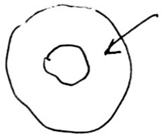
Re:

- For any $\mathcal{B}$ $\{ \prod_{i=1}^{k} e^{f_i} \mid k \in \mathbb{N},\ f_i \in \mathcal{B} \} = \mathcal{G}_0$
- If $\mathcal{B}$ commutative then $\exp(\mathcal{B}) = \mathcal{G}_0$

Note the distinction between $\exp x$, $x \in \mathbb{C}$ and $\exp(f)$

$f \in \mathcal{B}$.



$X \subseteq \mathbb{C}$ $\qquad f(z) = z \quad \forall z \in X$

is not in $\exp \mathcal{B}$ [$\mathcal{B}$ the Banach alg. of fcns defined on $X \to \mathbb{C}$ analytic in int (X) cont on X.]

whereas $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ (any $z \neq 0 \in \mathbb{C}$ $\exists \ln z$ — not unique —) but functions are not necessarily in $\exp(\mathcal{B})$. Otherwise, $\ln$ is "the same" with multivalued

Return to multip. functionals

Let $\mathcal{B}$ be a Banach algebra. A linear complex functional φ is multiplicative if $\varphi \not\equiv 0$ and $\varphi(fg) = \varphi(f)\varphi(g)$

Equivalently

1. $\varphi(fg) = \varphi(f) \cdot \varphi(g)$ $\qquad$ 2. $\varphi(1) = 1$.

We'll denote by $\mathcal{M}$ the set of all functionals on $\mathcal{B}$

We don't have to require continuity — it's automatic.

Thm if $\varphi \in \mathcal{M}$, then $\|\varphi\| = 1$

Indeed let as usual $K_\varphi = \ker \varphi = \{ f \in \mathcal{B} \mid \varphi(f) = 0 \}$

Note that $f \in \mathcal{B}$ can be decomposed as

$$f = \underbrace{f - \varphi(f) \cdot 1}_{k \in K_\varphi} + \underbrace{\varphi(f) \cdot 1}_{\lambda} = k + \lambda$$

By definition, $\|\varphi\| = \sup_{\|f\|=1} |\varphi f| = \sup_{\|f\|\neq 0} \frac{|\varphi f|}{\|f\|}$;

writing $f = k + \lambda$ $\quad k \in K_\varphi$, $\lambda \in \mathbb{C}$ $\quad \varphi f = \varphi k + \varphi \lambda = \lambda$

and thus $\quad \|\varphi\| = \sup_{\substack{k \in K_\varphi \\ \lambda \in \mathbb{C}}} \frac{|\lambda|}{\|k+\lambda\|}$ $\quad$ $\lambda = 0$ is not interesting

Let $\lambda \neq 0$ $\quad$ Then $\quad \|\varphi\| = \sup_{k \in K_\varphi} \frac{1}{\|k+1\|}$.

Note that $\|1 + k\| < 1 \Rightarrow k$ is invertible, thus $k \notin K_\varphi$

$\Rightarrow \|k+1\| \geq 1 \quad \forall\, k \in K_\varphi \Rightarrow \|\varphi\| \leq 1$

On the other hand $\quad \varphi(1) = 1 \quad \|1\| = 1 \Rightarrow \boxed{\|\varphi\| = 1}$

$\square$

Prop If $\mathcal{B}$ is a Banach algebra, then $\mathcal{M}$ is a w^* compact subset of $\mathcal{B}_1^*$

Proof $\quad$ Clearly $\mathcal{M} \subseteq \mathcal{B}_1^*$, and we only need to show that $\mathcal{M}$ is closed in $\mathcal{B}_1^*$.

Assume $\varphi_\alpha \to \varphi$ $\quad$ Then $\varphi_\alpha(fg) \to \varphi(fg)$

$$\varphi_\alpha(fg) = \varphi_\alpha(f)\,\varphi_\alpha(g) \to \varphi(f)\cdot\varphi(g), \qquad \varphi(fg) \overset{?}{=} \varphi(f)\cdot\varphi(g)$$

Thus $\mathcal{M}$ is Hausdorff and compact in w^*-topology

Recall that $\forall\, f \in \mathcal{B}$ $\exists\, \hat{f} : \mathcal{B}_1^*$ defined by $\hat{f}(\varphi) = \varphi(f)$. $\quad$ Let $\boxed{\Gamma(f) = \hat{f}}$; $\hat{f}$ cont. in w^* top.

