, since T is continuous, $\ker T$ is closed.

Proposition 5. If $T \in \mathcal{L}(\mathcal{H})$, then ker $T = (\operatorname{ran} T^*)^\perp$ and ker $T^* = (\operatorname{ran} T)^\perp$

Proof. By Proposition 2 it suffices to show the first part. Note that $x \in \ker T$ implies $(\forall y)\langle Tx, y\rangle = 0 = \langle x, T^*y\rangle$. In the opposite direction the proof is similar. □

Definition 6. An operator T is *bounded below* if $\exists \varepsilon > 0$ s.t. $(\forall x \in \mathcal{H}), \|Tx\| > \varepsilon\|x\|$.

Proposition 7. If $T \in \mathcal{L}(\mathcal{H})$ then T is invertible *iff* T is bounded below and has dense range.

Proof. First, if T is invertible, then $\|f\| = \|T^{-1}Tf\| \leqslant \|T^{-1}\|\|Tf\| \leqslant \|T^{-1}\|\|Tf\|$ implying $\|Tf\| > \|T^{-1}\|^{-1}\|f\|$, so T bounded below. The range of T must be $\mathcal{H}$, otherwise T would not be invertible.

Conversely, first note that the bound below implies injectivity: indeed, if $\|T(x-y)\| > \varepsilon\|(x-y)\|$, then clearly $x - y \neq 0 \Rightarrow Tx \neq Ty$. We need to show that $\operatorname{ran} T = \mathcal{H}$. Let now $y \in \mathcal{H}$ and $y_n \to y \in \mathcal{H}$ and let's look at $T^{-1}y_n = x_n$. Since $\|y_n - y_m\| = \|T(x_n - x_m)\| > \varepsilon\|x_n - x_m\|$, x_n is Cauchy, converging to some $x \in \mathcal{H}$. Clearly, by continuity $Tx = y$. Recalling that $Tx_n = y_n$ it follows that $\operatorname{ran} T = \mathcal{H}$. □

Corollary 8. If $T \in \mathcal{L}(\mathcal{H})$ s.t. T and T^* are bounded below, then T is invertible.

Proof. We only need to show that $\operatorname{ran} T$ is dense. But the closure of a space S is $(S^\perp)^\perp$ while $(\operatorname{ran} T)^\perp = \ker T^* = \{0\}$ (since T^* is bounded below). Since $0^\perp = \mathcal{H}$ the proof is complete. □

Definition 9. *Let $T \in \mathcal{L}(\mathcal{H})$. Then,*

1. T is normal if $T^*T = TT^*$.
2. T is self-adjoint, or *Hermitian* if $T = T^*$.
3. T is positive if $(\forall f \in \mathcal{H}), \langle Tf, f\rangle \geqslant 0$.
4. U is unitary if $UU^* = U^*U = I$.

The set

$$W(T) := \{\langle Tu, u\rangle : u \in \mathcal{H},\ \|u\| = 1\} \tag{28}$$

is called the numerical range of T.

Proposition 10. T is self-adjoint *iff* $W(T) \subset \mathbb{R}$. In particular, this is the case if T is positive.

Proof. If T is self-adjoint, then

$$\langle Tu, u\rangle \overset{\text{s-adj.}}{=} \langle u, Tu\rangle;\ \langle Tu, u\rangle \overset{\text{def.}}{=} \overline{\langle u, Tu\rangle} \Rightarrow \langle u, Tu\rangle \in \mathbb{R}$$

Conversely, iu $\langle Tu, u\rangle \in \mathbb{R}$, then

$$\langle Tu, u\rangle = \overline{\langle Tu, u\rangle} = \langle u, Tu\rangle \Rightarrow \langle Tf, f\rangle = \langle f, Tf\rangle \forall f \in \mathcal{H}$$

and by the polarization identity we have $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in \mathcal{H}$. □

Exercise What is the adjoint of $\mathcal{P}$ defined by $\mathcal{P}(f)(x) = \int_0^x f$? Show that $\mathcal{P}$ is *not* a normal operator *without* calculating $\mathcal{P}\mathcal{P}*$ and $\mathcal{P}^*\mathcal{P}$.

Exercise What is the spectrum of $\mathcal{P}\mathcal{P}^*$ (where P is as in the previous exercise)?

Proposition 11. If $T \in \mathcal{L}(\mathcal{H})$, then T^*T and TT^* are positive.

Proof.

$$\langle TT^* f\rangle = \langle T^* f, T^* f\rangle = \|T^* f\|^2$$

□

Proposition 12. If T is self-adjoint on $\mathcal{H}$ then $\sigma(T) \subset \mathbb{R}$.

Proof. With Γ the Gelfand transform, $\sigma(T) = \operatorname{ran}\Gamma(T) = \operatorname{ran}\Gamma(T^*) = \operatorname{ran}\overline{\Gamma(T)} \subset \mathbb{R}$. □

Direct proof. Since $T = T^*$ we only have to show that for $\lambda \notin \mathbb{R}$, $T + \lambda$ and $T + \overline{\lambda}$ are bounded below. If $\lambda = a + ib$ we can take $Q_1 = T - a$, note that Q_1 is also self-adjoint and show that $Q + ib, Q - ib$ are bounded below. Finally, if $b \neq 0$ by taking $Q = Q_1/b$ (also self-adjoint) it is enough to show that $Q \pm i$ are bounded below.

Now

$$\|(Q \pm i)u\|^2 = \langle (Q \pm i)u, (Q \pm i)u\rangle = \langle Qu, Qu\rangle + \langle u, u\rangle \geqslant 1$$

□

Corollary 13. *If T is a positive operator then $\sigma(T) \subset [0, \infty)$.*

Proof. By Proposition 10, T is self-adjoint, and by Proposition 11 $\sigma(T) \subset \mathbb{R}$. Let $\lambda = -a, a \geqslant 0$. We want to show that $T + a$ is bounded below:

$$\|T+a\| = \langle (T+a)u, (T+a)u\rangle = \langle Tu, Tu\rangle + 2\langle Tu, u\rangle + a^2\langle u, u\rangle \geqslant a^2\langle u, u\rangle \quad (29)$$

and thus $T + a = (T + a)^*$ is invertible. □

4 Spectral theorem for bounded normal operators (continuous functional calculus)

Theorem 2. [1] If N is a *normal* operator in a Hilbert space $\mathcal{H}$, then the C^*-algebra $\mathfrak{C}_N$ generated by N is commutative. The maximal ideal space M of $\mathfrak{C}_N$ is homeomorphic to $\sigma(N)$ and the Gelfand transform Γ is a * isometric isomorphism of $\mathfrak{C}$ onto $C(\sigma(N))$.

Note 1. For normal operators, since M is homeomorphic to $\sigma(N)$, by abuse of notation write that $\operatorname{ran}\Gamma_N = C(\sigma(N))$.

Proof. Since N and N^* commute, the polynomials in N, N^* form a commutative self-adjoint subalgebra C_1 of $\mathcal{L}(\mathcal{H})$ thus contained in the C^*-algebra generated by N. The closure of C_1 is clearly a C^*-algebra containing N, thus it equals $\mathfrak{C}_N$. In particular, $\mathfrak{C}_N$ is commutative.

Now we define the homeomorphism between $\mathfrak{C}$ and $C(\sigma(N))$ as follows: an element of M can be identified with a multiplicative functional φ. We want to map φ to a unique point in $\sigma(N)$. The natural way is to take $\varphi(N)$. That is, $\psi(\varphi) = \Gamma(N)(\varphi) =: \Gamma_N\varphi$. Note that $\operatorname{ran}\Gamma_N = \sigma(N)$ and thus ψ is onto. We need to check injectivity.

Assume $\psi(\varphi_1) = \psi(\varphi_2)$. It means that

$$\Gamma_N\varphi_1 = \Gamma_N\varphi_2 \Leftrightarrow \varphi_1(N) = \varphi_2(N) \tag{30}$$

and also

$$\Gamma_{N^*}\varphi_1 = \overline{\Gamma_N\varphi_1} = \overline{\Gamma_N\varphi_2} = \Gamma_{N^*}\varphi_2 \Leftrightarrow \varphi_1(N^*) = \varphi_2(N^*) \tag{31}$$

Thus, by linearity and multiplicativity, $\psi\varphi_1$ and $\psi\varphi_2$ take the same value on all polynomials in N, N^* and by its continuity, on all of $\mathfrak{C}_N$.

It remains to show bicontinuity, which simply follows from the map being one-to-one between Hausdorff spaces and from continuity of ψ which is easily checked:

$$\lim_{\beta\in B}\psi(\varphi_\beta) = \lim_{\beta\in B}\Gamma_N\varphi_\beta = \lim_{\beta\in B}\varphi_\beta(N) = \varphi(N) \tag{32}$$

by the continuity of φ. □

5 Functional calculus

We see that, for a normal operator T, Γ is an isomorphism between $\mathfrak{C}_T$ and a space of continuous functions on a domain in $\sigma(T) \subset \mathbb{C}$. If F is in $C(\sigma(T))$ we can simply define $F(T) := \Gamma^{-1}(F)$. We have already indirectly used the spectral theorem in Proposition 12.

[1] As we will see from the proof, the assumption that we are dealing with the algebra generated by an operator, rather than the one generated by some abstract element in an abstract C^*-algebra is not needed, so this theorem is more general.

It is easy to check that any polynomial P in T, which can be defined directly in $\mathfrak{C}_T$ coincides with $\Gamma^{-1}(P)$. In fact polynomials can be defined in any $\mathfrak{C}_T$ regardless of whether T is normal or not. But in both cases, it is very desirable to extend the functional calculus beyond these types of functions, as we'll see in a moment. For normal operators this is because we need to define projections and other L^∞ functions while $C(M)$ is already closed in $\| \|_\infty$; we will therefore need to weaken the topology.

Before that we will use this calculus to derive some fundamental properties of normal operators.

Proposition 14. *(i) Assume T is normal in $\mathcal{L}(\mathcal{H})$. Then, T is positive iff $\sigma(T) \in [0, \infty)$. (ii) T is self-adjoint iff $\sigma(T) \subset \mathbb{R}$.*

Proof. (ii) was proved in Proposition 12. For (i) recall from Theorem 2 that Γ is an isomorphism. We also know (this is general) $\sigma(T) = \operatorname{ran}\Gamma_T$. Since $\sigma(T) \subset [0, \infty)$ it follows that $\Gamma_T \geqslant 0$. Then, there exist (many [(2)]) $F \in C(\sigma(T))$ s.t. $\Gamma(T) = |F|^2 = F\overline{F} \Rightarrow T = \Gamma^{-1}(F)(\Gamma^{-1}(F))^*$ which we already proved is positive. If T is positive the result follows from Proposition 10 and Corollary 13. $\square$

Since the condition $\sigma(f) \in [0, \infty)$ makes sense in an abstract C^*-algebra, it allows us to *define* a *positive* element of a C^*-algebra as a normal element with the property $\sigma(f) \in [0, \infty)$.

5.1 Projections

These are very important tools in developing the spectral theorem.

Definition 15. A self-adjoint operator P is a projection if $P^2 = P$.

Note that, in particular, projectors are positive.

Proposition 16. Projections are into a one-to-one correspondence with closed subspaces of $\mathcal{H}$: $P\mathcal{H} = M$ is always a closed subspace of $\mathcal{H}$, and if M is a closed subspace and we decompose $x = x_M + x_\perp$ where $x_M \in M$ and $x_\perp \perp M$, then $x \mapsto Px = x_M$ is a projection.

Proof. Note that $P_\perp = (1-P)$ is also a projector, since $1-P$ is self-adjoint, and $(1-P)^2 = 1 - 2P + P = 1 - P$. Evidently, we can write $x = Px + (1-P)x = x_1 + x_2$, and $x_1 \perp x_2$ since $\langle x_1, x_2 \rangle = \langle Px, (I-P)x \rangle = \langle x, (Px - Px) \rangle = 0$. Note also that $x \in \operatorname{ran} Px$ is equivalent to $x = Px$, or $x \in \ker(P - I)$ which is closed. Thus $\operatorname{ran} P$ is closed and so is $\operatorname{ran}(1-P)$. Thus, $\operatorname{ran} P$ and $\operatorname{ran}(1-P)$ are mutually orthogonal closed subspaces of $\mathcal{H}$.

If now M is a closed subspace of $\mathcal{H}$, then we can perform the orthogonal decomposition $x = x_M + x_\perp$. Note that $x \mapsto x_M$ is a linear operator P_M of norm less than one and $P_M^2 = P_M$. We have $\langle P_M f, f \rangle = \langle f_M, f_M + f_\perp \rangle = \|f_M\|^2 \geqslant 0$ and thus P is self-adjoint. $\square$

[(2)] e.g. $\sqrt{\Gamma(T)}$.

Proposition 17. Let $M_1, ..., M_n$ be closed subspaces of $\mathcal{H}$, and let $P_1, ..., P_n$ be the associated projection operators. Then $P_i P_j = 0$ iff $M_i \perp M_n$ and $P_1 + ... + P_n = I$ iff the span of $M_1, ...M_n = \mathcal{H}$.

Proof. Assume $P_i P_j = 0$ if $i \neq j$. $x \in \text{ran } P_j$ iff $x = P_j x$ and also, by definition, if $x \in M_j$. But then $Px \perp M_i$ and thus $P_i P_j x = 0$. If M_j span $\mathcal{H}$, then by definition $x = x_1 + ... + x_n$ where $x_k \in M_k$ and the x_k are mutually orthogonal. This also means, by the definition of P_j that $(\forall x), x = P_1 x + ... + P_n x$ and thus $P_1 + ... + P_n = I$. We leave the converse as an exercise. □

Note now that for a finite-dimensional operator A with eigenvalues $\lambda_1, ..., \lambda_n$, the polynomial $\prod_1^j (A - \lambda_i)$ is a projector onto the span of eigenvectors and generalized eigenvectors corresponding to $\lambda_{j+1}, ..., \lambda_n$. In an infinite-dimensional context, a corresponding infinite product may make no sense, but the inverse image through Γ of the characteristic function of $[a, b] \subset \sigma(A)$ should be a projector, since it is real valued and equal to its square. Note however that, unless $\sigma(A)$ is disconnected or the topology is otherwise cooperating, characteristic functions are not continuous and Γ^{-1} of a characteristic function is not defined. This is one reason we need to extend functional calculus.

5.2 Square roots

The proof of Proposition 14 suggests that we can define the square root of a positive operator, and a canonical one if we insist the square root be positive.

Proposition 18. *If P is a positive operator in $\mathcal{L}(\mathcal{H})$, then there exists a unique* positive *operator Q s.t. $Q^2 = P$. Moreover, Q commutes with any operator commuting with P.*

Proof. As in the proof of Proposition 14 the fact that, if f is a continuous nonnegative function so is $\sqrt{f}$ allows us to define $Q = \sqrt{P} = \Gamma^{-1}(\sqrt{\Gamma(P)})$. If Q_1 is any positive operator s.t. $Q_1^2 = P$, then Q_1 commutes with P as seen by writing $Q_1^3 = (Q_1^2)Q_1 = Q_1(Q_1)^2$. Furthermore, a unique positive square root of P exists in the commutative algebra generated by Q which contains (in fact equals) the C^*-algebra generated by P; this is seen by looking through the isomorphism Γ at an equation of the form $F^2 = G^2$ where F and G are positive: they must clearly coincide. The last part is obvious since the C^*-algebra's generated by Q and P coincide. □

Corollary 19. *Let $T \in \mathcal{L}(\mathcal{H})$. Then T is positive* ***iff*** *there is an $S \in \mathcal{L}(\mathcal{H})$ s.t. $T = S^*S$.*

Proof. If T is positive, we can simply take $S = \sqrt{T}$. If $T = S^*S$ this follows from Proposition 11. □

Lemma 20. Let $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{B}$ the Banach algebra generated by T. Then the space of maximal ideals of T is homeomorphic to $\sigma(T)$ and $\Gamma(\mathcal{B})$ is contained in the closure in the sup norm of polynomials on $\sigma(T)$.

Proof. Let φ be a multiplicative functional. Then $\lambda = \varphi(T) \in \operatorname{ran}\Gamma = \sigma(T)$. As in the proof of the Gelfand-Naimark theorem let $\psi(\varphi) = \Gamma_T(\varphi) = \varphi(T) = \lambda$. Once more, since $\operatorname{ran}\Gamma_T = \sigma(T)$, ψ $P(x) = \sum_0^n c_n x^n$ is a polynomial, then $\Gamma_{P(T)}(\varphi) = \varphi(P(T)) = P(\lambda)$. If $\psi(\varphi_1) = \psi(\varphi_2)$, then, by definition $\Gamma_T(\varphi)_1 = \Gamma_T(\varphi)_2$, thus $\varphi_1(T) = \varphi_2(T)$ and $\varphi_1(P(T)) = \varphi_2(P(T))$ for any polynomial. Since by definition $P(T)$ are dense in $\mathcal{B}$, by continuity, $\varphi_1 = \varphi_2$ on $\mathcal{B}$. Continuity of ψ is shown as in the Gelfand-Naimark theorem. Now, $\|\Gamma(X)\|_\infty \leqslant \|X\|_{\mathcal{B}}$ and thus Γ maps $\mathcal{B}$ onto a subspace of the closure of polynomials on $\sigma(T)$ in the sup norm. $\square$

5.3 Example: the right shift

Example 21. 1. The right shift S_+ on $l^2(\mathbb{N})$ defined by $(x_1, x_2, x_3, ...) \to (0, x_1, x_2, ...)$ is clearly an isometry but $(1, 0, ...)$ is orthogonal to $\operatorname{ran} S$, and thus S is not invertible (note that $S_- := (x_1, x_2, x_3, ...) \mapsto (x_2, x_3, ...)$ is a left inverse, $S_- S_+ = I$ but $S_+ S_-$ is the projection to the subspace orthogonal on $e_1, \mathcal{H}_1^\perp$.

2. It is easy to check that $S_+^* = S_-$ and, by the above, S_+ is thus *not* a normal operator: $S_+^* S_+ = I$ while $S_+ S_+^*$ is the projection on $\mathcal{H}_1^\perp$.

3. Note also that the Banach algebra (*not* the C^*-algebra!) $\mathcal{B}$ generated by S_+ is commutative. The spectrum of S_+ is contained in $\overline{\mathbb{D}}$ since $\|S_+\| = 1$. The spectrum of S_+ is $\mathbb{D}$ (indeed, solving $S_+ s - \lambda s = u$ is equivalent to solving, coefficient by coefficient, the equation $zF - \lambda F = U$, in a space of analytic functions of the form $F = \sum_{n=0}^\infty s_k z^k$ with $\sum_{n=0}^\infty |s_k|^2 < \infty$. This implies that F is analytic in $\mathbb{D}$. If $\lambda \in \mathbb{D}$ it is clear that, with say $u = (1, 0, 0...)$ there is no analytic solution in $\mathbb{D}$. Since the spectrum is closed, the spectrum is thus $\overline{\mathbb{D}}$.

4. By Lemma 20 $\mathcal{B}$ is contained in the closure of polynomials in the sup norm on $\overline{\mathbb{D}}$, and this clearly consists of functions analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

5. Associating to $s \in l^2(\mathbb{N})$ an analytic function in the open unit disk, $\sum_n s_n z^n$ we see that S_+ becomes multiplication by z.

6. The maximal ideal space is the closed unit disk, and the shift is perhaps better seen through the Gelfand transform: given f, analytic with absolutely convergent Taylor series the right shift corresponds to $f(z) \mapsto zf(z)$ and this has an inverse, $f \to f/z$ which is defined on a proper subspace of analytic functions. Recall (and it is obvious) that Γ is not onto from $\mathcal{B}$ to $C(M)$. The left shift is $[f(z) - f(0)]/z$, which does not commute with multiplication by z, as expected.

7. By Lemma 20 $\Gamma(\mathcal{B})$ is contained in the closure in the sup norm of polynomials on $\overline{\mathbb{D}}$ which are the analytic functions on $\overline{\mathbb{D}}$ continuous on $\overline{\mathbb{D}}$, a space much smaller than $C(M) = C(\mathbb{D})$.

6 Partial isometries

Definition 22. An operator V on $\mathcal{H}$ is a partial isometry if for any $f \perp \ker V$ we have $\|Vf\| = \|f\|$. If, in addition $\ker(V)= \{0\}$, then V is an isometry. The initial space of a partial isometry is $(\ker V)^{\perp}$.

Note that in $\mathbb{C}^n$ any isometry is a unitary operator. In infinite dimensional spaces this not true in general (look, e.g., at S_-). The existence of a partial isometry between two closed subspaces S_1 and S_2 only depends on whether the subspaces have the same dimension (in terms of the cardinality of the basis).

Proposition 23. *Assume S_1, S_2, subsets of $\mathcal{H}$, have the same dimension. Then there is a partial isometry $V : S_1$ with range S_2.*

Proof. Let $\{e_\alpha\}_{\alpha\in A}$ be a basis in S_1 and $\{f_\alpha\}_{\alpha\in A}$ be a basis in S_2. For $x \in \mathcal{H}$ decompose $x = x_\perp + \sum_{\alpha\in A} x_\alpha e_\alpha$ where $x_\perp$ is orthogonal to S_1. Define now

$$Vx := \sum_{\alpha\in A} x_\alpha f_\alpha \tag{33}$$

□

6.1 Characterization of partial isometries

Lemma 24. For any operator A, $\ker A = \ker A^*A$. In one direction it's clear: $Af = 0 \Rightarrow A^*A = 0$. Conversely, if $A^*A = 0$, then

$$\langle A^*Af, f\rangle = \langle Af, Af\rangle = \|Af\|^2 = 0 \Rightarrow Af = 0 \tag{34}$$

Lemma 25. *If P is a positive operator, then $\langle Pf, f\rangle = 0 \Rightarrow Pf = 0$.*(3)

Proof. We write

$$\langle Pf, f\rangle = \langle (P^{1/2})^2 f, f\rangle = \langle P^{1/2}f, P^{1/2}f\rangle = 0 \Rightarrow P^{1/2}f = 0 \Rightarrow Pf = 0$$

□

Proposition 26. *Let V be an operator on the Hilbert space $\mathcal{H}$. The following are equivalent:*

1. *V is a partial isometry.*
2. *V^* is a partial isometry.*
3. *VV^* is a projection.*
4. *V^*V is a projection*

(3) Separated from the main proofs at the suggestion of I. Glogic.

If moreover V is a partial isometry, then VV^* is the projection onto the range of V and V^*V is the projection onto the initial space of V (S_1 in Proposition 23 above.)

Proof. Assume V is a partial isometry. For any $f \in \mathcal{H}$ we have

$$\langle (1 - V^*V)f, f\rangle = \langle f, f\rangle - \langle V^*Vf, f\rangle = \langle f, f\rangle - \langle Vf, Vf\rangle \geqslant 0 \tag{35}$$

and thus $1 - V^*V$ is a positive operator. By definition, if $f \perp \ker V$, then $\|Vf\| = \|f\|$ which means $\langle (1-V^*V)f, f\rangle = 0$. By Lemma 25 we have $V^*Vf = f$. Conversely, assume that V^*V is a projection and $f \perp \ker(V^*V) = \ker V$ by Lemma 24; then $V^*Vf = f$, thus

$$\|Vf\|^2 = \langle Vf, Vf\rangle = \langle V^*Vf, f\rangle = \|f\|^2$$

and thus V preserves the norm on $\ker(V^*V)^\perp$ implying that V is a partial isometry and 4 and 1 are equivalent. Similarly, 2 and 3 are equivalent.

Now we show that 3 and 4 are equivalent. We first note that $V(V^*V) = V$. Indeed, if $f \in \ker V = \ker V^*V$, then of course $V(V^*V)f = 0 = Vf$. If $f \perp \ker V^*V$, then, since V^*V is a projection, $= VV^*Vf = Vf = V$.

Hence,

$$(VV^*)^2 = V(V^*V)V^* = VV^*$$

The rest is a straightforward verification. □

7 Polar decomposition of bounded operators

An analog of the polar representation $z = \rho e^{it}$ familiar from complex analysis exists for bounded operators as well.

Note 2.

1. Recall that (T^*T) and TT^* are positive and thus have square roots, obviously positive too. As a candidate for ρ, note first that, if T is a normal operator, one can define $|T|$ simply by $\Gamma^{-1}(|\Gamma_T|)$. In general, we have two obvious candidates, $|T|^2 = T^*T$ and $|T|^2 = TT^*$. The right choice is T^*T however, as it preserves the norm of the image:

$$\|(T^*T)^{1/2}f\|^2 = \langle (T^*T)^{1/2}f, (T^*T)^{1/2}f\rangle = \langle Tf, Tf\rangle = \|Tf\|^2 \tag{36}$$

(this is of course stronger than $\|T\| = \|T^*\| = \|(T^*T)^{1/2}\| = \|(TT^*)^{1/2}\|$).

2. We first note that we can't expect to have a representation of the form $T = |T|U$ with U unitary since, for instance (see Example 21) $S_+^*S_+ = I = (S_+^*S_+)^{1/2}$ and thus the equality $S_+ = (S_+^*S_+)^{1/2}U$ as well as $S_+ = U(S_+^*S_+)^{1/2}U$ are impossible. However S_+ itself is an isometry (with nontrivial kernel).

Definition 27. *We naturally set $|T| := (T^*T)^{1/2}$.*

Note 3. $|T|$ does *not* behave like the absolute value of a functions. It is *false* that $|A| = |A^*|$ or $|A||B| = |AB|$, or even that $|A| + |B| \leqslant |A| + |B|$.

Theorem 3. Let $T \in \mathcal{L}(\mathcal{H})$. Then there exist partial isometries s.t.

1. $T = V|T|$;
2. $T = |T|W$.

Moreover a polar decomposition of the form $T = V_1 P$ is the one above if $\ker P = \ker V_1$; same with the decomposition $T = QV_2$ if $\operatorname{ran} W = (\ker Q)^\perp$.

Proof. We have already shown that $\|Tf\| = \||Tf|\|$. It is natural to define V on $\operatorname{ran}|T|$ by $V|T|f = Tf$, as this is what we want to achieve. If we extend V by zero on $(\operatorname{ran}|T|)^\perp = (\operatorname{ran} T)^\perp$, it is easy to see that V is a partial isometry.

We have, by definition $\ker V = (\operatorname{ran}|T|)^\perp = \ker|T|$.

For uniqueness, assume we have a decomposition $T = V_1 P$ as above. Since V_1, V_1^* are partial isometries and $\ker V_1 = \ker P = (\operatorname{ran} P)^\perp$, we have $V_1^* V_1 P = P$. It follows that

$$|T|^2 = T^*T = PV_1^*V_1P = P^2 \tag{37}$$

Uniqueness of the square root implies $P = |T|$ which in turn implies $V_1 = V$ on $\operatorname{ran}|T|$. But $(\operatorname{ran}|T|^\perp) = \ker|T| = \ker P = \ker V_1$, and thus $V_1 = V$ on $(\operatorname{ran}|T|^\perp)$ as well.

Uniqueness of 2 follows from 1 in a straightforward way, going through the same steps but with T^* instead of T, and is left as an exercise. ☐

Another general fact will be useful in the sequel.

Proposition 28. *If $A \in \mathcal{L}(\mathcal{H})$ is bounded below by a and is invertible, then $\|A^{-1}\| \leqslant a^{-1}$.*

Proof.

$$\|A^{-1}\| = \sup_x \frac{\|A^{-1}x\|}{\|x\|} = \sup_y \frac{\|y\|}{\|Ay\|} \leqslant a^{-1}$$

☐

8 Compact operators

We mention a useful general formula, whose proof is a simple verification. It is the non-commutative extension of $a^{-1} - b^{-1} = (b - a)a^{-1}b^{-1}$.

