Short introduction to eigenvalue problems (based on the notes of R.D. Costin

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1 General second order self-adjoint problems

Bounded operators. Recall that a matrix, or more generally, a bounded linear operator on a Hilbert space $\mathcal{H}$, is self-adjoint if by definition $T^* = T$ where the adjoint is defined as usual by $\langle Tx, y\rangle = \langle x, T^*y\rangle$.
Unbounded operators. The definition applies to unbounded linear operators T, such as operators based on d/dx that we will focus on; T is self-adjoint if, still, $T^* = T$. But the condition becomes considerably more subtle. First, d/dx is not be defined everywhere in L^2, but only on some *domain* $\mathcal{D}$, a dense set (such as C^∞) in L^2. The condition $\langle Tx, y\rangle = \langle x, T^*y\rangle$ (which is called merely **symmetry** or **formal self-adjointness** on $\mathcal{D}$, if T is unbounded) depends on $\mathcal{D}$. We assume T is defined on a *dense set* $\mathcal{D}$ in L^2. The *adjoint* T^* is defined as follows. Let g be s.t. $\langle Tf, g\rangle$ is a continuous linear functional on $\mathcal{D}$. By density, it can be then extended to a linear continuous functional on $\mathcal{H}$.

By the Riesz representation theorem then,

$$\langle Tx, y\rangle = \langle x, z\rangle \tag{1}$$

for some *unique* $z \in \mathcal{H}$. We *define* $T^*y = z$, or

$$\langle Tx, y\rangle = \langle x, T^*y\rangle; \quad \forall\, x \in \mathcal{D} \tag{2}$$

The domain of T^* is the set of all y for which (1) holds.

An operator is **self-adjoint** if by definition $T = T^*$. Note that this means that T is symmetric **and** that $\mathcal{D} = \mathcal{D}^*$. While symmetry is relatively easy to check, $\mathcal{D} = \mathcal{D}^*$ requires work. If for instance we define $T = id/dx$ on $\mathcal{D} = \{f \in C^\infty[0,1] : f(0) = f(1) = 0\}$, then T is symmetric, it can be seen by integration by parts:

$$\int_0^1 (if'(x))\overline{g(x)}dx = \int_0^1 f(x)(\overline{ig'(x)})dx \tag{3}$$

for all $g \in \mathcal{D}$. The problem is that (2) is valid also if we merely have, say, $g \in AC[0,1]$. In fact, in this example, the domain of T^* *is* $\mathcal{D}^* = AC[0,1]$.

Since we have $AC[0,1] \supsetneq \mathcal{D}$, $T^* \neq T$. Thus $\mathcal{D}$ is *too small*. And, of course, if we define T on $\mathcal{D}^*$, then it is not symmetric anymore. The domain $\mathcal{D}^*$ would be *too large* for T to be self-adjoint. *A* domain of self-adjointness of T is $\mathcal{D} = \{f \in AC[0,1] : f(0) = f(1)\}$. This, of course, requires a proof.

The question of self-adjointness is beyond the scope of this course. We will check the symmetry of the operators involved, and assume that a domain can be defined s.t. the operators are actually self-adjoint. This is *non-trivial* since such a domain might simply not exist. For instance, there is no such domain for id/dx on $[0,\infty)$ while we can find one on $\mathbb{R}$ and *many* on $[0,1]$.

2 Sturm-Liouville problems

General problem: Find the values of the constant λ for which the equation

$$\frac{1}{\rho(x)}\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)y + q(x)y + \lambda y = 0 \tag{4}$$

has non-identically zero solutions for $x \in [a,b]$ satisfying *boundary conditions* such as

$$y(a) = 0,\ y(b) = 0 \tag{5}$$

More general boundary conditions are described in the next subsection.

We shall see that the most general linear second order equation can be brought to the form (3).

This is an eigenvalue problem for the differential operator

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - q(x)$$

The operator L is formally self-adjoint in a *weighted* L^2 space: $L^2([0.L], \rho(x)dx)$ on a domain where the boundary values vanish, as we will see.

The scalar product is defined as usual by

$$\langle f, g\rangle := \int_0^L \overline{f(s)}g(s)\rho(s)ds \tag{6}$$

Assumption 1. *We will only analyze equation with real-valued coefficients.*

2.1 Homogeneous boundary conditions

These conditions are usually inherited from the PDEs which produced the ODE (15) by separation of variables. If the values on the boundary are not zero, substitutions can often be made to ensure zero values on the boundary: these are called *homogeneous boundary conditions*. These are the following

- Dirichlet conditions: $u(a) = 0,\ u(b) = 0,$
- Neumann conditions: $u'(a) = 0,\ u'(b) = 0,$

- Mixed Dirichlet-Neumann (Robin) conditions:

$$\begin{aligned} B_a[u] &\equiv \alpha u(a) + \alpha' u'(a) = 0 \\ B_b[u] &\equiv \beta u(b) + \beta' u'(b) = 0 \end{aligned} \tag{7}$$

The mixed conditions are the most general, and in fact they are the most general ensuring self-adjointness of L. Dirichlet and the Neumann conditions are clearly particular cases (if $\alpha' = 0 = \beta'$ we obtain Dirichlet conditions, and if $\alpha = 0 = \beta$ we obtain Neumann conditions). Therefore we work with the general mixed Dirichlet-Neumann conditions.

It must be assumed that **the boundary conditions are nontrivial**: at least one of the numbers α, α' is not zero (note that this condition can be written as $|\alpha| + |\alpha'| \neq 0$), and similarly, at least one of the numbers β, β' is not zero (i.e. $|\beta| + |\beta'| \neq 0$).

3 Examples of separation of variables leading to Sturm-Liouville eigenvalue problems

Many partial differential equations which appear in physics can be solved by separation of variables. Two examples are illustrated here.

3.1 Heat conduction in dimension one.

Consider a thin rod of length L, perfectly insulated. The temperature $u(x,t)$ at time t and position $x \in [0, L]$ satisfies the heat equation

$$u_t = \alpha\, u_{xx} \tag{8}$$

where α is a parameter (which depends on the rod).

Of course, the heat conduction depends on what happens at the two endpoints of the rod $x = 0$ and $x = L$ (boundary condition), and on the initial temperature distribution the rod (initial condition), which need to be specified.

3.1.1 Separation of variables

Looking for solutions in the form $u(x,t) = y(x)T(t)$ (with separated variables) and substituting in (7) we obtain

$$y(x)T'(t) = \alpha y''(x)T(t) \quad \text{therefore} \quad \frac{T'(t)}{\alpha T(t)} = \frac{y''(x)}{y(x)} = \text{constant} = -\lambda$$

and we obtain two ordinary differential equation which can be solved:

$$T'(t) = -\alpha\lambda T(t) \tag{9}$$

and

$$y'' + \lambda y = 0 \tag{10}$$

Of course, any admissible solution of (9) yields a solution of (7). Also clearly, this is not the most general solution as any linear combination of such solutions is still a solution. The question then is under what conditions can we recover the general solution of an initial condition or boundary value problem from equations of the type (9)? This leads to issues of completeness of the set of solutions of (9). Whenever we can solve the general PDE question in this way, we have effectively reduced a PDE to an ODE problem.

3.1.2 Boundary conditions.

1) Suppose both ends of the rod are kept at constant temperature zero: say $u(0,t) = 0$ and $u(L,t) = 0$ for all t. It follows that $y(0) = 0$ and $y(L) = 0$. Equation (9) with these boundary conditions is a Sturm-Liouville eigenvalue problem.

We saw that the eigenvalues of this problem are $\lambda_n = n^2$ $(n = 1, 2, \ldots)$ and the eigenfunctions are $y_n(x) = \sin(n\pi/L\, x)$.

2) Other constant boundary temperatures can be imposed, but these can be reduced to the case of zero boundary conditions. If, say, $u(0,t) = 0$ and $u(L,t) = T_0$ for all t, then substituting

$$u(x,t) = \frac{T_0}{L}\, x + v(x,t)$$

into (7) we find that $v(x,t)$ satisfies the heat equation and has zero boundary conditions: $v(0,t) = 0$ and $v(L,t) = 0$ for all t.

3) If one end radiates (say, $x = L$) then the boundary condition at $x = L$ is $u_x(L,t) = -hu(L,t)$ ($h > 0$ means heat loss due to radiation, $h = 0$ means there is no radiation) and $u(0,t) = 0$. This gives: $y(0) = 0$, $y'(L) + hy(L) = 0$ which together with equation (9) form another Sturm-Liouville eigenvalue problem.

3.1.3 Initial condition: need for completeness of eigenfunctions

The initial distribution of the temperature needs to be specified as well: $u(x,0) = u_0(x)$.

After finding the eigenvalues λ_n and eigenfunctions y_n of the appropriate Sturm-Liouville eigenvalue problem, equation (8) is solved yielding $T_n(t) = c_n e^{-\alpha\lambda_n t}$.

Since the heat equation is linear, then a superposition of solutions with separated variables: $u(x,t) = \sum_n c_n e^{-\alpha\lambda_n t} y_n(x)$ is again a solution.

Now it is the time to require that the initial condition be satisfied: $u(x,0) = u_0(x) = \sum_n c_n y_n(x)$. Assume $u_0 \in L^2[0,L]$; since we are in a compact domain, this is a physically quite reasonable condition.

If the eigenfunctions y_n are *complete* in $L^2[0,L]$, then indeed c_n exist, and are uniquely determined. The eigenfunctions will be shown to be orthogonal; assuming they have been normalized to have norm one, then $c_n = \langle y_n, u_0\rangle = \int_0^L y_n(x)u_0(x)dx$.

3.2 The vibrating string

Consider a vibrating string with space-dependent tension $T(x)$ and variable linear density $\rho(x)$, assumed to vibrate only due to the restoring tension.