Definition $\Gamma : \mathcal{B} \to C(\mathcal{M})$ is the Gelfand transform

Elementary properties of Γ

<u>Thm</u> If B is a Banach algebra and Γ is the Gelfand transform, then

1. Γ is an algebraic homomorphism
2. $\|\Gamma f\|_\infty \le \|f\|_B$ (the isomorphism is contractive)

<u>Proof</u>

1 is immediate. E.g. $\Gamma(fg)(\varphi) = \varphi(fg) = \varphi(f)\varphi(g)$
$= \Gamma(\varphi)\Gamma(g)$ etc.

2. $\sup_{\varphi \in M} |\Gamma_f(\varphi)| = \sup_{\varphi \in M} |\varphi(f)| \le \overbrace{\|\varphi\|}^{=1} \|f\|$

Note that, for a noncommutative B, Γ fails to capture the structure of B: $\Gamma(fg - gf) = 0$ though $fg - gf$ may not be zero

Commutative case M is large enough s.t. f is invertible in $B \iff \Gamma(f)$ is invertible in <u>$C(M)$</u> (note: <u>not only</u> in $\Gamma(B)$)

$\boxed{\text{Spectrum}}$ For B a Banach algebra we <u>define</u>, for $f \in B$

$\sigma_B(f) = \{\lambda \in \mathbb{C} \mid f - \lambda \text{ is not invertible}\}$

this is the <u>spectrum</u> of f.

Definition $\mathbb{C} \setminus \sigma_B(f) =: \rho_B(f)$ is the <u>resolvent set</u> of f

Before proceeding, we define vector valued, or Banach space valued analytic functions

Recall that $\|x\| = \sup_{\substack{\ell \in \mathcal{B}^* \\ \|\ell\|=1}} |\ell x|$

Note that ℓ is <u>any</u> linear functional, not necessarily multiplicative

Thus $\|x_n\| \to 0$ iff $\ell(x_n) \to 0$ <u>uniformly</u>

in ℓ.

Likewise $(x_n)_{n\in\mathbb{N}}$ is Cauchy iff $(\ell(x_n))_{n\in\mathbb{N}}$ is <u>uniformly</u> Cauchy.

<u>Def 1.</u> f is analytic in a domain $\mathcal{D} \subset \mathbb{C}$ (open, connected)

if by definition $\lim_{|h|\to 0} \frac{f(z+h)-f(z)}{h}$ exists $(\overset{\text{def}}{=} f'(z))$

for all $z \in \mathcal{D}$. The limit is understood as one in the Banach algebra $\mathcal{B}$ (norm limit)

<u>Def 2.</u> f is <u>weakly</u> analytic in $\mathcal{D}$ if $\forall \ell \in \mathcal{B}^*$

$\ell(f(z))$ is analytic for all $z \in \mathcal{D}$ <u>and</u> for all $\ell \in \mathcal{B}^*$

Thm f is ~~an~~ analytic iff f is weakly analytic

(note : uniformity in ℓ has been dropped)

In one direction it is obvious : if

$\left\| \frac{f(z+h) - f(z)}{h} - f'(z) \right\| \underset{h \to 0}{\to} 0$ then $\ell \left(\frac{f(z+h) - f(z)}{h} - f'(z) \right) \to 0$ (*)

and by linearity (*) is $\frac{\ell f(z+h) - \ell f(z)}{h} - \ell f'(z)$

In particular $(\ell f)' = \ell f'$.

Conversely, note first that ℓf is a usual analytic function. Thus

$$\frac{\ell f(z_0+h) - \ell f(z_0)}{h} - (\ell f)' = \frac{1}{2\pi i} \oint_C \ell(x(s)) \left\{ \left[\frac{1}{s-(z_0+h)} - \frac{1}{s-z_0} \right] \frac{1}{h} - \frac{1}{(s-z_0)^2} \right\} ds$$

where C is a simple closed curve in $\mathcal{D}$ with $z_0 \in \operatorname{int}(C)$

Now

$$\frac{1}{h} \left(\frac{1}{s-z_0+h} - \frac{1}{s-z_0} \right) = \frac{1}{(s-z_0+h)(s-z_0)}$$

and the integrand becomes

$$\ell(x(s)) \cdot \frac{1}{s-z_0} \left(\frac{1}{s-z_0+h} - \frac{1}{s-z_0} \right) = \frac{\ell(x(s))\, h}{(s-z_0)^2 (s-z_0+h)}$$

and the integral $\oint_c$ is

$$h \oint_c \frac{\ell(f(s))}{(s-z)^2(s-z+h)} ds \le \|\ell\| \|f\| \text{ Const} \cdot h \quad \text{as}$$

$h \to 0$ and the sequence

$$\frac{\ell(f(z+h_n)) - \ell(f(z))}{h_n} - (\ell f)'$$

is <u>uniformly</u> Cauchy thus

$$\left\| \frac{f(z+h) - f(z)}{h} - f'(z) \right\| \xrightarrow[h \to 0]{} 0$$