Proposition 29 (Second resolvent formula)**.** *Assume A and B are invertible operators. Then*

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} \tag{38}$$

Corollary 30. *If B is invertible and $\|(B - A)B^{-1}\| = \|I - AB^{-1}\| < 1$, then A is invertible and*

$$A^{-1} = B^{-1}(1 - (B - A)B^{-1})^{-1} \tag{39}$$

Proof. A calculation starting with (38) gives

$$A^{-1}(1-(B-A)B^{-1}) = B^{-1} \Rightarrow A^{-1} = B^{-1}(1-(B-A)B^{-1})^{-1} \tag{40}$$

This of course is a formal calculation, but is easily transformed into a rigorous proof inverting both sides of (39):

$$A = (1-(B-A)B^{-1})B \tag{41}$$

That this equality holds is an immediate calculation, and now invertibility of A and formula (40) are obvious. $\square$

Operators of the form

$$(Kf)(x) = \int_0^1 K(x,y)f(y)dy \tag{42}$$

are seen throughout analysis, for instance as solutions of differential equations. Assuming the original problem is not singular, often K is at least continuous for $(x,y) \in [0,1]^2$. We will look first at this case.

The operator K in (43) with $F \in C([0,1]^2$ is clearly bounded in $L^\infty[0,1]$ by $M = \sup_{[0,1]^2} K(x,y)$, and continuous. But Fredholm noticed that there is something more going on, and of substantial importance. Since K is continuous on $[0,1]^2$, it is uniformly continuous: for any ϵ there is a δ s.t. $|x-x'|+|y-y'| < \delta$ implies $|K(x,y) - K(x',y'| < \epsilon$ in $[0,1]^2$. This implies

$$|(Kf)(x) - (Kf)(x')| = \left| \int_0^1 [K(x,y) - K(x',y)]f(y)dy \right| \leqslant M\epsilon \tag{43}$$

Let B_R be the ball of radius R in $C[0,1]$ (43) implies that $K(B_R)$ is a set of equibounded, equicontinuous functions, and thus, by Ascoli-arzelà, precompact. That is, K maps a sequence of equibounded functions into a sequence having convergent subsequences. Fredholm showed that such operators have especially good features, including what we now know as the Fredholm alternative property. According to [5] p. 215, Fredholm's work "produced considerable interest among Hilbert and his school, and led to the abstraction of many notions we now associate with Hilbert space theory".

Definition 31. Let X, Y be Banach spaces. An operator $K \in \mathcal{L}(X,Y)$ is called compact if K maps bounded sets in X into precompact sets in Y.

Equivalently, K is compact if for any sequence $(x_n)_{n\in\mathbb{N}}$ the sequence $(Kx_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Integral operators of the type discussed at the beginning of this section are therefore compact.

Another important class of examples is provided by operators of *finite-rank* (called *degenerate* in [4]). These are by definition operators in $\mathcal{L}(X,Y)$ *with finite dimensional range.*

Proposition 32. Operators of finite rank are compact.

Proof. Writing the elements of KX in terms of a basis, $Kx = \sum_{k=1}^{n} x_n e_n$, $x_n \in \mathbb{C}$ and e_n linearly independent in KX, this is an immediate consequence of the fact that any bounded sequence in $\mathbb{C}^n$ has a convergent subsequence. $\square$

An important criterion (that was introduced as a *definition* of compact operators by Hilbert) is the following.

Theorem 4. A compact operator maps weakly convergent *sequences* into norm convergent ones.

Proof. Assume $x_n \stackrel{w}{\to} x$. The uniform boundedness theorem implies that $\|x_n\|$ are bounded. Denote by ℓ the elements of X^* and $y_n = Kx_n$. The sequence y_n is also weakly convergent to $y = Kx$ since by the definition of the adjoint ("transposed" in [4]) operator K' and its boundedness we have, with

$$|\ell y_n - \ell y| = |(K'\ell)(x_n - x)| \tag{44}$$

is convergent since $(K'\ell)$ is a continuous functional. Suppose to get a contradiction that y_n did not converge *in norm* to Kx. Then there would exist an $\varepsilon > 0$ sequence $y_{n_k} = Kx_{n_k}$ s.t. $\|y_{n_k} - y\| > \varepsilon$ for all $k \in \mathbb{N}$. Since K is compact, a subsequence of the y_{n_k} would converge in norm to some $\tilde{y} \neq y$. But this would imply that the subsequence also converges weakly to $\tilde{y}$, contradiction. $\square$

Proposition 33. *If X is reflexive, then the converse of Theorem 4 holds, and if B is the unit ball in X, then KB is compact.*

For reflexive spaces therefore, Hilbert's definition coincides with Definition 31.

Proof. Because X is reflexive, the unit ball in X is weakly compact (cf. [7] p.251) (because it is homeomorphic to the unit ball in the dual of X^*). We want to show KB is compact in norm. Let $\{x_n\}_n \in B$. Thus we can extract a weakly convergent subsequence x_{n_k}. But then, by assumption, Tx_{n_k} is norm-convergent. $\square$

Theorem 5. Let $K \in \mathcal{L}(X, Y)$, where X, Y are Banach spaces. Then

1. The norm limit of a sequence of compact operators is compact.
2. K is compact iff K' is compact.
3. If S_1, S_2 are bounded operators, and K is compact, then SK and KS are compact (that is, compact operators are an ideal in the space of bounded operators).

Note 4. In particular the norm limit of finite rank operators is compact.

Proof. 1. Let $(K_n)_{n\in\mathbb{N}}$ be the compact operator sequence and let $\{x_j\}_{j\in\mathbb{N}}$ be bounded (say, by one) and for each n extract a convergent subsequence x_{j_n} and by the diagonal trick we can extract a sub-subsequence $x_{j_{n_k}}$ s.t. $K_n x_{j_{n_k}} \to y_n$ for any n. We have $\|x_n\| \leqslant 1$ and $\sup_n K_n = M < \infty$ since $(K_n)_{n\in\mathbb{N}}$ is norm-convergent. Then an $\varepsilon/3$ argument shows that y_n converges. Then, in a similar way it follows that $K x_{j_{n_k}} \to y$.

2. Assume K is compact. Let $(\ell_n)_{n\in\mathbb{N}} \in X^*$ be bounded by 1. Then, ℓ_n are clearly equibounded and equicontinuous, as defined on the compact set KB in X. Then, by Ascoli-Arzelá there is a subsequence ℓ_{n_k} which converges uniformly on KB. Then

$$\sup_{u\in B}\|K'\ell_{n_k}(u) - K'\ell_{n_{k'}}(u)\| \stackrel{\text{def}}{=} \sup_{u\in B}|\ell_{n_k}(Ku) - \ell_{n_{k'}}(Ku)| = \sup_{y\in KB}|\ell_{n_k}(y) - \ell_{n_{k'}}(y)| \to 0 \quad (45)$$

This shows that $K'\ell_{n_k}$ is norm Cauchy, thus norm convergent.

3. This is straightforward.

□

From now on we focus on Hilbert spaces, a more frequent setting. In a Hilbert space we have a stronger connection between compact operators and finite rank ones than in Note 4.

Theorem 6. Let $\mathcal{H}$ be a separable Hilbert space. Then $K : \mathcal{H} \to \mathcal{H}$ is compact **iff** K is the norm limit of finite-rank operators.

Proof. In one direction, we proved this already. In the opposite direction, let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$. We show that K is the limit of F_{E_n} where E_n is the space generated by $e_1, ..., e_n$, $F = K$ on E_n and zero outside. If we let

$$\lambda_n = \sup\{\|Ku\| : \|u\| = 1 \text{ and } u \in E_n^{\perp}\} \quad (46)$$

clearly, λ_n is decreasing and positive, thus $\lambda_n \to \lambda$. If we show $\lambda = 0$, then the result is proved. Let $u_n \perp e_j \ \forall j \leqslant n$ be s.t. $\|Ku_n\| \geqslant \lambda/2$. The sequence $(u_n)_n$ converges weakly to zero. But then Ku_n converges to zero in norm, and thus $\lambda = 0$. □

We will then study finite rank operator in more detail.

8.1 Finite rank operators

Let F be a finite rank operator: the range of $FX = R$ is isomorphic to $\mathbb{C}^n$ for some n. We choose a basis $\{e_1, ..., e_n\}$ in R. We have $\langle e_i, Fx\rangle = \langle f_i, x\rangle$ where $f_i = F^* e_i$. By construction, none of the f_i can be zero.

Then for all $x \in \mathcal{H}$ we have

$$Fx = \sum_{i=1}^{n} \langle Fx, e_i \rangle e_i = \sum_{i=1}^{n} \langle x, f_i \rangle e_i \tag{47}$$

We can now apply Gram-Schmidt and make the f_i orthogonal, so we can assume that they are orthogonal to start with. We can write $f_i = \alpha_i u_i$ where $u_i = f_i / \|f_i\|$ form an orthonormal set, and we finally have the normal form of a finite rank operator as

$$Fx = \sum_{i=1}^{n} \alpha_i \langle x, u_i \rangle e_i \tag{48}$$

We will call the space U generated by $u_1, ..., u_n$ the *initial space* of F.

8.1.1 More about finite-rank operators

1. From (48) we see that F is zero on $U^{\perp}$. Let F_U be the restriction of F to U.

2. The image $E = F_U U$ has the same dimension as U, n.

3. Let $E' = U \oplus E$ and let $F_{E'} = F_{|_{E'}}$. Then clearly $F(E') \subset E'$ and thus in the decomposition $\mathcal{H} = E' \oplus E'^{\perp}$ we have

$$F = F_{|_{E'}} \oplus 0 \quad \dim E' \leqslant 2n \tag{49}$$

   In this sense, finite rank operators are square matrices on a finite dimensional subspace of $\mathcal{H}$ and zero outside it.

4. If F is finite-rank then F^* is of finite rank since $\operatorname{ran} F^* = (\ker F)^{\perp} = (U^{\perp})^{\perp} = U$.

5. From (48) we see that zero is always in the spectrum of F if $\dim \mathcal{H} > n$.

6. Finite rank operators form an ideal in $\mathcal{L}(\mathcal{H})$. Indeed, if F is of finite rank and $S \in \mathcal{L}(\mathcal{H})$, then clearly $\operatorname{ran}(FS) \subseteq \operatorname{ran} F$. In the opposite direction the proof is essentially the same: $(SF)^* = F^* S^*$ is of finite rank by the first part.

7. In analyzing the spectrum of finite rank operators, we note that (49) implies that

$$\lambda I - F = (\lambda I_{E'} - F_{|_{E'}}) \oplus \lambda I_{E'^{\perp}} \quad \dim E' \leqslant 2n \tag{50}$$

   where I_E is the identity on the closed subspace $E \subset \mathcal{H}$.

8.1.2 The Fredholm alternative, nonanalytic case

Theorem 7 (The Fredholm alternative)**.** (i) If K is a compact operator then $(I-K)^{-1}$ exists and is bounded **iff** $x = Kx$ has no nonzero solution, **iff** $(I-K)$ is bounded below.

(ii) If $\lambda \in \sigma(K)$ then either $\lambda = 0$ or it is an eigenvalue of K of finite multiplicity. The only possible accumulation point of $\sigma(K)$ is 0.

Note 5. We could derive this as a corollary of the analytic Fredholm alternative developed in the next section, which relies in fact on similar arguments, but for clarity of the presentation we prefer to prove Theorem 7 first.

Proof. We want to link invertibility of K to the invertibility of a finite rank operator. In general, an *additive* approximation, $K = F + \varepsilon, \|\varepsilon\|$ small should *not work* easily since, generally, the kernel of $I - K$ is bound to differ from the kernel of $I - F$ and the vectors where $I - K$ is not invertible will always depend on ε.

We look for a representation of the form $(I-g)(I+\varepsilon_1)$ with g of finite rank and ε_1 small. This is obtained from $K = F + \varepsilon$ for instance by factoring out to the right $I - \varepsilon$: [4] (factoring out $I - F$ would not have the same drawback as using additive approximations, it would change the kernel)

$$(I - K) = (I - F - \varepsilon) = (I - F(I - \varepsilon)^{-1})(I - \varepsilon) \tag{51}$$

With obvious notations, we get from (50),

$$I - K = \Big[(I_{E'} - g_{E'}) \oplus I_{E'^{\perp}}\Big](I - \varepsilon) \tag{52}$$

where $\varepsilon = F - K$ is an operator of small norm. This can also be obtained from the second resolvent formula (41), taking $A = I - K$ and $B = I - F$. Note that in (52) $(I_{E'} - g_{E'})$ is simply linear operator on the finite-dimensional space E'.

Now assume $I - K$ is bounded below. Then, $(I_{E'} - g_{E'})$, a finite dimensional operator, is bounded below, and thus invertible, implying by (52) is invertible.

In the opposite direction, if $I - F$ is not bounded below, then (52) implies that $I_{E'} - g_{E'}$ is not bounded below, thus $U = \ker(I_{E'} - g_{E'}) \neq \{0\}$. If $\{u_i\}, i = 1, ..., m$ generate U, then the kernel of $I - K$ is nonempty,

$$\ker(I - K) = U \oplus 0_{E'^{\perp}} \tag{53}$$

(ii) If $\lambda \neq 0$, then the equation $(\lambda - K)u = v$ is equivalent to $1 - K' = \lambda^{-1}v$ where $K' = \lambda^{-1}K$ is compact. The fact that only zero is a possible accumulation point follows from Corollary 36 below. The rest follows from (i). □

Corollary 34. Assume K is compact and $\ker(I - K) = \{0\}$. Then, $\operatorname{ran}(I - K) = \mathcal{H}$.

Proof. Indeed, if $\operatorname{ran}(I - K) \neq \mathcal{H}$, then $I - K$ is not invertible, and the result follows from Theorem 7. □

[4] Simplified calculation suggested by C. Xu

Corollary 35. *If K is compact, then $I - K$ has closed range.*

Proof. We have $\text{ran}\,(I - \varepsilon) = \mathcal{H}$ since it is a continuous bijection, while from (52)

$$\text{ran}\,(I - g) = \text{ran}\left[(I_{E'} - g_{E'}) \oplus I_{E'^{\perp}}\right]$$

clearly closed.

□

9 The analytic Fredholm alternative

Setting. Let $\mathcal{D}$ be a domain in $\mathbb{C}$, that is an open connected set. Let $K : \mathcal{D} \to \mathcal{L}(\mathcal{H})$ be an analytic operator-valued function s.t. $K(z)$ is compact for all $z \in \mathcal{D}$. Then the following alternative holds:

Theorem 8. Either

(i) $(I - K(z))^{-1}$ exists for no $z \in \mathcal{D}$, or

(ii) $(I - K(z))^{-1}$ exists in $\mathcal{D} \setminus S$ where the set S is discrete (no accumulation points in the *open* domain).

In case (ii), $(I - K(z))^{-1}$ is a meromorphic function of z whose residues at the poles z_s $(\lim_{z \to z_s} (z - z_s)^{k_s} (I - K(z))^{-1},\ k_s =$ order of the pole) are finite rank operators and $z \in S \Leftrightarrow K(z)u = u$ for some unit vector u.

Proof. We first prove the alternative in some neighborhood of $\mathcal{N}(z_0)$; from it, one can extended to the whole of $\mathcal{D}$ by a standard connectedness argument. Assume that $I - K$ is invertible at some z_0. Let $\mathcal{N}(z_0)$ be small enough s.t. $\|K(z) - K(z_0)\| \leqslant \varepsilon \forall z \in \mathcal{N}(z_0)$, choose F s.t. $\|F - K(z_0)\| < \varepsilon$. Then $\|K(z) - F\| \leqslant 2\varepsilon$. If $\varepsilon < 1/2$ we write the decomposition (52):

$$I - K(z) = \left[(I_{E'} - g_{E'}(z)) \oplus I_{E'^{\perp}}\right](I + \hat{\varepsilon(z)}) \tag{54}$$

where $\hat{\varepsilon(z)}$ is an operator of small norm if ε is small in $(0, 1/2)$. Since $I + \hat{\varepsilon}$ is analytic in $\mathcal{N}(z_0)$, so is g. We then see that

$$[I - K(z)]^{-1} = (I + \hat{\varepsilon(z)})^{-1}\left[(I_{E'} - g_{E'}(z))^{-1} \oplus I_{E'^{\perp}}\right] \tag{55}$$

Noting that $(I_{E'} - g_{E'}(z))^{-1}$ is meromorphic by the standard inversion formula, the rest is straightforward. □

Note 6. Assuming that $\det(I_{E'} - g_{E'}(z))$ has a zero of order k at $z = z_s$, then the finite rank operator at z_s is

$$F = (I + \hat{\varepsilon(z)})^{-1}[\lim_{z \to z_s} (z - z_s)^k (I_{E'} - g_{E'}(z))^{-1} \oplus 0] \tag{56}$$

Corollary 36. The only possible accumulation point in the spectrum of K is zero.

Proof. Indeed, other accumulation points would be essential singularities of $(K/z - I)^{-1}$ which is meromorphic. □

Theorem 9 (The Hilbert-Schmidt theorem)**.**

1. Let K be compact and self-adjoint on $\mathcal{H}$ (assumed separable and, to avoid trivialities, infinite dimensional). Then there is an orthonormal basis $\mathcal{E} = \{e_i\}_{i\in\mathbb{N}}$ for $\mathcal{H}$ s.t. $Ke_i = \lambda_i e_i$ and $\lambda_n \to 0$ as $n \to \infty$. In the basis $\mathcal{E}$, K is therefore diagonal. Each eigenvalue has finite degeneracy.

2. $\mathcal{H}$ can be written as
$$\mathcal{H} = \mathcal{H}_0 \bigoplus_k \mathcal{E}_k \tag{57}$$
where $K = 0$ on $\mathcal{H}_0$ (we could have $\mathcal{H}_0 = \{0\}$, and then $\mathcal{H}_0$ can be omitted from the decomposition) and $\mathcal{E}_k$ are finite dimensional spaces. On each $\mathcal{E}_k$, $\mathcal{H} = \lambda_k I_{E_k}$.

Proof. The spectrum of K cannot be empty, as we know. We also know that the spectrum is discrete and real, and it is one of eigenvalues e_i, except perhaps for 0.

1. (*) Assume first that $\sigma(K) = \{0\}$. Then, by Theorem 7 and the fact that K is self-adjoint, $\|K\| = r(K) = 0$. This case is trivial.

(**) We will now look at nonzero eigenvalues. By the usual trick
$$\langle e_i, Ke_k\rangle = \lambda_k\langle e_i, Ke_k\rangle = \langle Ke_i, e_k\rangle = \lambda_i\langle e_i, Ke_k\rangle \tag{58}$$
the eigenvectors corresponding to distinct eigenvalues are orthogonal to each-other. Let $\mathcal{H}_1$ be the space generated by all e_i. We claim that $\mathcal{H}_2 = (\mathcal{H}_1)^\perp$ is invariant under K. Indeed, if $f \perp e_j$, then
$$\langle Kf, e_j\rangle = \langle f, Ke_j\rangle = \lambda_j\langle f, e_j\rangle = 0 \tag{59}$$
Now let K_2 be the restriction of K to the Hilbert space $\mathcal{H}_2$. Clearly K_2 is also self-adjoint. K_2 cannot have any nonzero eigenvalue in $\mathcal{H}_2$ by the construction of $\mathcal{H}_1$, and (*) implies $K_2 = 0$ on $\mathcal{H}_2$. The rest of the proof is straightforward. □

9.1 Singular value decomposition

A decomposition similar to (48) holds for general compact operators. We could, in principle, pass it to the limit, but there is a shortcut.

Note first that $Kx = 0 \Leftrightarrow K^*Kx = 0$. That $Kx = 0 \Rightarrow K^*Kx = 0$ is obvious. In the opposite direction, it follows from the polar decomposition $K = U|K|$.

If K is compact, so is K^*K, and it is also self-adjoint. Let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of eigenvectors of K^*K corresponding to the eigenvalues μ_n. Since $K^*K \geqslant 0$, $\mu_n \geqslant 0$. We choose only the u_n' corresponding to $\mu_n > 0$, and let E be the space generated by the u_n'.There must exists at least one nonzero eigenvalue (otherwise $K^*K = 0 \Rightarrow K = 0$, in which case the result is obvious), so this space is nontrivial. Clearly $K^*K = 0$ on $E^\perp$, and then $K = 0$ on $E^\perp$ as well. Since

$$\langle Ku_n, Ku_m\rangle = \langle K^*Ku_n, u_m\rangle = \delta_{nm}\mu_n \tag{60}$$

(in particular $Ku_n \neq 0$) the Ku_n also form an orthogonal set, which can be made orthonormal by taking $v_n = \lambda_n Ku_n$, $\lambda_n = |\mu_n^{-1/2}|$, as seen from (60). If $\psi \in \mathcal{H}$, then we have

$$K\psi = K = K\sum_{u_n\in E}\langle\psi, u_n\rangle u_n = \sum_{u_n\in E}\lambda_n\langle\psi, u_n\rangle v_n \tag{61}$$

The λ_n are called **singular values of** K. The result is then,

Theorem 10 (Normal form of compact operators)**.** Let K be compact. Then there exist an orthonormal sets (not necessarily spanning $\mathcal{H}$) u_n, v_n s.t., with E the span of the u_n, (61) holds for all $\psi \in \mathcal{H}$, where the λ_n are the eigenvalues of $|K|$.

Exercise

(i) If K is compact, is $|K|$ necessarily compact?
(ii) If K^2 is compact, is K necessarily compact?

9.1.1 Trace-class and Hilbert-Schmidt operators

We mention another important class of operators, and give some results about it without proof (for proofs and other interesting results, see e.g. [5] §VI.6; the proofs are not difficult, but would take some time).

Let $e_1, e_2, ..., e_n, ...$ be a basis in $\mathcal{H}$. Let T be a positive operator and define the trace by $\operatorname{tr} T = \sum_{n\in\mathbb{N}}\langle e_n, Te_n\rangle$. A **trace-class operator** is one for which $\operatorname{tr}|T| < \infty$. A **Hilbert-Schmidt** operator is one for which $\operatorname{tr} T^*T < \infty$.

Theorem 11. Every trace-class, or Hilbert-Schmidt operator is compact. A compact K operator is trace-class iff $\sum\lambda_n < \infty$ where λ_n are the singular values of K and Hilbert-Schmidt if $\sum\lambda_n^2 < \infty$.

Theorem 12. Let (M, μ) be a measure space and $\mathcal{H} = L^2(M, d\mu)$. Then $T \in \mathcal{L}(\mathcal{H})$ is Hilbert-Schmidt iff there exists a $k \in L^2(M \times M, d\mu \otimes d\mu)$ s.t. for all $f \in \mathcal{H}$

$$(Tf)(x) = \int k(x, y)f(y)d\mu(y) \tag{62}$$

Moreover

$$\|T\|^2 = \int |k(x, y)|^2 d\mu(x)d\mu(y)$$

Note 7. *The conditions of the theorem above give a criterion of compactness of* T.

9.2 Application (O.C.)

Consider the operator K given by $(Kf)(x) = \int_0^x f$. Its adjoint in $L^2(0,1)$, K^*, is given by $(K^*f)(x) = -\int_1^x f$. Indeed,

$$\int_0^1 f(s)\left(\int_1^s (g(t)dt\right) ds = \int_0^1 \left[\left(\int_0^s f(u)du\right)' \int_1^s g(t)dt\right] ds$$
$$= -\int_0^1 g(s)\left(\int_0^s f(u)du\right) ds \quad (63)$$

Check that K, and therefore K^* are compact. Now, the operator

$$A = \frac{1}{2}i(K^* - K) \quad (64)$$

is then clearly self-adjoint. The spectrum is therefore discrete, and consists of finite degeneracy eigenvalues accumulating at zero, and the eigenvectors are complete in $L^2(0,1)$.

The spectral problem for A is

$$i\int_0^x h(s)ds + i\int_1^x h(s)ds = 2\lambda h \quad (65)$$

A solution of this equation is clearly C^∞ (by a smoothness bootstrapping argument, since h is given in terms of the integral of h). Thus we can differentiate (65) and obtain the equivalent problem

$$ih = \lambda h' = 0 \Leftrightarrow h' = -i\lambda^{-1}h = 0 \;\text{ with } h(0) = -h(1) = -\int_0^1 h \quad (66)$$

It is clear that the set of solutions of norm 1 consists exactly of

$$e^{(2n+1)ix}, \quad n \in \mathbb{N} \cup \{0\} \quad (67)$$

Therefore, the set $\{e^{(2n+1)ix}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(0,1)$.

Exercise Show that $\{e^{2n\pi ix}\}_{n\in\mathbb{Z}}$ is also an orthonormal basis in $L^2(0,1)$.

9.2.1 Almost orthogonal bases [4]

A small perturbation of an orthonormal basis is still a basis. This result is due to Paley and Wiener.

Theorem 13 (Paley-Wiener)**.** *Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis. Assume v_n are vectors s.t.*

$$\sum_{n=1}^{\infty} \|e_n - v_n\|^2 < M < \infty \quad (68)$$

and that v_n are linearly independent. Then v_n also form a basis in $\mathcal{H}$.

Proof. (Due to Birkhoff-Rota and Sz. Nagy) We take $x \in \mathcal{H}$ and write

$$x = \sum_{k=1}^{\infty} x_k e_k \tag{69}$$

Let now

$$B = G_N + R_N;\ G_N := \sum_{k=1}^{N} \langle \cdot, e_k \rangle v_k;\ R_N := \sum_{k=N+1}^{\infty} \langle \cdot, e_k \rangle v_k; \tag{70}$$

Let E_N be the span of $e_1, ...e_N$ and Then $B - I = G_N - I_{E_N} + R_N - I_{E_N^\perp}$. If we show that $\|R_N - I_{E_N^\perp}\| \to 0$ it follows that $B - I$ is a limit of finite rank operators, thus compact. We have

$$\|R_N x - x\|^2 \leqslant \left(\sum_{n=N+1}^{\infty} |x_k| \|e_k - v_k\| \right)^2 \leqslant \sum_{n=N+1}^{\infty} |x_k|^2 \sum_{n=N+1}^{\infty} \|e_n - v_n\|^2 \to 0 \tag{71}$$

Then $\operatorname{ran} B = \operatorname{ran}(I - (I - B)) = \mathcal{H}$ unless $\ker B \neq \{0\}$. But if $a \in \ker B$, then there is a nonzero sequence (a_n) s.t. $\sum a_k v_k = 0$, contradiction.

□

10 Closed operators, examples of unbounded operators, spectrum

1. We let X, Y be Banach space. Y' is a subset of Y.

2. We recall that bounded operators are closed. Note that T is closed iff $T + \lambda I$ is closed for some/any λ.

3.

   Proposition 37. *If $T : \mathcal{D}(T) \subset X \to Y' \subset Y$ is closed and injective, then T^{-1} is also closed.*

   Proof. Indeed, the graph of T and T^{-1} are the same, modulo switching the order. Directly: let $y_n \to y$ and $T^{-1} y_n = x_n \to x$. This means that $T x_n \to y$ and $x_n \to x$, and thus $Tx = y$ which implies $y = T^{-1} x$. □

4.

   Proposition 38. *If T is closed and $T : \mathcal{D}(T) \to X$ is bijective, then T^{-1} is bounded.*

   Proof. We see that T^{-1} is defined everywhere and it is closed, thus bounded. □

Corollary 39. If T is closed, then $\sigma(T) = \{z : (T-z) \text{ is not bijective}\}$. That is, the possibility $(T-z)^{-1} : Y \to \mathrm{dom}\,(T-z)$ is unbounded is ruled out *if T is closed.*

(Recall that $(T-z)^{-1}$ must exist on the whole of Y for $T-z$ to be invertible.)