3.2.1 Derivation of the equation

Denote by $y(x,t)$ the displacement at time t. (For each t, the graph of the function $x \mapsto y(x,t)$ represents the string.)

To deduce the equation of the motion we apply Newton's law on each small piece $[x, x+\Delta x]$. The force at each point x is the vertical component of $T(x)$: $T_V(x) = T(x)\sin\alpha$ where α is the angle between $T(x)$ and the x-axis. Assuming the oscillations are small then $\sin\alpha \approx \alpha \approx \tan\alpha = \frac{\partial y}{\partial x}$. Thus $T_V(x) \approx T(x)\frac{\partial y}{\partial x}$.

The force of the piece $[x, x+\Delta x]$ is

$$T_V(x+\Delta x) - T_V(x) \approx \frac{\partial T_V}{\partial x}\,\Delta x = \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right)\,\Delta x$$

On the other hand, mass times acceleration is $\rho(x)\Delta x\,\frac{\partial^2 y}{\partial t^2}$, therefore

$$\rho(x)\,\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right) \tag{11}$$

The motion depends on what happens at the endpoints of the string (the boundary conditions) and on its initial state (initial condition) which need to be specified for the equation (10).

3.2.2 Separation of variables

Looking for solutions of the form $y(x,t) = u(x)f(t)$ and plugging it into (10) it follows that

$$\rho(x)u(x)f''(t) = \frac{d}{dx}\left(T(x)\frac{du}{dx}\right)\,f(t)$$

therefore

$$\frac{f''(t)}{f(t)} = \frac{1}{\rho(x)u(x)}\,(T(x)u'(x))' = \text{constant} = -\lambda$$

and thus

$$f'' = -\lambda f \tag{12}$$

and

$$\frac{1}{\rho u}(Tu')' = -\lambda \tag{13}$$

We can rewrite (12) as

$$(Tu')' + \lambda\rho u = 0 \tag{14}$$

3.2.3 Boundary conditions.

Suppose the endpoints of the string are kept fixed: $y(0,t) = 0, \quad y(L,t) = 0$. Then this implies

$$u(0) = 0, \; u(L) = 0 \tag{15}$$

The problem (13), (14) is a Sturm-Liouville eigenvalue problem. As noted before, this is an eigenvalue/eigenfunction problem for the operator

$$-\frac{1}{\rho(x)}\frac{d}{dx}T(x)\frac{d}{dx}, \quad \text{in } \mathcal{H} = L^2([0,L], \rho(x)dx)$$

which is formally self-adjoint on

$$\{u \in \mathcal{H} \,|\, u', u'' \in \mathcal{H}, \; u(0) = 0, \; u(L) = 0\}$$

2) Another boundary conditions could be: while $x = 0$ is fixed, the endpoint $x = L$ is free. Then $u(0) = 0$ and $u'(L) = 0$ (there is no transverse force at $x = L$ to produce motion). Equation (13) together with the boundary conditions $u(0) = 0$, $u'(L) = 0$ is another example of a Sturm-Liouville eigenvalue problem.

3) Or, $x = 0$ is fixed, but the endpoint $x = L$ is tied to a spring that vibrates: $-Tu'(L) - ku(L) = 0$ (T is the string tension, k is the spring constant, the tail is accelerated up and down but there is no transversal force). In this case, the Sturm-Liouville eigenvalue problem consists of equation (13) together with the boundary conditions $u(0) = 0$, $Tu'(L) + ku(L) = 0$.

The appropriate Sturm-Liouville problem is solved, finding the eigenvalues λ_n and the corresponding eigenfunctions $u_n(x)$.

Remark. The eigenfunctions $u_n(x)$ are the *normal modes* of the string.

Then $f_n(t)$ can be found by solving (11): $f_n(t) = c_n \sin(\sqrt{\lambda_n}t) + d_n \cos(\sqrt{\lambda_n}t)$

3.2.4 Initial condition

To determine a unique solution the initial position of the string must be given: $u(x,0) = g(x)$ and the initial velocity $u_t(x,0) = v(x)$ (the equation is of order two in t!). It is then required that a superposition of the solutions $\sum_n f_n(t)u_n(x)$ satisfy the initial condition: $\sum_n f_n(0)u_n(x) = g(x)$ and $\sum_n f_n'(0)u_n(x) = v(x)$. If $u_n(x)$ form a complete set then c_n and d_n can be determined.

3.3 The wave equation

In particular, if $T(x)$ and $\rho(x)$ are constant then equation (10) becomes

$$y_{tt} = c^2 \, y_{xx} \qquad (\text{where } c^2 = T/\rho)$$

which is the wave equation.

The eigenfunctions u_n satisfy $u_n'' + \lambda_n u = 0$ and the appropriate boundary conditions. If these are $u(0) = 0, \; u(L) = 0$ then we showed that $u_n(x) = \sin(n\pi x/L)$ which are the normal modes of the string.

3.4 The symmetric form of a linear second order equation.

Consider again equations (30), but with $R(x)$ replaced by $R(x)+\lambda$ (where λ is a constant)

$$P(x)u'' + Q(x)u' + (R(x)+\lambda)u = 0 \tag{16}$$

(λ is singled out just to keep track on how it changes after the substitutions that follow).

We are interested in bringing (15) to a symmetric form:

$$\frac{1}{w(x)}\left(-\frac{d}{dx}p(x)\frac{du}{dx} + q(x)\right)u = \lambda u \tag{17}$$

or, expanded,

$$(pu')' + (-q+\lambda w)u = 0 \tag{18}$$

Expanding the left side of (16) we obtain

$$\frac{p}{w}u'' + \frac{p'}{w}u' + \left(-\frac{q}{w}+\lambda\right)u = 0$$

which must coincide with (15), therefore

$$\frac{p}{w} = P, \quad \frac{p'}{w} = Q, \quad \frac{q}{w} = -R$$

The first two equations imply that $p'/p = Q/P$ therefore

$$p(x) = \exp\left[\int \frac{Q(x)}{P(x)}\,dx\right] \tag{19}$$

Then since $w = p/P$ and $q = wR$ we obtain

$$w(x) = \frac{1}{P(x)}\,\exp\left[\int \frac{Q(x)}{P(x)}\,dx\right], \quad q(x) = -\frac{R(x)}{P(x)}\,\exp\left[\int \frac{Q(x)}{P(x)}\,dx\right] \tag{20}$$

3.4.1 Green's Identity and symmetry of the Sturm-Liouville operator

We show here that the problem (33) is indeed symmetric. Let us first show a general formula:

Lemma 1. *Green's identity:*

$$\int_a^b [(pu')'v - u(pv')'] = p(u'v-uv')\big|_a^b \tag{21}$$

Relation (20) follows easily using integration by parts:

$$\int_a^b [(pu')'v - u(pv')'] = \int_a^b (pu')'v - \int_a^b u(pv')'$$

$$= pu'v\big|_a^b - \int_a^b (pu')v' - upv'\big|_a^b + \int_a^b u'(pv') = p(u'v-uv')\big|_a^b$$

□

Theorem 1. *The operator*

$$L = \frac{1}{w(x)} \left(-\frac{d}{dx} p(x) \frac{d}{dx} \; + \; q(x) \right) \tag{22}$$

is self-adjoint in the weighted Hilbert space $\mathcal{H} = L^2([a,b], w(x)dx)$ *on the domain*

$$D = \{ u \in \mathcal{H} \,|\, u', u'' \in \mathcal{H}, \;\; B_a[u] = 0, \;\; B_b[u] = 0 \} \tag{23}$$

Proof. As discussed, we only show symmetry at this stage. By Assumption 1, it suffices to show symmetry on pairs of real-valued functions; symmetry then follows from Green's identity, which gives (noting that the terms containing q cancel each other):

$$\langle Lu, v\rangle - \langle u, Lv\rangle = \int_a^b \frac{1}{w} \Big[-(pu')' + qu \Big] \; v \, w \, dx \; - \int_a^b u \, \frac{1}{w} \Big[-(pv')' + qv \Big] \; w \, dx$$

$$= -p(u'v - uv')\Big|_a^b = p(a)\,(u'v - uv')\Big|_{x=a} - \, p(b)\,(u'v - uv')\Big|_{x=b} \tag{24}$$

We have that

$$p(a)(u'v - uv')\Big|_{x=a} = 0, \;\; p(b)(u'v - uv')\Big|_{x=b} = 0 \tag{25}$$

because $B_a[u] = 0 = B_a[v]$ and $B_b[u] = 0 = B_b[v]$.

Indeed, if $\alpha' = 0$ then $u(a) = 0 = v(a)$ therefore $(u'v - uv')\big|_{x=a} = 0$, and if $\alpha' \neq 0$ then $u'(a) = -\alpha/\alpha' u(a), \;\; v'(a) = -\alpha/\alpha' v(a)$ which substituted into the first relation of (24) gives again zero. The second relation of (24) follows in a similar way. □

Eigenvalues, eigenfunctions. Let again $\mathcal{D} \subset L^2([0.L], \rho(x)dx)$ be the domain of L; $\psi \in \mathcal{D}$ is an eigenfunction of L corresponding to the eigenvalue λ if

$$L\psi = \lambda\psi \tag{26}$$

Remark 1. *If L is symmetric, then the eigenvalues are real and eigenfunctions corresponding to different eigenvalues are orthogonal w.r.t. each-other.*

Proof. Indeed,

$$\lambda\langle\psi, \psi\rangle = \langle\psi, \lambda\psi\rangle = \langle\psi, L\psi\rangle = \langle L\psi, \psi\rangle = \overline{\lambda}\langle\psi, \psi\rangle \Rightarrow \lambda = \overline{\lambda} \tag{27}$$

and

$$\lambda_1\langle\psi_1, \psi_2\rangle = \langle L\psi_1, \psi_2\rangle = \langle\psi_1, L\psi_2\rangle = \lambda_2\langle\psi_1, \psi_2\rangle \Rightarrow \langle\psi_1, \psi_2\rangle = 0 \tag{28}$$

□

The question is when the ψ_k for an orthonormal *basis*, that is, when are they complete, in the sense that

$$\forall\, \varphi \in L^2\ \exists \{c_n\}_{n\in\mathbb{N}}\ s.t.\ \varphi = \sum_{k\in\mathbb{N}} c_n \psi_n \tag{29}$$

or equivalently in our setting,

$$(\forall\, k \in \mathbb{N}\, \langle \varphi, \psi_k \rangle = 0)\ \Rightarrow \varphi = 0 \tag{30}$$

This is for instance the case when L is self-adjoint and the *resolvent* $(L-z)^{-1}$, $z \in \mathbb{C}\backslash\mathbb{R}$ is compact. This is the case with the second order self-adjoint operators associated to Sturm-Liouville problems.

