C^* algebras, involutions

Let $\mathcal{A}$ be a Banach algebra. An involution is a mapping from $\mathcal{A}$ to $\mathcal{A}$ which satisfies

1. $f^{**} = f$
2. $(\alpha f + \beta g)^* = \bar{\alpha} f^* + \bar{\beta} g^*$
3. $(f g)^* = g^* f^*$

<u>Def</u> $\mathcal{A}$ is a C^* algebra if 1, 2, 3 hold and furthermore

4. $\|f\|^2 = \|f^* f\|^2$

<u>Lemma</u> In a C^*-algebra, involution is an isometry

Indeed : $\|T\|^2 = \|T^* T\| \leq \|T^*\| \, \|T\| \Rightarrow \|T\| \leq \|T^*\|$

$\|T^*\|^2 = \|T^{**} T^*\| = \|T T^*\| \leq \|T\| \|T^*\|$

$\Rightarrow \|T^*\| \leq \|T\|$

$\square$.

Thm 1) In a C^* algebra, unitary operators U have $\sigma(U) \subset \mathbb{T}$

2) Self-adjoint operators have real spectrum

Proof

1. Let $|\lambda| < 1$ then $U - \lambda$

$= U(1 - \underbrace{\lambda U^*}_{\| \; \| < 1})$ $\Rightarrow U-\lambda$ invertible

Let $|\lambda| > 1$; then $U - \lambda = -\lambda(1 - \underbrace{\frac{U}{\lambda}}_{\| \; \| < 1})$

invertible as well.

2. Let T be self-adjoint. Then $\sigma(T) \subset \mathbb{R}$

Proof. Let $U = e^{iT}$ Then $U^* = e^{-iT}$

(by expansion $\left(\sum \frac{(iT)^n}{n!}\right)^*$ and thus

$UU^* = U^*U = e^{i(T-T)} = 1$

Thus e^{iT} is unitary. Now the spectral theorem states $\sigma(e^{iT}) = e^{i\sigma(T)}$

thus, since $\sigma(e^{it}) \subset \mathbb{T} \Rightarrow \sigma(T) \subset \mathbb{R}$ $\triangleright$

If $\mathcal{A}^*$ is a C^* algebra and $\mathcal{A}_1^*$ is a self-adjoint subalgebra of $\mathcal{A}^*$, then $\forall f \in \mathcal{A}_1^*$, $\sigma_{\mathcal{A}_1^*}(f) =$

$= \sigma_{\mathcal{A}}(f)$

<u>Proof</u> It is clear that $\sigma_{\mathcal{A}_1^*}(f) \subset \sigma_{\mathcal{A}}(f)$

Need to show that if $(\lambda - T)^{-1}$ is the $\mathcal{A}^*$ inverse of $\lambda - T$, then $(\lambda - T) \in \mathcal{A}_1^*$

By denoting $\tilde{T} = \lambda - T$ we can assume w.l.o.g that $\lambda = 0$

Note that $[(T^{-1})^* T^*]^* = TT^{-1} = I$

and thus $(T^{-1})^* T^* = I^* = I$

hence, T is invertible iff T^* is invertible

and in particular TT^*, T^*T are then also invertible

We have $(T^*T)^* = T^*T$ is self-adjoint

Thus $\sigma(T^*T) \subset \mathbb{R}$

Now $\partial\sigma_{\mathcal{A}_1^*}(T^*T) = \partial\sigma_{\mathcal{A}}(T^*T)$ by Šilov

and furthermore $\sigma_{\mathcal{A}_1^*}(T^*T) = \sigma_{\mathcal{A}_1}(T) \cup \mathcal{B}$ where $\mathcal{B}$ are connected components of $\mathbb{C} \setminus \sigma_{\mathcal{A}}$

But here $\sigma \subset \mathbb{C}$ so these connected components are intervals (or points)

and $\partial \sigma_{A_1^*} = \partial \sigma_A$ forces $\sigma_{A_1^*}(T^*T) = \sigma_A(T^*T)$

(T^*T has an inverse in σ_A $(T^*T)^{-1}$

$$T^{-1} = T^{-1}(T^*T)^{-1}T^* = (T^*T)^{-1}T^*$$

0 is in spec

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Let A^* be commut C^* Algebra M max. ideal space of A^* then Γ is a $*$ isometric isomorphism of A^* onto $C(M)$

$\overline{\Gamma(T)} = \Gamma(T^*)$? $\|\Gamma(T)\| = \|T\|$?

$T \in A^* \Rightarrow$ $H = \frac{1}{2}(T + T^*)$ is s.a and

$K = \frac{T - T^*}{2i}$ are self adjoint ops in A^*

$T = H + iK$ $T^* = T - iK$

By Cor. 2.36 (B commutative Banach algebra and $f \in B \Rightarrow \sigma(f) = \text{range}\, \Gamma(f)$

$r(f) = \|\Gamma f\|_\infty$)

$\sigma(T)$ real $\Rightarrow$ $\Gamma(T)$ real

$$\overline{\Gamma(T)} = \overline{\Gamma(H) + i\Gamma(K)} = \Gamma(H) - i\Gamma(K) = \Gamma(T^*)$$

Isometry

$$\|T\|^2 \overset{\text{def}}{=} \|T^*T\| = \|(T^*T)(T^*T)\|^{\frac{1}{2}} = \|(T^*T)^2\|^{\frac{1}{2}}$$

$$= \cdots = \|(T^*T)^{2k}\|^{\frac{1}{2k}} = \lim_{k\to\infty} \|(T^*T)^{2k}\|^{\frac{1}{2k}}$$

$$= r(T^*T) = \|\Gamma(T^*T)\| = \|\Gamma(T^*)\Gamma(T)\|$$

$$= \|\Gamma(T)\|^2 \Rightarrow \|T\| = \|\Gamma(T)\|$$

Begin operators