Reminder: 1) the functional $\eta = \varphi \mapsto \varphi_x$ is continuous. If $\mathcal{A}$ is a self-adjoint subalgebra of $C(X)$, then η is onto from X onto $M_{\mathcal{A}}$ (the multipl. functionals on $\mathcal{A}$)

2) In general, $f \in \mathcal{B}$ (a commutative Banach algebra) then f is invertible in $\mathcal{B}$ iff Γf is invertible in $C(M_{\mathcal{B}})$

We plan to understand the self-adjoint subalgebras $\mathcal{A}$ of $C(X)$
Upshot: it will turn out that all such $\mathcal{A}$ are of the form $\mathcal{A} = \{f \in C(X) \mid f(S) = y_0\}$ for some fixed y_0 and S a set of the form $\theta^{-1}(Y)$ where Y is Hausdorff compact and $\theta : X \to Y$ is continuous

Proposition If $\mathcal{A}$ is a self-adjoint subalgebra of $C(X)$ containing 1, then Γ is an isometric isomorphism from $\mathcal{A}$ onto $C(M_A)$ [this property holds more generally, we will see this when studying C^*-algebras]

Proof We know already that $\|\Gamma(f)\|_\infty \le \|f\|_\infty$ and that all functionals on $\mathcal{A}$ are evaluation functionals at points in X
Assume $f \in \mathcal{A}$, $\|f\| = M$. By compactness $\exists X_m \in X$ s.t $|f(X_m)| = M \Rightarrow f(X_m) = Me^{i\varphi}$ Now

$$\|\Gamma f\|_\infty = \sup_{\varphi_x \in M_{\mathcal{A}}} |\varphi_x(f)| \ge |\varphi_{x_0}(f)| = M = \|f\|_\infty \ge \|\Gamma f\|_\infty$$

proving the isomorphism between $\mathcal{A}$ and $\Gamma(M_A) \subset C(M_A)$
We now show that $\Gamma(M_A) = C(M_A)$. Indeed: $\Gamma(M_{\mathcal{A}})$ is a self-adjoint $(\Gamma(\bar{f})) = \overline{\Gamma(f)})$ subalgebra of $C(M_A)$ containing 1 and separating points (for if $\varphi_1 \ne \varphi_2$ in M_A then by definition $\exists f \in \mathcal{A}$ s.t $\varphi_1(f) \ne \varphi_2(f) \iff \Gamma_f(\varphi_1) \ne \Gamma_f(\varphi_2)$ and Γf is continuous. Thus Stone-Weierstrass applies and

$$\boxed{\Gamma(M_A) = C(M_A)}$$

It remains to show that $\mathcal{A}$ is the set of functions in $C(X)$ taking the same value on some $S \subset X$ of the form described.

Since all functionals are evaluation ones, $\mathcal{M}_A \equiv Y \sim X'$ (in the sense of 1-1 correspondence, preserving topology)
$X' = \{[x] \mid x \in X\}$ $x_1 \in \{[x]\}$ iff def $\eta(x_1) = \eta(x_2)$

Indeed $x'_\alpha \to x' \in X' \Rightarrow f(x'_\alpha) \to f(x')$ $\forall f \in \mathcal{A}$ (continuous functions) $\Rightarrow \varphi_{x'_\alpha} \to \varphi_{x'}$

Conversely, assume $\varphi_{x_\alpha} \to \varphi_x$ and look at the set of all x'_α in $\eta^{-1}(x_\alpha)$, for each α. Since X is compact, $\exists\, x'_{\alpha\beta} \to x'$ thus $\varphi_{x'_{\alpha\beta}} \to \varphi_{x'}$ thus $x = x'$ and the map is bicontinuous

Finally, note that $\eta^* : C(X') \to C(X)$ defined as $\eta^*(h) = h \circ \eta$ is the inverse of Γ. Indeed

$(\eta^*(\Gamma(f)))(x) = \Gamma f(\eta x) = \hat{f}(\varphi_x) = \varphi_x(f) = f(x)$

and thus $\mathcal{M}_A \sim X'$. It is clear that $f(x_1) = f(x_2)$ if $x_1, x_2 \in [x]$ i.e. if $\eta(x_1) = \eta(x_2)$ from which the conclusion follows easily.

Application Wiener's theorem.