4 Second order linear ordinary differential equations

4.1 Recall some basic results.

A second order linear ordinary differential equation (ODE) has the form

$$P(x)u'' + Q(x)u' + R(x)u = 0 \tag{31}$$

Because the equation is linear, any linear combination of solutions is again a solution: if u_1, $u_2, \ldots, u_k$ are solutions of (30) and c_1, $c_2, \ldots, c_k$ are constants then $c_1u_1(x) + c_2u_2(x) + \ldots + c_ku_k(x)$ is also a solution of (30).

Assumptions. 1) It is assumed assume that the coefficients $P(x), Q(x), R(x)$ are continuous on an interval (a, b). (However, jump discontinuities can also be accommodated in a similar way, only requiring a bit more work.)

2) It is assumed that $P(x)$ does not vanish in (a, b). (Points where $P(x)$ is zero are singular, and solutions are usually very special at such points.)

3) It can be assumed without loss of generality that $P(x) > 0$ on (a, b). (Since $P(x)$ is never zero on (a, b), and it is continuous, then $P(x)$ is either positive on (a, b) or negative on (a, b). If $P < 0$ we multiply the equation by -1.)

General solution.

Recall that the general solution of (30) has the form

$$u(x) = C_1u_1(x) + C_2u_2(x)$$

where $C_{1,2}$ are arbitrary constants and $u_{1,2}$ are two solutions of (30) which are linearly independent, in the sense that the vectors $(u_1(x_0), u_1'(x_0))$ and $(u_2(x_0), u_2'(x_0))$ are linearly independent at some $x_0 \in (a, b)$ (equivalently, at any $x_0 \in (a, b)$). For example, the solutions with the initial conditions $u_1(x_0) = 1, u_1'(x_0) = 0$ and $u_2(x_0) = 0, u_2'(x_0) = 1$ are linearly independent.

An equivalent condition for two solutions to be linearly independent is that their Wronskian

$$W[u_1, u_2] = u_1'u_2 - u_1u_2'$$

satisfies $W(x_0) \neq 0$ at some $x_0 \in (a,b)$ (equivalently, at any $x_0 \in (a,b)$).

Recall that the Wronskian satisfies the differential equation

$$W'(x) = -\frac{Q(x)}{P(x)}\,W(x) \tag{32}$$

and therefore

$$W(x) = C\exp\left[\int -\frac{Q(x)}{P(x)}\,dx\right]$$

Existence and uniqueness of the solution to the initial value problem.

For $x_0 \in (a,b)$ and $c,d \in \mathbb{R}$ equation (30) has a unique solution $u(x)$ for $x \in (a,b)$ satisfying the initial conditions $u(x_0) = c$, $u'(x_0) = d$. This solution $u(x)$ is twice differentiable (moreover, u'' is continuous, as it is seen from (30)), and it depends continuously on the initial conditions.

The general solution depends on two parameters, so it makes sense that two conditions are required to determine these parameters. However, there is no a priori guarantee that solutions satisfying different problems, like boundary conditions, do exist.

4.1.1 Another way of writing $B_a[u]$, $B_b[u]$

Clearly if we multiply α and α' by the same constant, we obtain the same boundary condition $B_a[u]$, and similarly for β and β' in $B_b[u]$. It is sometimes convenient (and always possible!) to choose these in the form

$$\begin{aligned} B_a[u] &\equiv \cos(\theta_a)u(a) - \sin(\theta_a)p(a)u'(a) = 0 \\ B_b[u] &\equiv \cos(\theta_b)u(b) - \sin(\theta_b)p(b)u'(b) = 0 \end{aligned} \tag{33}$$

which are very suitable for Prüfer coordinates.

The transformation which brings (6) in the form (32) is the following: dividing α and α' by the quantity $\pm\sqrt{\alpha^2 + (\alpha'/p(a))^2}$ with the sign chosen to be opposite to the sign of α', we obtain $B_a[u] = 0$ in the form $\alpha_1 u(a) - \alpha_2 p(a)u'(a) = 0$ where $\alpha_1^2 + \alpha_2^2 = 1$ and $\alpha_2 \leq 0$ therefore there exists $\theta_a \in [0,\pi)$ such that $\alpha_1 = \cos(\theta_a)$ and $\alpha_2 = -\sin(\theta_a)$ (we choose $\theta_a < \pi/2$ if $\alpha_1 > 0$ and $\theta_a > \pi/2$ if $\alpha_1 < 0$). A similar transformation can be performed on B_b. Note that we can choose θ_b in $[n,(n+1)\pi)$ for any integer n (if n is even, we proceed as for the condition at $x = a$, while if n is odd we choose the opposite sign in front of $\pm\sqrt{\beta^2 + (\beta'/p(b))^2}$, namely the sign of β').

4.1.2 Singular boundary conditions

Other type of conditions which appear in applications are:

Periodic conditions: if $p(a) = p(b)$ then it can be required that the solutions be periodic:

$$u(a) = u(b), \quad \text{and } u'(a) = u'(b)$$

More generally:

$$\alpha_1 u(a) + \alpha_1' u'(a) + \beta_1 u(b) + \beta_1' u'(b) = 0$$
$$\alpha_2 u(a) + \alpha_2' u'(a) + \beta_2 u(b) + \beta_2' u'(b) = 0$$

If p vanishes at an endpoint, say $p(a) = 0$: then the boundary condition at $x = a$ is dropped.

4.2 Formulation of the homogeneous Sturm-Liouville problem

We will consider real-valued problems: the functions P, Q, R and the numbers $\alpha, \alpha', \beta, \beta'$ are real. In case complex valued functions are needed, then equations can be written and separately solved for the real and imaginary parts of these functions.

Note that with this reality assumption we have $p(x) > 0$ and $w(x) > 0$ (see (18), (19)).

Given the functions p, q, w continuous on $[a, b]$ and $p, w > 0$ on $[a, b]$, and $B_a[u]$, $B_b[u]$ (nontrivial) find the numbers λ so that the following problem has a nontrivial (i.e. nonzero) solution $u(x)$ on $[a, b]$:

$$\begin{cases} [p(x)u']' + [-q(x) + \lambda w(x)]u = 0 \\ \text{Boundary conditions at } x = a \text{ and } x = b \end{cases} \tag{34}$$

The boundary conditions are one of the following:

• *regular conditions:*

$$\begin{aligned} &B_a[u] \equiv \alpha u(a) + \alpha' u'(a) = 0 \qquad (|\alpha| + |\alpha'| \neq 0) \\ &B_b[u] \equiv \beta u(b) + \beta' u'(b) = 0 \qquad (|\beta| + |\beta'| \neq 0) \end{aligned} \tag{35}$$

• *singular conditions:*
if $p(b) = 0$:

$$B_a[u] \equiv \alpha u(a) + \alpha' u'(a) = 0 \qquad (|\alpha| + |\alpha'| \neq 0) \tag{36}$$

or, if $p(a) = 0$:

$$B_b[u] \equiv \beta u(b) + \beta' u'(b) = 0 \qquad (|\beta| + |\beta'| \neq 0) \tag{37}$$

• *periodic conditions:* (also singular) if $p(a) = p(b)$

$$\begin{aligned} C[u] &\equiv u(a) - u(b) = 0 \\ C'[u] &\equiv u'(a) - u'(b) = 0 \end{aligned} \tag{38}$$

The numbers λ are called eigenvalues, and the corresponding solutions - eigenfunctions.

4.3 Conclusions

Since L is formally self-adjoint on D, this implies that its eigenvalues (if any!) are real, and that eigenfunctions corresponding to different eigenvalues are orthogonal.

We will show that indeed, there exist infinitely many eigenvalues λ_n, and that the eigenfunctions u_n form a complete set in the Hilbert space $L^2([a, b], w(x)dx)$.

We will accomplish this program by studying the solutions of the differential equation.

It turns out that, in addition, λ_n can be ordered increasingly, and $\lambda_n \to \infty$, and that eigenfunctions u_n oscillate, and the larger n, the more rapid the oscillations.

5 Eigenfunctions associated to one eigenvalue

5.1 Regular problems

Lemma 2. *For regular problems (33), (34) the eigenspaces are one-dimensional: there is a unique (up to a scalar multiple) eigenfunction associated to each eigenvalue.*

Proof. We show that the eigenspace associated to one eigenvalue of (33) is one dimensional: any two (nonzero) solutions $u_1(x)$, $u_2(x)$ of (33) (for the same λ) are linearly dependent.

Assume, to get a contradiction, that u_1 and u_2 are linearly independent. Then the general solution of the differential equation in (33) is $u = C_1u_1 + C_2u_2$ with C_1, C_2 arbitrary constants. Since the boundary conditions are linear, it follows that $B_a[u] = C_1B_a[u_1]+C_2B_a[u_2] = 0$, $B_b[u] = C_1B_b[u_1]+C_2B_b[u_2] = 0$ therefore the boundary conditions are satisfied by any solution of the differential equation. For u the solution with $u(a) = 1, u(a) = 0$ we find that $\alpha = 0$ and u the solution with $u(a) = 0, u'(a) = 1$ it follows that $\alpha' = 0$, which contradicts the nontriviality assumption on B_a. (A similar argument can be made at $x = b$.)

