1 The Gelfand-Naimark theorem (proved on Feb 7)

Theorem 1. *If $\mathfrak{A}$ is a commutative Banach algebra and M is the maximal ideal space, of $\mathfrak{A}$ then the Gelfand map Γ is a *-isometric isomorphism of $\mathfrak{A}$ onto $C(M)$.*

Corollary 1. *If $\mathfrak{A}$ is a C^* algebra and $\mathfrak{A}_1$ is the self-adjoint subalgebra generated by $T \in \mathfrak{A}$, then $\sigma_{\mathfrak{A}}(T) = \sigma_{\mathfrak{A}_1}(T)$.*

Proof. Evidently, $\sigma_{\mathfrak{A}}(T) \subset \sigma_{\mathfrak{A}_1}(T)$. Let $\lambda \in \sigma_{\mathfrak{A}_1}$. Working with $T - \lambda$ instead of T we can assume that $\lambda = 0$. By the Gelfand theorem, T is invertible in $\sigma_{\mathfrak{A}_1}$ iff $\Gamma(T)$ is invertible in $C(M)$. Note that T^*T and TT^* are self-adjoint and if the result holds for self-adjoint elements, then it holds for all elements. Indeed $(T^*T)^{-1}T^*T = I \Rightarrow (T^*T)^{-1}T^*$ is a left inverse of T^* etc. Now $\mathfrak{A}_A$, the C^* sub-algebra generated by $A = T^*T$ is commutative and Theorem 1 applies. Since $\|T\|^2 = \|T^*T| \neq 0$ unless $T = 0$, a trivial case, $\|a\|_\infty = \|T^*T\| > 0$ where $a = \Gamma A$. Let M_A be the maximal ideal space for A. If A is not invertible, by Gelfand's theorem, $a(m) = 0$ for some $m \in M_A$, and for any ε, there is a neighborhood of m s.t. $|a(m)| < \varepsilon/4$. We take $\varepsilon < \|a\|$ and note that the set $S = \{x \in M_A : |a(x) \in (\varepsilon/3, \varepsilon/2)$ is open and disjoint from M_1. There is thus a continuous function g on M which is 1 on S and zero on M_A and zero on $\{m \in M_A : a(m) > \varepsilon\}$. We have $\|g\| = \|G\| = 1$ where $\Gamma G = g$ and clearly, $\|gA\|_{\mathfrak{A}_A} < \varepsilon$. But $\|GA\|_{\mathfrak{A}_A} = \|GA\|_{\mathfrak{A}}$. If A were invertible, then with $\alpha = \|A\|^{-1}$, then $\|G\| = \|GAA^{-1}\| \leq \alpha\|GA\| \leq \varepsilon\alpha \to 0$ as $\varepsilon \to 0$, a contradiction. □

2 Operators: Introduction

We start by looking at various simple examples. Some properties carry over to more general settings, and many don't. It is useful to look into this, as it gives us some idea as to what to expect. Some intuition we have on operators comes from linear algebra. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be linear. Then A can be represented by a matrix, which we will also denote by A. Certainly, since A is linear on a finite dimensional space, A is continuous. We use the standard scalar product on $\mathbb{C}^n$,

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i$$

with the usual norm $\|x\|^2 = \langle x, x \rangle$. The operator norm of A is defined as

$$\|A\| = \sup_{x \in \mathbb{C}^n} \frac{\|Ax\|}{\|x\|} = \sup_{x \in \mathbb{C}^n} \left\| A \frac{x}{\|x\|} \right\| = \sup_{u \in \mathbb{C}^n : \|u\| = 1} \|Au\| \tag{1}$$

Clearly, since A is continuous, the last sup (on a compact set) is in fact a max, and $\|A\|$ is bounded. Then, we say, A is bounded.

The spectrum of A is defined as

$$\sigma(A) = \{\lambda \mid (A - \lambda) \text{ is not invertible}\} \tag{2}$$

This means $\det(A - \lambda) = 0$, which happens iff $\ker\,(A - \lambda) \neq \{0\}$ that is

$$\sigma(A) = \{\lambda \mid (Ax = \lambda x) \text{ has nontrivial solutions}\} \tag{3}$$

For these operators, the spectrum consists exactly of the eigenvalues of A. This however is not generally the case for infinite dimensional operators.

• **Self-adjointness** A is symmetric iff

$$\langle Ax, y\rangle = \langle x, Ay\rangle \tag{4}$$

for all x and y. For matrices, symmetry is the same as self-adjointness, but this is another property that is generally true only in finite dimensional spaces. As an exercise, you can show that this is the case iff $(A)_{ij} = \overline{(A)_{ji}}$.

We can immediately check that all eigenvalues are real, using (4).

