$$= \varphi_z \left(\sum_{n=0}^{N} \alpha_n \chi_n \right)$$

for every function $\sum_{n=0}^{N} \alpha_n \chi_n$ in $\mathcal{P}_+$, it follows from Proposition 1.21 that ψ is continuous. ■

2.53 From Proposition 2.3 we know that the maximal ideal space of $C(\mathbb{T})$ is just $\mathbb{T}$. We have just shown that the maximal ideal space of the closed subalgebra A of $C(\mathbb{T})$ is $\overline{\mathbb{D}}$. Moreover, if φ_z is a multiplicative linear functional on A, and $|z| = 1$, then φ_z is the restriction to A of the "evaluation at z" map on $C(\mathbb{T})$. Thus the maximal ideal space of $C(\mathbb{T})$ is embedded in that of A. This example also shows how the maximal ideal space of a function algebra is, at least roughly speaking, the natural domain of the functions in it. In this case although the elements of A are functions on $\mathbb{T}$, there are "hidden points" inside the circle which "ought" to be in the domain. In particular, viewing χ_1 as a function on $\mathbb{T}$, there is no reason why it should not be invertible; on $\overline{\mathbb{D}}$, however, it is obvious why it is not—it vanishes at the origin.

Let us consider this example from another viewpoint. The element χ_1 is contained in both of the algebras A and $C(\mathbb{T})$. In $C(\mathbb{T})$ we have $\sigma_{C(\mathbb{T})}(\chi_1) = \mathbb{T}$, while in A we have $\sigma_A(\chi_1) = \overline{\mathbb{D}}$. Hence not only is the "A-spectrum" of χ_1 larger, but it is obtained from the $C(\mathbb{T})$-spectrum by "filling in a hole." That this is true, in general, is a corollary to the next theorem.

2.54 Theorem. (Šilov) If $\mathfrak{B}$ is a Banach algebra, $\mathfrak{U}$ is a closed subalgebra of $\mathfrak{B}$, and f is an element of $\mathfrak{U}$, then the boundary of $\sigma_{\mathfrak{U}}(f)$ is contained in the boundary of $\sigma_{\mathfrak{B}}(f)$.

Proof If $(f - \lambda)$ has an inverse in $\mathfrak{U}$, then it has an inverse in $\mathfrak{B}$. Thus $\sigma_{\mathfrak{U}}(f)$ contains $\sigma_{\mathfrak{B}}(f)$ and hence it is sufficient to show that the boundary of $\sigma_{\mathfrak{U}}(f) \subset \sigma_{\mathfrak{B}}(f)$. If λ_0 is in the boundary of $\sigma_{\mathfrak{U}}(f)$, then there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ contained in $\rho_{\mathfrak{U}}(f)$ such that $\lim_{n\to\infty} \lambda_n = \lambda_0$. If for some integer n it were true that $\|(f - \lambda_n)^{-1}\| < |1/(\lambda_0 - \lambda_n)|$, then it would follow that

$$\|(f - \lambda_0) - (f - \lambda_n)\| < 1/\|(f - \lambda_n)^{-1}\|$$

and hence $f - \lambda_0$ would be invertible as in the proof of Proposition 2.7. Thus we have $\lim_{n\to\infty} \|(f - \lambda_n)^{-1}\| = \infty$.

If λ_0 were not in $\sigma_{\mathfrak{B}}(f)$, then it would follow from Corollary 2.8 that $\|(f - \lambda)^{-1}\|$ is bounded for λ in some neighborhood of λ_0, which is a contradiction. ■

2.55 Corollary. If $\mathfrak{B}$ is a Banach algebra, $\mathfrak{U}$ is a closed subalgebra of $\mathfrak{B}$, and f is an element of $\mathfrak{U}$, then $\sigma_{\mathfrak{U}}(f)$ is obtained by adding to $\sigma_{\mathfrak{B}}(f)$ certain of the bounded components of $\mathbb{C}\backslash\sigma_{\mathfrak{B}}(f)$.