Therefore, u_1 and u_2 must be linearly dependent, hence scalar multiples of each other. □

5.2 When $p(x)$ vanishes at one endpoint

Suppose that $p(b) = 0$. Consider the the Sturm-Liouville problem (33), (36).

The Sturm-Liouville operator (21) is formally self-adjoint on the domain

$$D = \{u \in \mathcal{H} \,|\, u', u'' \in \mathcal{H},\ B_a[u] = 0\} \tag{39}$$

Indeed, relations (24) hold: at $x = b$ because $p(b) = 0$ and at $x = a$ with the same proof as for Theorem 21.

The eigenspaces are still one-dimensional, as the proof of Lemma 2 works.

The singular case with $p(a) = 0$ is similar.

5.3 Periodic problems

If $p(a) = p(b)$ the Sturm-Liouville eigenvalue problem (33), (37) is also formally self-adjoint: the operator (21) is formally self-adjoint on

$$D = \{u \in \mathcal{H} \,|\, u', u'' \in \mathcal{H},\ C[u] = 0,\ C'[u] = 0\} \tag{40}$$

since the last quantity in (23) is clearly zero for u, v in the domain (39).

However, for periodic boundary conditions the eigenspaces may be one, or two-dimensional.

Indeed, repeating the argument of §5.1 we find that if for an eigenvalue λ, if there are two independent eigenfunctions then all the solutions of the ODE are periodic (but the space of solutions of a second order ODE is a two-dimensional vector space). This means that the eigenspace is either one-dimensional or two-dimensional, and in the latter case we can choose two orthogonal eigenfunctions.

6 Fourier series

It is easy to solve the following periodic Sturm-Liouville problem:

$$u'' + \lambda u = 0, \qquad u(-\pi) = u(\pi),\ u'(-\pi) = u'(\pi) \tag{41}$$

It has the eigenvalues $\lambda_n = n^2,\ n = 0, 1, 2, \ldots$. For $n > 0$ there are two orthogonal eigenfunctions corresponding to $\lambda_n = n^2$: $\sin(nx)$ and $\cos(nx)$. For $n = 0$ the eigenfunction corresponding to $\lambda_0 = 0$ is the constant function.

It follows that these eigenfunctions are complete using the general theorem:

Theorem 2. Completeness of the eigenfunctions

Assume that the differential operator L in (3) *has the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ with $\lambda_n \to \infty$, and that each eigenspace is finite dimensional, spanned by a set of orthogonal eigenvectors u_n.*

Then the set of eigenvectors is complete: they form a basis for the Hilbert space $\mathcal{H}$.

This extends to self-adjoint operators with compact resolvent. The proof of the theorem is postponed until §10.

Applying Theorem 2 to $L = -\frac{d^2}{dx^2}$ in $\mathcal{H} = L^2[-\pi, \pi]$, which is formally self-adjoint on

$$D = \{u \in \mathcal{H} \,|\, u', u'' \in \mathcal{H},\ u(\pi) - u(-\pi) = 0,\ u'(\pi) - u'(-\pi) = 0\}$$

having the eigenvalues $0, 1, 1, 2^2, 2^2, \ldots$ it follows that its eigenfunctions 1, $\sin(nx)$, $\cos(nx)$ for $n = 1, 2, \ldots$ form an orthogonal basis for $L^2[-\pi, \pi]$, therefore:

Theorem 3. *Any function $f \in L^2[-\pi, \pi]$ can be expanded in a Fourier series:*

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{42}$$

where the Fourier coefficients a_n and b_n are given by the formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \ , \ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx,$$

While the series (41) converges to f in the L^2-norm (that is, in squared average), it is useful to know when the series converges point-wise (that is, for a fixed x, as a series of numbers):[(1)]

Theorem 4. Point-wise convergence of a Fourier series

A. *If f and f' are piecewise continuous on $[-\pi, \pi]$ then the series (41) converges for every x.*

B. *Let $c \in (-\pi, \pi)$.*

(i) If f is continuous at $x = c$ then the Fourier series (41) at $x = c$ converges to $f(c)$.

(ii) If f has a jump discontinuity at $x = c$ then the Fourier series (41) at $x = c$ converges to $[f(c-) + f(c+)]/2$.

C. *The behavior of the Fourier series at the points $x = \pi$ and $x = -\pi$ is seen in the following way. Continue $f(x)$ outside $[-\pi, \pi]$ by 2π-periodicity. If $f(\pi-) = f(-\pi+)$ then $x = \pm\pi$ are points of continuity, and the series (41) converges to $f(\pm\pi)$ for $x = \pm\pi$. Otherwise, $x = \pm\pi$ are points where there is a jump discontinuity and the series (41) converges to $[f(\pi-) + f(-\pi+)]/2$ for $x = \pm\pi$.*

The proof of Theorem 4 is not given here.

Recall that a function is called *even* if $f(-x) = f(x)$ for all x. For example the functions $1, x^2, x^4, x^8 - 3x^4, |x|, \cos(x)$ are even.

Recall that a function is called *odd* if $f(-x) = -f(x)$ for all x. For example the functions $x, x^3, x^5, x^9 - 2x, \sin(x), \tan x$ are odd.

Remark. *If $f(x)$ is an even function then all $b_n = 0$ and therefore f has a cosine-series.*

If $f(x)$ is an odd function then all $a_n = 0$ and therefore f has a sine-series.

Given any function $g(x)$ for $x \in [0, \pi]$, then g can be continued to $x \in [-\pi, \pi]$ by

1) requiring that the function on $[-\pi, \pi]$ be odd, that is continued as

$$g_{odd}(x) = \begin{cases} g(x) & \text{if } 0 < x < \pi \\ -g(-x) & \text{if } -\pi < x < 0 \end{cases}$$

(1) Note that piecewise continuous functions on $[-\pi, \pi]$ are necessarily in $L^2[-\pi, \pi]$.

in which case $g(x)$ has a sine-series, or by

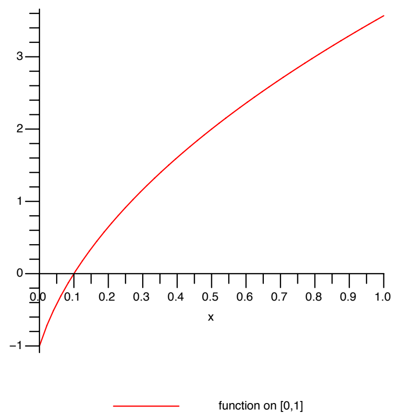
2) requiring that the function on $[-\pi, \pi]$ be even, that is continued as

$$g_{even}(x) = \begin{cases} g(x) & \text{if } 0 < x < \pi \\ g(-x) & \text{if } -\pi < x < 0 \end{cases}$$

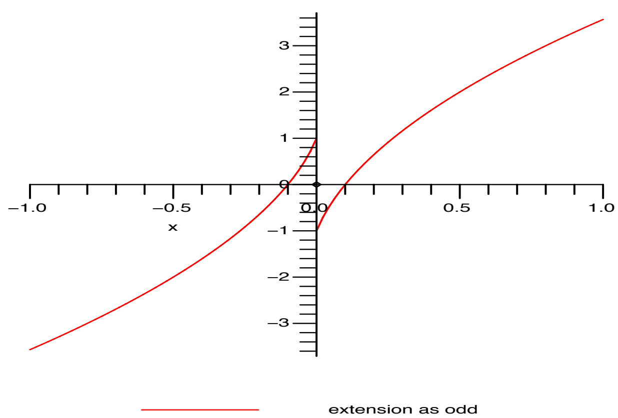
in which case $g(x)$ has a cosine-series.

(Recall that functions that differ by the value at one point, such as $x = 0$, are considered equal in L^2.)

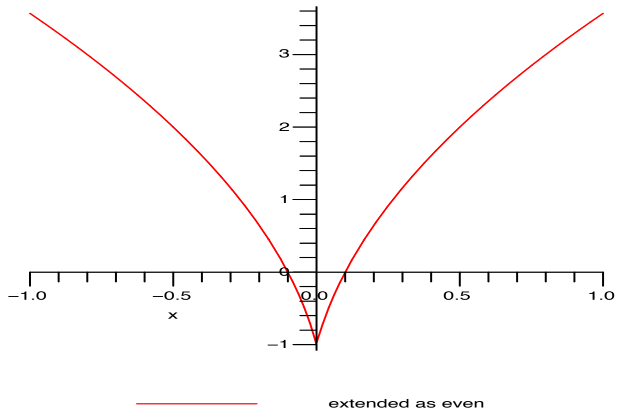
The figure show a function on $[0, 1]$



and its extension to $[-1, 1]$ as an odd function



and its extension to $[-1, 1]$ as an even function



Recall that the eigenfunctions of the Dirichlet homogeneous problem

$$u'' + \lambda u = 0, \;\; u(0) = 0, \;\; u(\pi) = 0 \tag{43}$$

are $\sin(nx)$, for $n = 1, 2, \ldots$ and they correspond to the eigenvalues $\lambda_n = n^2$. Theorem 2 does apply and we obtain that any function in $L^2[0, \pi]$ can be expanded in a sine-series. This series coincides with the Fourier series of the function continued to an odd function on $[\pi, \pi]$ as showed at point 1) above.

In conclusion, for a function f on $[a, b]$ we can write an expansion in a sine-series, or a cosine-series, or a general Fourier series. The general Fourier series is more appropriate if f is periodic on $[a, b]$ (i.e. $f(a) = f(b)$). If $f(x)$ has complex values, then the complex form of the Fourier series may be preferable:

$$\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n/(b-a)}$$

Note that this series is real-valued if $c_{-n} = \overline{c_n}$ for all n.

6.1 An example

Let us find the sine-series of function $f(x) = 1$ for $x \in [0, \pi]$.