As an example of an operator with unbounded left inverse we have $(\mathcal{P}f)(x) := \int_0^x f$. The left inverse is $f \mapsto f'$ is unbounded from $\mathrm{ran}\,\mathcal{P} \to L^2$; however, $\mathrm{ran}\,\mathcal{P} = AC \cap L^2 \subsetneq L^2$ (it is only a dense subspace of L^2).

Exercise

5. Show that if T is not closed but bijective between $\mathrm{dom}\,T$ and Y, there exist sequences $x_n \to x \neq 0$ such that $Tx_n \to 0$. (One of the "pathologies" of non-closed, and more generally, non-closable operators.)

6. Recall the definition of the spectrum: 5 on p. 6.

Spectrum

The spectrum of an operator plays a major role in characterizing it and working with it. In "good" cases, there exist unitary transformations that essentially transform an operator to a multiplication operator on the spectrum, an infinite dimensional analog of diagonalization of matrices.

For closed operators, there are thus two possibilities: (a) $T : \mathcal{D}(T) \to Y$ is not injective. That means that $(T-z)x = (T-z)y$ for some $x \neq y$, which is equivalent to $(T-\lambda)u = 0$, $u = x - y \neq 0$, or $\ker(T-\lambda) \neq \{0\}$ or, which is the same $Tu = \lambda u$ for some $u \neq 0$. This u is said to belong to the *point spectrum* of T. (b) $\mathrm{ran}\,(T-\lambda) \neq Y$. There are two cases here: (1) $\mathrm{ran}\,(T-\lambda)$ *is not dense.* This subcase is called the *residual spectrum.* Such would be the case of a finite rank operator in an infinite dimensional Hilbert space, and (ii) The range is dense but not equal to the whole target space. Assuming that such a λ is not an eigenvalue, we say that this part of the spectrum is the continuous spectrum.

Exercise

7. Operators which are not closable are ill-behaved in many ways. Show that the spectrum of such an operator must be the whole of $\mathbb{C}$.

Examples

1. Consider the operator X defined by $(Xf)(x) = xf(x)$ on $L^2([0,1])$. We noted already that $\sigma(X) = [0,1]$. You can show that there is no point spectrum or residual spectrum for this operator.

2. Consider now the operator X on $\{f \in L^2(\mathbb{R}) \mid xf \in L^2(\mathbb{R})\}$. Show that $\sigma(X) = \mathbb{R}$.

Exercises

3. Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces, and let $U : \mathcal{H} \to \mathcal{H}'$ be unitary. Let $T\mathcal{H} \to \mathcal{H}$ and consider its image UTU^*. Show that T and UTU^* have the same spectrum.

4. Show that $-i\partial$ densely defined on the functions in $L^2(\mathbb{R})$ so that f' exists and is in $L^2(\mathbb{R})$, and that it has as spectrum $\mathbb{R}$. For this, it is useful to

use item 3 above and the fact that $\mathcal{F}$, the Fourier transform is unitary between $L^2(\mathbb{R})$ and $L^2(\mathbb{R})$.

5. The spectrum depends very much on the domain of definition. In general, the larger the domain is, the larger the spectrum is. This is easy to see from the definition of the inverse.

6. The spectrum of unbounded operators, even closed ones, can be any closed set, including $\emptyset$ and $\mathbb{C}$.

7. Let $T_1 = \partial$ be defined on $\mathcal{D}(T_1) = \{f \in C^1[0,1] : f(0) = 0\}$ [(5)] with values in the Banach space $C[0,1]$ (with the sup norm). (Note also that $\operatorname{dom} T$ is dense in $C[0,1]$.) Then the spectrum of T_1 is empty.

   Indeed, to show that the spectrum is empty, note that by assumption $(\partial - z)\mathcal{D}(T_1) \subset C[0,1]$. Now, $(\partial - z)f = g$, $f(0) = 0$ is a linear differential equation with a unique solution

$$f(x) = e^{xz} \int_0^x e^{-zs} g(s) ds$$

   We can therefore check that f defined above is an inverse for $(\partial - z)$, by checking that $f \in C^1[0,1]$, and indeed it satisfies the differential equation. Clearly $\|f\| \leqslant const(z)\|g\|$.

8. What about our general C^* algebra proof that the spectrum cannot be empty? Also, $\sigma(T_1) = \emptyset$ implies that T_1 is closed, see the exercise above.

9. Clearly, at least when the spectrum is empty there is no analog of a Gelfand transform to determine the properties of an operator T from those of continuous functions on the spectrum.

10. At the "opposite extreme", $T_0 = \partial$ defined on $\mathcal{D}(T_0) = C^1[0,1]$, a dense subset of the Banach space $C[0,1]$, has as spectrum $\mathbb{C}$.

    Indeed, for any $z \in \mathbb{C}$, if $f(x;z) = e^{zx}$, then $T_0 f - zf = 0$.

    We note that T_0 is closed too, since if $f_n \to 0$ then $f_n - f_n(0) \to 0$ as well, so we can use 6 and 7 above.

11. The examples above show that a domain has to be specified together with an operator; T_0 and T_1 have very different behavior.

1. An interesting example is $\mathcal{E}$ defined by $(\mathcal{E}\psi)(x) = \psi(x+1)$. This is well defined and bounded (unitary) on $L^2(\mathbb{R})$. The "same" operator can be defined on the polynomials on $[0,1]$, an L^∞ dense subset of $C[0,1]$. Note that now $\mathcal{E}$ is unbounded.

2. $\mathcal{E} : P[0,1] \to \mathcal{E}P[0,1]$ is bijective and thus invertible in a function sense. But the inverse is unbounded as seen in a moment.

(5) $f(0) = 0$ can be replaced by $f(a) = 0$ for some fixed $a \in [0,1]$.

3. It is also not closable. Indeed, since $P[0,1] \subset P[0,2]$ and $P[0,2]$ is dense in $C[0,2]$, it is sufficient to take a sequence of polynomials P_n converging to a continuous nonzero function which vanishes on $[0,1]$. Then $P_n \to 0$ as restricted to $[0,1]$ while $P_n(x+1)$ converges to a nonzero function, and closure fails. In fact, P_n can be chosen so that $P_n(x+1)$ converges to any function that vanishes at $x = 1$.

Exercise 1. *Show that $T_2 = \partial$ defined on $\mathcal{D}(T_2) = \{f \in C^1[0,1] : f(0) = f(1)\}$ has spectrum exactly $2\pi i\mathbb{Z}$.*

It is also useful to look at the extended spectrum, on $\mathbb{C}_\infty$ We say that $\infty \in \sigma_\infty(T)$ if $(T-z)^{-1}$ is not analytic in a neighborhood of infinity.

11 Integration and measures on Banach spaces

In the following Ω is a topological space, $\mathcal{B}$ is the Borel σ-algebra over Ω, X is a Banach space, μ is a signed measure on Ω. Integration can be defined on functions from Ω to X, as in standard measure theory, starting with simple functions.

1. A simple function is a sum of indicator functions of measurable mutually disjoint sets with values in X:

$$f(\omega) = \sum_{j\in J} x_j \chi_{A_j}(\omega); \quad \operatorname{card}(J) < \infty \tag{72}$$

   where $x_j \in X$ and $\cup_j A_j = \Omega$.

2. We denote by $\mathcal{L}_s(\Omega, X)$ the linear space of simple functions from Ω to X.

3. We will define a norm on $\mathcal{L}_s(\Omega, X)$ and find its completion $B(\Omega, X)$ as a Banach space. We define an integral on $\mathcal{L}_s(\Omega, X)$, and show it is norm continuous. Then the integral on B is defined by continuity. We will then identify the space $B(\Omega, X)$ and find the properties of the integral.

4. $\mathcal{L}_s(\Omega, X)$ is a normed linear space, under the sup norm

$$\|f\|_\infty = \sup_{\omega\in\Omega} \|f(\omega)\| \tag{73}$$

5. We define $B(\Omega, X)$ the completion of $\mathcal{L}_s(\Omega, X)$ in $\|f\|_\infty$.

6. Check that, for a partition $\{A_j\}_{j=1,\dots,n}$ we have

$$\|f\|_\Omega = \max_{j\in J} \sup_{x\in A_j} \|f(x)\| =: \max_{j\in J} \|f\|_{A_j} \tag{74}$$

7. *Refinements.* Assume $\{A_j\}_{j=1,\dots,n}$ is partition and $\{A'_j\}_{j=1,\dots,n'}$ is a subpartition, in the sense that for any A_j there exists $A'_{j1}, \dots, A'_{jm}$ so that $A_j = \cup_{i=1}^m A'_{ji}$

8. The integral is defined on $\mathcal{L}_s(\Omega, X)$ as in the scalar case by

$$\int f d\mu = \sum_{j\in J} \mu(A_j) x_j \tag{75}$$

and likewise, the integral over a subset of $A \in \mathcal{B}(\Omega)$ by

$$\int_A f d\mu = \int \chi_A f d\mu \tag{76}$$

which is the natural definition since A is also a topological space with a Borel σ-algebra (the induced one) and with the same measure μ. Check that, is we choose $x_j' = x_j$ for each $A_j' \subset A_j$ then

$$\sum_j x_j \chi_{A_j} = \sum_j x_j' \chi_{A_j'} \tag{77}$$

and

$$\int_\Omega \sum_j x_j \chi_{A_j} d\mu = \int_\Omega \sum_j x_j' \chi_{A_j'} d\mu \tag{78}$$

9. Note that if A, B are disjoint sets in $\mathcal{B}(\Omega)$, then

$$\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu \tag{79}$$

10.

Lemma 40. *If $\{A_i'\}_{i=1,\dots,n'}$ is a subpartition of $\{A_i\}_{i=1,\dots,n}$ in the sense that $A_{i'}' \subset A_i$ for any i' and some i, and if $x_{i'} = x_i$ whenever $A_{i'}' \subset A_i$, then $\sum_{i'=1}^{n'} x_{i'} \chi_{A_{i'}} = \sum_{i=1}^{n} x_i \chi_{A_i}$.*

Proof. Since $\chi_{A+B} = \chi_A + \chi_B$, this is immediate. □

Lemma 41. *If $f_1, f_2 \in B(\Omega, X)$, then for any ϵ there is a (disjoint) partition $\{A_i\}_{i=1,\dots,n}$ of Ω so that for any $\omega_j \in A_j$ we have*

$$\|f_i - \sum_{i=1}^{n} f(\omega_j)\chi_{A_j}\|_X \leqslant \epsilon, \ i = 1, 2 \tag{80}$$

Proof. Taking as a partition a common refinement of the partitions for f_1 and f_2 which agree with f_1 and f_2 resp. within $\epsilon/2$ this is an immediate consequence of the previous two lemmas and the triangle inequality. □

Lemma 42. *Assume $|\mu|(\Omega) < \infty$ (otherwise choose $\Omega' \in \Omega$ so that $|\mu|(\Omega') < \infty$). The map $f \to \int f d\mu$ is well defined, linear and bounded in the sense*

$$\left\|\int_A f d\mu\right\| \leqslant \int_A \|f\| d|\mu| \leqslant \|f\|_{\infty,A} |\mu|(A) \tag{81}$$

where $|\mu|$ is the total variation of the signed measure μ, $|\mu| = \mu^+ + \mu^-$, where $\mu = \mu^+ - \mu^-$ is the Hahn-Jordan decomposition of μ.

Proof. All properties are immediate, except perhaps boundedness. We have

$$\left\| \int_A f d\mu \right\| \leqslant \sum_{j\in J} |\mu|(A_j)\|x_j\| = \int_A \|f\| d|\mu| \leqslant \|f\|_{\infty,A}|\mu|(A) \tag{82}$$

□

11. Thus $\int_A$ is a linear bounded operator from $\mathcal{L}_s(\Omega, X)$ to X and it extends to a bounded linear operator on from $B(\Omega, X)$ to X.

Lemma 43. *$f \in B(\Omega, X) \to \forall\,\epsilon$ there is a partition $\{A_i\}_{i=1,\dots,n}$ of Ω so that for* any *$\omega_j \in A_j$ we have*

$$\|f - \sum_j f(\omega_j)\chi_{A_j}\| \leqslant \epsilon \tag{83}$$

Proof. Choose a partition $\{A_i\}_{i=1,\dots,n}$ of Ω and x_j so that

$$\|f - \sum_j x_j \chi_{A_j}\| \leqslant \epsilon/2$$

This implies by the 6 above that

$$\|x_j - f(\omega_j)\| \leqslant \epsilon/2$$

for *all* $\omega_j \in A_j$. The rest follows from the triangle inequality. □

Exercise

Show that the same holds for a pair of functions f_1, f_2, namely there is a *common* partition $\{A_i\}_{i=1,\dots,n}$ of Ω so that

$$\|f_i - \sum_j f_i(\omega_j)\chi_{A_j}\| \leqslant \epsilon, i = 1, 2$$

12. Let T be a closed operator and $f \in B(\Omega, X)$ be such that $f(\Omega) \subset \mathcal{D}(T)$. Assume further that f is such that $Tf \in B(\Omega, X)$.

Theorem 14 (Commutation of closed operators with integration)**.** *Under the assumptions above we have*

$$T \int_A f d\mu = \int_A T f d\mu \tag{84}$$

Proof. Since $f \in B(\Omega, X)$, we have $\|f - \sum_{j=1}^{n} f(\omega_j)\chi_{A_j^{[n]}}\| \leqslant \epsilon_n$, $\epsilon_n \to 0$, for a sequence of refined partitions. We also have $Tf \in B(\Omega, X)$ and thus $\|Tf - \sum_{j=1}^{n} h_j \chi_{A_j^{[n]}}\| \leqslant \epsilon_n$ for a sequence of refinements $\chi_{A_j^{[n]}}$, that we can assume is the same as above. Thus $\|(Tf)_j - h_j\|_{A_j^{[n]}} \to 0$ uniformly. On the other hand on A_j, uniformly in j, $\|(Tf) - y_j\| \to 0$. But $\|(Tf) - y_j\| \to 0$ means in particular that $\|(Tf(\omega_j)) - y_j\| \to 0$ *uniformly in* j. It follows that $T \sum f(\omega_j)\chi_{A_j^{[n]}}$ converges and $\sum f(\omega_j)\chi_{A_j^{[n]}}$ converges to f; thus $T \sum f(\omega_j)\chi_{A_j^{[n]}}$ converges **to** $\boldsymbol{Tf}$. On the other hand, for any simple function f_s, the definition of the integral implies immediately that

$$\int_A Tf_s = T \int_A f_s \tag{85}$$

Since the integral is continuous and we have $Tf_n \to Tf$ we have $\int Tf_n \to \int Tf$ while $\int Tf_n = T \int f_n$. So $\int f_n \to \int f$ and $T \int f_n = \int Tf_n$ converges, thus to $T \int f$. □

Note 8. *Check, from the definition of the integral, that if* $B(\omega)$ *is a bounded operator,* $T : D(T) \to X$ *is any operator and* $u \in D(T)$*, then*

$$\int_A B(\omega)Tud\mu(\omega) = \left(\int_A B(\omega)d\mu(\omega)\right)Tu \tag{86}$$

Exercise 1. *Formulate and prove a theorem allowing to differentiate under the integral sign in the way*

$$\frac{d}{dx}\int_a^b f(x,y)dy = \int_a^b \frac{\partial}{\partial x} f(x,y)dy$$

Corollary 44. *In the setting of Theorem 14, if* T *is bounded we can drop the requirement that* $Tf \in B$.

Proof. Tf_m is a simple function, and it converges to Tf. The rest is immediate. □

1. An important case of Corollary 44 is that for any $\varphi \in X^*$, we have

$$\varphi \int_A f d\mu = \int_A \varphi f d\mu \tag{87}$$

2. We recall a corollary of the Hahn-Banach theorem (see [5], p.77):

   Corollary 45. *If* $\varphi(x) = \varphi(y)$ $\forall \varphi \in X^*$*, then* $x = y$.

3.

   Definition 46. *The last results allow us to transfer many properties of the usual integral to the vector setting.*

   For instance, if A *and* B *are disjoint, then*

$$\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu \tag{88}$$

Proof. By Corollary 45 , (89) is true iff for any φ we have

$$\int_{A\cup B} \varphi f d\mu = \int_A \varphi f d\mu + \int_B \varphi f d\mu \tag{89}$$

which clearly holds.

□

Note 9. Note that the integral we constructed this way is a Lebesgue integral restricted to a subset of an "analog of L^1". For the L^1 extension see §**??**

Recall that f is measurable between two topological spaces endowed with the Borel sigma algebras, if the preimage of a measurable set is measurable.

Proposition 47. *If $f \in B(\Omega, X)$ then f is measurable.*

Proof. Usual proof, f is a uniform limit of measurable functions, f_m. □

Theorem 15. *The function f is in $B(\Omega, X)$* **iff** *f is measurable and $f(\Omega)$ is relatively compact.*

Proof. Recall that in a metric space a set is *totally bounded*[6] iff it is precompact. Let $f \in B(\Omega, X)$ implies $f(\Omega)$. We have $f = f_m + O(\epsilon)$ for some $f_m = \sum_{j=1}^{n_m} \chi_{A_j} x_j$. Thus for any ϵ, the whole range of f is within ϵ of some finite set $x_1, ..., x_{N_m}$, the definition of totally bounded.

Now, if $f(\Omega)$ is precompact, then it is totally bounded and $f(\Omega)$ is within ϵ of a set $\{x_1, ..., x_N\}$. Out of it, it is easy to construct a simple function approximating f within ϵ. □

Corollary 48. *Let $\Omega' \subset \Omega$ be compact and denote by $C(\Omega', X)$ the continuous $X-$valued functions supported on Ω'. Then $C(\Omega', X)$ is a closed subspace of $B(\Omega, X)$.*

Proof. If $f \in C(\Omega', X)$, then $f(\Omega')$ is compact, since Ω' is compact and f is continuous. Measurability follows immediately, since the preimage of open sets is open. □

4. If $f_n \in B(\Omega, X)$, $f_n \to f$ in the sup norm, then $\int f_n d\mu \to \int f d\mu$. This is clear, since $\int_A$ is a continuous functional.

[6] In a metric space, this is the case if the set can be covered by finitely many balls of radius ϵ, for all ϵ.

12 Extension (Bochner integration)

A theory of integration similar to that of Lebesgue integration can be defined on the measurable functions from Ω, X to Y ([2])

The starting point are still simple functions. Convergence can be understood however in the sense of L^1. We endow simple functions with the norm

$$\|f\|_1 = \int_A \|f(\omega)\| d|\mu|$$

and take the completion of this space. Convergence means: $f_n \to f a.e.$ and f_n are Cauchy in $L_1(A)$. Then $\lim \int f_n d\mu$ is by definition $\int_A f d\mu$.

Integration is continuous, and then the final result is $L^1(A)$.

Then, the usual results about dominated convergence, Fubini, etc. hold.

13 Analytic vector valued functions

Let X be a Banach space. Analytic functions : $\mathbb{C} \to X$ are functions which are, locally, given by convergent power series, with coefficients in X.

1. More precisely, let $\{x_k\} \subset X$ be such that $\limsup_{n\to\infty} \|x_n\|^{1/n} = \rho < \infty$. Then, for $z \in \mathbb{C}$ $|z| < R = 1/\rho$ the series

$$S(z, x_k) = \sum_{n=0}^{\infty} x_k z^k \tag{90}$$

converges in X (because it is *absolutely convergent* [(7)]). (There is an interchange of interpretation here. We look at (90) also as a series over X, with coefficients z^k.)

2. **Abel's theorem**. Assume $S(z)$ converges for $z = z_0$. Then the series converges uniformly, together with all formal derivatives on D_r where $D_r = \{z : |z| \leqslant r\}$, if $r < |z_0|$.

Proof. This follows from the usual Abel theorem, since the series $\sum_{n=0}^{\infty} \|x_k\||z|^k =: S(|z|, \|x_k\|_k)$ converges (uniformly) in D_r. □

13.1 Functions analytic in an open set $\mathcal{O} \in \mathbb{C}$

1. Let $\mathcal{O} \in \mathbb{C}$ be an open set. The space of X-valued analytic functions $H(\mathcal{O}, X)$ is the space of functions defined on $\mathcal{O}$ with values in X such

[(7)] That is, $\sum_{n=0}^{\infty} \|x_k\||z|^k$ converges.

that for any $z_0 \in \mathcal{O}$, there is an $R(z_0) \neq 0$ and a power series $S(z; z_0)$ with radius of convergence $R(z_0)$ such that

$$f(z) = S(z; z_0, x_k) =: \sum_{k=0}^{\infty} x_k(z_0)(z - z_0)^k \quad \forall z, |z - z_0| < R(z_0) \tag{91}$$

2. If $\mathcal{O} = \mathbb{C}$, then we call f entire.

   Proposition 49. *An analytic function is continuous.*

   Proof. We are dealing with a uniform limit on compact sets of continuous functions, $\sum_{k=0}^{N} x_k(z_0)(z - z_0)^k$. □

   Corollary 50. *Let $\mathcal{O}$ be precompact. Then $H(\mathcal{O}, X) \subset B(\Omega, X)$.*

   Proof. This follows from Proposition 49 and Corollary 48. □

3. Let now X be a Banach algebra with identity.

   Lemma 51. *The sum an product of two series $s(z; z_0, x_k)$ and $S(z; z_0, y_k)$ with radia of convergence r and R respectively, is convergent in a disk D of radius at least* $\min\{r, R\}$.

   Proof. By general complex analysis arguments, the real series $s(\|z\|; z_0, \|\|x_k\|)$ and $S(\|z\|; z_0, \|\|x_k\|)$ converge in D and then so does $s(\|z\|; z_0, \|\|x_k\|) + S(\|z\|; z_0, \|\|x_k\|)$ etc. □

   Corollary 52. *The sum and product of analytic functions, whenever the spaces permit these operations, is analytic.*

4. Let $\mathcal{O}$ a relatively compact open subset of $\mathbb{C}$. We can introduce a norm on $H(\mathcal{O}, X)$ by $\|f\| = \sup_{z \in \mathcal{O}} \|f(z)\|_X$.

5. Let us recall what a rectifiable Jordan curve is: This is a set of the form $\mathcal{C} = \gamma([0, 1])$ where $\gamma : [0, 1] \to \mathbb{C}$ is in CBV (continuous functions of bounded variation), such that $x \leqslant y$ and $\gamma(x) = \gamma(y) \Rightarrow (x = y$ or $x = 0, y = 1)$ (that is, there are no nontrivial self-intersections; if $\gamma(0) = \gamma(1)$ then the curve is closed). (Of course, we can replace $[0, 1]$ by any $[a, b]$, if it is convenient.) Then γ' exists a.e., and it is in L^1. Thus $d\gamma(s) = \gamma'(s)ds$ is a measure absolutely continuous w.r.t ds. As usual, we define positively oriented contours, the interior and exterior of a curve etc.

6. We can define complex contour integrals now. Note that if f is analytic, then it is continuous, and thus $f(\gamma) : [0, 1] \mapsto \mathbb{C}$ is continuous, and thus in $B(\mathcal{O}, X)$. Then, by definition,

$$\int_{\mathcal{C}} f(z)dz := \int_0^1 f(\gamma(s))\gamma'(s)ds =: \int_0^1 f(\gamma(s))d\gamma(s) \tag{92}$$

7.

Proposition 53. Let Y be a Banach space, $T : X \to Y$ continuous (i.e. bounded), and $f \in H(\mathcal{O}, X)$. Then $Tf \in H(\mathcal{O}, Y)$.

Proof. Let $z_0 \in \mathcal{O}$. Then there is a disk $\mathbb{D}(z_0, \epsilon)$ such that for all $z \in \mathbb{D}(z_0, \epsilon)$ we have $\|f(z) - \sum_{k=0}^{N} x_k(z - z_0)^k\| \to 0$, as $n \to \infty$. then,

$$\|Tf(z) - \sum_{k=0}^{N} Tx_k(z - z_0)^k\| \leqslant \|T\|\|f(z) - \sum_{k=0}^{N} x_k(z - z_0)^k\| \to 0$$

and the result follows. □

8. In particular, f analytic implies φf analytic for any $\varphi \in X^*$.

9. As a result of (87) we have, for any $\varphi \in X^*$

$$\varphi \int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}} \varphi f(z)dz \tag{93}$$

Proposition 54. *If $f \in B(S^1, X)$ (S^1 =the unit circle) and $z \in \mathbb{D}^1$ (the open unit disk), then*

$$F(z) = \oint_{S^1} f(s)(s - z)^{-1}ds \tag{94}$$

is analytic in $\mathbb{D}_1$.

Proof. As usual, we pick $z_0 \in \mathbb{D}^1$, let d =dist(d, S^1) and take the disk $\mathbb{D}^{d/2}(z_0)$. We write $(s - z) = (s - z_0)^{-1}/(1 - (z - z_0)/(s - z_0)) =: (s - z_0)^{-1}/(1 - x)$. Using

$$1/(1 - x) = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

we get

$$\begin{aligned} &\oint_{S^1} f(s)(s - z)^{-1}ds - \oint_{S^1} f(s)(s - z_0)^{-1}ds \\ &= \sum_{k=0}^{n}(z - z_0)^k \oint_{S^1} \frac{f(s)ds}{(s - z_0)^{k+1}} + (z - z_0)^{n+1} \oint_{S^1} \frac{f(s)ds}{(s - z)(s - z_0)^{n+1}}ds \end{aligned} \tag{95}$$

We now check that the last integral has norm $\leqslant 2^{-n}C$ where C is independent of n, and the result follows. □

10. The last few results allow us to transfer the results that we know from usual complex analysis to virtually identical results on strongly analytic vector valued functions.

11.

Proposition 55. *The function f is analytic* **iff** *it is* weakly analytic*, that is φf is analytic for any $\varphi \in X^*$.*

Proof 1. Let $\varphi \in X^*$ be arbitrary. Then $\varphi f(s)$ is a scalar valued analytic function, and then

$$\varphi f(z) = \oint_{\bigcirc} \varphi f(s)(s-z)^{-1} ds = \varphi \oint_{\bigcirc} f(s)(s-z)^{-1} ds \tag{96}$$

if the circle around z is small enough. Since this is true for all φ, we thus we conclude that

$$f(z) = \oint_{\bigcirc} f(s)(s-z)^{-1} ds \tag{97}$$

and by Proposition 54, f is analytic. □

Proof 2. Consider the family of operators $f_{zz'} := (f(z) - f(z'))/(z - z')$ on $\mathcal{O}^2 \setminus D$ where D is the diagonal $(z, z) : z \in \mathcal{O}$. Then $|\varphi f_{zz'}| < C_\varphi$ for all $\varphi \in X^*$. Now we interpret $f_{zz'}$ as a family of functionals on X^{**}, indexed by z, z'. By the uniform boundedness principle, $\|f_{zz'}\|_{X^{**}} \leqslant B < \infty$ is bounded with the bound independent of z, z', and by standard functional analysis $\|f_{zz'}\|_{X^{**}} = \|f_{z,z'}\|_X \leqslant B$. Then, $f(z)$ is continuous, and thus integrable. But then, since $\oint_\Delta \varphi f(s) ds = 0$ it follows that $\oint_\Delta f(s) ds = 0$ for all $\Delta \in \mathcal{O}$, and thus f is analytic. □

12.

