Lemma If $f \in L^\infty$ then $\sigma(f) = \text{essran}(f)$

Proof If $\lambda \notin \text{essran}(f)$ then $\exists\ \varepsilon$ $|f-\lambda| \geq \varepsilon$ a.e

or equiv. $\left|\frac{1}{f-\lambda}\right| \leq \frac{1}{\varepsilon} = M$ a.e. thus $\frac{1}{f-\lambda} \in L^\infty$.

Conversely, if $(f-\lambda)$ $\lambda \in$ essran f, then

$\forall \varepsilon$ $\exists\ S_\varepsilon \subset L^\infty$ s.t $|f-\lambda| < \varepsilon$ with

>0 measure. This implies $\frac{1}{|f-\lambda|} > M$

with >0 probability for all M, thus $\frac{1}{f-\lambda} \notin L^\infty$

Thm If M is the max ideal space of L^∞ then

Γ is an isometric isomorphism of L^∞ onto $C(M)$

Proof Recall that $\Gamma(\bar{f}) = \overline{\Gamma(f)}$. The image of

L^∞ under Γ is a self-adjoint subalgebra

of $C(M)$ containing $1 = \Gamma(1)$

It also separates points since $\varphi_1 \neq \varphi_2 \in M$

$\Rightarrow \varphi_1(f) \neq \varphi_2(f)$ for some $f \in L^\infty$ $\Rightarrow$

$\hat{f}(\varphi_1) \neq \hat{f}(\varphi_2)$ for $f \in L^\infty$ and we

learn that $\hat{f}$ is continuous.

Thus (Stone - Weierstrass) $\boxed{\Gamma(L^\infty) = C(M)}$

(we have also shown $\sigma(f) = \text{essran} f = \|\Gamma f\|$

$\Downarrow$

$\|\Gamma f\| = \sigma(f) \Rightarrow r(\text{essran} f) = \|f\|$

②

This is typically the Stone Čech compactification.

Another interesting example; Šilov's theorem

Let $\mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$ and for $n \in \mathbb{Z}$

let $\chi_n(z) = z^n$. Then $\chi_0 = 1$, $\chi_{-n} = \overline{\chi_n}$

$\chi_{n+m} = \chi_n \cdot \chi_m$. The functions

$$P_N = \sum_{n=-N}^{N} \alpha_n \chi_n, \quad \alpha_n \in \mathbb{C} \quad \text{are called}$$

trig polynomials.

Note that $\mathcal{P}$ is a self-adjoint subalgebra of $C(\mathbb{T})$ containing 1 and separating points.

Thus $\overline{\mathcal{P}} = C(\mathbb{T})$; $\overline{\quad}^{\text{trivial}}$ $M = \mathbb{T}$, $C(\mathbb{T}) \sim C(\mathbb{T})$

What about half infinite sums?

Let $\mathcal{P}_+ = \{ \sum_{n \geq 0} \alpha_n \chi_n \mid \alpha_n \in \mathbb{C} \}$

what is $\overline{\mathcal{P}_+}$ (Note that this is not self-adjoint; the elements of $\mathcal{P}$ are not invertible, in general

Still, $\overline{\mathcal{P}_+}$ is a Banach algebra.

What is M ?

Note that if $w \in \mathbb{C}$ $\sum_{n=0}^{N} \alpha_n w^n$ is in $\mathbb{C}$

as well, and that $|w| < 1 \Rightarrow$

$$\Rightarrow \quad P_N(w) = \frac{1}{2\pi} \int_0^{2\pi} \sum \alpha_n X_n(e^{it}) \frac{1}{1 - we^{-it}} dt \quad (*)$$

(Cauchy)

Clearly, for $w<1$ $P_N(w)$ is multiplicative and the integral formula $(*)$ shows that it is bounded. It thus

$\varphi_w = w \to \mathcal{P}_N(w)$

extends to $\overline{P_N(w)}$ and it is multiplicative

in $\overline{P_N(w)} = A$

Let $\overline{\mathbb{D}} = \{ z \mid |z| \leq 1 \}$ and M the max ideal space of A. We claim that $M \sim \mathbb{D}$

First, $\forall z \in \mathbb{D}$ and $f \in A$, $f(w)$ is well defined by the closure of φ_w, and is multiplicative

Conversely, let φ be <u>any</u> multipl. functional on $\mathcal{A}$. Since $\|\varphi\| = 1$ we have $|\varphi(f)| \leq 1$ $\forall f$. In particular, $\varphi(\chi_1) \in \overline{\mathbb{D}}$ and
$z_0 :=$

$$\varphi(P_N) = \sum \alpha_n z_0^n = \varphi_{z_0}(P_N)$$

By continuity and closure, $\varphi(f) = f(z_0)$

Note that the correspondence $\varphi_z \leftrightarrow z \in \mathbb{D}$ is one-to-one since $z_1 \neq z_2$ $\Leftrightarrow \varphi_{z_1} \neq \varphi_{z_2}$ (as applied to $\chi_1 \equiv z$)