Let's first study another interesting $\mathcal{B}$, $\ell^1(\mathbb{Z})$ with convolution instead of product

$$\ell^1 = \{(x_n)_{n\in\mathbb{Z}} \mid \sum |x_n| < \infty\} \qquad (f * g)(k) = \sum_{j=-\infty}^{\infty} f_j\, g_{k-j}$$

We first show that this is a Banach algebra. Linearity, commutativity associativity etc are left as straightforward exercises once we have shown that $\|f * g\| \le \|f\|\,\|g\|$

$$\sum_n \left|\sum_j f_j g_{n-j}\right| \leq \sum_n \sum_j |f_j||g_{n-j}| =$$

$$= \sum_j |f_j| \sum_n |g_{n-j}| = \sum_j |f_j| \sum_n |g_n| = \|f\| \|g\|$$

(How did Fubini apply?)

<u>Identity</u> If $e_n = (\ldots 0, \ldots, 1, \ldots)$ then
$\uparrow$ position $n \in \mathbb{Z}$

$(e_k * e_\ell)_n = \sum e_{k,j} \, e_{\ell, n-j}$ and there is only one nonzero term in the sum, when $j = k$ and $n - j = \ell$ that is, $n = k + \ell$. We have established

$e_k * e_\ell = e_{k+\ell}$

This shows that $e_0 = I$ and $e_m^{-1} = e_{-m}$.

(It is easy to check that $e_0 * f = f \ \forall f$)

For this $\mathcal{B}$, Γ <u>is not</u> an isomorphism (what conditions fail in the previous theorems?)

Note that, if $z \in \mathbb{T} = \partial \mathbb{D}$ and $x \in \ell^1$, then $\sum |x_n z^n|$ converges so $f = \sum x_n z^n$ is an absolutely convergent Fourier series. By dominated convergence you can show that f is continuous.

Given f of the form $f \sum x_n e^{in\varphi}$ - Fubini shows that $\frac{1}{2\pi} \int_0^{2\pi} f(\varphi) e^{-in\varphi} \, d\varphi = x_n$

thus there is a 1-1 corresp between functions in $C(\mathbb{T})$ with abs convergent Fourier series and elements in ℓ^1.

<u>Note</u> It is <u>false</u> that $f \in C(\mathbb{T}) \Rightarrow$ Fourier series converges! (let alone uniformly)

$$\boxed{\text{Thus } Y = \{f \in C(\mathbb{T}) \mid \text{Fourier series is a.c}\} \subsetneq C(\mathbb{T})} \quad \text{(A)}$$

let's show that $\mathbb{T}$ is the set of multiplicative functionals.

Indeed, let $\varphi \in M$. Then $\varphi(e_1) \in \mathbb{C} = z_1$

we have $\|e_1\| = 1 \geq |\varphi(e_1)| = |z_1| = \frac{1}{|z_1^{-1}|} = \frac{1}{|\varphi(e_{-1})|} \geq \frac{1}{|e_{-1}|} \geq 1$

thus $z_1 \in \mathbb{T}$. Note that $x \in \ell^1 \Rightarrow$

$\Rightarrow x = \sum x_n e_n$ where the sum is absolutely convergent.

Note also that $\varphi(e_n) = \varphi(\underbrace{e_1 * \ldots * e_1}_{n \text{ times}}) = \varphi(e_1)^n = z_1^n$

and by continuity, $\varphi(x) = \sum x_n \varphi(e_n) = \sum x_n z_1^n$

thus $\mathbb{T}$ is the set of multiplicative functionals $\mathcal{M}_{\ell^1}$

By (A) we see that $\Gamma(\ell^1)$ is <u>not</u> onto $C(\mathbb{T})$ (what fails in the assumptions of some previous theorems?)

<u>Exercise</u> Also, show that $\|\Gamma f\|_\infty < \|f\|_\infty$ for some f.

To invert Γ we simply use the Fourier transform

$$\Gamma x = f = \sum x_n e^{in\varphi} \Rightarrow x_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\varphi} f(\varphi)\, d\varphi$$

as before

Theorem (Wiener)

If $F \in C(\mathbb{T})$ has an absolutely convergent Fourier series and $F(x) \neq 0$ $\forall x$, then $\frac{1}{F}$ has a convergent Fourier series too.

Proof $F \neq 0 \Rightarrow F$ is invertible in $C(\mathbb{T})$ / let $F = \Gamma(x)$ and $y = x^{-1}$ which exists (x invertible in B iff Γx invertible in $C(M)$)

Now $\Gamma(y) \in Y$ and $\Gamma(Y) F = 1$ $\square$