Corollary 56. *f is analytic in a region $\mathcal{R}$ iff f is strongly differentiable at all $z \in \mathcal{R}$ ($\lim_{h\to 0} h^{-1}(f(z + h - f(z))$ exists in norm $\forall z \in \mathcal{R}$) iff it is weakly differentiable in z.*

Proof. Assume that f is weakly differentiable. This implies that for any $\varphi \in X^*$ we have φf is analytic, which in turn implies that for a small enough circle around z we have

$$\varphi f = \oint_{\bigcirc} \frac{\varphi f(s)}{s-z} ds = \varphi \oint_{\bigcirc} \frac{ds f(s)}{s-z} \tag{98}$$

which means

$$f(z) = \oint_{\bigcirc} \frac{f(s)}{s-z} ds \tag{99}$$

and, again by Proposition 54, f is analytic. The rest is left as a simple exercise. □

14 Functions defined on $\mathcal{B}$

Analytic functional calculus allows us to define *analytic* functions of operator that are not normal, and more general Banach algebra elements, and prove a spectral mapping theorem for these analytic functions.

Clearly, right from the definition of an algebra, for any polynomial P and any element $a \in \mathcal{B}$, $P(a)$ is well defined, and it is an element of the algebra. A rational function $R(a) = P(a)/Q(a)$ can be also defined *provided that the zeros of Q are in $\rho(a)$*, with obvious notations, by

$$R(a) = (p_1 - a) \cdots (p_m - 1)(q_1 - a)^{-1} \cdots (q_n - a)^{-1}$$

(since we have already shown the resolvents are commutative, the definition is unambiguous and it has the expected properties.

We can define analytic functions of a as follows.

A natural way is to start with Cauchy's formula,

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(s)ds}{s - z} \tag{100}$$

and replace z by a. We have to ensure that (a) The integral makes sense and it is useful. For that we choose the contour so that a "is inside" that is, $\mathcal{C}$ should be outside the spectrum.

(b) on the other hand, f should be analytic in the interior of $\mathcal{C}$, $\sigma(a)$ included. We can see that by looking at Exercise 7, where a would be z. If we don't assume analyticity of f on the spectrum of z, we don't get an analytic function.

So, let's make this precise. Let a be given, with spectrum $\sigma(a)$.

*

Consider the set of functions analytic in an open set $\mathcal{O}$ *containing* $\sigma(a)$.

For a generalization see 5 below.

Proposition 57. *Let $\mathcal{C}$ be a Jordan curve in $\mathcal{O} \setminus \sigma(a)$. Then $f(s)R_s(a)$ is continuous thus integrable, and the function*

$$f(a) := \frac{1}{2\pi i} \oint_{\mathcal{C}} f(s)(s - a)^{-1} ds \tag{101}$$

is well defined and it is an element of $\mathcal{B}$ (in fact, it is also an element of the Banach algebra generated by $R(a)$).

Exercise 1. ** Prove Proposition 57 above.*

We thus **define** $f(a)$ by (101).

(a) Note also that $\mathcal{O}$ was not assumed to be connected. We have defined analyticity in terms of local Taylor series. More generally, we consider an open set $\mathcal{O}_1 \supset \sigma(a)$, connected or not. If $\mathcal{C} = \mathcal{C}_1 = \partial\mathcal{O}_1$ consists of a finite number of rectifiable Jordan curves, then the definition (101) is still meaningful. Clearly, a function analytic on a disconnected open set is simply any collection of analytic functions, one for each connected component, and no relation needs to exist between two functions belonging to different components. Cauchy's formula still applies on $\mathcal{C}_1$. This will allow us to define projectors, and some analytic functions of unbounded operators.

(b) We are technically dealing with classes of equivalence of functions of a, where we identify two elements f_1 and f_2 if they are analytic on a common subdomain and coincide there. But this just means that f_1 and f_2 are analytic continuations of each-other, and we make the choice of not distinguishing between a function and its analytic continuation. So we'll write $f(a)$ and not $[f](a)$ where $[f]$ would be the equivalence class of f.

Exercise 2. Let $\Omega \supset \sigma(a)$ be open in $\mathbb{C}$.

(i) Show that if $f_n \to f$ in the sup norm on Ω, where f_n are analytic, then $f_n(a) \to f(a)$. In particular the mapping $[z \mapsto f(z)] \mapsto f(a)$ is continuous.

(ii) Show that the map $f \to f(a)$ is continuous from $H(\Omega)$, with the sup norm, into X.

Remark 10. Let $\rho \notin \sigma(a)$ and $f(z) = 1/(\rho - z)$. Then $f(a) = (\rho - a)^{-1}$. Indeed $f(x)(\rho - x) = (\rho - x)f(x) = 1$ and thus, by 14, we have $f(a)(\rho - a) = (\rho - a)f(a) = 1$, that is $f(a) = (\rho - a)^{-1}$.

More generally, if P and Q are polynomials and the roots $q_1, ..., q_n$ of Q are outside $\sigma(a)$, then $(P/Q)(a) = P(a)(a - q_1)^{-1} \cdots (a - q_k)^{-1}$. [8].

(c) Thus we can write

$$(s - a)^{-1} = \frac{1}{s - a}$$

Exercise 3. Show that $f \to f(a)$, for fixed a, defines an algebra homomorphism between $H(\Omega)$ and its image.

13. Show that the integral only depends on the homotopy class of $\mathcal{C}$ in $\mathcal{O}\backslash\sigma(a)$. So in this sense, $f(a)$ is canonically defined.

14. A polynomial $P(a)$ defined through (100) coincides with the direct definition. By linearity it suffices to check this on monomials z^n. Then we can

[8] Note however that there is no immediate extension in general of the *local* Taylor theorem $f(x) = \sum f^{(k)}(x_0)(x - x_0)^k$, since $\|x_0 - a\|$ would be required to be arbitrarily small to apply this formula for every analytic f, which in turn wold imply $a = x_0$.

choose a disk $\mathbb{D}_r$ around zero with $r > r(a)$ Then

$$a^n = \frac{1}{2\pi i}\oint_{\partial \mathbb{D}_r} \frac{s^n}{s(1-a/s)} ds \tag{102}$$

Indeed, $(1-a/s)^{-1}$ has a convergent expansion in $1/s$ where $|s| > r(a)$. Then the integral can be expanded convergently, as usual, and the result coincides with a^n.

15.

Proposition 58. (i) $(fg)(a) = f(a)g(a)$ (ii) in particular $f(a)g(a) = g(a)f(a)$. The set of functions f of a, where f is analytic in a neighborhood of $\sigma(a)$ forms a commutative Banach algebra.

Proof. (i) Indeed let Int$\mathcal{C} \supset \mathcal{C}'$. Note first that

$$\oint_{\mathcal{C}'} \frac{g(s')}{(s'-s)} ds' = 0$$

since s is outside $\mathcal{C}'$. Using this fact, the first resolvent formula and Fubini, we get

$$\begin{aligned}
f(a)g(a) &= \frac{1}{-4\pi^2}\oint_{\mathcal{C}}\oint_{\mathcal{C}'} f(s)(s-a)^{-1}g(s')(s'-a)^{-1}ds'ds \\
&= \frac{1}{-4\pi^2}\oint_{\mathcal{C}}\oint_{\mathcal{C}'} \frac{f(s)g(s')}{s'-s}\left(\frac{1}{s-a}-\frac{1}{s'-a}\right)ds'ds \\
= \frac{1}{-4\pi^2}&\oint_{\mathcal{C}}\oint_{\mathcal{C}'} \frac{f(s)g(s')}{(s'-s)(s-a)}ds'ds + \frac{1}{4\pi^2}\oint_{\mathcal{C}}\oint_{\mathcal{C}'} \frac{f(s)g(s')}{(s'-s)(s'-a)}ds'ds \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}'} \frac{f(s')g(s')}{s'-a}ds' = \frac{1}{2\pi i}\oint_{\mathcal{C}'} \frac{(fg)(u)}{u-a}du = (fg)(a) \qquad (103)
\end{aligned}$$

(ii) is immediate. Note that $\Gamma = f(a) \mapsto f(z), z \in \sigma(a)$ is a multiplicative functional, thus continuous, of norm one. □

16. We can use Proposition 55 and Corollary 56 to prove in a very simple way that vector-valued analytic functions satisfy many of the properties of usual analytic functions.

    For instance:

    (a) $\int_{\mathcal{C}}$ does not depend on the parametrization of $\mathcal{C}$, but on $\mathcal{C}$ alone.

    (b) If we take a partition of $\mathcal{C}$, $\mathcal{C} = \cup_{i=1}^N \mathcal{C}_i$ then

$$\int_{\mathcal{C}} f(z)dz = \sum_{i=1}^N \int_{\mathcal{C}_i} f(z)dz \tag{104}$$

(c) We have Cauchy's formula: Assume $\mathcal{O}$ is a precompact open set in $\mathbb{C}$, $\mathcal{C}$ a a closed, positively oriented contour in $\mathcal{O}$ and if $z \notin \mathcal{C}$, where $\mathcal{C}$ is a closed, positively oriented contour, then

$$\oint_{\mathcal{C}} f(s)(s-z)^{-1} ds = 2\pi i f(z) \chi_{\mathrm{Int}\mathcal{C}}(z) \tag{105}$$

Proof. Note that $f(s)(s-z)^{-1}$ is analytic in z, for any s. Thus it is continuous, integrable, etc. We have for any $\varphi \in X^*$,

$$\oint_{\mathcal{C}} \varphi f(s)(s-z)^{-1} ds = 2\pi i \varphi f(z) \chi_{\mathrm{Int}\mathcal{C}}(z) \tag{106}$$

□

(d) Liouville's theorem: If f is analytic in $\mathbb{C}$ and bounded, in the sense that $\|f(\mathbb{C}, X)\| \subset K \in \mathbb{R}$ where K is compact, then f is a constant.

Proof. Indeed, it follows that $\varphi(f)(z) = \varphi(f(z)$ is entire and bounded, thus constant. Hence, $\varphi(f)(z) - (\varphi f)(0) = 0 = \varphi(f(z) - f(0))$. Since φ is arbitrary, we have $f(z) = f(0)\ \forall z$. □

(e) Morera's theorem. Let $f : (\mathcal{O}, X) \to Y$ be continuous (it means it is single-valued, in particular), and assume that $\int_\Delta f(s) ds = 0$ for every triangle in Ω. Then, f is analytic.

Proof. Indeed, φf is continuous on $\mathcal{O}$ and we have $\oint_\Delta \varphi f ds = 0$ for every triangle in Ω. But then φf is a usual analytic function, and thus

$$\varphi f(z) = \frac{1}{2\pi i} \oint_{\bigcirc} \varphi f(s)(s-z)^{-1} ds$$

By Proposition 54, the right side is analytic. □

(f) (Removable singularities) If f is analytic in $\mathcal{O} \setminus a$ and $(z-a)f = o(z-a)$ then f extends analytically to $\mathcal{O}$.

Proof. This is true for φf. □

Corollary 59. *$H(\mathcal{O}, X)$ is a linear space; if X is a Banach algebra then $H(\mathcal{O}, X)$ is a Banach algebra.*

Proof. Straightforward verification. □

We can likewise define double integrals, as integrals with respect to the product measure. If $f \in B(\Omega, X_1 \times X_2)$, then $f(\cdot, x_2)$ and $f(x_1, \cdot)$ are in $B(\Omega, X_2)$ and $B(\Omega, X_1)$ respectively, and Fubini's theorem applies (since it applies for every functional). Check this.

Exercise 4. The generalized Mergelyan's theorem states the following:

Theorem 16 (Mergelyan)**.** *Let $K \subset \mathbb{C}$ be compact and suppose $\mathbb{C}^* \setminus K$ has only finitely many connected components. If f in $C(K)$ is holomorphic in the interior of K and if $\epsilon > 0$, then there is a rational function $r(z)$ with poles in $\mathbb{C}^* \setminus K$ such that $\max_{z\in K} |f(z) - r(z)| < \epsilon$.*

"Assume that K is such a compact set and a is an element of a Banach algebra s.t. $\|a^2\| = \|a\|^2$. Then, the Gelfand transform of the Banach algebra *generated by the rational functions of a* whose denominators do not vanish in $\sigma(a)$ consists of the functions which are continuous on K and analytic in its interior."

True or false?

14.1 Functions analytic at infinity

1. f is analytic at infinity if f is analytic in $\mathbb{C} \setminus K$ for some compact set K (possibly empty) and f is bounded at infinity. Equivalently, $f(1/z)$ is analytic in a punctured neighborhood of zero and bounded at zero. Then $f(1/z)$ extends analytically uniquely by $f(\infty)$.

Cauchy formula at ∞

2. Let f be analytic at infinity and $\mathcal{C}$ positively oriented about infinity, which by definition means that the neighborhood of infinity is to the left of the curve as we traverse it. (That is, $\mathcal{C}$ is **negatively oriented** if seen as a curve around 0). Then

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(s)}{s-z} ds = -f(\infty) + \chi_{Ext\mathcal{C}}(z) f(z) \tag{107}$$

Proof. This follows from the scalar case, which we recall. Let f be analytic in $\mathbb{C}_\infty \setminus K$ We have

$$J = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(s)}{s-z} ds = -\frac{1}{2\pi i}\oint_{1/\mathcal{C}} \frac{f(1/t)}{t^2(1/t-z)} dt \tag{108}$$

where $1/\mathcal{C}$ is the positively oriented closed curve $\{1/\gamma(1-t) : t \in [0,1]\}$ (where we assumed a standard parametrization of γ, and we can assume that $\gamma \neq 0$, since otherwise f is analytic at zero, and the contour can be homotopically moved away from zero). There is a change in sign from $ds = -dt/t^2$ and we also note that $z \to 1/z$ changes the orientation of the curve, so *it becomes positively oriented in τ*. Then, with $g(t) = f(1/t)$ and writing

$$\frac{1}{t - zt^2} = \frac{1}{t} + \frac{z}{1-zt} = \frac{1}{t} + \frac{1}{1/z - t}$$

$$J = -\frac{1}{2\pi i}\oint_{1/\mathcal{C}} \frac{g(t)}{t} + \frac{1}{2\pi i}\oint_{1/\mathcal{C}} \frac{g(t)}{t-1/z} = -g(0) + \chi_{Ext\mathcal{C}}(z) f(z) \tag{109}$$

□

Exercise 5. Let f be analytic in a neighborhood of the spectrum of a and at infinity. Let $\mathcal{C}$ be a simple closed curve outside $\sigma(a)$, positively oriented about ∞. Then we can define

$$f(a) = -f(\infty) + \frac{1}{2\pi i}\oint_{\mathcal{C}} f(s)(s-a)^{-1}ds$$

This allows for defining functions (analytic at infinity) of unbounded operators having a nonempty resolvent set. The properties are very similar to the case where f is analytic on $\sigma(a)$.

14.2 Spectrum of $f(a)$

The spectrum of an operator, or of an element of a Banach Algebra is very robust, in that it "commutes" with many operations.

Proposition 60. *Let f be analytic on $\sigma(a)$. Then $\sigma(f(a)) = f(\sigma(a))$.*

Proof. The proof essentially stems from the fact that $f \mapsto f(a)$ is linear and multiplicative.

In one direction assume let $f_0 \notin f(\sigma(a))$. We want to show that $f_0 \notin \sigma(f(a))$. Thus, $f(z) - f_0$ does not vanish for any $z \in \sigma(a)$, a compact set, and therefore it does not vanish on some open set $\mathcal{O} \supset \sigma(a)$. This implies that

$$g(z) = \frac{1}{f_0 - f(z)}$$

is analytic in $\mathcal{O} \supset \sigma(a)$ and we have

$$g(z)(f_0 - f(z)) = (f_0 - f(z))g(z) = 1$$

which implies

$$g(a)(f_0 - f(a)) = (f_0 - f(a))g(a) = 1$$

and thus $f_0 \notin \sigma(f(a))$.

Now let $f_0 \notin \sigma(f(a))$. We show that $f_0 \notin f(\sigma(a))$. Assume the contrary, that $\exists z_0 \in \sigma(a)$ s.t. $f(z_0) = f_0$. We will show that $a - z_0$ is invertible, a contradiction.

Consider the function

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

This function has one removable singularity in $\mathcal{O}$, namely, at z_0, and thus g extends to an analytic function in $\mathcal{O}$, which we will still denote by g.

We have

$$g(z)(z - z_0) = (z - z_0)g(z) = f(z) - f(z_0)$$

implying

$$g(a)(a - f_0) = (a - f_0)g(z) = f(a) - f(z_0)$$

But f_0 was assumed not to be in $\sigma(f)$ and thus $f(a) - f(z_0)$ is invertible. Let $h = [f(a) - f(z_0)]^{-1}$. Then,

$$[hg(a)](a - f_0) = (a - f_0)[hg(a)] = 1$$

the contradiction we mentioned.

□

Exercise 6. *Prove the following corollary.*

Corollary 61. *Let $\mathcal{B}$ be a Banach algebra, $a \in \mathcal{B}$ and $\sigma(a) = K$. Assume further that f, g are analytic in the open set $\mathcal{O} \supset K$.*

Then, $\Big[f(a) = g(a)\Big] \Leftrightarrow \Big[f(z) = g(z)\ \forall z \in K\Big]$. In particular, if card$(K) = \infty$, then $f \equiv g$.

14.3 Behavior with respect to algebra homeomorphisms

Proposition 62. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Banach algebras with identity and $H : \mathcal{A}_1 \to \mathcal{A}_2$ a Banach Algebra homeomorphism.

(i) Then $\sigma_{\mathcal{A}_2}(H(a)) := \sigma(H(a)) \subset \sigma(a)$.

In particular, if H is an isomorphism, then the spectrum is conserved: $\sigma(H(a)) = \sigma(a)$.

(ii) If f is analytic in a neighborhood of $\sigma(a)$, then $H(f(a)) = f(H(a))$.

Proof. (i) This is because $(a-\alpha)g(a) = g(a)(a-\alpha) = 1$ implies $H((a-\alpha)g(a)) = H(g(a)(a-\alpha)) = 1$, which, in view of the assumptions on H means $H(a) - \alpha)H(g(a)) = (H(g(a))(H(a) - \alpha)) = 1$.

(ii) We know that H is continuous and it thus commutes with integration. If we write

$$f(a) = \frac{1}{2\pi i} \oint f(s)(s - a)^{-1}$$

we simply have to show that $H(s - a)^{-1} = (s - H(a))^{-1}$ which holds by (i).

□

14.4 Composition of operator functions

Proposition 63. *Let f be analytic in an open set $\Omega \supset \sigma(a)$ and g be analytic in $\Omega' \supset f(\sigma(a))$. Then $g(f(a))$ exists and it equals $(g \circ f)(a)$.*

Proof. That $g(f(a))$ exists follows from Proposition 60 and the definition of f and g. We have, for suitably chosen contours (find the conditions!)

$$g(f(a)) = \oint_{\mathcal{C}_1} \frac{g(s)}{s - f(a)} ds \tag{110}$$

while on the other hand

$$\frac{1}{s - f(a)} = \oint_{\mathcal{C}_2} \frac{1}{s - f(t)} \frac{1}{t - a} dt \tag{111}$$

and thus

$$g(f(a)) = \oint_{\mathcal{C}_1} g(u)du \oint_{\mathcal{C}_2} \frac{1}{u - f(s)} \frac{1}{s-a} ds \tag{112}$$

We also have

$$(g \circ f)(a) = \oint_{\mathcal{C}_1} \frac{g(f(s))}{s-a} dt = \oint_{\mathcal{C}_1} \frac{1}{s-a} \oint_{\mathcal{C}_3} \frac{g(u)}{u - f(s)} du \tag{113}$$

The rest follows easily from the uniform bounds on the integrand, which allow interchange of orders of integration. □

Remark 11. In particular, this proves again, in a different way that $(fg)(a) = f(a)g(a)$. Indeed, since $2fg = (f+g)^2 - f^2 - g^2$, it is enough to show that $f^2(a) = f(a)^2$. But this follows from Proposition 63 with $g(x) = x^2$. (Note that $f(\sigma(a))$ is a bounded set, $f(a)$ is a bounded operator, and the contours in Proposition 63 can be chosen so that no use of Proposition 60 is needed).

Exercise 7. ** Let K be a compact set in $\mathbb{C}$. Find a Banach algebra and an element a so that its spectrum is exactly K. Hint: look at $f(z) = z$ restricted to a set.*

14.5 Extended spectrum

We say that ∞ is in the extended spectrum, σ_∞, of an operator if $(z-T)^{-1}$ is not analytic at infinity.

Proposition 64. *If T is closed, then ∞ is not an essential singularity of $(z - T)^{-1}$ iff T is bounded (in the latter case ∞ is a removable singularity and $(z - T)|_{z=\infty} = 0.$).*

Proof. We have already shown that T bounded implies $(z-T)^{-1}$ is analytic *in* $1/z$ for large z, thus, by definition ∞ is not a singularity of R_z.

In the opposite direction, assume that $R_z := (z-T)^{-1}$ has at most a pole at infinity and then let k be the largest power of z so that the Laurent coefficient A is nonzero. We have

$$R_z = z^k A + z^{k-1} B + \cdots \tag{114}$$

On the other hand,

$$1 = (z-T)R_z = zR_z - TR_z$$
$$TR_z = zR_z - 1 \tag{115}$$

We first show that $k \leqslant -1$. Indeed, assume $k \geqslant 0$. Then, $z^{-k-1}R_z = A/z + \cdots \to 0$. On the other hand

$$z^{-k-1}(zR_z - 1) \to A \Rightarrow A = \lim_{z\to\infty} z^{-k-1}(zR_z - 1) = \lim_{z\to\infty} Tz^{-k-1}R_z = \lim_{z\to\infty} T0 = 0$$

contradiction. Thus $k \leqslant -1$ and by (114) $R_z \to 0$, and TR_z converges, by (115). Since T is closed, $zR_z - 1 \to 0$ meaning $A = 1$.

One more step. Now $z^2(R_z - 1) \to B$ and $zR_z \to 1$ implies $TzR_z \to B$, while $zR_z u \to u$ for all u and since the range of $zR_z = \mathrm{dom}\,(T)$ we have $Tv = Bv$, and thus T is bounded.

$\square$

In particular for any closed operator (bounded or unbounded) the extended spectrum is nonempty.

15 Projections, spectral projections

This is an important ingredient in understanding operators and in spectral representations.

Note. The spectrum $\sigma(T)$ does not have to be connected for the spectral theorem to hold. Indeed, if $\sigma(T) \subset K_1 + K_2$ (disjoint union) it means there exist $\mathcal{O}_1 \supset K_1$ and $\mathcal{O}_2 \supset K_1$ open sets, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. A function analytic on $\mathcal{O}_1 \cup \mathcal{O}_2$ is any pair of functions $(f_1, f_2) =: f$ so that f_i is analytic in $\mathcal{O}_i$, $i = 1, 2$. Check that, if $\mathcal{C}_i \in \mathcal{O}_i$, then $f(z) = \oint_{\mathcal{C}_1 + \mathcal{C}_2} f(s)/(s - z)ds$ and thus the definition of $f(a)$ in this case is the same,

$$f(a) = \oint_{\mathcal{C}_1 + \mathcal{C}_2} f(s)(s - a)^{-1} ds$$

Let us first look at operators. Assume for now the operator is bounded, though we could allow unbounded operators too.

Definition 65. Let $\mathcal{A}$ be a Banach algebra, and $P \in \mathcal{A}$, P is a projector if $P^2 = 1$ [(9)]. Perhaps the simplest example, and a relevant one as we shall see is a characteristic function χ_A in L^∞.

General properties.

1. Assume $P \in \mathcal{L}(X)$ is a projector. Then PX is a closed subspace of X and P is the identity on PX.

   Proof. Assume $Px_k \to z$. Continuity of P implies $P^2 x_k$ converges to Pz, thus $z = Pz$, and thus $z \in PX$. We have $x = Py \Rightarrow Px = P^2 y = Py = x$ and thus $Px = x$. $\square$

2. If T and P commute, and $X_P = PX$ then $TX_P \subset X_P$, that is T can be restricted to X_P. Indeed, $TX_P = TPX_P = PTX_P \subset X_P$.

Let us first look at operators. Assume for now the operator is bounded, though we could allow unbounded operators too.

As we have seen in 6 on p. 30, $\lambda \in \sigma(T)$ iff $T - \lambda$ is not bijective.

Assume now that the spectrum of T is not connected. Then $\sigma(T) = K_1 + K_2$ where K_1 and K_2 are compact, nonempty, and disjoint (of course, K_1 and K_2 could be further decomposable).

---

(9) There is no good notion of self-adjointness in Banach spaces.

Theorem 17 (Elementary spectral decomposition)**.** *Let $T \in \mathcal{L}(X)$ an operator such that $\sigma(T) = K_1 + K_2$, $K_{1,2}$ as above.*

Then there exist nonzero closed subspaces of X, X_1 and X_2 so that

(i) $X = X_1 + X_2$, $X_1 \cap X_2 = \{0\}$ (X is isomorphic to $X = X_1 \oplus X_2$).

(ii) $TX_i \subset X_i$ and $\sigma_{X_i}(T) \subset K_i$.

Proof. 1. Let $\mathcal{O}_{1,2} \supset K_{1,2}$ be two open disjoint sets in $\mathbb{C}$, $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, and let $f_i(z) = \chi_{\mathcal{O}_i}(z)$. Note that $f_i(z)$ are *analytic* in $\mathcal{O}$ (the parts are disjoint; f is differentiable in each piece).

2. Let $P_i = \chi_{\mathcal{O}_i}(T)$. Since $\chi^2_{\mathcal{O}_i}(z) = \chi_{\mathcal{O}_i}(z)$, we have $P_i^2 = P_i$, that is, P_i are projectors.

3. Similarly, we have $P_1P_2 = \chi_1(T)\chi_2(T) = (\chi_1\chi_2)(T) = 0$.

4. We have $P_1 + P_2 = 1$ (the identity). Indeed $\chi_{\mathcal{O}_1} + \chi_{\mathcal{O}_2} = \chi_{\mathcal{O}}$. Then,

$$\oint_{\mathcal{C}_1\cup\mathcal{C}_2} \frac{\chi_{\mathcal{O}_1}(s) + \chi_{\mathcal{O}_2}(s)}{s-a} ds = \oint_{\mathcal{C}_1\cup\mathcal{C}_2} \frac{\chi_{\mathcal{O}}(s)}{s-a} ds = \oint_{\mathcal{C}_1\cup\mathcal{C}_2} \frac{1}{s-a} ds = 2\pi i$$

(recall the calculation (102)).

5. Let $X_i = P_iX$. These are closed subspaces of X by 1.

6. If $x \in P_1X$ and $x \in P_2X$, then $x = P_1x_1 = P_2x_2$. We multiply the latter equality by P_1. We get $P_1x = P_1^2x_1 = P_1x_1 = P_1P_2x_2 = 0$. Similarly, $P_2x = 0$. Thus $x = P_1x + P_2x = 0$ and $X_1 \cap X_2 = \{0\}$.

7. Since $1 = P_1 + P_2$, any $x \in X$ can be written as $P_1x + P_2x = x_1 + x_2$, $x_i \in X_i$.

8. We have $\sigma(P_iT) = \sigma(\chi_i(T)Ide(T))$ (where, here, $Ide(z) = z$) $= (z\chi_i(z))(\sigma(T)) = K_i$.

□

16 Analytic functional calculus for unbounded, closed operators with nonempty resolvent set

As we have seen, the spectrum of an unbounded closed operator can be any closed set (that it is necessarily closed we will see shortly), including $\mathbb{C}$ and $\emptyset$. (The extended spectrum is never empty though, as we have seen). If the spectrum of T is the whole of $\mathbb{C}$, little more can be said in general. Will shall assume from this point on that $\rho(T) \neq \emptyset$.

Also, calculus with better behaved operators (normal, self-adjoint) is richer, and we will later focus on that.

If an operator is closed, then it is invertible iff it is bijective, 6, on p. 30.

