Let $\mathcal{B}$ be a Banach algebra and $\mathcal{G}$, $\mathcal{G}_\ell$, $\mathcal{G}_r$ be the set of invertible, left-invertible, right invertible elements of $\mathcal{B}$. We proved last time that $\mathcal{G}$, $\mathcal{G}_\ell$, $\mathcal{G}_r$ are open sets.

<u>Thm</u> Let $\mathcal{B}$ be a Banach algebra, $\mathcal{G}$ as above and let $\mathcal{G}_0$ be the connected component of 1. Then $\mathcal{G}_0$ is an open, closed normal subgroup of $\mathcal{G}$. The cosets of $\mathcal{G}_0$ are the connected components of $\mathcal{G}$, and $\mathcal{G}/\mathcal{G}_0$ is discrete

<u>Proof</u> ~~Take $f \in \mathcal{G}_0$~~. Note that $\mathcal{G}_0$, as a component of locally connected set, is both open and closed ("clopen") Take $f \in \mathcal{G}_0$ ~~and~~ Since f is invertible (in $\mathcal{G}$) $f = F^{-1}$ for some F $(= f^{-1})$ and thus $f\mathcal{G}_0 = F^{-1}(\mathcal{G}_0)$ where $F^{-1}(g) = F^{-1}g = fg$. multiplication is continuous thus $F^{-1}(\mathcal{G}_0) \underset{\text{is}}{=} f\mathcal{G}_0$ open and closed too (and connected) Since $f \in \mathcal{G}_0 \cap f\mathcal{G}_0$ the intersection is nonempty $\mathcal{G}_0$ is already a ~~c~~ maximally connected set containing 1 and thus $f\mathcal{G}_0 = \mathcal{G}_0$

Therefore, $f, g \in \mathcal{G}_0 \Rightarrow fg \in \mathcal{G}_0$ likewise, $gf \in \mathcal{G}_0$ ~~Similar~~ and $f^{-1}\mathcal{G}_0 = \mathcal{G}_0 \Rightarrow \mathcal{G}_0$ is a group

Take now $F \in \mathcal{G}$ and consider the set $F\mathcal{G}_0F^{-1}$ Note that $F(\cdot)F^{-1}$ is a cont. function, one to one with inverse $F^{-1}(\cdot)F$. By the same argument as above, since $I \in \mathcal{G}_0 \cap F\mathcal{G}_0F^{-1}$, we have $\mathcal{G}_0 = F\mathcal{G}_0F^{-1}$ and $\mathcal{G}_0$ is a <u>normal</u> subgroup

Then, the factor group $\mathcal{G}/\mathcal{G}_0$ exists and since $\mathcal{G}_0$ is clopen $\mathcal{G}/\mathcal{G}_0$ is discrete.

Finally, reasoning as before, $\forall F \in \mathcal{G}$ $F\mathcal{G}_0$ is clopen therefore a connected component of $\mathcal{G}$:

$\mathcal{G} = \bigcup_{F \in \mathcal{G}} F\mathcal{G}_0$ (where $\phi \neq F\mathcal{G}_0 \cap F'\mathcal{G}_0 \Leftrightarrow F\mathcal{G}_0 = F'\mathcal{G}_0$)

<u>Definition</u> $\mathcal{G}/\mathcal{G}_0 = \Lambda_B$ is called the <u>abstract index group</u> and the natural homomorphism between $\mathcal{G}$ and Λ_B is called abstract index.

What is $\mathcal{G}_0$ for $C(X)$ where X is Hausdorff compact?

<u>Prop</u> for $C(X)$, $\mathcal{G}_0$ is the set of all functions homotopic to 1 in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

Note first that both $\mathcal{G}$ and $\mathcal{G}_0$ consist of functions with <u>range is in $\mathbb{C}^*$</u>. They are both locally pathwise ~~arcwise~~ connected since f close to g, $\|f-g\| < \varepsilon$ $\Rightarrow$ g is invertible for small ε if f is invertible.

By a general thm in topology, $\mathcal{G}_0$ connected & $\mathcal{G}_0$ locally pathwise connected $\Rightarrow$ $\mathcal{G}_0$ is pathwise connected

(proof · Take the ~~union~~ set of all f pathwise conn. to 1
· show it is clopen (and nonempty) thus $= \mathcal{G}_0$)

This proves the claim

<u>Prop</u> The cosets $F\mathcal{G}_0$ are the homotopy classes of ~~the~~ cont. maps from X to $\mathbb{C}^*$

<u>Proof</u> $\mathcal{G}_0$ is pathwise connected $\Rightarrow$ $F\mathcal{G}_0$ is also

Note connection with $\pi^1(X)$ and $H^1(X, \mathbb{Z})$ (Douglas)

Relation with exponential map. Define $e^f = \sum \frac{f^n}{n!}$ ($\exists$ since $\sum \frac{\|f\|^n}{n!} < \infty$)

Note If $[f, g] \neq 0$ it usually is false that $e^{f+g} = e^f e^g$!

