Rem. $\eta_x = x \to \hat{x}$ $\hat{x}(\varphi) = \varphi_x$ is continuous. The Gelfand transform $\Gamma : \mathcal{B} \to M$ $\Gamma(x) = \hat{x}$.

- We know $\|\Gamma f\|_\infty = r(f) \leq \|f\|$ $\sigma(f) = \operatorname{Ran}(\Gamma(f))$
- f is invertible in $\mathcal{B}$ iff Γf is invertible in $C(M)$
- $\|f^2\| = \|f\|^2 \Leftrightarrow \|\Gamma f\| = \|f\|$ Γ is an isometry

The self-adjoint subalgebras of $C(x)$ containing 1 are exactly classes of functions in $C(X)$ w. the property $f(x) = f(x')$ $\forall x, x'$ s.t. $\eta(x) = \eta(x')$

Application Wiener's theorem

Let's take a more nontrivial example. Consider $\ell^1(\mathbb{Z})$ with $*$ as product:

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f_k \, g_{n-k}$$

We showed last time that $\|f * g\| \leq \|f\| \|g\|$ and in particular $*$ is well defined, and $\ell^1(+, *)$ forms a Banach algebra (commutative

The elements $e_n = (\ldots 0 \ldots 0 \underbrace{\ldots 1}_{n} \ldots 0 \ldots)$ play a special role.

We see that $\|e_n\| = 1$. Also $(e_\ell * e_k)(n) =$

$= \sum e_{\ell, j} \, e_{k, n-j}$ and $e_{\ell, j} \, e_{k, n-j} \neq 0$

iff $j = \ell$ and $n - j = n - \ell = k$ i.e. $n = \ell + k$

Then $e_n^{-1} = e_{-n}$. It is easy to check that $e_0 * f = f \ \forall f$ thus e_0 is the unit element

It is also easy to check that the following decomposition is convergent (and unique):

$$f = \sum_{k\in\mathbb{Z}} f_k e_k \quad \text{(convergent in norm)}$$

Let's find all φ and Γ. Note that

$\forall \varphi$, $\varphi(e_1) \in \mathbb{C}$ (obviously) But

furthermore $1 \geq |\varphi(e_1)| = \left|\frac{1}{\varphi(e_{-1})}\right| = \frac{1}{|\varphi(e_{-1})|} \geq 1$

Thus $|\varphi(e_1)| = 1$, $\varphi(e_1) \in \mathbb{T} \ \forall \varphi$ $\varphi(e_1) = e^{i\theta} = z$

$\varphi = \varphi_\theta$

Now by continuity, $\forall f$

$$\varphi(f) = \sum f_k \varphi(e_k). \quad \text{Now } e_k = \underbrace{e_1 * e_1 * \dots * e_1}_{k}$$

$$\Rightarrow \varphi(e_k) = \varphi(e_1)^k = z^k$$

$$\Rightarrow \varphi(f) = \sum f_k e^{ik\theta}$$

φ sends f into the Fourier transform evaluated at θ.

The space of multiplicative functionals can be identified with $\mathbb{T}$

That for any θ_1 $\sum e^{ik\theta_1} f_k$ is continuous we can check it is multiplicative thus of norm 1 (Fubini + dominated convergence)

We will also see that, in general $\|f^2\| < \|f\|^2$ so Γ is not an isometry

$\Gamma(\theta) = \sum e^{ik\theta} f_k$ (*) $\Rightarrow$ $\hat{}$ is the Fourier transform

Evidently (by dominated convergence (d.c.)) (*) is continuous in θ and again by d.c. and Fubini we have

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} (\mathcal{F}f)(\theta)\, d\theta$$

From Gelfand–Naimark (studied later) we conclude Γ is not onto. $\Gamma(\ell^1) \subsetneq C(\mathbb{T})$

$\Gamma(\ell^1)$ is in fact the space of fcns $\in C^0$ which have absolutely convergent $\mathcal{F}$. Let $\Gamma(\ell^1) = \ell^0_a$

what is known about pointwise convergence of $\mathcal{F}$ series of cont fcns?

Carleson, 1966 → A) Continuous fcns have a.e convergent $\mathcal{F}$

Katznelson – B) $\forall E$ $m(E) = 0$ $\exists f \in C(\mathbb{T})$ s t $\mathcal{F}(f)$

Kahane 1972 diverges on E

In particular, $\mathcal{B}$ implies that $\exists f \in C(\mathbb{T})$ $\sum |\hat{f}_k| = \infty$ ("most" are like this)

<u>Theorem</u> (Wiener) If g has an absolutely convergent $\mathcal{F}$ series, so does $\frac{1}{g}$ if $\frac{1}{g}$ exists.