Evidently, the domain of $T - z$ is the same as the domain of T, so $T - z; z \in \mathbb{C}$ share a common domain. So $z \in \sigma(T)$ iff $T - z$ is not bijective from $D(T)$ to

Y. With this remark, the proof in the first resolvent formula goes through essentially without change.

Assume in the following that $X = Y$.

Proposition 66 (First resolvent formula, closed unbounded case)**.** *(i) $\mathbb{C} \setminus \sigma_\infty(T) = \rho(T)$ is open (possibly empty).*

(ii) Assume $\rho(T) \neq \emptyset$ and $(s,t) \in \rho(T)$. Then,

$$R_s(T) - R_t(T) = (t-s)R_t(T)R_s(T) \tag{116}$$

In particular R_s and R_t commute.

Proof. We assume that $\rho(T) \neq \emptyset$. We start with (116), assuming for the moment that there are at least *two* elements, $s, t \in \rho(T)$. Of course, this will follow once we have proved that $\rho(T)$ is open. The proof is not circular, check this.

Once more, since T is closed, $s \in \rho(T)$ **iff** $(s - T)$ is bijective between $D(T) \subset X$ and X. In particular, $x \in D(T)$ is zero iff $(s-T)x = 0$. By the definition of the resolvent, $R_s(T)$ is a bounded bijection between Y and $D(T)$. So is $R_t(T)$. In particular, $[R_s(T) - R_t(T)]x$ and $R_t(T)R_s(T)x \in D(T)$. If we apply $(t - T)$ to both sides we get

$$\{s-T+(t-s)\}R_s-(t-T)R_t = (t-s)R_s \;\; (s-t)(t-T)R_tR_s = (s-t)R_s \tag{117}$$

so the equality checks $\forall x \in X$.

Suppose R_t exists, $\|R_t\| = m$, $\epsilon < 1/m$, and $|s-t| < \epsilon$. *Define* R_s from the first resolvent formula, solved formally first:

$$R_s - (t-s)R_sR_t = R_s(1-(t-s)R_t) = R_t \Rightarrow R_s = R_t(1-(t-s)R_t)^{-1} \tag{118}$$

Exercise 1. Define $R_s := R_t(1-(t-s)R_t)^{-1}$. Find the domains and ranges of $(s-T)R_s$ and $R_s(s-T)$ and show that R_s is the resolvent of $(s-T)$.

□

16.1 Analytic functions of unbounded, closed operators

Since in this section we specifically deal with unbounded operators, we shall will always have $\infty \in \sigma_\infty(T)$, see Proposition 64.

In the following, we **assume that** $\sigma_\infty(T) \neq \mathbb{C}_\infty$, to be able to define nontrivial analytic functions on $\sigma_\infty(T)$.

Note 12. Note that a function analytic on $\sigma_\infty(T)$ is therefore analytic in an open set $\mathcal{O}_\infty$ containing ∞. Clearly, the complement in $\mathbb{C}$ of $\mathcal{O}_\infty$ is compact. So f is analytic in $\text{Ext}(K)$ for some K and on the *rest of the spectrum* of T, necessarily contained in a compact set. We can break $\sigma_{\mathbb{C}}(T)$ in a (possibly infinite, possibly consisting of just one set) disjoint union of connected compact sets. Each connected component K_α is contained in a connected $\mathcal{O}_\alpha$, where all

$\mathcal{O}_\alpha$ are disjoint. By the finite covering theorem, f is analytic in $\text{Ext}(K)$ and in $\cup_{i=1}^n \mathcal{O}_i$ where $\mathcal{O}_i$ are open and connected. Now, for each of these, we can assume that the boundary is a polygonal arc (check!). In fact, by a similar construction, we can take the boundary to be an analytic curve. How?

Recall that, from (107), if $z \in Ext\ \mathcal{C}$ we have

$$f(z) = f(\infty) + \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(s)}{s-z} ds \tag{119}$$

1. An identity: If f is analytic at infinity, and $0 \in \text{Ext}_\infty(\mathcal{C})$ that is, to the right of $\mathcal{C}$ traversed clockwise, where $\mathcal{C}$ is a curve or system of curves so that f is analytic in $\text{Int}_\infty \mathcal{C}$ then we have

$$\frac{1}{2\pi i} \oint s^{-1} f(s) ds = -f(\infty) \tag{120}$$

   (The left side is *not* necessarily $f(0)$ which could be undefined! Even if defined, there are typically other singularities of f in $\text{Ext}_\infty(\mathcal{C})$.) To prove (120), note that

$$f(z)/z = 0 + \frac{1}{2\pi i} \oint \frac{f(s)}{s(s-z)} ds = -\frac{1}{2\pi i z} \oint \frac{f(s)}{s} ds + \frac{1}{2\pi i z} \oint \frac{f(s)}{s-z} ds$$
$$= -\frac{1}{2\pi i z} \oint \frac{f(s)}{s} ds + \frac{1}{z} \left(-f(\infty) + f(z)\right) \tag{121}$$

   and the conclusion follows after multiplication by z.

2. Let f be analytic on $\mathcal{O} \supset K = \sigma_\infty(T)$. We take $\Delta \subset \mathcal{O}$ be an open set between $\mathcal{O}$ and K , such that its boundary consists of a finite number of nonintersecting simple Jordan curves, positively oriented with respect to infinity. We can always reduce to this case as explained at the beginning of the section.

3. Analytic functions on the spectrum are now functions analytic at infinity as well. Let f be such a function.

4. Therefore, we define for such an f and a (multi) contour $\mathcal{C}$ such that the spectrum of T lies in $Ext\ \mathcal{C}$, or in the language of item 2 above the spectrum F is in Δ and $\mathcal{C} = \partial\Delta$,

$$f(T) = f(\infty) + \frac{1}{2\pi i} \oint_{\mathcal{C}} f(s)(s-T)^{-1} ds \tag{122}$$

Note that this is a bounded operator, by the definition of the spectrum and the properties of integration. Note also that now we don't have the luxury to define polynomials first, etc.

Note 13 (Independence of contour)**.** *Assume $\mathcal{C}_1$ is a contour homotopic to $\mathcal{C}_2$ in $\mathcal{D}_1$, inFig. 2. Then*

$$\int_{\mathcal{C}_1} f(s)(s-T)^{-1}ds = \int_{\mathcal{C}_2} f(s)(s-T)^{-1}ds \tag{123}$$

(As usual, this can be checked using functionals.)

If φ is a linear functional in X^*, since $(s-T)^{-1}$ is analytic outside the spectrum of T, $\mathcal{C}_i$, and $\mathcal{C}_1$ and $\mathcal{C}_2$ are as above, we have

$$\int_{\mathcal{C}_1} f(s)\varphi(s-T)^{-1}ds = \int_{\mathcal{C}_2} f(s)\varphi(s-T)^{-1}ds \tag{124}$$

That is,

$$\varphi\int_{\mathcal{C}_1} f(s)(s-T)^{-1}ds = \varphi\int_{\mathcal{C}_2} f(s)(s-T)^{-1}ds \tag{125}$$

and therefore, the contour of integration is immaterial in the definition of $f(T)$, modulo homotopies.

Proposition 67. *Let f and g be analytic in $\mathcal{O}\supset\sigma_\infty(T)$. Then $(fg)(T) = f(T)g(T) = g(T)f(T)$.*

Proof. The proof, using the first resolvent formula, is very similar to the one in the bounded case and left as an exercise. □

Proposition 68. *(i) Assume f is as in item 3, and satisfies $f(\infty) = 0$. Then $f(T)X \subset D(T)$.*

(ii) f be as above and let $g = zf(z)$. Then clearly g is analytic at infinity. We have $g(T) = Tf(T) = f(T)T$. That is, also, $(zf(z))(T) = Tf(T)$.

Recall the fundamental result of commutation of closed operators with integration, Theorem 14.

For convenience, we repeat it here:

Theorem (14, p. 34 above) Let T be a closed operator and $a \in B(\Omega, X)$ be such that $a(\Omega) \subset \mathcal{D}(T)$. Assume further that T is measurable, in the sense that $Ta \in B(\Omega, X)$. Under the assumptions above we have

$$T\int_A a(\omega)d\mu(\omega) = \int_A Ta(\omega)d\mu(\omega) \tag{126}$$

(In particular $\int_A a(\omega)d\mu(\omega) \in D(T)$ if $a(\omega) \in D(T)$ for all ω.) Check that the following satisfies all requirements w.r.t. T.

$$a(s) = (s-T)^{-1}$$

See also Note 8.

Proof. We can assume that the contour of integration does not pass through 0; in case it did, then 0 would be in the domain of analyticity of f and we can deform the contour around zero.

By definition we have

$$\begin{aligned} g(T) = g(\infty) + \frac{1}{2\pi i}\oint_{\mathcal{C}} sf(s)(s-T)^{-1}ds \\ = g(\infty) + \frac{1}{2\pi i}\oint_{\mathcal{C}} (s-T+T)f(s)(s-T)^{-1}ds \\ = g(\infty) + \frac{1}{2\pi i}\oint_{\mathcal{C}} s^{-1}g(s)ds + \frac{1}{2\pi i}\oint_{\mathcal{C}} Tf(s)(s-T)^{-1}ds \\ = \frac{1}{2\pi i}\oint_{\mathcal{C}} Tf(s)(s-T)^{-1}ds = T\frac{1}{2\pi i}\oint_{\mathcal{C}} f(s)(s-T)^{-1}ds \end{aligned} \quad (127)$$

where we used (120) and Theorem 14. □

Corollary 69. *If $\rho \in \rho(T)$ and $f(z) = 1/(\rho - z)$, then $(\rho - T)^{-1} = f(T)$, and we can again write*

$$(\rho - T)^{-1} = \frac{1}{\rho - T}$$

Proof. Evidently, by a change of T, we can assume that $\rho = 0$. Take $f(z) = 1/z$. We see that $f(\infty) = 0$, Proposition 68 applies and thus

$$1 = (zf(z))(T) = Tf(T) = f(T)T$$

□

Proposition 70. *Let $T \in \mathcal{C}(X)$ and $f \in H(\sigma_\infty(T), \mathbb{C})$. Then, $\sigma_\infty(f(T)) = f(\sigma_\infty(T))$.*

Proof. The proof is similar to the one in the bounded case. If $\rho \notin f(\sigma(T))$ then $\rho - f(z)$ is invertible on $\sigma(T)$, and let the inverse be g. Then (note that g is always a bounded operator),

$$(\rho - f(z))g(z) = g(z)(\rho - f(z)) = 1 \Rightarrow g(T)(\rho - f(T)) = (\rho - f(T))g(T) = 1$$

and thus $\rho \notin \sigma(f(T))$. Conversely, assume that $f(\sigma_0) \in f(\sigma_\infty(T))$, but that $f(T) - f(\sigma_0)$ was invertible. Without loss of generality, by shifting T and f we can assume $\sigma_0 = 0 \in \sigma_\infty(T)$ and $f(0) = 0$. Then, $f(z) = zg(z)$ with g analytic at zero (and on $\sigma_\infty(T)$)). Then $g(\infty) = 0$ and thus, since $f(T)$ was assumed invertible, we have

$$f(T) = Tg(T) = g(T)T, \text{ or} 1 = T[g(T)f^{-1}(T)] = [g(T)f^{-1}(T)]T \quad (128)$$

so that T is invertible and thus $0 \notin \sigma_\infty(T)$, a contradiction. □

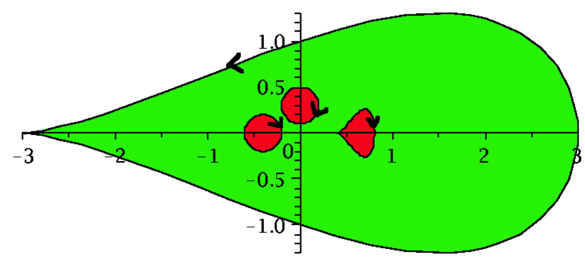


Figure 1: Analyticity near zero

16.2 *Analyticity at zero and at infinity: a discussion*

Consider an open set $\mathcal{D} \subsetneq \mathbb{C}_\infty$, including a neighborhood of infinity and let g be analytic in $\mathcal{D}$. (Functions analytic on $\mathbb{C}_\infty$ are constant.) We then take a point in $\mathbb{C}_\infty \setminus \mathcal{D}$ –say the point is zero– and make an inversion: define $f(z) = g(1/z)$ defined in $D_1 = 1/\mathcal{D} := \{1/z : z \in \mathcal{D}\}$. Then f is analytic in $\mathcal{D}_1$. If $\mathcal{D}_2 \subset \mathcal{D}_1$ is a multiply connected domain whose boundary is a finite union of disjoint, simple Jordan curves, then Cauchy's formula still applies, and we have

$$f(z) = \frac{1}{2\pi i} \oint_{\partial \mathcal{D}_2} \frac{f(s)}{s - z} ds \tag{129}$$

where the orientation of the contour is as depicted. Green regions are regions of analyticity, red ones are excluded regions. The first region is an example of a relatively compact $\mathcal{D}$. Cauchy's formula (129) applies on the boundary of the domain, the integral gives $f(z)$ at all $z \in \mathcal{D}_2$. The orientation of the curves must be as depicted.

The second domain is $\mathcal{D} = \{1/z : z \in \mathcal{D}_2\}$. It is the domain of analyticity of $g(z) = f(1/z)$ and it includes ∞. Here we can apply Cauchy's formula at infinity to find g in the green region.

The orientation of the contours is the image under $z \to 1/z$ of the original orientations.

If the spectrum of an operator is contained in the green region in the second figure (infinity included, clearly) then the contours should be taken as depicted.

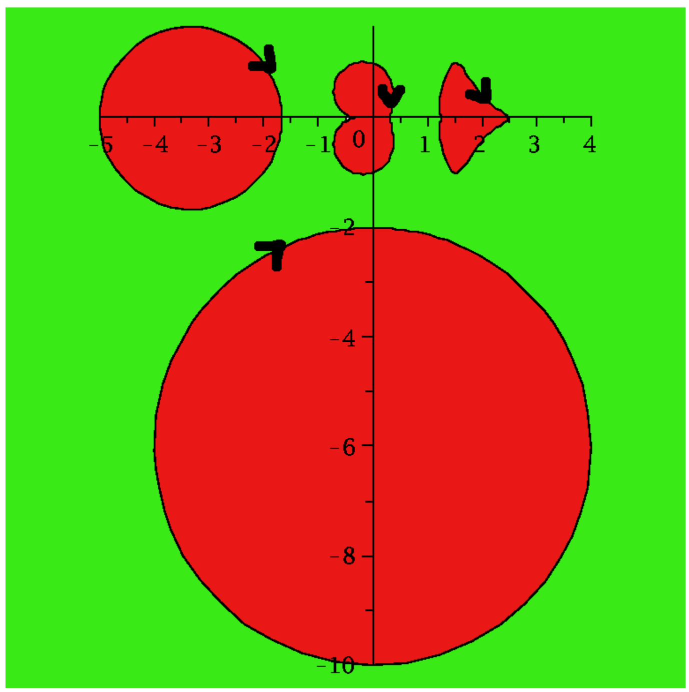


Figure 2: Analyticity near ∞

Assume as before that $\sigma_\infty(T) \neq \mathbb{C}_\infty$. Then there is a $z_0 \in \mathbb{C} \setminus \sigma_\infty(T)$ and we assume without loss of generality that $z_0 = 0$. We can define, as before, the set $\sigma_1 = 1/\sigma_\infty(T) = \{1/z : z \in \sigma_\infty(T)\}$.

Then a function f is analytic on $\mathcal{D} \supset \sigma_\infty(T)$ iff $f(1/z)$ is analytic on $\sigma_1(T)$. The spectrum $\sigma_\infty(T)$ is contained in $\mathcal{D}$ iff $1/\mathcal{D} \supset 1/\sigma_\infty(T)$. A curve, or set of curves, gives the value of $f(T)$ iff the curves are so chosen that $1/\sigma_\infty(T)$ is contained in the domain defined by the curves.

If f is nontrivial and analytic on $\sigma_\infty(T)$, then

Proposition 71. *The Banach algebra of analytic functions on $\mathcal{O}$ with the sup norm is isomorphic to the algebra of bounded operators $f[T]$, in the operator norm.*

Proof. Linearity, continuity etc are proved as before. Multiplicativity could also be proved by density, taking say polynomials in $1/(z - z_0)$, $z_0 \notin \sigma(T)$ as a dense set.

Alternatively, it could be proved directly from the definition (122) by Fubini.

In any case, the analysis is rather straightforward and we leave all details to the reader. □

Finally,

Proposition 72. *Assume $\sigma_\infty(T) = K + S$ (disjoint union) where K is compact in $\mathbb{C}$. Let $\mathcal{O} \supset K$ be open, relatively compact and disjoint from S, and (w.l.o.g.) with rectifiable boundary $\mathcal{C}$. Then*

$$P_K = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{1}{z - T} dz \tag{130}$$

defines a projector such that

(i) $P_K(X) \subset D(T)$, $T(P_K(X)) \subset P_K(X)$.

(ii) $\sigma(T|_{P_K(X)}) = K$.

(iii) T restricted to $P_K(X)$ is bounded.

Proof. (Compare with §15.) Let $P = P_K, \chi = \chi_K$, First, we note that $P = \chi(T)$ and χ is analytic on $\sigma_\infty(T)$, $\chi(\infty) = 0$ and thus, by Proposition 68 (i), PX is in $D(T)$. $P^2 = P$ since $\chi^2 = \chi$. Thus P is a projector, as in §15. By Prop. 68 (ii) we have $TP = (z\chi)(T) = (\chi z)(T) = PT$. Thus $TP = TP^2 = PTP$, and $TPX \subset PX$, and thus $TP|_{PX} \subset PX$. We further have $\sigma_\infty(TP) = \chi(\sigma_{\infty(T)}) = K$. Since $\infty \notin \sigma_\infty(TP)$, TP is bounded.

□

16.3 Bounded self-adjoint and normal operators on a Hilbert space $\mathcal{H}$

Note 14. Assume A is bounded on the Hilbert space $\mathcal{H}$ and self-adjoint, that is $\langle Ax, y\rangle = \langle x, Ay\rangle$ $\forall x, y \in \mathcal{H}$ (this condition is not enough if A is unbounded). Then, by the Gelfand transform theory, $\sigma(A) = K \subset \mathbb{R}$.

For a bounded operator to be self-adjoint, it suffices that it is symmetric, that is $\langle Ax, y\rangle = \langle x, Ay\rangle$. Note that if A is self-adjoint, then $\|A^2\| = \|A\|^2$.

From our excursion in Banach algebras we know that $\|A^2\| = \|A\|^2$ [(10)] and, more generally, $\|A^n\| = \|A\|^n$. Recall also that $\|A\| = \sup|\sigma(A)| = R(A)$. which holds more generally for normal operators.

17 Unbounded operators: adjoints, self-adjoint operators etc.

In this section we work with operators in Hilbert spaces.

1. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces (we will most often be interested in the case $\mathcal{H} = \mathcal{K}$), with scalar products $\langle,\rangle_{\mathcal{H}}$ and $\langle,\rangle_{\mathcal{K}}$.

2. $T : D(T) \subset \mathcal{H} \to \mathcal{K}$ is **densely defined** if $\overline{D(T)} = \mathcal{H}$.

3. Assume T is densely defined.

4. The adjoint of T is defined as follows. We look for those y for which
$$\exists\ v = v(y) \in \mathcal{H} \text{ s.t.} \forall\ x \in D(T),\ \langle y, Tx\rangle_{\mathcal{K}} = \langle v, x\rangle_{\mathcal{H}} \tag{132}$$
Since $D(T)$ is dense, such a $v = v(y)$ is unique.

5. We define $D(T^*)$ to be the set of y for which $v(x)$ exists $\forall\ x \in D(T)$, and define $T^*(y) = v$. Note that $Tx \in \mathcal{K}$, $y \in \mathcal{K}$, $T^*y \in \mathcal{H}$.

6. Check that $D(T^*)$ is a linear space and $T^*(a_1y_1 + a_2y_2) = a_1T^*(y_1) + a_2T^*(y_2)$ that is, T^* is linear.

7. **Definition.** We write $T_1 \subset T_2$ iff $D(T_1) \subset D(T_2)$, and $T_1 = T_2$ on $D(T_1)$.

   Exercise 1. *Show that $T_1 \subset T_2 \Rightarrow T_2^* \subset T_1^*$. (A short proof is given in Corollary 77.)*

8.

   Proposition 73. *In the setting of 1 and 3, T^* is closed.*

   Proof. Let $y_n \to y$ and $T^*y_n \to v$. Then, for any $x \in D(T)$ we have
$$\langle y_n, Tx\rangle =: \langle T^*y_n, x\rangle \to \langle v, x\rangle = \lim\langle y_n, Tx\rangle = \langle y, Tx\rangle$$
   Thus, for any $x \in D(T)$ we have $\langle y, Tx\rangle = \langle v, x\rangle$ and thus, by definition, $y \in D(T^*)$ and $T^*y = v$. ☐

[(10)] Direct proof: with $\|u\| = 1$, we have
$$\sup_{\|u\|=1} \|A^2\| \geqslant \sup_{\|u\|=1} \|\langle A^2u, u\rangle\| = \sup_{\|u\|=1} \|\langle Au, Au\rangle\| = \sup_{\|u\|=1} \|Au\|^2 = \|A\|^2 \tag{131}$$
. Since $\|A^2\| \leqslant \|A\|^2$, the result follows.

9.

Proposition 74. *Let T and $D(T)$ be as in 1 and 3. Then,*

(i) For any $\alpha \in \mathbb{C}$, we have $(\alpha T)^* = \overline{\alpha} T^*$.

(ii) If $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, then T^* is the usual adjoint, everywhere defined, and for two such operators, we have $(T_1 + T_2)^* = T_1^* + T_2^*$.

Proof. Exercise. □

Proposition 75. *(i) Let T_i and $D(T_i)$ be as in 1 and 3. Assume furthermore that $D(T_1) \cap D(T_2)$ is dense. Then $(T_1 + T_2)^* \supset T_1^* + T_2^*$ (this means that the domain is larger, and wherever all adjoints make sense, we have $(T_1 + T_2)^* = T_1^* + T_2^*$).*

(ii) Let $T_1 : \mathcal{H} \to \mathcal{K}_1$, $T_2 : \mathcal{H} \to \mathcal{K}_2$ and assume $D(T_2)$ as well as $D(T_2 T_1)$ are dense. Then, $(T_2 T_1)^* \supset T_1^* T_2^*$.

Proof. We define $T_1 + T_2$ on $D(T_1) \cap D(T_2)$. Let $x \in D(T_1) \cap D(T_2)$ and assume that $y \in D(T_1)^* \cap D(T_2)^*$. This is, by definition, the domain of $T_1^* + T_2^*$. Then

$$\begin{aligned} \langle y, (T_1 + T_2)x \rangle &= \langle y, T_1 x + T_2 x \rangle = \langle y, T_1 x \rangle + \langle y, T_2 x \rangle \\ &= \langle T_1^* y, x \rangle + \langle T_2^* y, x \rangle = \langle T_1^* y + T_2^* y, x \rangle =: \langle (T_1 + T_2)^* y, x \rangle \end{aligned} \quad (133)$$

and thus $y \in D(T_1 + T_2)^*$, and $(T_1 + T_2)^* = T_1^* + T_2^*$.

(ii) Let $x \in D(T_2 T_1)$ (note that, by definition, $D(T_2 T_1) \subset D(T_1)$) and $w \in D(T_1^* T_2^*)$. Then,

$$\langle T_1^*(T_2^* w), x \rangle = \langle (T_2^* w), T_1 x \rangle = \langle w, T_2 T_1 x \rangle \quad (134)$$

and thus $w \in D(T_2 T_1)^*$ etc. □

17.1 Example: An adjoint of d/dx

There are much shorter ways to obtain the results in this section, but for practicing with operators and adjoints, we will proceed here "the hard way".

1. Let $T = d/dx$ be defined on $C^1[0,1]$. We need to see for which y do we have $\langle y, Tf \rangle = \langle w, f \rangle \ \forall f \in D(T)$, that is, $\forall f \in D(T)$

$$\int_0^1 y(s) \frac{df(s)}{ds} ds = \int_0^1 w(s) f(s) ds$$

Let $h(s) = \int_0^s w(t)dt$. Then, $h \in AC[0,1]^{L_2} := \{AC[0,1] \cap \{f : f' \in L^2[0,1]\}\}$ and we can integrate by parts

$$\int_0^1 y(s) \frac{df(s)}{ds} ds = \int_0^1 \frac{dh(s)}{ds} f(s) ds = f(1)h(1) - \int_0^1 h(s) \frac{df(s)}{ds} ds \quad (135)$$

hence

$$\int_0^1 (y(s) + h(s)) \frac{df(s)}{ds} ds = f(1)h(1) \tag{136}$$

Let $v \in C[0,1]$ be arbitrary and take $f = \int_1^x v$. Then $f \in C^1[0,1]$, $f' = v$ and $f(1) = 0$. Then,

$$\int_0^1 (y(s) + h(s)) v(s) ds = 0 \tag{137}$$

Note that the set of such v is dense. We can also set, for each $v \in C[0,1]$ $f = \int_1^x v + 1$. As before, $f \in C^1[0,1]$, $f' = v$ but now $f(1) = 1$. Thus, also on a dense set,

$$\int_0^1 (y(s) + h(s)) v(s) ds = h(1) \tag{138}$$

Thus $h(1) = 0$. Also, by density, say from (137), $y(s) + h(s) = 0$. Now h is by construction in $\mathrm{AC}[0,1]^{L_2} := \{AC[0,1] \cap \{f : f' \in L^2[0,1]\}\}$, and so is then y, $h(0) = 0$ and $h(1) = 0$ implies $y(1) = y(0) = 0$. Thus, $D(T^*) = AC[0,1]_{01}^{L_2}$ where the subscript 01 indicates that the function vanishes at both ends. Clearly, on $D(T^*)$ we have $(d/dx)^* = -d/dx$.

17.2 Symmetric does not mean self-adjoint

One property we certainly want to preserve is that $\sigma(A) \subset \mathbb{R}$ for a self-adjoint operator. Note that $Ti = d/dx$ defined on $\{f \in C^1[0,\infty)_0 \cap L^2(\mathbb{R}^+) : f' \in L^2(\mathbb{R}^+)\}$, the subscript 0 indicating that the functions vanish at zero, is symmetric on its domain, just by integration by parts. But note that, for $\lambda > 0$, $i\lambda \in \sigma(T)$, since $f' - \lambda f = g$, $f(0) = 0$ has the unique solution $f(x) = e^{\lambda x} \int_0^x e^{-\lambda s} g(s) ds$, which is not everywhere defined (apply it to $e^{-\lambda s}$. So $i\lambda \in \sigma(T)$, and in fact, all λ with $\mathrm{Im}\,\lambda > 0$ are in the spectrum.

17.3 Operations on graphs and graph properties

From now on, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. Recall that $\mathcal{H} \otimes \mathcal{K}$ is a Hilbert space under the scalar product

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle \tag{139}$$

1. Recall that the graph of an operator T is the set of pairs

$$\mathbb{G}(T) = \{(x, Tx) : x \in D(T)\} \subset \mathcal{H} \otimes \mathcal{K}$$

Remember also that an operator T is closed iff $\mathbb{G}(T)$ is a closed set, and it is closable iff the closure of $\mathbb{G}(T)$ is the graph of an operator.

