Clarification

G-M

If $\mathcal{B}$ is a Banach algebra with division then there is a unique isometric isomorphism from $\mathcal{B}$ onto $\mathbb{C}$ {not stated clearly enough:

Note This is an isomorphism between $\mathcal{B}$ and $\mathbb{C}$ seen as a Banach algebra over $\mathbb{C}$

An isomorphism is then unique $\mathcal{J}(\alpha z) = \alpha \mathcal{J}(z)$

rules out complex conjugation

- Showed that for a commut Banach algebra, $\mathcal{B}$ multipl. functionals are in 1-1 corresp with maximal 2-sided ideals of $\mathcal{B}$

In the case of $\mathbb{C}$ $\{0\}$ is a maximal ideal

- $f$ invertible in $\mathcal{B}$ iff $\Gamma(f)$ is invertible in $C(M)$ $\sigma(f) = \operatorname{ran} \Gamma(f)$

- $M$ nonempty

- $\|\Gamma f\|_\infty = r(f) \leq \|f\|$

- $\Gamma$ is algebra homomorphism . An example → Fourier transform

## Spectral mapping theorem

Assume $f(z) = \sum c_n z^n$ converges for $|z| < R$

and let $f \in \mathcal{B}$ (a Banach algebra) be s.t.

$\|f\| < R$. Then $\sum c_n f^n = g(f)$ is absolutely

convergent. Note that by homomorphism

and continuity, $\Gamma(g(f)) = g(\Gamma f)$

Proposition $\sigma(g(f)) = g(\sigma(f))$

Proof $\sigma(g(f)) = \operatorname{ran} \Gamma(g(f)) =$

$= \operatorname{ran}(g(\Gamma f)) = g(\operatorname{ran} \Gamma f) =$

$= g(\sigma(f))$ $\square$

Corollary $\Gamma$ is an isometry (isomorphic

on image) iff

$\|f\|^2 = \|f^2\| \quad \forall f \in \mathcal{B}$

indeed $\|f\| \geq \|\Gamma f\| = r(f) = \lim \|f^{2^n}\|^{\frac{1}{2^n}} = \|f\|$

Let $\mathcal{B}$ be a closed subalgebra of $C(X)$,

<u>Proposition</u> Let $\eta : X \to M_{\mathcal{B}}$ given by

$x \to \varphi_x$ $\qquad \varphi_x(f) = f(x)$. Then $\eta$ is

continuous

<u>Proof</u> Let $\{x_\alpha\}_{\alpha \in J}$ $\quad x_\alpha \to x$ $\quad$ Then $\forall f$ $f(x_\alpha) \to f(x)$

$\Rightarrow \quad \varphi_{x_\alpha}(f) \to \varphi_x(f) \quad \forall f \quad \Rightarrow \quad x \to \varphi_x$ is

continuous. Denote this map by $\eta$.

But in general, $\eta$ is not <u>1-1 nor onto</u>

if $\mathcal{B} = \mathcal{A}$ (a self-adjoint algebra of functions) then <u>$\eta$ is onto</u>

<u>Proof</u> Fix $\varphi \in M_{\mathcal{A}}$ and for every $f$ let

$k_f = \{x \mid \varphi(f) = f(x)\}$. Then the $k$'s

have the finite intersection property, that

is, $\forall f_1, \ldots, f_n \quad k_{f_1} \cap \ldots \cap k_{f_n} \neq \emptyset$

Indeed to get a cd. assume $k_{f_1} \cap \ldots \cap k_{f_n} = \emptyset$

Then $\forall x \ \exists f_k \quad f_k(x) \neq \varphi(f_k)$

Since $M_u$ is compact, $K_{f_i}$ are compact

if $K_{f_1} \cap .. \cap K_{f_n} = \phi$ then $\forall x \in X$

$\exists k,\ f_k(x) \neq 0.$

Construct the function

$$g = \sum (f_k(x) - \varphi(f_k))(\bar{f}_k(x) - \overline{\varphi(f_k)})$$

Note that $g > 0$ Want to show $g$ is invertible. Is this automatic? — No bc invertibility in $C(X) \not\Rightarrow$ invertibility in $B$ immediately — this requires a proof

Note that $X$ compact and $g > 0 \Rightarrow \exists \varepsilon > 0$

$|g(x)| \geq \varepsilon$. Then $\| 1 - \frac{g}{\|g\|_\infty} \| \leq \| 1 - \frac{|\varepsilon|}{\|g\|_\infty} \| < 1$

and thus $g$ is invertible by geometric series, bound to converge in $\mathcal{B}$

But this leads to a contradiction bc.