Prop If $[f, g] = 0$ then $e^{f+g} = e^f e^g$. Proof: "copy" the usual one

$\boxed{\text{The log}}$ Prop. If $\|1 - f\| < 1$ then $f \in e^{\mathcal{B}}$

Proof Let $g = 1 - f$ Then $\|g\| < 1$ and $\sum \frac{g^n}{n} = -\log(1-g) \}$. Notation

check that $\exp(\log(1-g)) = 1-g$. $\square$

Theorem $\mathcal{G}_0$ is the set $\Pi = \{ e^{f_1} e^{f_2} \dots e^{f_k} \mid f_i \in \mathcal{B},\ k \in \mathbb{N} \}$

Proof Note first that $e^{\lambda f}$ is cont in λ, $e^{1 \cdot f} = e^f$, $e^{0 f} = 1$

Thus Π is connected (in fact path connected). Also $[f, f] = 0$ $\Rightarrow e^{f-f} = e^f e^{-f} = 1 \Rightarrow e^f$ invertible and connected to 1 $\Rightarrow \Pi \subseteq \mathcal{G}_0$. Note also that Π is open since a neighborhood of 1 $\{ f : \|1 - f\| < 1 \}$ is in Π and Π is a group. Thus Π is an open, connected subgroup of $\mathcal{G}_0$. Note also that $F_1 g_0 = F_2 g_1$ for some $g_0, g_1 \in \Pi$ and $F_1, F_2 \in \mathcal{G}_0$ implies $F_2^{-1} F_1$ (and $F_1^{-1} F_2$) $\in \Pi$ and the cosets $F\Pi$ are either disjoint or identical. Furthermore $\{ F\Pi \mid F \in \mathcal{G}_0 \} = \mathcal{G}_0$ and $\cup F\Pi$ provides a disjoint open covering of $\mathcal{G}_0$. $\mathcal{G}_0$ is connected and thus there is only one component $1 \cdot \Pi$ and $\Pi = \mathcal{G}_0$.

2.16 Let X be a compact Hausdorff space and let $\mathcal{G}$ denote the invertible elements of $C(X)$. Hence a function f in $C(X)$ is in $\mathcal{G}$ if and only if $f(x) \neq 0$ for all x in X, that is, $\mathcal{G}$ consists of the continuous functions from X to $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Since $\mathcal{G}$ is locally arcwise connected, a function f is in $\mathcal{G}_0$ if there exists a continuous arc $\{f_\lambda\}_{\lambda\in[0,1]}$ of functions in $\mathcal{G}$ such that $f_0 = 1$ and $f_1 = f$. If we define the function F from $X \times [0, 1]$ to $\mathbb{C}^*$ such that $F(x, \lambda) = f_\lambda(x)$, then F is continuous, $F(x, 0) = 1$ and $F(x, 1) = f(x)$ for x in X. Hence f is homotopic to the constant function 1. Conversely, if g is a function in $\mathcal{G}$ which is homotopic to 1, then g is in $\mathcal{G}_0$. Similarly, two functions g_1 and g_2 in $\mathcal{G}$ represent the same element of $\Lambda = \mathcal{G}/\mathcal{G}_0$ if and only if g_1 is homotopic to g_2. Thus Λ is the group of homotopy classes of maps from X to $\mathbb{C}^*$.

2.17 Definition. If X is a compact Hausdorff space, then the first cohomotopy group $\pi^1(X)$ of X is the group of homotopy classes of continuous maps from X to the circle group $\mathbb{T}$ with pointwise multiplication.

2.18 Theorem. If X is a compact Hausdorff space, then the abstract index group Λ for $C(X)$ and $\pi^1(X)$ are naturally isomorphic.

Proof We define the mapping Φ from $\pi^1(X)$ to Λ as follows: A continuous function f from X to $\mathbb{T}$ determines first an element $\{f\}$ of $\pi^1(X)$ and second, viewed as an invertible function on X, determines a coset $f + \mathcal{G}_0$ of Λ. We define $\Phi(\{f\}) = f+\mathcal{G}_0$. To show, however, that Φ is well defined we need to observe that if g is a continuous function from X to $\mathbb{T}$ such that $\{f\} = \{g\}$, then f is homotopic to g and hence $f+\mathcal{G}_0 = g+\mathcal{G}_0$. Moreover, since multiplication in both $\pi^1(X)$ and $\mathcal{G}$ is defined pointwise, the mapping Φ is obviously a homomorphism. It remains only to show that Φ is one to-one and onto.

To show Φ is onto let f be an invertible element of $C(X)$. Define the function F from $X \times [0, 1]$ to $\mathbb{C}^*$ such that $F(x, t) = f(x)/|f(x)|^t$. Then F is continuous, $F(x, 0) = f(x)$ for x in X, and $g(x) = F(x, 1)$ has modulus one for x in X. Hence, $f + \mathcal{G}_0 = g + \mathcal{G}_0$ so that $\Phi(\{g\}) = f + \mathcal{G}_0$ and therefore Φ is onto.

If f and g are continuous functions from X to $\mathbb{T}$ such that $\Phi(\{f\}) = \Phi(\{g\})$, then f is homotopic to g in the functions in $\mathcal{G}$, that is, there exists a continuous function G from $X \times [0, 1]$ to $\mathbb{C}^*$ such that $G(x, 0) = f(x)$ and $G(x, 1) = g(x)$ for x in X. If, however, we define $F(x, t) = G(x, t)/|G(x, t)|$, then F is continuous and establishes that f and g are homotopic in the class of continuous functions from X to $\mathbb{T}$. Thus $\{f\} = \{g\}$ and therefore Φ is one-to-one, which completes the proof. ■

The preceding result is usually stated in a slightly different way.

2.19 Corollary. If X is a compact Hausdorff space, then Λ is naturally isomorphic to the first Čech cohomology group $H^1(X, \mathbb{Z})$ with integer coefficients.

Proof It is proved in algebraic topology (see [67]) that $\pi^1(X)$ and $H^1(X,\mathbb{Z})$ are naturally isomorphic. ∎

These results enable us to determine the abstract index group for simple commutative Banach algebras.

2.20 Corollary. The abstract index group of $C(\mathbb{T})$ is isomorphic to $\mathbb{Z}$.

Proof The first cohomotopy group of $\mathbb{T}$ is the same as the first homotopy group of $\mathbb{T}$ and hence is $\mathbb{Z}$. ∎

We now return to the basic ...