Gelfand Mazur's theorem, proved last time except for uniqueness

<u>Theorem</u> If $\mathcal{B}$ is a division algebra (every nonzero element is invertible) then there is a unique isometric isomorphism between $\mathcal{B}$ and $\mathbb{C}$ ($\mathbb{C}$ <u>seen as an algebra over $\mathbb{C}$</u>)

<u>Uniqueness.</u> If ϕ_1 ϕ_2 are isomorphisms, $\mathcal{B} \overset{\phi_{1,2}}{\longleftrightarrow} \mathbb{C}$ then $F = \phi_1 \circ \phi_2^{-1}$ is an isometric automorphism of $\mathbb{C}$ as a Banach algebra over $\mathbb{C}$. Since $F(1x) =$
$= F(1)\,F(x) = F(x)$ then $F(1) = 1$. $F(\alpha) = F(\alpha \cdot 1) =$
$= \alpha F(1) = \alpha \Rightarrow$ F is the identity

$\square$

<u>Quotient algebras</u> Let $\mathcal{B}$ be a Banach algebra and assume $\mathcal{J}$ is a closed 2 sided ideal of $\mathcal{B}$. In particular, $\mathcal{J}$ is a closed subspace of $\mathcal{B}$ and $\mathcal{B}/\mathcal{J}$ is an algebra ($[g] \in \mathcal{B}/\mathcal{J}$ if $[g] = \{g + k \mid k \in \mathcal{J}\}$, and it is a <u>Banach</u> algebra.

Indeed, by the definition of $\mathcal{B}/\mathcal{J}$ as a Banach space $\|[g]\| = \inf_{k \in \mathcal{J}} \|g + k\|$

Now, $\|[1]\| = \inf_{k \in \mathcal{J}} \|1 + k\| \geq 1$ (otherwise k would be invertible and could not be in $\mathcal{J}$)

Since $0 \in \mathcal{J}$, $\|[1]\| = 1$

Now, $\|[f][g]\| \overset{\text{by def}}{=} \|[fg]\| = \inf_{k \in \mathcal{J}} \|fg + k\| \leq$

$\leq \inf_{k_1, k_2 \in \mathcal{J}} \|(f+k_1)(g+k_2)\| \leq \inf_{k_1, k_2} \|f+k_1\| \|g+k_2\|$

$= \left(\inf_{k \in \mathcal{J}} \|f+k\|\right)\left(\inf_{k \in \mathcal{J}} \|g+k\|\right) = \|[f]\| \|[g]\|$

Proposition If $\mathcal{B}$ is a <u>commutative</u> Banach algebra, then the set M of linear multiplicative functionals on $\mathcal{B}$ is in a 1-1 correspondence with the set of maximal ideals in $\mathcal{B}$.

<u>Proof</u> Let first $\varphi \in M$. Then $\ker \varphi \equiv K_\varphi$ is a two sided maximal ideals, $\mathcal{J}$ of $\mathcal{B}$:

if $f \in M$ then of course $fg, gf \in M$ $\forall g \in \mathcal{B}$.

To show it is maximal, take $h \notin \mathcal{J} \equiv K_\varphi$. Then, by definition $\varphi(h) = \alpha \neq 0$. We can then also assume w.l.o.g $\varphi(h) = 1$ since $\frac{h}{\alpha} \in \mathcal{B} \setminus \mathcal{J}$

But then, $1 = 1 - h + h = k + h$ $k \in \mathcal{J}$

shows that the span of $\mathcal{J}$ and h $= \mathcal{B}$

In the opposite direction, let $\mathcal{J}$ be a maximal ideal of $\mathcal{B}$ and $h \in \mathcal{J}$; then of course h is noninvertible, otherwise $\mathcal{J} = \mathcal{B}$. But then $\|1+h\| \geq 1$. This inequality shows that $\overline{\mathcal{J}}$ is also a maximal ideal since $1 \notin \overline{\mathcal{J}}$. Therefore, by maximality, $\mathcal{J} = \overline{\mathcal{J}}$

Then, as we saw, $\mathcal{B}/\mathcal{J}$ is an algebra. We claim it is a <u>division algebra</u>.