<u>Proof</u> We have $g \in \Gamma(\ell^1) = \mathcal{C}_a^0$. By Gelfand, $g \neq 0$ on $\mathbb{T}$ $\Rightarrow$ $\frac{1}{g}$ exists in $C(\mathbb{T})$ and thus is $\Gamma(\ell_1)$!

<u>Compactifications</u> Consider now L^∞ of some set X locally compact, not compact.

Check that L^∞ forms a Banach algebra w.r.t. $\| \|_\infty$ and usual multiplication.

It turns out, non-intuitively that $L^\infty(X) \sim C(M)$ for some compact M ! (M = max ideal space.)

Definition: essential range (essran)

If f is measurable on X then

$$\text{essran} f = \{\lambda \in \mathbb{C} \mid \mu\{x : |f(x) - \lambda| < \varepsilon\} > 0 \ \forall \varepsilon > 0\}$$

<u>Lemma</u> $f \in L^\infty \Rightarrow \text{essran} f$ is compact in $\mathbb{C}$.

<u>Proof</u> First show it is open

Let $m = \|f\|_\infty$ and take $m_1 < m$ $\quad m_1 > \|f\|$

Then $\not\exists \varepsilon > 0$ $\{|f(x) - m_1| > \varepsilon\}$ a.e.

thus $\left| f(x) - m_1 - \frac{\varepsilon}{2} \right| > \frac{\varepsilon}{2}$ a.e.

and thus $(\text{essran} f)^c$ is open

essran f is closed.

Let $m_2 > \|f\| \to \exists \varepsilon, |f - m_2| > \varepsilon$ a.e.

and thus the set essranf is bounded

by $\|f\|_\infty$.

We want to show that $\exists \lambda$, $\|f\| = \lambda$

and $\lambda \in$ essranf

For simplicity normalize f $\quad f \to \frac{f}{\|f\|} = f_1 \quad \|f_1\| = 1$

Assume, to get a cd that $\forall \lambda \in \mathbb{T}$ $\exists \varepsilon$

s.t $|f(x) - \lambda| > \varepsilon$ $\quad \forall x.$ This is an open

($O_\lambda = |f(x) - \lambda| > \frac{\varepsilon}{2}$ in some nbd) of $\mathbb{T}$

thus $\exists$ finite cover $\bigcup_{n \le m} O_{\lambda_n}$

thus, with $\varepsilon_m = \min_{n \le m} \varepsilon_n$ we have

$|f(x) - \mathbb{T}| > \varepsilon_n$ $\quad$ Thus if $|z| > 1 - \frac{\varepsilon}{2}$

$\mu\{x \in X \mid f(x) = z\} = 0 \Rightarrow z \notin$ essranf

Lemma If $f \in L^\infty$, then $\sigma(f) = \text{ess ran } f$

proof if $\lambda \notin \text{ess ran } f$ then $\left|\frac{1}{f-\lambda}\right| < M$ for some M a.e. and $\frac{1}{f-\lambda}(f-\lambda) = 1$

Conversely if $\lambda \in \text{ess ran } f$ then $\forall M$ $\{x : |f-\lambda| < \frac{1}{M}\} > 0 \Rightarrow \left\|\frac{1}{f-\lambda}\right\| = \infty$

Thm If M is the max ideal space of L^∞, then Γ is an isometric isomorphism of L^∞ onto $C(M)$

Proof Recall that $\Gamma(\bar{f}) = \overline{\Gamma(f)}$. The image $\Gamma(L^\infty)$ is a self-adjoint subalgebra of $C(M)$ containing 1. Then, recall that $\sigma(f) = \text{Ran}(\Gamma f) = \|\Gamma f\|$

$\text{ess ran } f \Rightarrow \|\Gamma f\| = \|f\|$

thus Γ is an isometric isomorphism between L^∞ and $\Gamma(L^\infty)$. Furthermore $\Gamma(L^\infty)$ separates pts (by def. $\varphi_1 \neq \varphi_2 \Rightarrow \exists f$ $\varphi_1(f) \neq \varphi_2(f) \Leftrightarrow \hat{f}(\varphi_1) \neq \hat{f}(\varphi_2)$ and $\hat{f}$ is continuous (we proved that)

By Stone - Weierstrass, $\Gamma(L^\infty(X)) = C(M)$

This compactification is Gelfand's compactification (in many instances it is the Stone - Čech compactification)