Exercise 2. *If T is closable, then* $\mathbb{G}(\overline{T}) = \overline{\mathbb{G}(T)}$

17.3.1 A formula for the adjoint

1. Let us note something simple but very important. Assume $y \in D(T^*)$. Then

$$\langle w, x\rangle_{\mathcal{H}} - \langle y, Tx\rangle_{\mathcal{K}} = 0 \Rightarrow (-T^*y, y) = V(y, T^*y) = (-w, y) \perp (x, Tx) \tag{140}$$

where $V : \mathcal{K} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ is defined by

$$V\ (x, y) = (-y, x) \tag{141}$$

Exercise 3. *Compute the adjoint of V (in $\mathcal{K} \otimes \mathcal{H}$) and show that $VV^* = V^*V = I$, that is, V is unitary. Then $V(E^\perp) = (V(E))^\perp$ for any subspace of $\mathcal{K} \otimes \mathcal{H}$. Clearly also, if mathcalH $= \mathcal{K}$, then $V^2 = -1$.*

Thus there is a simple link between $\mathbb{G}(T)$ and $\mathbb{G}(T^*)$:

Lemma 76 (Graph of the adjoint)**.** *We have*

$$\mathbb{G}(T^*) = V[\mathbb{G}(T)]^\perp$$

Proof. Of course, $(a, b) \perp \mathbb{G}(T)$ **iff**

$$\forall x \in D(T) : \quad (a, b) \perp (x, Tx) \Leftrightarrow \langle a, x\rangle = \langle -b, Tx\rangle$$

that is, by definition, **iff** $b \in D(T^*)$ and $b = -T^*a$. That is,

$$\mathbb{G}(T)^\perp = \{(-T^*a, a) : a \in D(T^*)\}$$

and

$$V(\mathbb{G}(T))^\perp = \{(a, T^*a) : a \in D(T^*)\} = \mathbb{G}(T^*) \tag{142}$$

□

Corollary 77. $T_1 \subset T_2 \Rightarrow T_2^* \subset T_1^*$.

Proof. This follows from the reformulation above. Alternatively, $D(T_1) \subset D(T_2)$ and $T_1 = T_2$ on $D(T)$ means that $D(T_2)^* \subset D(T_1)$ and then for $y \in D(T_2)^*$ and $x \in D(T_1)$ we have $\langle T_2^*y, x\rangle = \langle y, T_1x\rangle = \langle T_1^*y, x\rangle$ etc. □

From now on, **we assume** for simplicity that $\mathcal{H} = \mathcal{K}$.

2.

Theorem 18. *Let T be densely defined. Then,*

(i) T^* is closed.

(ii) T is closable **iff** *$D(T^*)$ is dense, and then $\overline{T} = T^{**}$.*

(iii) If T is closable, then the adjoints of T and of $\overline{T}$ coincide.

Proof. (i) By (142), T^* is closed (since its graph is closed).

(ii) Assume $D(T^*)$ is dense. Then T^{**} is well defined. By (142) its graph is given by

$$\mathbb{G}(T^{**}) = (V\mathbb{G}(T^*))^\perp = (V(V(\mathbb{G}(T)))^\perp)^\perp \\ = (V^2\mathbb{G}(T)^\perp)^\perp = (\mathbb{G}(T)^\perp)^\perp = \overline{\mathbb{G}(T)} \quad (143)$$

and thus $\overline{\mathbb{G}(T)}$ is the graph of an operator. Since T^* is densely defined, T^{**} exists, and by Lemma 76 the left side of (143) is $\mathbb{G}(T^{**})$. Conversely, assume $D(T^*)$ is not dense. Let $y \in D(T^*)^\perp$ $(y \neq 0)$; then $(y, 0)$ is orthogonal on all vectors in $\mathbb{G}(T^*)$, so that $(y, 0) \in (\mathbb{G}(T^*))^\perp$. Then, $V(y, 0) = (0, y) \in V(\mathbb{G}(T^*))^\perp = \overline{\mathbb{G}(T)}$ by (143), and thus $\overline{\mathbb{G}(T)}$ is not the graph of an operator.

(iii) We have by definition $\mathbb{G}(\overline{T}) = \overline{\mathbb{G}(T)}$ and thus

$$\mathbb{G}(\overline{T}^*) = V[(\overline{\mathbb{G}(T)})^\perp] = V(\mathbb{G}(T)^\perp) = \mathbb{G}(T^*) \quad (144)$$

since $E^\perp = \overline{E}^\perp$ for any E.

□

Proposition 78. *Assume T is as in 4. Then (i)* $\ker(T^*) = \operatorname{ran}(T)^\perp$ *and (ii)* $\ker(T)^\perp = \overline{\operatorname{ran}(T^*)}$.

Proof. The proof is straightforward; it can be done from the graph approach or directly by interpreting the equality $\langle T^*y, x\rangle = 0 = \langle y, Tx\rangle$ where y spans $\ker(T^*)$ and x spans $D(T)$, or x spans $\ker(T)$ and y spans $D(T^*)$. For instance

$$y \in \ker(T^*) \;\Leftrightarrow\; (y \in D(T^*) \text{ and } T^*y = 0) \\ \Leftrightarrow \langle T^*y, x\rangle = 0 = \langle y, Tx\rangle \;\forall x \in D(T) \quad (145)$$

□

17.4 Self-adjoint operators

1. Assume now that $\mathcal{K} = \mathcal{H}$.

2. **Definition: Symmetric (or Hermitian) operators.** Let T be densely defined on Hilbert space. T is symmetric if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $\{x, y\} \subset D(T)$. This means that $T \subset T^*$.

3. A symmetric operator is always closable, by Theorem 18 since T^* is densely defined, since $T^* \supset T$, by 2.

4. If T is closed, then $\mathbb{G}(T) = \mathbb{G}(\overline{T})$ and thus by Theorem 18,

$$T = T^{**} \quad (146)$$

5. Note that, if T is symmetric, then $\langle Tx, x\rangle \in \mathbb{R}$, since

$$\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$

   and also, for any $x \in D(T)$, $\|(T \pm i)x\|^2 = \|Tx\|^2 + \|x\|^2$, since

$$\|(T \pm i)x\|^2 = \langle Tx, Tx\rangle + \langle x, x\rangle \pm \langle Tx, ix\rangle \pm \langle ix, Tx\rangle = \langle Tx, Tx\rangle + \langle x, x\rangle \tag{147}$$

6.

   Lemma 79. *If T is* **symmetric and** *(assume already)* **closed***, then (i)* ran $(T \pm i)$ *is closed and (ii) $(T \pm i)$ are injective.*

   Proof. (ii) Injectivity is immediate from (147) above.

   (i) Let now $(T \pm i)x_n$ be a convergent sequence, or, which is the same, a Cauchy sequence. But then, by (147) Tx_n and x_n are both Cauchy sequences. Then x_n converges to x; since T is closed, $x \in D(T)$, and $Tx_n \to Tx$.

   Exercise 4. *Note that $\|(T \pm i)x\|$ and the norm on the (closed) graph $\mathbb{G}(T)$ coincide and that the space $(0, ran(T \pm i))$ is an orthogonal projection ($P = P^*$) of $\mathbb{G}(T \pm i)$. Provide an alternative proof of (i) based on this.*

   Definition: Self-adjoint operators. T is self-adjoint if $T = T^*$. It means $D(T) = D(T^*)$, and T is symmetric.

   □

7. It follows that **self-adjoint operators are closed.**

8. For self-adjoint operators we have from 4 that $T = T^* = T^{**}$.

   Lemma 80. *If T is* **closed and symmetric** *and T^** **is symmetric***, then T is self-adjoint.*

   Proof. By 2, $T \subset T^*$. Since T^* is symmetric, $T^* \subset T^{**}$ But $T^{**} = T$ since T is closed, thus $T^* \subset T$, hence $T^* = T$. □

17.5 The *minimal* and *maximal* operators based on d/dx: T_0 and T_m

Note 15. The maximal domain of classical differentiation within $L^2[0, 1]$ defined as the functional inverse of the bounded operator $\int^x$ (see also the interesting discussions in the chapter "The Fundamental Theorem of Calculus" in Rudin, Real and Complex analysis), [(11)] is $D_m := AC[0, 1]^{L_2} := \{f \in AC[0, 1] : f' \in L^2[0, 1]\}$. Indeed, $L^2 \subset L^1$ and thus ran $\int^x := \{\int^x g : g \in L^2\} = D_m$. We let T_m be id/dx with $D(T_m) = D_m$.

[(11)] There are other possible definitions, s.a. through Fourier transform, and the domain of the operator will depend on the definition.

Define

$$T_0 = id/dx \text{ on } D(T_0) = AC[0,1]_{01}^{L_2} := \{f \in AC[0,1], f' \in L^2[0,1] : f(0) = f(1) = 0\} \tag{148}$$

If $\{x, y\} \subset D(T_0)$, then a simple integration by parts shows that $\langle y, T_0 x\rangle = \langle T_0 y, x\rangle$, and thus $T_0 \subset T_0^*$.

The range of T_0 consists of the functions of average zero, since $\int_0^1 Ty = y|_0^1 = 0$ and conversely $y = g'$ has a solution g if $\int_0^1 y = 0$. Another way to phrase it is

$$\text{ran} T_0 = \mathbb{C}^\perp \tag{149}$$

Note 16. *Similarly,* $\text{ran}\,(T_0 \pm i) = (e^{\mp ix}\mathbb{C})^\perp$.

Lemma 81. *We have*

$$T_0^* = id/dx \quad on \quad D(T_0^*) = AC[0,1]^{L_2} \tag{150}$$

First proof. Elementary analysis implies that f' convergent and $f_n'(0) = 0$ implies $f_n \to f$ for some f and $f(0) = 0$ (likewise $f(1) = 0$). Thus T_0 is closed. T_m is the maximal domain for id/dx and $T_m = T_0^*$ (easy to check), T_m is not symmetric thus T_0 is not self-adjoint (nor does it admit self-adjoint extensions as we will see).

□

Second proof. Let τ be the operator in §**??**; we showed there that $\tau^* = T_0$. Thus T_0 is closed. We also proved before that τ is closed. Then $\tau = \tau^{**} = T_0^*$.

□

Direct proof. Assume that $y \in D(T_0^*)$. Then, there exists a $v \in L^2[0,1]$ such that for all $x \in D(T_0)$, we have

$$\langle y, T_0 x\rangle = \langle v, x\rangle \tag{151}$$

Since $v \in L^2[0,1]$, we have $v \in L^1[0,1]$, and thus $h(x) = \int_0^x v(s)ds \in AC[0,1]^{L_2}$, and $h(0) = 0$. With $u = \overline{x}$, it follows from (151) by integration by parts that

$$\int_0^1 y(s)\frac{du(s)}{ds} = -\int_0^1 \frac{dh(s)}{ds}u(s)ds = \int_0^1 h(s)\frac{du(s)}{ds}$$

Let $\varphi = y - h$. By (149), we have

$$\int_0^1 \varphi(s)w(s)ds = 0 \quad \forall w \in \mathbb{C}^\perp$$

This means that $\varphi \in \mathbb{C}^{\perp\perp} = \mathbb{C}$. Thus y is an element of $AC[0,1]_0^{L_2}$ plus an arbitrary constant, which simply means $y \in AC[0,1]^{L_2} = D(T_0^*)$. □

By essentially the same argument as in §**??**, we have $T_0^{**} = id/dx$ on $D(T_0)$. Thus T_0 is closed. We note that T_0^* is not symmetric. In some sense, T_0 is too small, and then T_0^* is too large.

Definition: Normal operators. The definition of an unbounded normal operator is essentially the same as in the bounded case, namely, operators which commute with their adjoints.

More precisely: an operator T which is **closed and densely defined** is normal if $TT^* = T^*T$. The questions of domain certainly become very important. The domain if T^*T is $\{x \in D(T) : Tx \in D(T^*)\}$. (Similarly for TT^*.) Thus, the operator T in (148) is not normal. Indeed, $D(T^*T)$ consists of functions in $AC[0,1]^{L_2}$ vanishing at the endpoints, with derivative in $AC[0,1]^{L_2}$, while $D(TT^*)$ consists of functions in $AC[0,1]^{L_2}$, with the derivative, in $AC[0,1]^{L_2}$, vanishing at the endpoints.

Definition: Essentially self-adjoint operators. A **symmetric** operator is essentially self-adjoint if its closure is self-adjoint. In the opposite direction, if T is closed, then a **core** for T is a set $D_1 \subset D(T)$ such that $\overline{T|_{D_1}} = T$.

Note 17. We will show that T^*T and TT^* are selfadjoint (under domain conditions, of course), for any **closed and densely defined** T. If an operator T is normal, then in particular $D = D(T^*T) = D(TT^*)$, and since D is dense and T is closed, we see that D is a core for both T and T^*.

Note 18. By Corollary 77, if $T_1 \subset T_2$ and T_i are self-adjoint, then $T_1 = T_2$. Indeed, $T_1 \subset T_2 \Rightarrow T_1 = T_1^* \supset T_2^* = T_2$.

Lemma 82. *(i) If T is essentially self-adjoint, then there is a unique self-adjoint extension, $\overline{T}$. (Phrased differently, if $S \supset T$ is self-adjoint, then $S = T^{**}$.)*

(ii) (Not proved now). Conversely, if T has only one self-adjoint extension, then it is essentially self-adjoint.

Proof. (i) By definition $\overline{T}$ is selfadjoint. Let S be any self-adjoint extension of T. By 7 above, S is closed. We have $T \subset S \Rightarrow \overline{T} \subset \overline{S} = S$. By the note 17.5 above $\overline{T} = S$. □

Corollary 83. *A self-adjoint operator is* **uniquely specified by giving it on a core.**

Exercise 5. ** Consider the operator T in §17.2. Is T closed? What is the adjoint of T? What is the range of $T \pm i$?*

1. A symmetric operator is essentially self-adjoint if and only if its adjoint is self-adjoint.

   Indeed, assume T is e.s.a. and symmetric. Then T^{**} is s.a., and coincides with all its successive adjoints. In particular $T^* = \overline{T^*} = T^{***} = T^{**}$.

   Conversely, if T^* is s.a., then $\overline{T} := T^{**} = T^*$ which is self-adjoint.

Lemma 84. *If T is self-adjoint, then* $\ker(T \pm i) = \{0\}$.

Proof. Compare with Lemma 79. □

Theorem 19 (Basic criterion of self-adjointness)**.** *Assume that T is* **symmetric**.

Then the following three statements are equivalent:

(a) T *is self-adjoint*

(b) T *is closed and* $\ker(T^* \pm i) = \{0\}$.

(c) $\operatorname{ran}(T \pm i) = \mathcal{H}$. *More generally, if* $(a, b) \in \mathbb{R}^2$ *and* $b \neq 0$ *then* $\operatorname{ran}(T + a + ib) = \mathcal{H}$

(d) $\sigma(T) \subset \mathbb{R}$

Of course (we can take) $D(T \pm i) = D(T)$. Remember the operator T in §17.2: it follows from (c) that T is not s.a.

Proof. (a)⇒ (b) is simply Lemma 84.

(b)⇒ (c): T is symmetric, thus $T^* \supset T$ is densely defined and closed, thus, Proposition 78 above implies the result. The generalization is immediate, since aT/b is symmetric and closed as well. This implies in particular, since T is closed, that $T + a + ib$ is invertible, proving (c).

(c) ⇒ (a) Let $y \in D(T^*)$. We want to show that $y \in D(T)$. By definition, for such a y,

$$\langle y, (T + i)x\rangle = \langle (T + i)^* y, x\rangle$$

Since $\operatorname{ran}(T - i) = \mathcal{H}$, there is an $s \in D(T)$ s.t.

$$(T + i)^* y = (T - i)s$$

for some $s \in D(T)$. Thus, since $x \in D(T)$ and since T is symmetric, we have

$$\langle y, (T + i)x\rangle = \langle v, x\rangle = \langle (T - i)s, x\rangle = \langle s, (T + i)x\rangle \qquad (152)$$

Now we use the fact that $\operatorname{ran}(T + i) = \mathcal{H}$ to conclude that $s = y$. But s was in $D(T)$ and the proof is complete. □

Corollary 85. *Let T be symmetric. Then, the following are equivalent:*

1. *T is essentially self-adjoint.*

2. $\ker(T^* \pm i) = \{0\}$.

3. $\operatorname{ran}(T \pm i)$ *are dense.*

Example 86. We know already that T_0 is not s.a., nor e.s.a since it is already closed. Let's however pretend we don't have this argument, and use instead the above criteria.

The range of T_0 (see (149)) and Corollary 85 show that T_0 is symmetric (thus closed since $D(T_0)$ is dense), but **not** essentially self-adjoint. Also, we note that $\ker (T_0^* \pm i) \neq \{0\}$. Indeed,

$$i\frac{d\varphi}{dx} \pm i\varphi = 0 \Rightarrow \varphi = Ce^{\mp x}$$

17.5.1 All self-adjoint extensions of id/dx on $[0, 1]$

Lemma 87. *If $T = id/dx$ defined on a dense domain in $AC[0,1]^{L_2}$ then T is self-adjoint* **iff** *$D(T) = AC[0,1]^{L_2}_{\varphi}$ where $AC[0,1]^{L_2}_{\varphi} = \{f \in AC[0,1], f' \in L^2 : f(0) = e^{i\varphi}f(1), \varphi \in [0, 2\pi)\}$.*

Proof. We need $\operatorname{ran}(T \pm i) = \mathcal{H}$ which means, by solving the ODEs $(T \pm i)f = g$ that

$$\forall g \in L^2[0,1], \exists C = C(g) \text{ s.t. } Ce^{\pm x} + \int_0^x g(s)ds \in D(T) \tag{153}$$

solves the ODE for any C so the range is dense. This is one condition. The other one is of course symmetry, and we have C to try to arrange this. By integration by parts $\int if\overline{g} = \int \overline{fig}$ iff $\forall f, g \in D(T)$, denoting $f_0 = f(0)$ etc. we have

$$f_0\overline{g_0} + f_1\overline{(-g_1)} = 0 \tag{154}$$

or

$$\langle u, \sigma v\rangle_{\mathbb{C}} = 0, \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \forall (u, v) \in \mathcal{E} \tag{155}$$

where $\mathcal{E}$ is a subspace of $\mathbb{C}^2$ and $\langle u, \sigma v\rangle_{\mathbb{C}}$ is the usual scalar product in $\mathbb{C}^2$. Clearly $\sigma^2 = 1$, σ is self-adjoint and thus by the polarization identity it is equivalent to satisfy (155) when $v = u$. Since for $a \neq 0$ $(a, 0) \notin \mathcal{E}, \mathcal{E} \neq \mathbb{C}^2$ and it is a zero or one dimensional subspace. If it is zero-dimensional, then $\mathcal{E} = \{0\}$, which would give us $T = T_0$, $D(T_0) = AC([0,1])_{01}$ on which we know T is not self-adjoint. Let then $(1, \alpha)$ generate $\mathcal{E}$. It is immediate to check that $|\alpha| = 1$ and thus $\alpha = e^{i\varphi}$ for some φ. That this φ is independent of the f having $f_{0,1}$ given follows from the fact that otherwise $|f_1^2| = e^{i(\varphi_1 - \varphi_2)}|f_1|^2$. ☐

T_0 is called the minimal operator associated to id/dx, T_0^* is the maximal operator. We see that T_0^* has as spectrum $\mathbb{C}_\infty$ (since e^{as} is an eigenvalue for any a); T_0^* is already a bit "too large").

18 Extensions of symmetric operators; Cayley transforms

Overview: The Cayley transform of a bounded self-adjoint operator, $U = \frac{T-i}{T+i}$ is unitary (by functional calculus). This will be the case of unbounded s.a.

operators as well. Note meanwhile that $(z-i)/(z+i)$ maps $\mathbb{C}$ into $\mathbb{D}$. Conversely, we will show that unitary operators such that $1-U$ is injective generate self-adjoint operators. Thus self-adjoint operators can be fully analyzed if we simply understand unitary operators! Of course, more careful proofs are needed, and they follow below.

Partial isometries are used to measure self-adjointness, find whether self-adjoint extensions exist, and find these extensions.

1. Let $H^{\pm}$ be closed subspaces of a Hilbert space $\mathcal{H}$.

2. **Definitions**

   (a) Recall that a unitary transformation U_p from H^+ to H^- is called a **partial isometry**.

   (b) The dimensions of the spaces $(H^{\pm})^{\perp}$, finite or infinite, are called deficiency indices.

3.

   Lemma 88. *Let U_p by a partial isometry. Then U_p extends to a unitary operator on $\mathcal{H}$ iff the deficiency indices of U_p are equal.*

   Proof. This is straightforward: assume the deficiency indices coincide. Two separable Hilbert spaces are isomorphic **iff** they have the same dimension. Let $U_{\perp}$ be any unitary between $(H^{\pm})^{\perp}$. Let $U = U_p$. We write $\mathcal{H} = H^+ \oplus (H^+)^{\perp}$ and $\mathcal{H} = H^- \oplus (H^-)^{\perp}$ and define $U = U_p \oplus U_{\perp}$. Clearly this is a unitary operator.

   Conversely, let U is a unitary operator that extends U_p. Then $U(H^{+\perp}) = (U(H^+))^{\perp} = (U_p(H^+))^{\perp} = H^{-\perp}$, and in particular $H^{+\perp}$ and $H^{-\perp}$ have the same dimension.

   □

4. In the steps below, T **is closed and symmetric.**

5. By Lemma 79 $H^{\pm} = \operatorname{ran}(T \pm i)$ are closed, and $T \pm i$ are one-to-one onto from $D(T)$ to $H^{\pm}$.

   Lemma 89. *The operator $(T-i)(T+i)^{-1}$ is well defined and a partial isometry between H^+ and H^-.*

   Note 19. *The partial isometry $(T-i)(T+i)^{-1}$ is known as* **the Cayley transform** *of T.*

   By 5, $U_p = (T-i)(T+i)^{-1}$ is well defined on H^+ with values in H^-, and also, for any $u^- \in H^-$ there is a unique $f \in D(T)$ such that $(T-i)f = u^-$. With $u^+ \in H^+$, we have $U_p u^+ = u^-$. Since $\|u_-\| = \|(T-i)f\|$ and $(T+i)^{-1}u^+ = f$, that is $u^+ = (T+i)f$, we have $\|u_{\pm}\|^2 = \|f\|^2 + \|Tf\|^2$.

Lemma 90. *$x \in D(T)$ **iff** $x = y - U_p y$ for some $y \in H^+$, i.e., $D(T) =$ $\operatorname{ran}(1 - U_p)$.*

Proof. Note that on the range of $T + i$ we have

$$1 - U_p = (T + i)(T + i)^{-1} - (T - i)(T + i)^{-1} = 2i(T + i)^{-1}$$

and the result follows easily. □

18.1 Duality between self-adjoint operators and unitary ones

Lemma 91. *(i) If U is unitary and $\ker(1 - U) = \{0\}$ [12], then there exists a self-adjoint T such that $U = (T - i)(T + i)^{-1}$.*

(ii) Conversely, if T is symmetric, then T is self-adjoint **iff** *its Cayley transform is unitary on $\mathcal{H}$.*

Proof. 1. We start with (ii). Since T is s.a., $T \pm i$ are invertible, and in particular $\operatorname{ran}(T \pm i) = \mathcal{H}$, and $\sigma(T) \subset \mathbb{R}$. By the above, $(T - i)(T + i)^{-1}\mathcal{H} \to \mathcal{H}$, since $D(T - i)^{-1} = \operatorname{ran}(T + i) = \mathcal{H}$. We have already shown that $(T - i)(T + i)^{-1}$ is an isometry wherever defined, thus in this case it is unitary. Conversely, $U = (T - i)(T + i)^{-1}$ is defined on $\mathcal{H}$, thus $\operatorname{ran}(T + i) = \mathcal{H}$ and it since it is an isometry, it follows that $(T - i)\mathcal{D}(T) = (T - i)(T + i)^{-1}\mathcal{H} = \mathcal{H}$ thus $\operatorname{ran}(T \pm i) = \mathcal{H}$.

2. We now prove (i). Note that $\ker(1 - U) = \ker(1 - U^*)$. Indeed, $x = Ux \Leftrightarrow U^*x = x$. Let $T = i(1 + U)(1 - U)^{-1}$. Then T is defined on $\operatorname{ran}(1 - U)$, which is dense in $\mathcal{H}$, since $\operatorname{ran}(1 - U)^{\perp} = \ker(1 - U^*) = \{0\}$.

   Now we want to see that for all $\{x, y\} \subset D(T)$ we have $\langle x, Ty\rangle = \langle Tx, y\rangle$.

3. Note first that $\operatorname{ran}(1 - U) = (1 - U)\mathcal{H} = -U(U^* - 1)\mathcal{H} = U\operatorname{ran}(1 - U^*)$.

4. Let $x \in D(T)$. Symmetry: $x \in D(T)$ is, by the above, the same as

$$x = U(U^* - 1)z = (1 - U)z; \quad (\text{and } z = (1 - U)^{-1}x) \tag{156}$$

   Thus, to determine the adjoint, by (156) we analyze the expression $\langle x, Ty\rangle$ where we first show T is symmetric. Let $x, y \in D(T)$. Then,

$$\langle x, Ty\rangle := \langle U(U^* - 1)z, i(1 + U)(1 - U)^{-1}y\rangle$$
$$\langle Uz, i(U - 1)(1 + U)(1 - U)^{-1}y\rangle = \langle Uz, -i(1 + U)y\rangle = \langle i(1 + U^*)Uz, y\rangle$$
$$= \langle i(U + 1)z, y\rangle = \langle i(U + 1)(1 - U)^{-1}x, y\rangle = \langle Tx, y\rangle \tag{157}$$

[12] Or, equivalently, $\operatorname{ran}(1 - U)$ is dense. This condition is needed to ensure $D(T)$ is dense, see Lemma 90.

5. Let's check the range of $T+i$. We have $D(T) = \operatorname{ran}(1-U)$, and thus

$$(T+i)D(T) = (T+i)(1-U)\mathcal{H} = i(1+U)(1-U)^{-1}(1-U)\mathcal{H}+i(1-U)\mathcal{H} = i(1+U)\mathcal{H} + i(1-U)\mathcal{H} = 2i\mathcal{H} = \mathcal{H} \quad (158)$$

Likewise, $\operatorname{ran}(T-i) = \mathcal{H}$.

□

18.2 Von Neumann's Theorem on self-adjoint extensions

Theorem 20. Let T be a closed, densely defined, symmetric operator. Then T has a self-adjoint extension *iff* the deficiency indices of its Cayley transform are equal.

Proof. (i) Assume T_e is a self-adjoint extension of T. Let $U = (T-i)(T+i)^{-1}$ and $U_e = (T_e - i)(T_e + i)^{-1}$. By Lemma 91, U_e is unitary. We want to show that U_e is an extension of U. Remember $U : H^+ \to H^-$. Let $x \in H^+$, then $x = (T+i)f = (T_e+i)f$ and $U_e x = (T_e-i)f = (T-i)f = (T_e-i)(T_e+i)^{-1}x = Ux$. Since U admits a unitary extension, then the deficiency indices are equal, see Lemma 88.

(ii) Conversely, assume that $H_\pm$ have the same dimension. Then there is a unitary U_e extending U. We first need to show that $1-U_e$ is injective. If this were not the case, and $z \in \ker(1-U_e)$, thus, as before, $z \in \ker(1-U_e^*)$ thus $z \in \operatorname{ran}(1-U_e)^\perp \subset \operatorname{ran}(1-U)^\perp = D(T)^\perp = \{0\}$.

The rest of the proof is quite similar to that of (i). Let $T_e = i(1+U_e)(1-U_e)^{-1}$, a self-adjoint operator. We want to show that $T_e \supset T$. U_e is defined on $\operatorname{ran}(T+i)$ and $D(T_e) = \operatorname{ran}(1-U_e)$. If $x \in D(T)$, then $x = (1-U)y = (1-U_e)y$ for some y and thus $Tx = i(1+U)(1-U)^{-1}x = i(1+U_e)(1-U_e)^{-1}x = T_e x$ □

Corollary 92. Let T be symmetric and closed. Then T has a self-adjoint extension iff $\ker(T^*-i)$ and $\ker(T^*+i)$ have the same dimension, that is, iff $\operatorname{ran}(T \pm i)^\perp$ have the same dimension.