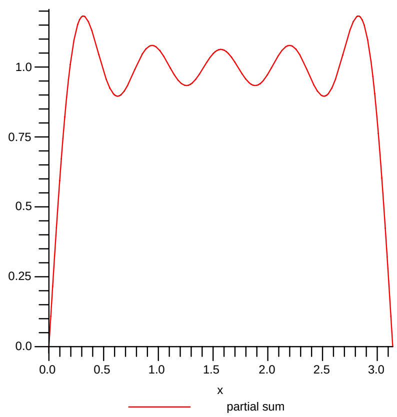
It can be checked that $b_n = 0$ for n even, and its sine-series is

$$\frac{4}{\pi} \sin(x) + \frac{4}{3\pi} \sin(3\,x) + \frac{4}{5\pi} \sin(5\,x) + \frac{4}{7\pi} \sin(7\,x) + \frac{4}{9\pi} \sin(9\,x) + \ldots$$

For each $x \in (0, \pi)$ the sine-series converges to 1 by Theorem 4 **B.**(i). To understand the value at which the series converges at the end points $x = 0$ and $x = \pi$ we first continue the function to an odd function on $[-\pi, \pi]$, which is $f_{odd}(x) = 1$ for $x \in (0, \pi]$ and $f_{odd}(x) = -1$ for $x \in [-\pi, 0)$. Using Theorem 4 **B.**(i) the series at $x = 0$ converges to $[f_{odd}(0-) + f_{odd}(0+)]/2 = 0$ and by Theorem 4 **C.** at $x = \pi$ the series converges to $[f_{odd}(-\pi+) + f_{odd}(\pi-)]/2 = 0$.

Of course, by substituting directly $x = 0$ or $x = \pi$ in the series we see that all its terms are zero.

The picture shows the plot of the sum of the first five nonzero terms.



Note the large overshoot of the partial sum at the at the jump discontinuities at $x = 0$ and $x = \pi$: this is called Gibbs phenomenon. This behavior of truncates of Fourier series gives rise to artifacts in signal processing.

6.2 Problems on infinite intervals

If we have eigenvalue problems on infinite intervals (like $[a, +\infty)$ or $\mathbb{R} = (-\infty, +\infty)$) a similar theory can be developed, only series need to be replaced by integrals.

For example, consider the differentiation operator in $L^2(\mathbb{R})$: $L = \frac{d}{dx}$. Since

$$\frac{d}{dx} e^{ikx} = ike^{ikx}$$

then the function e^{ikx} is a *generalized* eigenfunction of L (it must be called "generalized" because it does not belong to the Hilbert space $L^2(\mathbb{R})$).

However, any $f \in L^2(\mathbb{R})$ can be developed in terms of these generalized eigenfunctions:

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} F(k)\, dk$$

(the inverse Fourier transform, up to a multiple) to be compared to the Fourier series for $f \in L^2[-\pi, \pi]$

$$f = \sum_{k=-\infty}^{\infty} f_k e^{ikx}$$

7 Abel's Theorem

The following results is (31) in disguise:

Theorem 5. Abel's Theorem

Let u, v be two solutions of the second order linear differential equation

$$[p(x)u']' + [-q(x) + \lambda w(x)]u = 0 \tag{44}$$

Then

$$p(u'v - uv') = \text{constant}$$

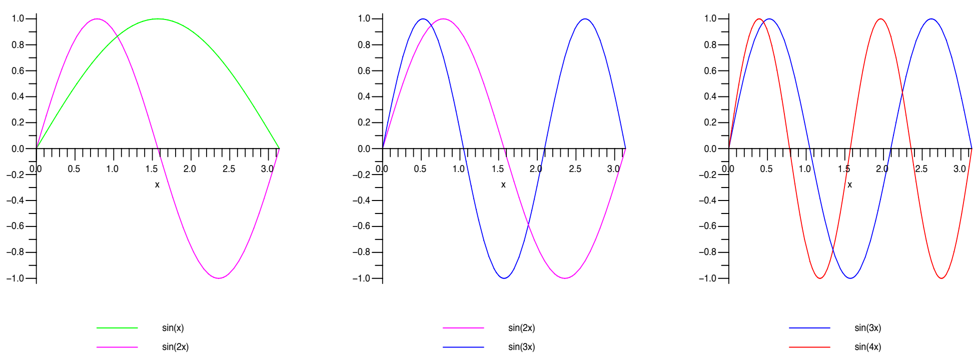
Proof. Equation (43) expanded is $pu'' + p'u' + (-q + \lambda w)u = 0$. For $W = W[u, v] = u'v - uv'$ relation (31) implies that $W' = -p'/p\, W$ so $\ln W = -\ln p + const$ which implies $pW = const$. $\square$

Note that once a solution u of (43) another independent solution v can be found by solving $p(u'v - uv') = c$, which is a first order differential equation, linear nonhomogeneous for v.

8 Sturm's Oscillation Theorems

8.1 A simple example

It is always useful to consider simple examples (or, "toy models") for the more complicated system studied. A simple example (and exactly solvable!) of a Sturm-Liouville problem is obtained for for $w(x) = 1$, $p(x) = 1$, $q(x) = 0$, and $\alpha' = 0, \beta' = 0$, and $[a, b] = [0, \pi]$, namely the problem (42). We studied this equation and we found the eigenvalues $\lambda_n = n^2$ $(n = 1, 2, \ldots)$ and the eigenfunctions $u_n = \sin(nx)$.



The figures show, on the same plot, u_n and u_{n+1} for $n = 1, 2, 3$. Please note how their zeros interlace: between two consecutive zeros on u_n there is a zero of u_{n+1}. Sturm's Comparison Theorem shows that this is a general feature. (Please note that the numbers of zeros is linked to the number of oscillations.)

8.2 Sturm's Comparison Theorem

Theorem 6. *Consider two solutions $u(x)$ and $v(x)$ of two equations*

$$[p(x)u']' + g(x)u = 0 \tag{45}$$

and

$$[p(x)v']' + h(x)v = 0 \tag{46}$$

where $p(x) > 0$ on $[a,b]$.

(i) If $g(x) < h(x)$ on $[a,b]$ then $v(x)$ oscillates more rapidly than $u(x)$: between any two zeros of $u(x)$ there is a zero of $v(x)$.

(ii) If $g(x) < 0$ on $[a,b]$ then $u(x)$ does not oscillate: it vanishes at most once on $[a,b]$.

Sturms's Comparison Theorem applied to $g(x) = -q(x) + \lambda w(x)$ with $w > 0$ shows that if λ is negative enough there cannot be solutions of (44) which vanish at both endpoints a, b. However, once there is a solution vanishing at $x = a$ and $x = b$, then the higher λ the more zeros solutions have (i.e. more oscillations) and that zeros of solutions for different λs do interlace.

Proof of Sturm's Comparison Theorem

(i) Multiplying (44) by v, (45) by u and subtracting it is found that

$$\frac{d}{dx}\left[p(u'v - uv')\right] = (h-g)uv$$

therefore by integration

$$p(u'v - uv')\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} (h-g)uv \tag{47}$$

Choose $x_1 < x_2$ be two consecutive zeros of u. Assume, to get a contradiction, that v has no zero on (x_1, x_2). Then v has a constant sign on (x_1, x_2), say $v > 0$ (otherwise replace v by $-v$). We can also assume $u > 0$ on (x_1, x_2). Then the right-hand side of (46) is positive, while the left-hand side is negative since it equals

$$\underbrace{p(x_2)}_{>0}\underbrace{u'(x_2)}_{\leq 0}\underbrace{v(x_2)}_{>0} - \underbrace{p(x_1)}_{>0}\underbrace{u'(x_1)}_{\geq 0}\underbrace{v(x_1)}_{>0} \leq 0$$

which is a contradiction.

(ii) For $h(x) = 0$: since $(pv')' = 0$ then the general solution of (45) is $c_1 + c_2\int_a^x (1/p)$. Choosing $v = 1 + \int_a^x (1/p)$ we have $v > 0$ hence v has no zero. Therefore by (i) u has at most one zero. $\Box$.

9 Existence of eigenvalues: The Prüfer transformation

The differential equation (in formally self-adjoint form) is

$$(p(x)u')' + g(x)u = 0 \qquad \text{with } g(x) = -q(x) + \lambda w(x) \tag{48}$$

Assume that $p(x) > 0$ and that p', g' are continuous.

Sturm's Comparison Theorem shows the oscillatory character of the solutions of (47). Polar coordinates in the phase space are then natural.

9.1 The Prüfer system

The Prüfer transformation consists of writing the (phase-space like) quantities u and $v = pu'$ in polar coordinates:

$$u(x) = r(x)\,\sin\theta(x), \quad u'(x) = \frac{r(x)}{p(x)}\,\cos\theta(x) \tag{49}$$

The equation (47) is $v' + gu = 0$, therefore

$$\frac{d}{dx}\,(r\cos\theta) + gr\sin\theta = 0 \tag{50}$$

The second equation is obtained from $u' = r/p\cos\theta$ which expanded gives

$$r'\sin\theta + r\theta'\cos\theta = r/p\cos\theta \tag{51}$$

On the other hand, expanding (49) we obtain

$$r'\cos\theta - r\theta'\sin\theta + gr\sin\theta = 0 \tag{52}$$

Solving (50), (51) for r', θ' we obtain

$$r' = \frac{1}{2}\left(\frac{1}{p} - g\right)\,r\sin 2\theta \tag{53}$$

and

$$\theta' = g\sin^2\theta + \frac{1}{p}\cos^2\theta \equiv F(x,\theta) \tag{54}$$

(multiplying (50) by $\sin\theta$, multiplying (51) by $\cos\theta$ and adding them up we obtain (52), while multiplying (50) by $\cos\theta$, multiplying (51) by $-\sin\theta$ and adding them up we obtain (53)).