$\varphi(g^{-1})\varphi(g) = 0 = 1$

Let again $\mathcal{B}$ be a Banach algebra of continuous functions on $X$

Note that if $u \in \mathcal{B}$ and $u$ is real-valued then $\Gamma u$ is real-valued too. Indeed

$u - \lambda$ is invertible $\forall \lambda \in \mathbb{C} \setminus \mathbb{R}$ and thus the same is true for the function $\Gamma u$ and thus $\operatorname{ran} \Gamma u \subset \mathbb{R}$

Then $\Gamma(u + iv) = \Gamma(u) + i\Gamma(v) \to \Gamma(\bar{f}) = \overline{\Gamma(f)}$

<u>Prop</u> Assume $\mathcal{A}$ is a closed self-adjoint subalgebra of $C(X)$ containing the constant fcn. 1

Then $\Gamma$ is an isometric isomorphism between $\mathcal{A}$ and $C(M_{\mathcal{A}})$.

$\to$ we know $M_{\mathcal{A}}$ consists of $\varphi_x$

<u>Proof</u> Clearly $\Gamma(\mathcal{A})$ is a closed subalgebra of $C(M_{\mathcal{A}})$ and $\Gamma(\mathcal{A})$ is also self-adjoint since $\Gamma(\bar{f}) = \overline{\Gamma(f)}$

Note also that $\Gamma(\mathcal{A})$ separates points. Indeed

Let $\varphi_1 \neq \varphi_2$ in $M_{\mathcal{A}}$. Then, by definition $\exists f$ $\varphi_1(f) \neq \varphi_2(f) \Rightarrow \hat{f}(\varphi_1) \neq \hat{f}(\varphi_2)$
but $\hat{f}$ is continuous

Thus $\Gamma(A) = C(M_A)$, by Stone - Weierstrass

Remains to show isometry.

Assume $\|f\|_\infty = m$. Then $\exists x_0$ $|f(x_0)| = m$

$f(x_0) = me^{i\varphi}$

Then $|\varphi_{x_0}(f)| = |f(x_0)| = \|f\|_\infty$

But $\|\Gamma f\| = \sup_{\varphi \in M_A} \Gamma_f(\varphi) \geq |\Gamma_f(\varphi_{x_0})| = \|f\|$

Study carefully algebras of functions

First a general lemma

<u>Lemma</u> Let $X$ $Y$ be compact Hausdorff and $\theta : X \to Y$ continuous. Consider the map

$\theta^*$ from $C(Y)$ onto $C(X)$

defined by $\theta^* f = f \circ \theta$.

Then $\theta^*$ is an algebraic isometric isomorphism from $C(Y)$ onto $C(\Pi_y)$ where

$\Pi_y = \{ \theta^{-1}(y) \}$

<u>Proof</u> That $|\theta^* f|_X = |f\|_Y$ is clear

It is also clear that $f \circ \theta$ is constant on each $\Pi_y$.

It remains to take an $g \in C(X)$ constant on $\Pi_y$ and show that $g = h(\theta(g))$ for some continuous $h$ on $Y$

First we can define $h(y) = g(\Pi_y)$ unambiguously.

Only have to show continuity

Let $y_\alpha \to y$ and $x_\alpha$ all $\theta^{-1}(y_\alpha)$

Then $\{x_\alpha\}$ contains convergent subnets.

$x_{\alpha_\beta} \to x \Rightarrow g(x_{\alpha_\beta}) \to g(x)$

$\theta(x_{\alpha\beta}) \to \theta(x)$ $\theta(x_{\alpha\beta}) = y_\alpha \to y$

$g(\Pi y_\alpha) \to g(\Pi_y)$

<u>Prop</u> Let $A$ be a closed <u>self-adjoint</u> subalgebra of $C(X)$ and define $\eta$ as before, $\eta(x) = \varphi_x \in M_{\mathcal{U}}$

Then $\eta^*$ is inverse of $\Gamma$, that is $((\eta^* \circ \Gamma) f)(x) \equiv \Gamma f(\eta x) = f(x)$

$\Rightarrow \eta^* \circ \Gamma$ is identity.

$\Gamma$ is isometric isomorphism between $A$ and $C(M_A)$, $\eta^*$ maps $M_A$ onto $A$