Exercise 1. *Show that the symmetric operator in §17.2 has* **no** *self-adjoint extension.*

19 Spectral theorem: various forms

We first formulate the various forms of this theorem, then apply it on a number of examples, and then prove the theorem.

19.1 Bounded operators

Theorem 21 (Functional calculus form (I))**.** Let A be a bounded self-adjoint operator on $\mathcal{H}$, a separable Hilbert space. Then there exists a finite measure space $\{M, \mu\}$, a unitary operator from $\mathcal{H}$ onto $L^2(M, d\mu)$ and a **bounded** function F so that the image of A under U is the operator of multiplication by F. That is $A = U^{-1}\overset{\times}{F}U$, where $(\overset{\times}{F}f)(\omega) =: F(\omega)f(\omega)$.

Equivalently (the equivalence is simple, we'll show it later).

Theorem 22 (Functional calculus form (II))**.** Let A be a bounded self-adjoint operator on $\mathcal{H}$, a separable Hilbert space. Then there exists a decomposition

$$\mathcal{H} = \oplus_{n=1}^{N}\mathcal{H}_n, \quad N \leqslant \infty$$

so that, on each H_n, A is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\mu_n)$ for some finite measure μ_n (meaning $A = U^{-1}\overset{\times}{x}U$ where $\overset{\times}{x}(f(x)) = xf(x)$). That is, A is unitarily equivalent to multiplication by x. The measures depend on A, of course. *The measures are thus non-canonical.* Also, for any analytic f, $f(A) = U^{-1}\overset{\times}{F}U$.

The measures $d\mu_n$ are called spectral measures.

19.2 Unbounded operators

Theorem 23. Let A be a self-adjoint operator on $\mathcal{H}$, a separable Hilbert space. Then there exists a Radon measure space $\{M, \mu\}$, a unitary operator from $\mathcal{H}$ onto $L^2(M, d\mu)$ and a function F such that the image of A under U is the operator of multiplication by F. That is $A = U^{-1}\overset{\times}{F}U$. Also, $\psi \in D(A)$ *iff* $F(\omega)(U\psi)(\omega) \in L^2(M, d\mu)$).

(Furthermore, the measure space can be arranged so that $F \in L^p(d\mu)$ for any $p \in [0, \infty)$.)

19.2.1 Spectral projection form

Let A be selfadjoint. There is a family of orthogonal projections P associated to A, with the following properties. For every measurable set S in $\mathbb{R}$ there is a projection P_S (the projection on the part S of the spectrum of A). They have the following properties.

1. $P_\emptyset = 0, P_\mathbb{R} = I$.
2. If $S = \cup_1^\infty S_n$ (all sets being measurable) and $S_n \cap S_m = \emptyset$ for $n \neq m$, then $P_S = s - \lim \sum_1^\infty P_n$ where $s - \lim$ is the strong limit.
3. $P_{S_1}P_{S_2} = P_{S_1 \cap S_2}$.

Theorem 24 (Spectral projection form. This form is the same for bounded or unbounded operators)**.** Let A be a self-adjoint operator. Then there exist spectral projections as above, such that

$$A = \int_\mathbb{R} \lambda dP_\lambda$$

For now, we understand (iv) in the sense

$$\langle f, Ag\rangle = \int_{\mathbb{R}} \lambda d\langle f, P_\lambda g\rangle$$

where $d\langle f, P_\lambda g\rangle$ is a usual measure, as it is straightforward to check using the properties of the family P.

Note 20. It is important to emphasize that, **unlike** the other forms of the spectral theorem, **this representation is canonical:** given A there is a unique family of projections with the properties above.

Returning to our functional calculus, if g is analytic on the spectrum of A (contained in $\mathbb{R}$, of course, containing infinity if the operator is unbounded), the image of $g(A)$ is the function $g(F)$. But note that now we can define $e^{itA} = \int_{\mathbb{R}} e^{it\lambda} dP_\lambda$ for an unbounded A, while e^{ix} is certainly not analytic at ∞.

20 Proof of the functional calculus form

20.1 Cyclic vectors

Consider a bounded self-adjoint operator A, a vector ψ in $\mathcal{H}$ and the vectors $\{A^n\psi\}_{n\in\mathbb{N}}$. If we take all linear combinations of $A^n\psi$ and then its closure, the "span" of $A^n\psi$ denoted by $\bigvee_{n=1}^{\infty} A^n\psi$, is a Hilbert space $\mathcal{H}_\psi$. If ψ is such that $\mathcal{H}_\psi = \mathcal{H}$, then ψ is a cyclic vector. Not all operators A have cyclic vectors.

Exercise 1. Which self-adjoint matrices have cyclic vectors?

Note that we can take any vector, form $\mathcal{H}^{[0]} = \mathcal{H}_\psi$ and then pick a vector in $\mathcal{H}_\psi^\perp$ and construct $\mathcal{H}^{[1]}$, "etc." (see the exercise below for the "etc.")

Exercise 2. Show (for instance using Zorn's Lemma) that every separable Hilbert space can be written as

$$\mathcal{H} = \oplus_{n=1}^{N} \mathcal{H}_{\psi_N}, \quad N \leqslant \infty \tag{159}$$

where ψ_j is cyclic for $\mathcal{H}_{\psi_j}$.

Now, to prove Theorem 22 it suffices thus to prove it in each $\mathcal{H}_{\psi_N}$, or, w.l.o.g., assume that ψ is already cyclic for $\mathcal{H}$. Of course, we can assume $\|\psi\| = 1$.

We can define a functional L, first on analytic functions f on $\sigma(A)$ by

$$L(f) = \langle f(A)\psi, \psi\rangle \tag{160}$$

L is bounded since

$$|L(f)| \leqslant \|f(A)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(A)\} = \|f\|_\infty \tag{161}$$

(see Prop. 60 above, p. 46, (131), and Exercise 4 on p. 58.

On the other hand, this is a positive functional, since if f is positive [13] on $\sigma(A)$ then $f = g^2$ with g analytic too and real valued, and thus

$$\langle f(A)\psi, \psi\rangle = \langle g(A)^2\psi, \psi\rangle = \langle g(A)\psi, g(A)\psi\rangle = \|g(A)\psi\|^2 \geqslant 0 \tag{162}$$

Since analytic functions on a compact set in $\mathbb{R}$, $\sigma(A)$, are dense in $C(\sigma(A))$ (polynomials are already dense) and the functional $f \mapsto \langle f(A)\psi, \psi\rangle$ is positive and of norm one on a dense subset, it extends to $C(\sigma(A))$ and, by the Riesz-Markov theorem, there is a measure $d\mu_\psi$ so that

$$\langle f(A)\psi, \psi\rangle = \int_{\sigma(A)} f d\mu_\psi(x) \tag{163}$$

Note the dependence on ψ; see Note 30 below for a physical interpretation of $\mu_\psi(x)$. For simplicity however, we drop the subscript $\psi : \mu = \mu$. We can now consider $\mathcal{H}_A = L^2(\sigma(A), d\mu)$. We want to define a unitary transformation between $\mathcal{H}$ and $\mathcal{H}_A$. (Note that up to now we showed bounds between $\mathcal{H}$ and $L^\infty(\sigma(A))$.)

It is more convenient to define U on $\mathcal{H}_A$ with values in $\mathcal{H}$. The vectors in $\mathcal{H}_A$ are of course functions on $\sigma(A)$. From (163) we expect φ to be mapped into $\varphi(A)\psi$. Define thus for f analytic (not necessarily real-valued) on $\sigma(A)$,

$$Uf = f(A)\psi \tag{164}$$

Remember (see p. 54) that

$$f(A)^* = f^*(A) \text{ and for } x \in \mathbb{R} \text{ we have } f^*(x) = \overline{f(x)} \tag{165}$$

Therefore

$$\int |f|^2 d\mu \overset{(163)}{=} \langle f(A)(f(A))^*\psi, \psi\rangle = \|f(A)\psi\|^2 = \|Uf\|^2 \tag{166}$$

Thus the transformation U, defined on a dense subset of $L^2(\sigma(A), d\mu)$ extends to an isometry from $L^2(\sigma(A), d\mu)$ into $\mathcal{H}$.

We only have to check that indeed the image $U^{-1}AU$ of A is the operator of multiplication by the variable. We have for analytic f,

$$Uf := f(A)\psi \Rightarrow U^{-1}A\underline{Uf} = U^{-1}A\underline{f(A)\psi} = U^{-1}\underline{Af(A)\psi}$$
$$= U^{-1}\underline{(\overset{\times}{x}f_\times)(A)\xi} = U^{-1}U\overset{\times}{x}\overset{\times}{f} = \overset{\times}{x}f \tag{167}$$

Since (167) holds on a dense set, the proof is complete.

Note 21. Whether ψ is cyclic or not, formulas (164) and (166) above allow us to define any L^2 functions of A. Indeed, analytic functions (or even polynomials) are dense in L^2. If $f_n \to h$ in the sense of L^2, then $f_n(A)\psi$ converges to (by

[13] Not merely nonnegative.

the definition of $f_n(A)$) $h(A)\psi$. Thus we can define, given ψ arbitrary, $h(A)\psi$ for any $h \in L^2$. It is easy to check that h is bounded and linear: Indeed, if ψ is cyclic for the whole of $\mathcal{H}$ there is essentially nothing to show. If it is not, then it is correctly defined and bounded on each $\mathcal{H}_n$ with values in $\mathcal{H}_n$ while $\mathcal{H}_n$ are mutually orthogonal. Simply define $h(a)x = \sum h(A)x_n$ where $(x_n)_{n\leqslant N}$ is the orthogonal decomposition of x. Now, if $h = \chi_S$, the characteristic function of a measurable set, then $\chi_S(A)$, is, of course, a spectral projection.

Note 22. Note that (161) shows that if we take the closure in *the operator norm* of $\mathcal{A}(A) := \{f(A) : f$ analytic on $\sigma(A)\}$ then we get $C(A)$ the continuous functions on $\sigma(A)$. In Note 21 we have taken the *strong-limit closure* of $\mathcal{A}(A)$, since convergence of f_n is only used pointwise– convergence of $f_n\psi$ for any ψ. This gives us in particular $M(A)$, the bounded measurable functions on $\sigma(A)$, applied to A. Compare also with what we have done in §11. The apparent similarity can be misleading.

Note 23. Note also that we have made substantial use of the fact that A was self-adjoint to define $M(A)$ (for instance, in using (165)).

21 Examples: The Laplacian in $\mathbb{R}^3$

Most of this section is based on [6], where more results, details, and examples can be found.

In a number of cases, the unitary transformation mapping an unbounded self-adjoint operator T to a multiplication operator is explicit. Then, we can use the exercise below to find all information about T.

Exercise 1. *Show that, if U is unitary between $\mathcal{H}_1$ and $\mathcal{H}_2$, then T is self-adjoint on $D(T) \subset \mathcal{H}_1$ iff $UTU^{-1} = UTU^*$ is self-adjoint in $UD(T)$.*

As an example, consider defining $-\Delta$ in $\mathbb{R}^3$. Define it first on $D_0(\Delta) = C_0^\infty$ where it is symmetric. Let U be the Fourier transform, a unitary operator. It is in fact easier to work with another set of functions, still dense in $L^2(\mathbb{R}^3)$ as can be easily checked, which admit an explicit Fourier image:

$$D_1 = \left\{ e^{-\mathbf{x}^2/2} P(\mathbf{x}) : P \text{ polynomial} \right\} \tag{168}$$

This is Fourier transformed to

$$UD_1 = \left\{ e^{-\mathbf{k}^2/2} P(\mathbf{k}) : P \text{ polynomial} \right\} \tag{169}$$

Exercise 2. *Check that D_1 is dense in $L^2(\mathbb{R}^3)$. Check that on D_1 T is symmetric and that $UT(f) = |\mathbf{k}^2|Uf$, a multiplication operator.*

We will denote $|\mathbf{k}^2|$ by k^2 and Uf by $\hat{f}$.

Exercise 3. *Check that k^2 is self-adjoint on*

$$D(k^2) = \{f \in L^2(\mathbb{R}^3) : k^2 f \in L^2(\mathbb{R}^3)\} = \{f : (k^2+1)f \in L^2(\mathbb{R}^3)\} \tag{170}$$

Thus, we have

Proposition 93. $-\Delta$ *is self-adjoint on* $U^{-1}D(k^2)$. *Call this operator* H_0.

The characterization of $D(H_0)$ is simplest through Fourier transform. This gives another dimension to the need for **Sobolev spaces** etc.

Let $u \in D(H_0)$ and $\hat{u}$ be its Fourier transform. A direct space characterization is more difficult, though we might simply say that u together with all second order partial derivatives exist *as weak derivatives, in distributions* (equivalently, defined in L^2 as as inverse Fourier transforms of $k_i k_j \hat{u}$), and the derivatives are in L^2.

We can see some classical properties of elements of $u \in D(H_0)$. We can show that u is bounded and uniformly continuous as follows. Noting that both $\hat{u}$ and $k^2\hat{u}$ are in L^2 it follows immediately that $(\alpha^2 + |k|^2)\hat{u} \in L^2$ for any α. This is, essentially by definition the **Sobolev space** $H^2(\mathbb{R}^3)$. Let $k = |\mathbf{k}|$ and dk be the Lebesgue measure on $\mathbb{R}^3$.

For boundedness, we use Cauchy-Schwarz:

$$\left(\int_{\mathbb{R}^3} |\hat{u}| dk\right)^2 \leqslant \int_{\mathbb{R}^3} \frac{dk}{(k^2+\alpha)^2} \int_{\mathbb{R}^3} [(k^2+\alpha)|\hat{u}|]^2 dk = \frac{\pi^2}{\alpha} \|(H_0+\alpha)u\|^2 < \infty \tag{171}$$

Note also that

$$|e^{i\mathbf{k}\cdot\mathbf{x}} - e^{i\mathbf{k}\cdot\mathbf{y}}| \leqslant \max\{2, k|\mathbf{x}-\mathbf{y}|\} \tag{172}$$

Similarly, we can calculate that, if $\beta < 1/2$, then

$$|u(\mathbf{x}) - u(\mathbf{y})| \leqslant C_\beta |\mathbf{x}-\mathbf{y}|^\beta \left(\alpha^{-(1/2-\beta)}\|H_0 u\| + \alpha^{3/2+\beta}\|u\|\right) \tag{173}$$

which easily implies that u is Hölder continuous with any exponent $< 1/2$. However, u is not even once classically differentiable!

The inequalities (171) and (173) are special cases of Sobolev inequalities.

For more details and examples, see [6], pp. 299.

Here is a very useful and general criterion of self-adjointness, in the unbounded case. First, a definition:

Definition 94. *(i) An operator A is relatively bounded with respect to T ($T-$bounded) if $D(A) \supset D(T)$ and for some a, b and all $u \in D(T)$ we have*

$$\|Au\| \leqslant a\|u\| + b\|Tu\| \tag{174}$$

An equivalent condition is that for some (different) a', b' we have

$$\|Au\|^2 \leqslant (a')^2\|u\|^2 + (b')^2\|Tu\|^2 \tag{175}$$

The equivalence of (174) *and* (175) *is left as an exercise (details are given in [6], p. 287, but the exercise is simple enough). (ii) The $T-$bound is defined as the greatest lower bound of the b for which there is an a so that* (174) *holds, or equivalently the greatest lower bound of the b' for which there is an a' so that* (175) *holds.*

Theorem 25 (Kato-Rellich)**.** *Let T be self-adjoint. If A is symmetric and $T-$bounded with $T-$bound $b' < 1$ then $T + A$ is* **self-adjoint***. This is the case in particular if A is bounded.*

Proof. First, it is clear that $D(T + A) = D(T)$ and $T + A$ is symmetric. We will show that (c) in Theorem 19 holds. Without loss of generality we take also $a' > 0$. Recall (147). This implies immediately that

$$\|Ax\| \leqslant \|(b'T \mp ia')x\|, \quad (x \in D(T)) \tag{176}$$

Denote $c' = a'/b'$ and $(T \mp ic')x = y$, and recall Theorem 19 (b) (c). This implies that

$$\mathcal{R}(\mp ic') \stackrel{\text{def}}{=} (T \mp ic')$$

exist and are bounded. Thus

$$\|A\mathcal{R}(\mp ic')y\| \leqslant b'\|y\|, \quad (x \in D(T)) \tag{177}$$

In particular (again by Theorem 19 (b) (c)) this means that

$$B_{\pm} = -A\mathcal{R}(\mp ic') \text{ are bounded and } \|B_{\pm}\| < b' < 1 \tag{178}$$

Therfore, by the standard Neumann series argument, $(1 - B_{\pm})^{-1}$ exist and are bounded (with norm $\leqslant 1/(1 - b')$) and thus $(1 - B_{\pm})$ are bounded and one-to-one. Note now that $T \mp ic'$ is one to one, so is $(1 - B_{\pm})$ and thus their product is one-to-one.

$$(1 - B_{\pm})(T \mp ic') = T \mp ic' + A\mathcal{R}(\mp ic')(T \mp ic') = T \mp ic' + A \tag{179}$$

Thus $\text{ran}\,(T + A \mp ic') = \text{ran}\,\frac{1}{c'}(T + A) \mp i = \mathcal{H}$ and thus $T + A$ is self-adjoint. □

Example The operator $-\Delta + V(\mathbf{x})$ is self-adjoint on $D(-\Delta)$ for any bounded real function V. This follows trivially from Theorem 25.

In one dimension, say on $L^2[0, 1]$, for any bounded, measurable, real f, $id/dx + f(x)$ (or $-d^2/dx^2 + f(x)$) are self-adjoint on any domain on which id/dx (or $-d^2/dx^2$) is self-adjoint. Find self-adjointness domains for $-d^2/dx^2$.

More generally, one can show (see [6]) the following.

Proposition 95 ([6], p. 302)**.** *Consider functions of the form $q = q_0 + q_1$ where $q_0 \in L^\infty(\mathbb{R}^3)$ and $q_1 \in L^2(\mathbb{R}^3)$. Then*

$$-\Delta + q \tag{180}$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ and self-adjoint on $D(-\Delta)$.

Example $q = 1/r$, $r = |\mathbf{x}|$ satisfies the assumptions of Proposition 95. Thus the Hamiltonian of the Coulomb atom,

$$-\Delta + e/r \tag{181}$$

is self-adjoint on $D(H_0)$.

22 A few more facts about compact operators

Proposition 96. *If A, B are as in Theorem* **??** *and A^{-1} is compact and B is bounded, then $(A+B)^{-1}$ is compact.*

This is immediate from the second resolvent formula.

Corollary 97. *Let $T = id/dx + V(x)$ on $AC[0,1]$ with periodic boundary conditions, where $V(x)$ is real valued and bounded. Then T is self-adjoint with compact resolvent.*

Proof. This follows from Theorem 25 and the second resolvent formula, Theorem **??**. □

23 Equivalence with Theorem 21

For that we simply take N copies of $\mathbb{R}$, take their disjoint union $M = \cup_{n=1}^{N}\mathbb{R}$ and on each copy we take a vector of norm 2^{-n}. Then we take the measure $\mu = \oplus\mu_n$ on M, and clearly $\mu(M) < \infty$. The rest is immediate.

24 Extension to Borel functions of A

Since A corresponds to multiplication by a function F, if h is any bounded Borel measurable function on $\sigma(A)$ we define $h(A) = Uh \circ FU^{-1}$, whenever $h \circ F$ exists in the underlying Hilbert space. In particular, if h is chosen to be χ_S, the characteristic function of a subset S, then $P_S = U\chi_S U^{-1}$ is an orthogonal projection, in fact a spectral measure. We want to see that

$$A = \int x dP(x)$$

Let S be an arbitrary measurable subset of $\sigma(A) \subset \mathbb{R}$ and define, for each pair $x, y \in \mathcal{H}$,

$$\nu(S) = \langle P_s x, y\rangle$$

Then,

25 Defining self-adjoint operators

Suppose A is symmetric, and that we can find a unitary transformation U that maps $\mathcal{H}$ into $L^2(\mathbb{R}, d\mu)$ in such a way that A is mapped onto $a\cdot$ for all $\psi \in D(A)$, and that means that $a\cdot$ is initially defined on $U(D(A))$ which is also dense, since U is unitary. Then, clearly, $(A \pm i)^{-1}$ are mapped to $(a \pm i)^{-1}$, since $(A \pm i)(A \pm i)^{-1} = \mathbf{1}$ on $D(A)$. It is clear that $U_a = (a \cdot -i)(a \cdot +i)^{-1}$, the image of $U_A = (A-i)(A+i)^{-1}$ is extends to a unitary operator on $\overline{UD(A)} = U\mathcal{H}$. But this does not mean that it was unitary to start with, since it was only defined on $\{(a+i)f : f \in UD(A)\}$ which may not be dense in $L^2(\mathbb{R}, d\mu)$.

Nevertheless, U_a extends to a unitary operator, and thus U_A extends to a unitary operator, and thus A has *some* self-adjoint extension A_1, (canonical wrt this particular construction...). What is the domain of A_1? This can be written in terms of U_A: $D(A) = \text{ran}\,(U_A - 1)$, which is simply $U^{-1}\text{ran}\,(u_a - 1) = D(a\cdot)$. On the other hand, this is simple to calculate, it consists of the functions g such that

$$D(a) = \left\{ g \in L^2(\mathbb{R}, d\mu) : \int_{\mathbb{R}} |g|^2 a^2 da < \infty \right\} \tag{182}$$

Exercise 1. Apply these arguments to $A = id/dx$ defined originally on C_0^∞, using as U the discrete Fourier transform and, as usual k instead of a as a discrete variable. What is $UD(A)$? On $UD(A)$ $k\cdot$ is simply multiplication by k. The domain of the adjoint is *larger* than the set of sequences $\{c_k\}_{k\in\mathbb{Z}}$ with $kc_k \in L^2$. What is it? Find a self-adjoint extensions of A. Can you find more than one?

Thus,

$$D(A) = U^{-1} \left\{ g \in L^2(\mathbb{R}, d\mu) : \int_{\mathbb{R}} |g|^2 a^2 da < \infty \right\} \tag{183}$$

26 Spectral measures and integration

We have seen that finitely many isolated parts of the spectrum of an operator yield a "spectral decomposition": The operator is a direct sum of operators having the isolated parts as their spectrum. Can we allow for infinitely many parts?

We also saw that, if the decomposition of the spectrum is $K_1 + K_2$, where $\chi_1\chi_2 = 0$, then the decomposition is obtained in terms of the spectral projections $P_i = \chi_i(T)$ where $P_1P_2 = 0$ and $P_1 + P_2 = I$. We can of course reinterpret (probability-)measure theory in terms of characteristic functions. Then a measure becomes a functional on a set of characteristic functions such that $\chi_i\chi_j = 0$ (the sets are disjoint) then

$$\chi(\bigcup_{i=1}^{\infty} A_i) = \sum_{k=1}^{\infty} \chi_i$$

and

$$\chi(A_1 \cap A_2) = \chi_1\chi_2$$

In terms of measure, we then have

$$\mu(\sum_{k=1}^{\infty} \chi_k) = \sum_{k=1}^{\infty} \mu(\chi_k)$$

$$\mu(1) = 1$$

(continuity of the functional) etc.

In the following, we let $E_i := E(A_i)$.

Note 24. 1. In a Hilbert space it is natural to restrict the analysis of projectors to orthogonal projections. Indeed, if $\mathcal{H}_1 \subsetneq \mathcal{H}$ is a closed subspace of $\mathcal{H}$, itself then a Hilbert space, then any vector x in $\mathcal{H}$ can be written in the form $x_1 + x_2$ where $x_1 \in \mathcal{H}_1$ and $x_2 \perp x_1$. So to a projection we can associate a natural orthogonal projection, meaning exactly the operator defined by $(x_1 + x_2) \to x_1$.

2. An orthogonal projection P is a self-adjoint operator. Indeed,

$$\langle Px, y\rangle = \langle x_1, y_1 + y_2\rangle = \langle x_1, y_1\rangle = \langle x_1 + x_2, y_1\rangle = \langle x, Py\rangle$$

3. Conversely, if P is self-adjoint, and $x = x_1 + x_2$, where $Px_1 = x_1$, then

$$\langle x_1, x_2\rangle = \langle Px, (1-P)x\rangle = \langle x, P(1-P)x\rangle = 0$$

Let Ω be a topological space, $\mathcal{B}_\Omega$ be the Borel sets on Ω and $\mathcal{A}(\mathcal{H})$ be the *self-adjoint* bounded operators on the Hilbert space $\mathcal{H}$: $\forall (x,y), \langle Tx, y\rangle = \langle x, Ty\rangle$. Thus, with $A_i \in \mathcal{B}_\Omega$, the spectral projections $E := \mathcal{X}(T)$ should satisfy

1.
$$(\forall i \neq j, A_i \cap A_j = \emptyset) \;\Rightarrow\; E(\bigcup_{i=1}^{\infty} A_i)x = \sum_{k=1}^{\infty} E_i x$$

2.
$$E((A_1 \cap A_2) = E_1 E_2$$

3.
$$E(\Omega) = I; \quad E(\emptyset) = 0$$

Definition 98. *A projector valued spectral measure on $\mathcal{B}_\Omega$ with values in $\mathcal{A}(\mathcal{H})$ is a map $E : \mathcal{B}_\Omega \to \mathcal{H}$ with the properties 1,2,3 above.*

Note 25. *$E(A)$ is always a projector in this case, since $E(A) = E(A \cap A) = E(A)^2$ and $E(A)E(B) = E(B)E(A)$ since they both equal $E(A \cap B)$.*

Proposition 99 (Weak additivity implies strong additivity)**.** *Assume 2. and 3. above, and that for all $(x,y) \in \mathcal{H}^2$ we have*

$$(\forall i \neq j, A_i \cap A_j = \emptyset) \;\Rightarrow\; \langle E(\bigcup_{i=1}^{\infty} A_i)x, y\rangle = \langle \sum_{k=1}^{\infty} E_i x, y\rangle$$

Then 3. above holds.

Remark 26. *Note a pitfall: we see that*

$$E(\bigcup_{i=1}^{\infty} A_i)x = \sum_{k=1}^{\infty} E_i x \tag{184}$$

for all x. But this does not mean that $\sum_{i=1}^{N} E_i x$ converge normwise!

Note 27. If $\langle v_n, y\rangle \to \langle v, y\rangle$ for all y and $\|v_n\| \to \|v\|$ then $v_n \to v$. This is simply since

$$\langle v_n - v, v_n - v\rangle = \langle v_n, v_n\rangle - \langle v_n, v\rangle - \langle v, v_n\rangle + \langle v, v\rangle$$

Proof. Note first that $E(A_i)x$ form an orthogonal system, since

$$\langle E(A_1)x, E(A_2)x\rangle = \langle x, E(A_1)E(A_2)x\rangle = \langle x, E(A_1 \cap A_2)x\rangle = 0 \tag{185}$$

$$\langle \sum_{i=1}^N E_i x, \sum_{i=1}^N E_i x, \rangle = \sum_{i=1}^N \langle E_i x, E_i x, \rangle$$

while

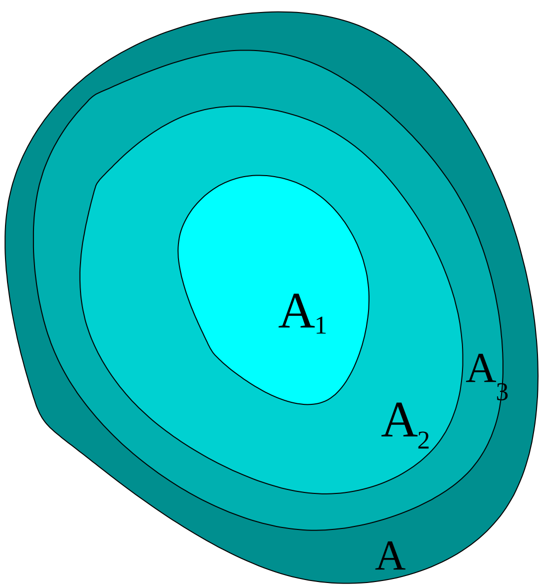
$$\sum_{i=1}^N E_i x$$

converges. Thus $\sum_{i=1}^N \|E_i x\|$ converges too and thus $\| \sum_{i=1}^N E_i x - \sum_{i=1}^\infty E_i x\| \to 0$.