Remarkably, equation (53) is a first order equation for $\theta(x)$, independent of $r(x)$! With $\theta(x)$ determined by (53) we can integrate (52) to obtain $r(x)$:

$$r(x) = K\,\exp\int_a^x \frac{1}{2}\left(\frac{1}{p(s)} - g(s)\right)\,\sin 2\theta(s)\,ds$$

Note that $r(x) \neq 0$. Note that we can take the constant $K = 1$ (otherwise we divide $u(x)$ by K).

It is easy to see (from (48)) that the boundary conditions (32) become

$$\theta(a) = \theta_a,\;\; \theta(b) = \theta_b \tag{55}$$

9.2 Behavior of $\theta(x)$ and the zeros of $u(x)$

Choosing λ large enough assume that $g(x) > 0$ on $[a, b]$, so that oscillations are possible.

(A) Note that (from (48))

$$u(x) = 0 \quad \text{if and only if } \sin\theta = 0 \text{ so if and only if } \theta = k\pi,\ k \in \mathbb{Z}$$

(B) On the other hand $\theta'(x) = F(x, \theta) > 0$, so $\theta(x)$ is an increasing function. In fact, the larger g, the larger θ' (the rate of increase of θ).

(C) There can be no accumulation of zeros of $u(x)$ as the distance between two successive zeros is no smaller that a positive number d.

Why: Denoting $M_F = \max\{F(x, \theta)\,|x \in [a, b],\ \theta \in [0, 2\pi]\}$ we have $\theta'(x) < M_F$. Therefore, if $x_1 < x_2$ are two successive zeros of $u(x)$ then we have $\theta(x_1) = k\pi$ and since $\theta(x_2) = (k + 1)\pi$ and $\theta(x_2) - \theta(x_1) = \theta'(c)(x_2 - x_1)$ for some $c \in (a, b)$ it follows that

$$x_2 - x_1 = \frac{1}{\theta'(c)}(\theta(x_2) - \theta(x_1)) \geq \frac{\pi}{M_F} \equiv d \tag{56}$$

9.3 Boundary conditions and existence of eigenvalues

We showed that the Sturm-Liouville eigenvalue problem (33) becomes, in Prüfer coordinates: determine the values λ so that the following boundary value problem has a nontrivial solution:

$$\theta' = (-q(x) + \lambda w(x)) \sin^2\theta + \frac{1}{p(x)} \cos^2\theta \tag{57}$$

$$\theta(a) = \theta_a \in [0, \pi) \tag{58}$$

$$\theta(b) = \theta_b \in [n\pi, (n + 1)\pi) \tag{59}$$

To solve the problem, first solve the equation (56) with the initial condition (57). We obtain $\theta(x; \lambda)$ depending on the parameter λ, chosen large enough so that $-q(x) + \lambda w(x) > 0$ on $[a, b]$.

The solution $\theta(x; \lambda)$ has the following properties:

1) it is continuous in λ (since g depends continuously on the parameter λ) and increasing in λ (since θ' is increasing in λ);

2) for x fixed $\lim_{\lambda\to\infty} \theta(x; \lambda) = +\infty$ (since θ' is increasing in λ, without any bound);

3) for x fixed $\lim_{\lambda\to-\infty} \theta(x; \lambda) = 0$ (because for large λ equation (53) is approximately $\theta' \approx \lambda w(x) \sin^2\theta$ with the solution $\cot\theta \approx -\lambda \int_c^x w$. In the limit $\lambda \to -\infty$ then $\cot\theta \to 0$).

Therefore, for some $\lambda = \lambda_n$, condition (58) is satisfied.

We thus find an increasing sequence of eigenvalues λ_n, with $\lambda_n \to \infty$.

9.4 The distance between two consecutive zeros of the eigenfunction u_n

Using the formula (55) for $g(x) = -q(x) + \lambda_n w(x)$ we obtain that the distance between two consecutive zeros of the eigenfunction u_n is no smaller that $d_n = \pi/M_n$ where

$$M_n = \max\{[-q(x) + \lambda_n w(x)] \sin^2 \theta + \frac{1}{p(x)} \cos^2 \theta \,\big|\, x \in [a, b], \, \theta \in [0, 2\pi]\}$$

(Note that this estimate can be used only when p does not vanish at the end-points of the interval $[a, b]$.)

10 Completeness

Consider the problem (33). We showed that there exists a sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ with $\lambda_n \to \infty$ and that the corresponding eigenfunctions u_n are orthogonal. The hypothesis of Theorem 2 are thus satisfied. It only remains to prove Theorem 2.

10.1 Proof of Theorem 2

As in finite dimensions, the eigenvalues of this formally self-adjoint operator can be calculated using the maximin principle. (We cannot speak about a minimax since there is no maximum eigenvalue).

Recall that, using the Rayleigh quotient

$$R[u] = \frac{\langle u, Lu \rangle}{\langle u, u \rangle}$$

we have

$$\lambda_1 = \min R[u]$$

then

$$\lambda_2 = \min\{R[u] \,|\, \langle u_1, u \rangle = 0\}, \quad \lambda_3 = \min\{R[u] \,|\, \langle u_1, u \rangle = 0, \langle u_2, u \rangle = 0\} \ \ldots$$

or, with the notation $W_1 = Sp(u_1),\ W_2 = Sp(u_1, u_2),\ \ldots W_n = Sp(u_1, u_2, \ldots, u_n) \ldots$ then

$$\lambda_2 = \min_{u \in W_1^\perp} R[u],\ \lambda_2 = \min_{u \in W_2^\perp} R[u], \ldots, \lambda_{n+1} = \min_{u \in W_n^\perp} R[u] \ \ldots$$

Let u_n be eigenfunctions: $Lu_n = \lambda_n u_n$ which by assumption satisfy $u_n \perp u_k$ if $n \neq k$ (note that this is automatic if $\lambda_n \neq \lambda_k$ since L is formally self-adjoint). We can assume $\|u_n\| = 1$.

To prove completeness of the eigenfunctions u_n we first show that any f in the domain D of the operator L can be expanded in terms of u_n, in other words that the space $S = Sp(u_1, u_2, u_2, \ldots)$ is dense in D. Then since D was assumed to be dense in $\mathcal{H}$ it follows that S is dense in $\mathcal{H}$, therefore $u_1, u_2, u_3, \ldots$ form a basis for the Hilbert space (see details in §10.2 below).

To show that S is dense in D for any arbitrary $f \in D$, form the series

$$\sum_{n=1}^{\infty} f_n u_n, \quad \text{with } f_n = \langle u_n, f\rangle \tag{60}$$

and show that this series converges to f. Note that the partial sums of the series (59) belong to S:

$$f^{[N]} \equiv \sum_{n=1}^{N} \langle u_n, f\rangle u_n \in S$$

We show that the error when approximating f by partial sums

$$h^{[N]} = f - f^{[N]} \tag{61}$$

goes to zero as $N \to \infty$, that is

$$\|h^{[N]}\| \to 0 \ \text{ as } N \to \infty$$

Since $h^{[N]} \in W_N^{\perp}$ then

$$\lambda_{N+1} = \min_{u \in W_n^{\perp}} R[u] \ \le R[h^{[N]}] = \frac{\langle h^{[N]}, Lh^{[N]}\rangle}{\|h^{[N]}\|^2}$$

and therefore

$$\|h^{[N]}\|^2 \le \frac{1}{\lambda_{N+1}} \langle h^{[N]}, Lh^{[N]}\rangle \tag{62}$$

Now expand, using (60),

$$\langle h^{[N]}, Lh^{[N]}\rangle = \langle f, Lf\rangle - \sum_{n=1}^{N} f_n \langle u_n, Lf\rangle - \sum_{n=1}^{N} f_n \langle f, Lu_n\rangle + \sum_{n,m=1}^{N} f_n f_m \langle u_n, Lu_m\rangle$$

$$= \langle f, Lf\rangle - \sum_{n=1}^{N} 2 f_n \lambda_n \langle u_n, f\rangle + \sum_{n=1}^{N} f_n^2 \lambda_n = \langle f, Lf\rangle - \sum_{n=1}^{N} f_n^2 \lambda_n$$

Since $\lim_n \lambda_n = +\infty$ then the eigenvalues are positive staring with a certain rank p: suppose $\lambda_n \ge 0$ for $n > p$. Then (for $N > p$)

$$\langle f, Lf\rangle - \sum_{n=1}^{N} f_n^2 \lambda_n = \langle f, Lf\rangle - \sum_{n=1}^{p} f_n^2 \lambda_n - \sum_{n=p+1}^{N} f_n^2 \lambda_n \le \langle f, Lf\rangle - \sum_{n=1}^{p} f_n^2 \lambda_n \tag{63}$$

Using (62) in (61) we obtain

$$\|h^{[N]}\|^2 \le \frac{1}{\lambda_{N+1}} \left(\langle f, Lf\rangle - \sum_{n=1}^{p} f_n^2 \lambda_n \right) \to 0 \ \text{ as } N \to \infty$$

which completes the proof of the convergence and of completeness. $\square$

10.2 If S is dense in D and D is dense in $\mathcal{H}$ then S is dense in $\mathcal{H}$

Intuitively: the statement that D is dense in $\mathcal{H}$ means that for any $f \in \mathcal{H}$ we can find an $f_D \in D$ as close to f as we wish. Similarly, if S is dense in D then we can find $f_S \in S$ as close to f_D as we wish. By the triangle's inequality: if f_D is close to f, and f_S is close to f_D, then f_S is close to f.

So, let $f \in \mathcal{H}$. We want (for an arbitrary $\varepsilon > 0$) to find $f_S \in S$ so that $d(f, f_S) < \varepsilon$. But we can certainly find $f_D \in D$ so that $d(f, f_D) < \varepsilon/2$ and for that f_D we can certainly find $f_S \in S$ so that $d(f_D, f_D) < \varepsilon/2$.