Indeed, $g \notin \mathcal{J} \Rightarrow \exists\, \alpha \neq 0$ st $\alpha g + k = 1$ for some $k \in \mathcal{J}$. Hence $[\alpha][g] = [1] = 1$ and g is invertible

Hence $\mathcal{B}/\mathcal{J} \underset{\text{isometric}}{\overset{\text{isomorph}}{\sim}} \mathbb{C}$

Let $\psi : \mathcal{B} \to \mathcal{B}/\mathcal{J}$ $\quad \psi(g) = [g]$ and $\phi(\mathcal{J}) = \mathbb{C}$ be the isomorphism. Then $\phi \circ \psi$ is a linear multiplicative functional.

We check that $\mathcal{J} = \ker \varphi$. Indeed $k \in \mathcal{J}$ $\Rightarrow [k] = 0 \Rightarrow \varphi(k) = 0$ and this is already a maximal ideal $\Rightarrow$ $\boxed{\ker \varphi = \mathcal{J}}$

It remains to show that the correspondence $\phi \to \ker \phi$ is 1-1

Indeed if $\phi_1 \neq \phi_2 \Rightarrow \exists f \quad \varphi_1(f) \neq \varphi_2(f)$

Note that for any f, $f - \varphi_i(f) \in \ker \phi_{1,2}$

and thus $\psi = f - \varphi_1(f) - (f - \phi_2(f)) \in \ker \phi_{1,2}$

However $\varphi_2(\psi) = 0 = \varphi_2(f) - \varphi_1(f)$ contradiction.

Commutativity was only used in the last step to ensure that $\mathcal{B}/\mathcal{J}$ is a <u>commutative</u> algebra

<u>Theorem</u> (Gelfand) Let $\mathcal{B}$ be a commutative Banach algebra, M the set of its maximal ideals and $\Gamma : \mathcal{B} \to C(M)$ be the Gelfand transform

Then

1. M is nonempty
2. Γ is an algebra homomorphism
3. $\|\Gamma f\|_\infty \le \|f\|_\infty$
4. f is invertible in $\mathcal{B}$ iff $\Gamma(f)$ is invertible in <u>$C(M)$</u>

<u>Note</u>. The fact that invertibility is in $C(M)$ and not just in $\Gamma(\mathcal{B})$ is key to many applications.

We showed already 2, 3. For 1 note that $\forall f \in \mathcal{B}$ $\exists \lambda \in \mathbb{C}$ $\lambda \in \sigma(f)$, that is, $(f-\lambda)$ is not invertible. But then $\{g(f-\lambda) \mid g \in \mathcal{B}\}$ is an ideal, which is always contained in a maximal ideal.

4 Assume f^{-1} exists in $\mathcal{B}$ and let $H = \Gamma(f^{-1})$. Obviously $H\,\Gamma(f) = 1$ and thus Γ is invertible in $C(M)$

Assume now that f is not invertible in $\mathcal{B}$. Then,
$J_1 = \{fg \mid g \in \mathcal{B}\}$ is an ideal, and it is proper since $1 \notin J_1$. Then J_1 is contained in a maximal ideal J and there is a φ s.t. $\ker \varphi = J$. But Then $\varphi(f) = 0$ by construction. But then $\Gamma_f(\varphi) = \varphi(f) = 0$ and Γ_f is not invertible in $\underline{C(M)}$,

<u>Corollary</u> If $\mathcal{B}$ is a commutative Banach algebra and $f \in \mathcal{B}$ then $\sigma(f) = \text{range}(\Gamma f)$ and $r(f) = \|\Gamma f\|_\infty$.
spectral radius

Proof $\lambda \in \sigma(f)$ iff $\Gamma_{f-\lambda} = \Gamma f - \lambda$ is not invertible in $C(M)$. But then $\lambda \in \text{range}(\Gamma f)$. Now $\max\limits_{\lambda \in \sigma(f)} |\lambda| =$

$$= \max_{\lambda \in \text{ran}(f)} |\lambda| = \|\Gamma f\|_\infty .$$