□

Lemma 100. *If $A_j \uparrow A$ then $E_j x \to E(A)x$.*

Proof. Same as in measure theory. $A_{j+1} \setminus A_j$ are disjoint, their sum to N is A_{N+1} and their infinite sum is A, see figure. □



27 Integration wrt projector valued measures

Let f be a simple function defined on $S \in \mathcal{B}_\Omega$. This is a dense set in $\mathcal{B}(\Omega, \mathbb{C})$.

$$f(\omega) = \sum_{i=1}^{N} f_j \chi(S_j) \tag{186}$$

As usual, $S_i \cap S_j = \emptyset$ and $\cup S_i = S$. Then we naturally define

$$\int_S f(\omega) dE(\omega) = \sum_{i=1}^{N} f_j E(S_j) \tag{187}$$

This is the usual way one defines integration wrt a scalar measure.

Note 28. We note an important distinction between this setting and that of §11: here we integrate scalar functions wrt operator valued measures, whereas in §11 we integrate operator-valued functions against scalar measures.

Note 29. [(14)] In §11 as well as in the few coming sections, we restrict ourselves to integrating uniform limits of simple functions. The closure in the sup norm of simple functions are the measurable bounded functions, which is not hard to verify, so this is a more limited integration than Lebesgue's (e.g., L^1 is not accessible in this way.)

It is also interesting to note the following theorem:

Theorem 26 (Šikić, 1992)**.** A function f is Riemann integrable iff it is the uniform limit of simple functions, where the supporting sets A_i are Lebesgue measurable with the further property that $\mu(\partial A_i) = 0$.

Without the restriction on $\mu(\partial A_i)$, the closure of simple functions in the sup norm is the space of bounded measurable functions (exercise).

Proposition 101. *The definition (187) is independent of the choice of* S_j*, for a given* f*.*

Proof. Straightforward. □

Proposition 102. *The map* $f \to \int_S f$ *is uniformly continuous on simple functions, and thus extends to their closure. We have*

$$\left\| \int_S f(\omega) dE(\omega) \right\| \leqslant \sup_S |f(\omega)| \tag{188}$$

Proof. We can assume that the sets S_i are disjoint. Then the vectors $E_i x$ are orthogonal and

$$\left\| \int_S f(\omega) dE(\omega) \right\|^2 = \sum_{i=1}^{N} f_i \|E(S_i)x\|^2 \leqslant \sup_S |f|^2 \sum_{i=1}^{N} \langle E_i x, x\rangle \leqslant \sup_S |f|^2 \|x\|^2$$

□

(14) Pointed out by Sivaguru Ravisankar.

Consider then the functions in $B(\Omega, \mathbb{C})$ which are uniform limits of simple functions. The integral extends to $B(\Omega, \mathbb{C})$ by continuity and we write

$$\int_S f(s)dE(s) := \lim_{n\to\infty} \int_S f_n(s)dE(s) \tag{189}$$

Definition 103. *A C^*-algebra is a Banach algebra, with an added operation, conjugation, which is antilinear and involutive ($(f^*)^* = f$) and $\|f^* f\| = \|f\|^2$.*

Exercise 1. *Show that $\|f^*\| = \|f\|$.*

Proposition 104. *$f \mapsto \int_S f(\omega)dE(\omega)$ is a C^*-algebra homeomorphism.*

Proof. We only need to show multiplicativity, and that only on characteristic functions, and in fact, only on very simple functions, where $N = 1$, and the coefficients are one. We have

$$\int \chi(S_1)\chi(S_2)dE(S) = E(S_1 \cap S_2) = E(S_1)E(S_2) \tag{190}$$

Exercise 2. *Complete the details.*

□

27.1 Spectral theorem for (unbounded) self-adjoint operators

Theorem 27. *Let A be self-adjoint. There is a spectral measure on the Borel σ-algebra of $\mathbb{R}$ such that*

$$Ax = \int_{\mathbb{R}} tdE(t)\, x := \lim \int_{-k}^{k} tdE(t)\, x \tag{191}$$

for all $x \in D(A)$. Furthermore,

$$D(A) = \{x \in \mathcal{H} : \int_{\mathbb{R}} t^2 d\langle E(t)x, x\rangle < \infty\} \tag{192}$$

We know that self-adjoint operators are in a one-to-one correspondence with unitary operators. Assuming we have the spectral theorem for bounded normal operators and u is a unitary operator, then we have

$$U = \int_C udE(u) \tag{193}$$

where C is the unit circle. Then, with $A = i(1 + U)(1 - U)^{-1}$ we expect

$$A = \int_C i\frac{1+u}{1-u}dE(u) \tag{194}$$

which we can formally write

$$A = \int_{\mathbb{R}} td(E \circ \varphi)(t) = \int_{\mathbb{R}} tdE'(t) \tag{195}$$

where

$$\varphi(t) = \frac{t-i}{t+i} \tag{196}$$

and where $E'(S) = E(\varphi(S))$ is a new spectral measure.

We have to ensure that the theory of integration holds, and that it is compatible with changes of variable, and to interpret the singular integrals obtained.

28 Spectral representation of self-adjoint and normal operators

We first state the main results, which we will prove in the sequel.

Theorem 28 (Spectral theorem for bounded normal operators)**.** *Let N be bounded and normal (that is, $NN^* = N^*N$). Then there is spectral measure E defined on the Borel sets on $\sigma(N)$, see Definition 98, such that*

$$N = \int_{\sigma(N)} zdE(z) \tag{197}$$

Theorem 29 (Uniqueness of the spectral measure)**.** *Let N be a normal, bounded operator.*

(i) Assume $E^{[1]}$ and $E^{[2]}$ are spectral measures on the Borel sets on $\sigma(N)$ such that

$$N = \int_{\sigma(N)} zdE^{[1]} = \int_{\sigma(N)} zdE^{[2]} \tag{198}$$

Then $E^{[1]} = E^{[2]}$.

(ii) More generally, if $\Omega \in \mathbb{C} \supset \sigma(N)$ is a closed set and $E^{[3]}$ is a spectral measure on Ω such that

$$N = \int_{\sigma(N)} zdE^{[3]} \tag{199}$$

Then $E(\Omega \setminus \sigma(N)) = 0$ and the Borel measure induced by $E^{[3]}$ on $\sigma(N)$ coincides with E, the spectral measure of N.

What can the spectrum of a normal operator be? Let $\mathcal{D}$ be a compact set in $\mathbb{R}^2$, and consider $L^2(\mathcal{D})$. Consider the operator $Z := f(x,y) \mapsto (x+iy)f(x,y)$. Then $Z^* = f(x,y) \mapsto (x-iy)f(x,y)$, and Z is a normal operator. Clearly, if $(x_0,y_0) \notin \mathcal{D}$ and $\lambda = x_0 + iy_0$, then $1/(x+iy-\lambda)$ is bounded on $\mathcal{D}$ and it is the inverse of $(x+iy-\lambda)$; otherwise, if $(x_0,y_0) \in \mathcal{D}$, then $(x+iy-\lambda)$ has no nonzero lower bound, and thus is not invertible. Then $\sigma(Z) = \mathcal{D}$.

28.1 Spectral theorem in multiplicative form

Definition 105. *A vector $\psi \in \mathcal{H}$ is* **cyclic** *for the (bounded) operator A if the closure of the set of linear combinations of vectors of the form $A^n\psi$, $n \in \mathbb{N}$ (denoted by $\bigvee_{n=1}^{\infty} A^n\psi$) is $\mathcal{H}$.*

In this formulation, self-adjoint operators are unitarily equivalent with multiplication operators on a direct sum of $L^2(\sigma(A)), d\mu_n$ with respect to certain measures. If A has a cyclic vector, then the measure is naturally defined in the following way. We know how to define bounded measurable functions of A (for instance taking limits of polynomials). Then, $\langle\psi, f(A)\psi\rangle = \int_{\sigma(A)} f(\omega)d\mu_\psi(\omega)$.

Note 30. In Quantum mechanics, if o (angular momentum, energy, etc.) is an observable, then it is described by a *self-adjoint* operator O. The spectrum of O consists of all possible measured values of the quantity o. An eigenvector ψ_a of O is a state in which the measured observable has the value $o = a$ with probability one.

If ψ is not an eigenvector, then the measured value is not uniquely determined, and for an ensemble of measurements, a number of values are observed, with different probabilities. The average measured value of o or *the expected value of* o, in an ensemble of particles, each described by the wave function ψ is $\langle\psi, O\psi\rangle$.

Imagine that the measured quantity is $\chi_{[a,b]}(o)$. That is, if say o is the energy, then $\chi_{[a,b]}(o)$ is a filter, only letting through particles with energies between a and b (a spectrometer only lets through photons in a certain frequency range, and since $E = h\nu$, certain energies).

Then, $\langle\psi, \chi_{a,b}(E)\psi\rangle$ is the expected number of times the energy falls in $[a, b]$, that is, the density of states. This is a physical interpretation of $d\mu_\psi$.

Theorem 30. *Assume A is a bounded self-adjoint operator with a cyclic vector ψ. Then there is a unitary transformation U that maps $\mathcal{H}$ onto $\mathcal{H}_A = L^2(\sigma(A), d\mu_\psi)$ and A into the multiplication operator $\Lambda = f(\lambda) \mapsto \lambda f(\lambda)$:*

$$UAU^{-1} = \Lambda \tag{200}$$

Note 31. This unitary transformation is non-canonical. But cyclicity means that the transformation $\varphi \mapsto O\psi$ is ergodic, mixing, and the measures are expected to be equivalent, in some sense.

Clearly, uniqueness cannot be expected. $L^2([0, 1])$ with respect to dx and with respect to $f(x)dx$ $\alpha < 0 < f < \beta$ are unitarily equivalent, and the unitary transformation is $g \mapsto \sqrt{f}g$. X, multiplication by x is invariant, since $UXU^{-1} = X$. The measure is determined up to measure equivalence: two measures are equivalent iff they have the same null sets. Instead, the spectral projections are unique.

In case A does not have a cyclic vector, then $\mathcal{H}$ can be written as a finite or countable direct sum of Hilbert spaces $\mathcal{H} = \oplus_{n=1}^{N}\mathcal{H}_n$, $N \leqslant \infty$ such that

$A|_{\mathcal{H}_n}$ has a cyclic vector ψ_n. Then, as before, there is a unitary transformation $U : \mathcal{H} \to \oplus_{n=1}^{N} L^2(\mathbb{R}, d\mu_n)$ so that

$$(UAU^{-1})_{\mathcal{H}_n} = \Lambda \tag{201}$$

Let F be the operator $g \mapsto fg$ where $f, g \subset L^2(M, d\mu)$. As a corollary,

Proposition 106. *Let A be bounded and self-adjoint. Then, there exists a finite measure space (M, μ) ($\mu(M) < \infty$), a bounded function f on M and a unitary operator $U : \mathcal{H} \to L^2(M, d\mu)$ so that*

$$UAU^{-1} = F \tag{202}$$

Like all other measures, μ_ψ can be decomposed uniquely into three disjoint measures: $\mu_{pp} + \mu_{ac} + \mu_{sing}$. Let μ be a finite measure on $\mathbb{R}$.

(i) μ_{pp} (pure point). A measure is pure point, μ_{pp} if, by definition, for any $B \in \Omega$ we have $\mu(B) = \sum_{\omega \in B} \mu(\omega)$. We note that the support of μ_{pp} is a countable set, since the sum of any uncountable family of positive numbers is infinite.

(ii) μ_{ac}. A measure is absolutely continuous with respect to the Lebesgue measure dx if, by definition, $d\mu = f(x)dx$ where $f \in L^1(dx)$.

(iii) A measure is singular with respect to the Lebesgue measure if there is a set S of full Lebesgue measure ($\mathbb{R} = S + S'$ where S' is a set of measure zero, and $\mu(S) = 0$. Thus μ_{sing} is concentrated on a zero Lebesgue measure set.

(iv) Pure point measures are clearly singular. To distinguish further, we say that a measure μ is continuous if $\mu(x) = 0$ for any point x.

(v) μ_{sing}. A measure is singular continuous if μ is continuous and singular with respect to the Lebesgue measure.

Decomposition theorem. Any measure μ on $\mathbb{R}$ has a canonical decomposition

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sing} \tag{203}$$

$$L^2(\sigma(A), d\mu_\psi) = L^2(\sigma(A), d\mu_{pp}) \oplus L^2(\sigma(A), d\mu_{ac}) \oplus L^2(\sigma(A), d\mu_{sing}) \tag{204}$$

where the norm square in the lhs space is the sum of the square norms of the three rhs spaces. We can then decompose $\varphi \in L^2(\sigma(A), d\mu_\psi)$ in $(\varphi_1, \varphi_2, \varphi_3)$, in the usual orthogonal decomposition sense.

Then, through the unitary transformation, we have

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing} \tag{205}$$

where

$$H_{pp} := U^{-1} L^2(\mathbb{R}, d\mu_{pp})$$

etc. How is A acting on $\mathcal{H}_{pp}$? Assume μ_{pp} is concentrated on the points $x_1, ..., x_n, ...$ We see that $UAU^{-1} = F$. Here we take $g_k = \delta(x = x_k)$. Then, $Fg_k = x_k g_k$, and also any $g \in L^2(\mathbb{R}, d\mu_{pp}) = \sum_{n=1}^{\infty} g_k g(x_k)$ and thus $AU^{-1}g_k = x_k U^{-1} g_k$, and for any $\psi \in H_{pp}$ $\psi = \sum c_k \varphi_k$ where $\psi_k = U^{-1} g_k$.

Thus, H_{pp} is a Hilbert space where the restriction of A has a complete set of eigenvectors (relative to H_{pp}).

In Quantum Mechanics, these are the bound states, or eigenstates. For instance, for the Hydrogen atom, we have infinitely many bound states, or orbitals, of increasing size. Here, the *energy* operator is unbounded though,

$$H = -\Delta + \frac{1}{r}$$

Likewise, H_{ac} consists of *scattering states*. For the Hydrogen atom, these would be energies too large for the electron to be bound, and it travels to infinity.

H_{sing} is in some sense non-physical, and the struggle is to show it does not exist.

28.2 Multiplicity free operators

Definition 107. *A bounded self-adjoint operator A is multiplicity-free if, by definition, A is unitarily equivalent with Λ on $L^2(\mathbb{R})$.*

Proposition 108. *The following three statements are equivalent:*

1. *A is multiplicity-free*
2. *A has a cyclic vector*
3. *$\{B : AB = BA\}$ is an Abelian algebra*

Exercise 1. *Let A be a diagonal matrix. Show by direct calculation that 2. and 3. hold iff the eigenvalues are distinct.*

Exercise 2. *Does the operator X of multiplication by x on $L^2[0, 1]$ have cyclic vectors? How about id/dx?*

28.3 Spectral theorem: continuous functional calculus

Assume N is normal and $\mathfrak{S} = \sigma(N)$. Let $C(\mathfrak{S})$ be the C^*-algebra of continuous functions on $\mathfrak{S}$ in the sup norm.

Let on the other hand $C^*(N)$ be the C^*-algebra generated by N. This is the norm closure of the set $\{f(N), f(N^*) : f \text{ analytic}\}$. Since it is a closure in "sup" norm, we get continuous functions in this way. When we need measurable functions, we have to take a strong (not norm) closure.

Theorem 31 (Spectral theorem: continuous functional calculus)**.** *There is a unique isomorphism between $C(\mathfrak{S})$ and $C^*(N) \subset \mathcal{L}(\mathcal{H})$ such that $\varphi(x \to x) = N$. Furthermore, the spectrum of $\check{g} \in C(\mathfrak{S})$ is the same as the spectrum of $g(N)$ and if $N\psi = \lambda\psi$, then $f(N)\psi = f(\lambda)\psi$.*

We consider $C_{\mathfrak{S}} = C(\mathfrak{S})$ in the sup norm, and the closure C_N of $P(N, N^*)$ in the operator norm. Let f be an analytic function. We write $\varphi(f)(N) = f(N)$

and $\varphi(\overline{f})(N) = f^*(N^*)$. It is easy to check that this is an algebra homeomorphism. Furthermore, we have

$$\|P(N)\|_{\mathcal{L}(\mathcal{H})} = \sup_{\lambda\in\mathfrak{S}} |P(\lambda)| = \|P\|_\infty \tag{206}$$

so φ is an isometry. In particular, φ is one-to-one on its image. The domain of φ, $D(\varphi)$, contains all functions of the form $P(z)$ and $\overline{P}(z)$ where P is a polynomial, and by the Stone-Weierstrass theorem, the closure of D is $C(\mathfrak{S})$. Thus, φ extends by continuity to $C(\mathfrak{S})$. On the other hand, since φ is one-to-one isometric, $\varphi(C(\mathfrak{S})) = C^*(N)$, the closure of polynomials in N, N^* in the operator norm.

28.4 Spectral theorem, measurable functional calculus form

Now that we have defined continuous functions of N, we can extend calculus to measurable functions by taking strong limits, in the following way.

Recall that:

Lemma 109. *T_n is a sequence of operators which converges strongly iff $\langle\psi, T_n\varphi\rangle$ converge for all ψ, φ. By the polarization identity, T_n converges strongly iff $\langle\psi, T_n\psi\rangle$ converges for all ψ.*

Proof. This is Theorem VI.1 in [5]. □

We then take the closure of $C^*(N)$ in the strong limit. This is the von Neumann, or W^*-algebra. Strong closure of $C^*(N)$ corresponds to essup-type closure of $C(\sigma(N)$, giving Borel functions. This essup is however calculated with respect to an infinite system of measures, one for each pair of vectors in the original Hilbert space. For every $\psi_1, \psi_2 \in \mathcal{H}$ and $g \in C(\mathfrak{S})$, consider the application $g \mapsto \langle\psi_1, g(N)\psi_2\rangle$. This is a linear functional on $C(\mathfrak{S})$, and thus there is a complex Baire measure so that

$$\langle\psi_1, g(N)\psi_2\rangle = \int_{\mathfrak{S}} g(s)d\mu_{\psi_{12}}(s) \tag{207}$$

(cf. Theorem IV.17 in [5]). (alternatively, we could use the polarization identity and work with positive measures generated by $\langle\varphi, g(N)\varphi\rangle$. The closure contains all pointwise limits of uniformly bounded sequences.

Proposition 110. *If $g_n \to f$ pointwise and $|g_n| < C\ \forall n$, then $g_n(N)$ converges strongly to an element in $W^*(N)$, which we denote by $f(N)$. We have $\|f(N)\| = \|f\|_\infty$.*

Proof. Let $\mathcal{B}(\mathfrak{S})$ be the bounded Borel functions on $\mathfrak{S}$.

We take a sequence g_n converging in the sense of the Theorem to $f \in \mathcal{B}(\mathfrak{S})$. Then the rhs of (209) converges, by dominated convergence, and so does the left side. Then, the sequence of $g_n(T)$ converge strongly, let the limit be T. We then have,

$$\langle\varphi, T\psi\rangle = \int_{\mathfrak{S}} f(s)d\mu_{\psi_{12}}(s) \tag{208}$$

and it is natural to denote T by $f(N)$. Likewise, we could have defined $f(N)$ by

$$\langle \psi, f(N)\psi \rangle = \int_{\mathfrak{S}} f(s) d\mu_\psi(s) \tag{209}$$

with $\psi_2 = \psi$ and ψ_1 spanning $\mathcal{H}$.

We have an isometric correspondence Φ between operators $g(T)$, and bounded measurable functions. The isometric isomorphism extends to $W^*(N)$.

Note that the collection of spectral measures only depends on N.

Remark 32. *We note that $f(N)$ is self-adjoint iff f is real-valued. Indeed, in this case we have $f(N) \leftrightarrow f(z) = \overline{f(z)} \leftrightarrow f^*(N^*) = f(N^*)$ since f is real.*

□

Now we can define spectral measures! It is enough to take $E(S) = \chi_S(N)$.

By Remark 32, we have that $E(S)$ are self-adjoint. They are projections, since $P^2 = P$, orthogonal if $S_1 \cap S_2 = \emptyset$. In fact, $\{E(S) : S \in \mathrm{Bor}(\mathfrak{S})$ is a spectral family, the only property to be checked is sigma-additivity, which follows from the same property of characteristic functions, and continuity of Φ.

What is the integral wrt dE? We have, for a simple function $f = \sum f_k \chi(S_k)$, by definition,

$$\int f dE = \sum_k f_k E(S_k) = \Phi^{-1} \sum f_k \chi(S_k) = \Phi^{-1}(f) = f(N)$$

We thus get the projection-valued measure form of the spectral theorem

Theorem 32 (Spectral theorem for bounded normal operators)**.** *Let N be bounded and normal (that is, $NN^* = N^*N$). Then there is a spectral measure E defined on the Borel sets on $\sigma(N)$, see Definition 98, such that*

$$N = \int_{\sigma(N)} z dE(z) \tag{210}$$

Furthermore, if $f \in B(\mathfrak{S})$, we have

$$f(N) = \int_{\sigma(N)} f(z) dE(z) \tag{211}$$

28.5 Changes of variables

Let now $u(s)$ be measurable from $\mathfrak{S}$ to the measurable set $\Omega \in \mathbb{C}$, taken as a measure space with the Borel sets. Then, $E(u^{-1}(O))$ is a spectral family, on $Bor(\Omega)$. It is defined in the following way: if $O \in \Omega$ is measurable, then $u^{-1}(O) = S \in \mathfrak{S}$ is measurable, $\chi(S)$ is well defined, and so is $E_1(O) :=$

$\Phi^{-1}(\chi(u^{-1}(O)))$. We have, for a simple function f on Ω: $f = f_k$ on O_k, by definition,

$$\int_\Omega f_k dE_1 = \sum f_k E_1(O_k) = \sum f_k E(u^{-1}(O_k)) = \sum f_k E(S_k) = \int_{\mathfrak{S}} \tilde{f} dE \tag{212}$$

where $\tilde{f} = f_k$ if $s \in S_k$ or, which is the same, $u(s) \in O_k$. In other words, $\tilde{f} = f \circ u$. Thus,

$$\int_{u(\mathfrak{S})} f(t) dE_1(t) = \int_{u(\mathfrak{S})} f(t) d(E \circ u^{-1})(t) \int_{\mathfrak{S}} f(u(s)) dE(s) \tag{213}$$

In particular, assuming u is one-to-one, taking f to be the identity, we have

$$\int_{\mathfrak{S}} u(s) dE(s) = \int_{u(\mathfrak{S})} t dE_1(t) = \int_{u(\mathfrak{S})} t d(E \circ u^{-1})(t) \tag{214}$$

This form of the spectral measure theorem is still a form of functional calculus, it allows to define Borel functions of N, in particular characteristic functions of N which are projections, and the Hilbert space is, heuristically, $\bigoplus_{\lambda \in \sigma(N)} dE(\lambda)(\mathcal{H})$, but this not does not yet really relate the action of N on $\mathcal{H}$ to multiplication by n on $\sigma(N)$. And in fact, $\mathcal{H}$ is not, in general, isomorphic to $L^2(\sigma(N))$ in such a way that N becomes multiplication by N, but in fact to a direct sum, maybe infinite, of such spaces.

Remark 33. *Let N be normal. Then $A = \frac{1}{2}(N + N^*)$ and $B = -\frac{i}{2}(N - N^*)$ are self adjoint, commute with each-other and $N = A + iB$. Thus the spectral theorem for normal operators follows from the one on self-adjoint operators, once we deal with families of commuting ones.*

Let first A be a self-adjoint, operator.

Proposition 111. *If A is multiplicity-free, then there is a measure μ so that $\mathcal{H}$ is isomorphic with $L^2(\sigma(A), d\mu)$ in such a way that the unitary equivalence U has the property $UAU^{-1} = a\cdot$ where $a\cdot$ is the operator of multiplication by a.*

Proof. If f is, say, continuous, then $f(A)\psi$ is dense in $\mathcal{H}$. We then define U on this dense set: $U(f(A)\psi) = f$. We have

$$\|f(A)\psi\| = \langle f(A)\psi, f(A)\psi\rangle = \langle f(A)\overline{f}(A)\psi, \psi\rangle = \langle f\overline{f}(A)\psi, \psi\rangle = \int |f|^2 d\mu_\psi = \|f\|_2 \tag{215}$$

so $f(A)\psi = g(A)\psi$ iff $f = g[d\mu_\psi]$. This is a point is where the type of measure μ_ψ, which depends of course on A, is important. The same equality shows that $\|Uf(A)\psi\| =: \|f\|_2 = \|f(A)\psi\|$ so that U is an isometry, on this dense set, so U extends to an isometry on $\mathcal{H}$. Clearly, $UAf(A)\psi = af(a)$, so $UAU^{-1} = a\cdot$.

In general, we can "iterate" the construction: we take any φ and look at the closure $\mathcal{H}_1$ of the orbit of $g(A)\varphi$; if this is not the whole of $\mathcal{H}$, then we take φ' in the orthogonal complement of $\mathcal{H}_1$ and consider the orbit of that. This construction requires transfinite induction (Zorn's lemma). For separable spaces we can be more constructive, but this is not the main point. Then, $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ and on each, $\mathcal{H}_n$, the theorem holds, with a different measure.

□

Let A be selfadjoint, unbounded (the bounded case has been dealt with). Then $N = (A - i)^{-1}$ is a normal operator, and the spectrum of N is $1/(z - i)(\sigma(A)) = \Omega$, a piece of a circle.

We have

$$N = \int_{\Omega} n dE(n) = \int_{\sigma(A)} (t - i)^{-1} dE_1(t) \tag{216}$$

We define the operator of multiplication by t on $L^2(\sigma(A))$. To it we attach an operator A:

Theorem 33. *Let A be self-adjoint. There is a spectral measure $E : Bor(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$, so that*

(i)

$$D(A) = \left\{ x \in \mathcal{H} : \int_{\mathbb{R}} t^2 d\langle E(t)x, x\rangle < \infty \right\}$$

(ii) For $x \in D(A)$ we have

$$Ax = \left(\int_{\mathbb{R}} t dE \right) x$$

References

[1] R. G. Douglas, Banach Algebra Techniques in Operator Theory, Springer, 2nd ed., (1998).

[2] N. Dunford and J.T. Schwartz, Linear Operators, Part I: General Theory, Interscience, New York (1960.)

[3] Function Theory of One Complex Variable: Thir (Hardcover) by R. E. Greene and S. G. Krantz, Function Theory of One Complex Variable, American Mathematical Society; 3 edition (2006).

[4] P. D. Lax, Functional Analysis, Wiley (2002).

[5] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic Press, New York, 1972).

[6] T Kato, *Perturbation Theory for Linear Operators*, Springer Verlag (1995).

[7] E. Megginson An Introduction to Banach Space Theory, Springer (1998).

[8] W. Rudin, Real and Complex Analysis, McGraw-Hill (1987).