The corresp. $z \to \varphi_z$ is one-one, to show it is bycontinuous we only need to show it is continuous

Let $z_\alpha \to z$ Then $\varphi_{z_\alpha}(P_N) = P_N(z_\alpha) \to P_N(z)$ $\forall P_N$ Thus, by density and continuity

$$\varphi_{z_\alpha} \to \varphi_z$$

Thus $M(\mathcal{A}) = \overline{\mathbb{D}}$ (while $M(C(\mathbb{T})) = \mathbb{T}$)

there are, in a sense, more points where the elements of $\mathcal{A}$ can be evaluated

We see that $\sigma_{C(\mathbb{T})}(f) \subsetneq \sigma_{A_+}(f)$

<u>Theorem</u> (Šilov) If A is a Banach algebra and A_1 is a closed subalgebra of A then for $f \in A_1$ $\partial\sigma_{A_1}(f) \subset \partial\sigma_A(f)$

<u>Proof</u> If $f-\lambda$ is invertible in A_1, then it is invertible in A, obviously, thus $\partial\sigma_{A_1}(f) \supset \partial\sigma_A(f)$
We only have to show the converse: $\partial\sigma_{A_1}(f) \subset \partial\sigma_A(f)$

Assume $\lambda_0 \in \partial\sigma_{A_1}(f)$. Then $\exists\ \lambda_1, \dots, \lambda_n, \dots$
$\in \sigma_{A_1}(f)$ s.t. $\lambda_n \to \lambda_0$.

Assume, to get a contradiction that $\|f-\lambda_n\|^{-1}$ were bounded. Say by $\dfrac{1}{|\lambda_0 - \lambda_n|}$

Then $|f-\lambda_0| = |f - \lambda_n + \lambda_n - \lambda_0| =$

$$= (f-\lambda_n)\Big(1 + \underbrace{(\lambda_n - \lambda_0)\cdot\frac{1}{f-\lambda_n}}_{\|\ \ \| < 1}\Big)$$

$\Rightarrow f-\lambda_0$ invertible, contradiction.

Thus $\|f-\lambda_n\|^{-1} \to \infty$ also in A

thus $\lambda_0 \in \sigma(A) \supset \partial\sigma(A)$.

Corollary $\sigma_{\mathcal{A}_1}$ and $\sigma_{\mathcal{A}}$ are compact sets in $\mathbb{C}$

We can take the connected component decomposition of $\sigma_{\mathcal{A}}$ and the one of $\sigma_{\mathcal{A}_1}$. Since $\partial\sigma_{\mathcal{A}_1} = \partial\sigma_{\mathcal{A}}$, then $\sigma_{\mathcal{A}_1} = \sigma_{\mathcal{A}} \cup$ connected components of $\mathbb{C} \setminus \sigma_{\mathcal{A}}$ ($\sigma_{\mathcal{A}_1}$ is obtained from $\sigma_{\mathcal{A}}$ by „filling holes")

C^* algebras and involutions

Let $\mathcal{A}$ be a Banach algebra. An involution on $\mathcal{A}$ is a mapping $f \to f^*$ which satisfies,

1. $f^{**} = f \quad \forall f \in \mathcal{A}$
2. $(\alpha f + \beta g)^* = \bar{\alpha} f^* + \bar{\beta} g^*$
3. $(f g)^* = g^* f^*$

Definition. A B^* algebra is a Banach Algebra in which 1,2,3 hold and also

4. $\|f^* f\| = \|f\|^2 \quad \forall f.$

An element f in an algebra is called

- unitary if $ff^* = f^* f = I$
- self-adjoint if $f = f^*$
- normal if $ff^* = f^* f$.

(Thus unitary and self-adj elements are always normal)

<u>Lemma</u> In a C^*-algebra $\|T\| = \|T^*\|$ (involution is an isometry)

Indeed $\|f\|^2 = \|f^* f\| \le \|f^*\| \, \|f\|$ (1)

$\|f^*\|^2 = \|f^* f^{**}\| = \|f^*\| \, \|f\|$ (2)

(1) implies $\|f\| \le \|f^*\|$; 2 implies $\|f^*\| \le \|f\|$

<u>Theorem</u> In a C^*-algebra the spectrum of any s.a. T element is real; Unitary: $\sigma(U) \subset \mathbb{T}$

<u>Proof</u> Let $U = e^{iT}$. Then, from the series $U^* = e^{-iT^*} = e^{iT}$ Thus U is unitary $\Rightarrow$ $\|U\| = \|UU^*\| = \|1\| = 1$

Thus we proved $\sigma(U) \subset \overline{\mathbb{D}}$

Now, if $|\lambda| < 1$ then $U - \lambda$

invertible

$= U^*(1 - \lambda U^*)$

invertible.

Thus $\sigma(U) \subset \mathbb{T}$. By the spectral theorem, valid in any Banach space,

$\sigma(e^{iT}) = e^{i\sigma(T)}$ $\Rightarrow$ $\sigma(T) \subset \mathbb{R}$ $\square$