$\sigma(U) = \exp(i\sigma(H))$ by Corollary 2.37, we see that the spectrum of H must be real. ∎

4.28 Theorem. If $\mathfrak{B}$ is a C^*-algebra, $\mathfrak{A}$ is a closed self-adjoint subalgebra of $\mathfrak{B}$, and T is an element of $\mathfrak{A}$, then $\sigma_{\mathfrak{A}}(T) = \sigma_{\mathfrak{B}}(T)$.

Proof Since $\sigma_{\mathfrak{A}}(T)$ contains $\sigma_{\mathfrak{B}}(T)$, it is sufficient to show that if $T-\lambda$ is invertible in $\mathfrak{B}$, then the inverse $(T-\lambda)^{-1}$ is in $\mathfrak{A}$. We can assume $\lambda = 0$ without loss of generality. Thus, T is invertible in $\mathfrak{B}$, and therefore T^*T is a self-adjoint element of $\mathfrak{A}$ which is invertible in $\mathfrak{B}$. Since $\sigma_{\mathfrak{A}}(T^*T)$ is real by the previous theorem, we see that $\sigma_{\mathfrak{A}}(T^*T) = \sigma_{\mathfrak{B}}(T^*T)$ by Corollary 2.55. Thus, T^*T is invertible in $\mathfrak{A}$ and therefore

$$T^{-1} = (T^{-1}(T^*)^{-1})T^* = (T^*T)^{-1}T^*$$

is in $\mathfrak{A}$ and the proof is complete. ∎

We are now in a position to obtain a form of the spectral theorem for normal operators. We use it to obtain a "functional calculus" for continuous functions as well as to prove many elementary results about normal operators.

Our approach is based on the following characterization of commutative C^*-algebras.

4.29 Theorem. (Gelfand-Naimark) If $\mathfrak{A}$ is a commutative C^*-algebra and M is the maximal ideal space of $\mathfrak{A}$, then the Gelfand map is a $*$-isometric isomorphism of $\mathfrak{A}$ onto $C(M)$.

Proof If Γ denotes the Gelfand map, then we must show that $\overline{\Gamma(T)} = \Gamma(T^*)$ and that $\|\Gamma(T)\|_\infty = \|T\|$. The fact that Γ is onto will then follow from the Stone-Weierstrass theorem.

If T is in $\mathfrak{A}$, then $H = \frac{1}{2}(T+T^*)$ and $K = (T-T^*)/2i$ are self-adjoint operators in $\mathfrak{A}$ such that $T = H + iK$ and $T^* = H - iK$. Since the sets $\sigma(H)$ and $\sigma(K)$ are contained in $\mathbb{R}$, by Theorem 4.27, the functions $\Gamma(H)$ and $\Gamma(K)$ are real valued by Corollary 2.36. Therefore,

$$\overline{\Gamma(T)} = \overline{\Gamma(H) + i\Gamma(K)} = \Gamma(H) - i\Gamma(K) = \Gamma(H - iK) = \Gamma(T^*),$$

and hence Γ is a $*$-map.

To show that Γ is an isometry, let T be an operator in $\mathfrak{A}$. Using Definition 4.26, Corollary 2.36, Theorem 2.38, and the fact that T^*T is self-adjoint, we have

$$\|T\|^2 = \|T^*T\| = \left\|(T^*T)^{2^k}\right\|^{1/2^k} = \lim_{k\to\infty}\left\|(T^*T)^{2^k}\right\|^{1/2^k}$$
$$= \|\Gamma(T^*T)\|_\infty = \|\Gamma(T^*)\Gamma(T)\|_\infty = \||\Gamma(T)|^2\|_\infty = \|\Gamma(T)\|_\infty^2.$$

Therefore Γ is an isometry and hence a $*$-isometric isomorphism onto $C(M)$. ∎