By the triangle's inequality then

$$d(f, f_S) < d(f, f_D) + d(f_D, f_S) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

which proves the claim.

11 More examples of separation of variables

11.1 The wave equation

Vibrations of a string and propagation of waves is modeled by the wave equation

$$u_{tt} = c^2\, u_{xx} \tag{64}$$

It can be easily checked that if $h_{1,2}$ are two arbitrary twice differentiable functions then

$$u(x,t) = h_1(x - ct) + h_2(x + ct) \tag{65}$$

satisfies the wave equation and therefore (64) is the general solution of the wave equation. Note that c represents the speed of propagation of the wave.

Let us solve (63) for $x \in [0, L]$ and $t > 0$. For this we need the boundary conditions at $x = 0$ and at $x = L$, which we take for simplicity to be the homogeneous Dirichlet problem:

$$u(0,t) = 0, \qquad u(L,t) = 0 \tag{66}$$

and we also need initial conditions, at $t = 0$. Since (63) is second order in t we need the initial positions and the initial velocity. Therefore the conditions are

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{67}$$

11.1.1 Separation of variables

Looking for solutions of (63) in the form $u(x,t) = X(x)T(t)$ the PDE becomes

$$T''(t)X(x) = c^2 T(t)X''(x) \;\text{ therefore }\; \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda$$

(we include c^2 with T just for the convenience of solving the equation for X). The boundary conditions (65) imply that

$$X(0) = 0, \quad X(L) = 0 \tag{68}$$

We solved the Sturm-Liouville problem

$$X''(x) + \lambda X(x) = 0$$

with (67); the solutions are

$$\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2 \ldots, \quad X_n(x) = \sin\frac{n\pi x}{L} \tag{69}$$

Next solve

$$T''(t) + c^2\lambda_n T(t) = 0$$

which gives

$$T(t) = T_n(t) = a_n \cos\frac{cn\pi t}{L} + b_n \sin\frac{cn\pi t}{L} \tag{70}$$

(note the undetermined constants a_n, b_n). We obtained the solutions

$$u(x,t) = u_n(x,t) = \left[a_n \cos\frac{cn\pi t}{L} + b_n \sin\frac{cn\pi t}{L}\right] \sin\frac{n\pi x}{L} \tag{71}$$

which represent the modes of vibration: the mode u_n has frequency $\omega_n = \frac{1}{2\pi}\frac{cn\pi}{L} = \frac{c}{2L}$ cycles per unit time, and it is called *the nth harmonic.* The harmonic with $n = 1$ is called *the fundamental frequency, or first harmonic.* In the case of the vibrating sting all the harmonic frequency are multiples of the fundamental one.

11.1.2 Superposition

Since the PDE (63) is linear, then any sum of solutions (70) is again a solution, so let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\frac{cn\pi t}{L} + b_n \sin\frac{cn\pi t}{L}\right] \sin\frac{n\pi x}{L} \tag{72}$$

11.1.3 Initial conditions

We now require that the solution (71) satisfies (66):

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x) \tag{73}$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{cn\pi}{L} \sin\frac{n\pi x}{L} = g(x) \tag{74}$$

Assuming that $f, g \in L^2[0, L]$ we can expand them as a sine-series (recall that $\sin \frac{n\pi x}{L}$ are eigenfunctions of a formally self-adjoint operator satisfying the hypothesis of the Completeness Theorem 2), and

$$a_n = \frac{\langle \sin \frac{n\pi x}{L}, f\rangle}{\| \sin \frac{n\pi x}{L} \|^2}, \quad b_n = \frac{L}{cn\pi} \frac{\langle \sin \frac{n\pi x}{L}, g\rangle}{\| \sin \frac{n\pi x}{L} \|^2}$$

Since

$$\| \sin \frac{n\pi x}{L} \|^2 = \int_0^L \sin^2 \frac{n\pi x}{L}\, dx = \frac{1}{2} \int_0^L \left(1 - \cos \frac{2n\pi x}{L}\right) dx = \frac{L}{2}$$

we obtain

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx \tag{75}$$

11.1.4 Connection with the general solution

To see how the solution (71), (74) is related to the general solution (64) use the trigonometric formulas $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$ and $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$ to rewrite (71) as

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} \Bigg[& \frac{a_n}{2} \left(\sin \frac{(x+ct)n\pi t}{L} + \sin \frac{(x-ct)n\pi t}{L} \right) \\ & + \frac{b_n}{2} \left(\cos \frac{(x-ct)n\pi t}{L} + \cos \frac{(x+ct)n\pi t}{L} \right) \Bigg] \\ & \equiv \frac{1}{2} \left[F(x+ct) - F(x-ct) \right] \end{aligned}$$

where

$$F(x) = \sum_{n=1}^{\infty} \left(a_n \sin \frac{nx\pi t}{L} + b_n \cos \frac{nx\pi t}{L} \right)$$

Note that $u(x, 0) = F(x)$ therefore $F(x)$ is the initial condition.

Note that the graph of $F(x - ct)$ (for any fixed t) is the same as the graph of $F(x)$ (the initial shape), only shifted to the right by ct: it represents the initial "wave" traveling to the right with speed c. Similarly, $F(x + ct)$ represents the initial "wave" traveling to the left with speed c.

11.2 Laplace's equation

The two-dimensional heat equation (modeling the temperature distribution in a lamina) is

$$u_t = \alpha^2 \left(u_{xx} + u_{yy} \right)$$

The stationary solutions (for which $u(x, y, t) \equiv u(x, y)$) satisfy Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{76}$$

It is clear that in order to solve (75) we need information about the temperature on the boundary of the lamina.

Consider as example a lamina in the shape of a semi-infinite vertical strip: $0 \le x \le L$ and $y \ge 0$ and the Dirichlet problem: we need the temperature along the segment $(x, 0)$ with $0 \le x \le L$, along the half lines $(0, y)$, $y \ge 0$ and (L, y), $y \ge 0$ and for $y \to +\infty$:

$$\begin{array}{ll} u(x,0) = f(x) & \text{for } 0 \le x \le L \\ u(0,y) = 0 & \text{for } y \ge 0 \\ u(L,y) = 0 & \text{for } y \ge 0 \\ \lim_{y\to+\infty} u(x,y) = 0 & \text{for } 0 \le x \le L \end{array} \tag{77}$$

11.2.1 Separation of variables

Looking for solutions of (75) in the form $u(x, Y) = X(x)Y(y)$ the PDE becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0 \quad \text{therefore } \frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = -\lambda$$

Solving for $X(x)$ we obtain (68) then solving for $Y(y)$:

$$Y(y) = Y_n(y) = a_n \exp(-n\pi y/L) + b_n \exp(n\pi y/L)$$

The last condition in (76) implies that $b_n = 0$.

11.2.2 Solving the problem by superposition

Since the equation (75) is linear, then a sum of the solutions with separated variables is again a solution:

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y) = \sum_{n=1}^{\infty} a_n \exp(-n\pi y/L) \sin\frac{n\pi x}{L}$$

Requiring that the first condition in (76) be satisfied we obtain

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x)$$

If $f \in L^2[0, L]$ then a_n are found by (74).

11.3 The vibrating rod

Transverse vibrations of a homogeneous rod is described by the equation

$$u_{xxxx} + u_{tt} = 0 \tag{78}$$

Assume that the rest position of the rod is for $0 \le x \le \pi$.

As before, we solve by separation of variable: $u(x,t) = X(x)T(t)$ which gives

$$\frac{X^{(IV)}(x)}{X(x)} = \frac{-T''(t)}{T(t)} = \lambda$$

The boundary conditions come from the system studied. For example, they could be
1) free ends: $X'' = X''' = 0$ for $x = 0$ and $x = \pi$, or
2) supported ends: $X = X'' = 0$ for $x = 0$ and $x = \pi$, or
3) clamped ends: $X = X' = 0$ for $x = 0$ and $x = \pi$, or
4) $X' = X''' = 0$ for $x = 0$ and $x = \pi$, or
5) periodicity: $X(0) = X(\pi)$, $X'(0) = X'(\pi)$, $X''(0) = X''(\pi)$, $X'''(0) = X'''(\pi)$.

Let us solve the problem with free ends:

$$X^{(IV)} = \lambda X(x) = 0, \quad X'' = X''' = 0 \text{ for } x = 0,\ x = \pi$$

Can we expect a complete set of eigenfunctions?

Consider the operator $L = \frac{d^4}{dx^4}$ in $\mathcal{H} = L^2[0,\pi]$ on the domain

$$D = \{f \in \mathcal{H} \,|\, f', f'', f''', f^{(IV)} \in \mathcal{H},\ f''(0) = f'''(0) = 0,\ f''(\pi) = f'''(\pi) = 0\}$$

Is the operator self-adjoint on D?

For $f, g \in D$ calculate

$$\begin{aligned}
\langle Lf, g\rangle &= \int_0^\pi f^{(IV)}(x)g(x)\,dx = f'''g\big|_0^\pi - \int_0^\pi f^{'''}(x)g'(x)\,dx \\
&= -\int_0^\pi f^{'''}(x)g'(x)\,dx = f''g'\big|_0^\pi + \int_0^\pi f^{''}(x)g''(x)\,dx \\
&= \int_0^\pi f^{''}(x)g''(x)\,dx = f'g''\big|_0^\pi - \int_0^\pi f^{'}(x)g'''(x)\,dx \qquad (79)\\
-\int_0^\pi f^{'}(x)g'''(x)\,dx &= fg'''\big|_0^\pi + \int_0^\pi f(x)g^{(IV)}(x)\,dx = \langle f, Lg\rangle
\end{aligned}$$

therefore L is indeed formally self-adjoint on D.

Note also that L is positive semidefinite, so the eigenvalues λ are nonnegative. Indeed we have from (78) that

$$\langle Lf, f\rangle = \int_0^\pi |f^{''}(x)|^2\,dx \geq 0$$

and $\langle Lf, f\rangle = 0$ implies $f^{''}(x) = 0$ hence $f(x) = a + bx \in D$.