We can also check that eigenvectors x_1, x_2 corresponding to distinct eigenvalues λ_1, λ_2 are orthogonal, since

$$\langle Ax_1, x_2\rangle = \lambda_1\langle x_1, x_2\rangle = \langle x_1, Ax_2\rangle = \lambda_2\langle x_1, x_2\rangle \tag{5}$$

More generally, we can choose an orthonormal basis consisting of eigenvectors u_n of A. We write these vectors in matrix form,

$$U = \begin{pmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n2} \\ \cdots & \cdots & \cdots & \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{pmatrix} \tag{6}$$

and note that

$$UU^* = U^*U = I \tag{7}$$

where I is the identity matrix. Equivalently,

$$U^* = U^{-1} \tag{8}$$

We have

$$AU = A\begin{pmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & \cdots & u_{n2} \\ \cdots & \cdots & \cdots & \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} Au_1 & Au_2 & \cdots & Au_n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{pmatrix} = U\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} =: UD \tag{9}$$

where D is a diagonal matrix. In particular

$$U^*AU = D \tag{10}$$

which is a form of the spectral theorem for A. It means the following. If we pass to the basis $\{u_j\}$, that is we write

$$x = \sum_{k=1}^{n} c_k u_k \tag{11}$$

we have

$$c_k = \langle x, u_k \rangle \tag{12}$$

that is, since $\tilde{x} = (c_k)_k$ is the new representation of x, we have

$$\tilde{x} = U^* x \tag{13}$$

We also have

$$Ax = \sum_{k=1}^{n} c_k A u_k = \sum_{k=1}^{n} c_k \lambda_k u_k = D\tilde{x} =: \tilde{A}\tilde{x} \tag{14}$$

another form of (10). This means that after applying U^* to $\mathbb{C}^n$, the new A, $\tilde{A}$ is diagonal (D), and thus is *it acts multiplicatively.*

• **In infinite dimensional spaces:**

◦ Eq. (1) stays as a definition but the sup may not be attained.

◦ Definition (2) stays.

◦ Property (3) will not be generally true anymore.

◦ (4) will still be valid for bounded operators but generally false for unbounded ones.

◦ (5) stays.

◦ (14) properly changed, will be the spectral theorem.

Let us first look at $L^2[0,1]$; here, as we know,

$$\langle f, g \rangle = \int_0^1 f(s)\overline{g(s)} ds$$

We can check that X, defined by

$$(Xf)(x) = xf(x)$$

is symmetric. It is also bounded, with norm ≤ 1 since

$$\int_0^1 s^2 |f(s)|^2 ds \leq \int_0^1 |f(s)|^2 ds \tag{15}$$

The norm is exactly 1, as it can be seen by choosing f to be the characteristic function of $[1-\epsilon, \epsilon]$ and taking $\epsilon \to 0$. Note that unlike the finite dimensional case, the sup is *not attained: there is no f s.t.* $\|xf\| = 1$.

What is the spectrum of X? We have to see for which λ $X - \lambda$ is not invertible, that is the equation

$$(x - \lambda)f = g \tag{16}$$

does not have L^2 solutions for all g. This is clearly the case iff $\lambda \in [0,1]$.

But we note that $\sigma(X)$ has no eigenvalues! Indeed,

$$(x-\lambda)f = 0 \Rightarrow f = 0 \forall x \neq \lambda \Rightarrow f = 0 \; a.e, \Rightarrow f = 0 \text{ in the sense of } L^2 \tag{17}$$

Finally, let us look at X on $L^2(\mathbb{R})$. The operator stays symmetric, wherever defined. Note that now X is unbounded, since, with χ the characteristic function,

$$x\chi_{[n,n+1]} \geq n\chi_{[n,n+1]} \tag{18}$$

and thus $\|X\| \geq n$ for any n. By Hellinger-Toeplitz, proved in the sequel see §3.3, X cannot be everywhere defined. Specifically, $f = (|x|+1)^{-1} \in L^2(\mathbb{R})$ whereas $|x|(|x|+1)^{-1} \to 1$ as $x \to \pm\infty$, and thus Xf is not in L^2. What is the domain of definition of X (domain of X in short)? It consists of all f so that

$$f \in L^2 \text{ and } xf \in L^2 \tag{19}$$

It is easy to check that (19) is equivalent to

$$f \in L^2(\mathbb{R}^+, (|x|+1)dx) := \{f : (|x|+1)f \in L^2(\mathbb{R}) \tag{20}$$

This is not a closed subspace of L^2. In fact, it is a dense set in L^2 since C_0^∞ is contained in the domain of X and it is dense in L^2. X is said to be densely defined.

3 Bounded and unbounded operators

1. Let X, Y be Banach spaces and $D \subset X$ a *linear* space, not necessarily closed.

2. A linear operator is a linear map $T : D \to Y$.

3. D is the domain of T, sometimes written $\text{Dom}(T)$, or $\mathcal{D}(T)$.

4. The *range* of T, $\text{ran}(T)$, is simply $T(D)$.

5. The *graph of* T is

$$\mathbb{G}(T) = \{(x, Tx) | x \in \mathcal{D}(T)\}$$

   The graph will play an important role, especially in the theory of unbounded operators.