To calculate the eigenfunctions we find the general solution of the ODE.

For $\lambda \neq 0$ denote $\nu = \lambda^{1/4}$ and then

$$X(x) = c_1\cos(\nu x) + c_2\sin(\nu x) + c_3e^{\nu x} + c_4e^{-\nu x} \qquad (80)$$

and for $\lambda = 0$

$$X(x) = d_1 + d_2 x + d_3 x^2 + d_4 x^4$$

which belongs to D if $d_3 = d_4 = 0$. Hence the eigenspace corresponding to $\lambda = 0$ is two-dimensional, consisting of linear functions $X_0(x) = d_1 + d_2 x$.

Imposing the boundary condition in (79) we obtain a linear system of four equations for the four constants $c_1, \ldots, c_4$. The condition that this system has a nontrivial solution is that its determinant be zero. Calculation of this determinant yields the condition

$$\cosh \nu\pi \, \cos \nu\pi = 1 \tag{81}$$

which determines λ. However, (80) is a transcendental equation for ν (we cannot solve it explicitly). We can deduct that there is a sequence of solutions ν tending to $+\infty$ in the following way. Rewrite (80) as

$$\cosh x = \frac{1}{\cos x}, \qquad (x = \nu\pi) \tag{82}$$

The function $\cosh x$ is increasing and greater than 1 for $x > 0$. The graph of the right-hand side of (81) has vertical asymptotes. On intervals $(-\pi/2 + 2n\pi, 2n\pi]$ (n integer) it decreases from $+\infty$ to 1, and on intervals $[2n\pi, \pi/2 + 2n\pi)$ it increases from 1 to $+\infty$. Therefore, in each of these intervals there is a solution of equation (81). On the other intervals $1/\cos x$ is negative and equation (81) has no solutions.

A sequence of nonnegative eigenvalues λ_n do exist and $\lambda_n \to \infty$.The Completeness Theorem applies and the eigenfunctions are complete in $L^2[0\pi]$.

The equation is then solved using specified initial conditions.

12 An introduction to the Fourier transform

12.1 The space $L^2(\mathbb{R})$

Consider any function in the vector space of continuous functions which are zero outside a closed interval:

$$C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \,|\, f \text{continuous}, \ f(x) = 0 \text{ for all } x \notin [a, b] \text{ for some } a < b\}$$

(they are called *continuous functions with compact support*). If $f \in C_0$ vanished outside some $[a, b]$ then clearly it has finite $L^2(\mathbb{R})$-norm since

$$\|f\|^2 = \int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \int_a^b |f(x)|^2 \, dx < \infty$$

The space $L^2(\mathbb{R})$ is defined as the completion of C_0 with respect to the $L^2(\mathbb{R})$-norm

$$\|f\| = \left(\int_{-\infty}^{+\infty} |f(x)|^2 \, dx \right)^{1/2}$$

and it is a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} \overline{f(x)} g(x)\, dx$$

(which is finite by the Cauchy-Schwartz inequality).

Remark *If* $\int_{-\infty}^{+\infty} |f(x)|^2\, dx < \infty$ *then necessarily* $\lim_{x\to\pm\infty} f(x) = 0$.

The Hilbert space $L^2(\mathbb{R})$ has many features common to $L^2[a, b]$; for example, if two functions differ at only a number of points (finitely many, or countably many) are considered equal in L^2.

One novel feature is that, for a function, even continuous on $\mathbb{R}$, to belong to $L^2(\mathbb{R})$, this function needs *to decay to zero fast enough for* $x \to \pm\infty$ (so that the improper integral converges).

For example, consider functions decaying like a power, say $f(x) = 1/(1 + |x|^a)$. For which a is such a function in $L^2(\mathbb{R})$?

For large x, $f(x) \sim x^{-a}$ and $|f(x)|^2 \sim x^{-2a}$ which integrated gives a multiple of x^{1-2a}. The improper integral converges if $1 - 2a < 0$, therefore for $a > 1/2$.

Other examples of functions belonging to $L^2(\mathbb{R})$ are $e^{-|x|},\ e^{-x^2}$.

12.2 The Fourier Transform

For $f \in C_0$ its Fourier transform, $\mathcal{F}f$, is defined as

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\, dx \tag{83}$$

It can be shown that $\|\hat{f}\| = \|f\|$ (Parseval's identity) and that the Fourier transform can be extended as a unitary operator from $L^2(\mathbb{R})$ to itself, and that its inverse is:

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \hat{f}(\xi)\, d\xi \tag{84}$$

Note the similarity with Fourier series, only we have an integral instead of a series: recall $f = \sum_n e^{inx} f_n$.

Note also the similarity with matrix multiplication: if $U_{\xi,x} = e^{-i\xi x}$ and $f = f_x$ then (82) is like $\sum_x U_{\xi,x} f_x$ while the inverse of U, which equals its adjoint, is $U^*_{x,\xi} = \overline{U_{\xi,x}} = e^{i\xi x}$.

Remark. Some books define the Fourier transform (82) without the prefactor $\frac{1}{\sqrt{2\pi}}$. In this case, the transform is no longer a unitary operator, and the inverse (83) must have the prefactor $\frac{1}{2\pi}$.

12.3 The Fourier transform diagonalizes the differentiation operator.

Indeed, consider the linear operator $\frac{d}{dx}$ in $L^2(\mathbb{R})$, defined on the domain

$$D = C_0^1 \equiv \{f \in C_0 \,|\, f' \in C_0\}$$

Just like for finite intervals, the space C_0^1 is dense in $L^2(\mathbb{R})$.

For $f, g \in C_0^1$ we have

$$\langle \frac{d}{dx} f, g\rangle = \int_{-\infty}^{\infty} \overline{f'(x)} g(x)\, dx = \overline{f(x)} g(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \overline{f(x)} g'(x)\, dx = -\langle f, \frac{d}{dx} g\rangle$$

and therefore $\frac{d}{dx}$ is skew-symmetric (hence its eigenvalues - if any!- are purely imaginary).

We can see that $e^{ix\xi}$ (for $\xi \in \mathbb{R}$) are eigenfunctions of $\frac{d}{dx}$. However they are *generalized eigenfunctions* , since they do not belong to the Hilbert space $L^2(\mathbb{R})$, and we have expansions as integrals (82) rather than series. Indeed,

$$\frac{d}{dx} e^{ix\xi} = i\xi e^{ix\xi}$$

Moreover, integrating by parts we find that

$$(\mathcal{F} \frac{d}{dx} f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f'(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\xi e^{-ix\xi} f(x)\, dx = i\xi (\mathcal{F} f)(\xi)$$

or

$$\widehat{\frac{df}{dx}} = i\xi \hat{f} \tag{85}$$

or $\mathcal{F} \frac{d}{dx} \mathcal{F}^{-1}$ is the operator of multiplication by $i\xi$ - hence it is diagonal!.

12.4 An example

Problem: solve the initial value problem for the heat equation on the line

$$u_t = \alpha u_{xx}, \quad \text{for } x \in \mathbb{R},\ t > 0 \tag{86}$$

with

$$u(x, 0) = u_0(x) \in L^2(\mathbb{R}) \tag{87}$$

Take the Fourier transform in x in the heat equation: denoting

$$\hat{u}(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} u(x, t)\, dx$$

we obtain

$$\hat{u}_t = \alpha \widehat{u_{xx}}$$

which gives using (84)

$$\hat{u}_t = -\xi^2 \alpha \widehat{u}$$

which is an ODE in t whose general solution is

$$labelhatusol\hat{u}(\xi, t) = F(\xi)e^{-\xi^2 \alpha t} \tag{88}$$

Using the initial condition (86) we see that we must have $F(\xi) = \widehat{u_0}(\xi)$ and taking the inverse Fourier transform in (**??**) we obtain the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi - \xi^2 \alpha t} \widehat{u_0}(\xi)\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi - \xi^2 \alpha t} \int_{-\infty}^{\infty} e^{-iy\xi} u_0(y)\, dy\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} K(x,y,t)\, u_0(y)\, dy$$

where

$$K(x,y,t) = \int_{-\infty}^{\infty} e^{ix\xi - \xi^2 \alpha t - iy\xi}\, d\xi$$

which is easily calculated by completing the squares:

$$\xi^2 \alpha t - i(x-y)\xi = \left(\xi \sqrt{\alpha}\sqrt{t} - i\frac{x-y}{2\sqrt{\alpha}\sqrt{t}} \right)^2 + \frac{(x-y)^2}{4\alpha t}$$

therefore

$$K(x,y,t) = \frac{1}{\sqrt{4\pi\alpha t}}\, e^{-\frac{(x-y)^2}{4\alpha t}}$$

and therefore the solution to (85), (86) is

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\alpha t}} u_0(y)\, dy$$

13 Appendix

13.1 Basic facts about vector spaces

Vector spaces are modeled after the familiar vectors in the line, plane, space etc. abstractly written as $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3, \ldots, \mathbb{R}^n, \ldots$; these are vector spaces over the scalars $\mathbb{R}$. It turned out that there is a great advantage to allow for complex coordinates, and then we may also look at $\mathbb{C}$, $\mathbb{C}^2$, $\mathbb{C}^3, \ldots, \mathbb{C}^n, \ldots$; these are vector spaces over the scalars $\mathbb{C}$.

In general:

Definition 3. *A* vector space *over the scalar field F is a set V endowed with two operations, one between vectors: if $x, y \in V$ then $x + y \in V$, and one between scalars and vectors: if $c \in F$ and $x \in V$ then $cx \in V$ having the following properties:*
- commutativity of addition: $x + y = y + x$

- *associativity of addition:* $x + (y + z) = (x + y) + z$
- *existence of zero: there is an element* $0 \in V$ *so that* $x + 0 = x$ *for all* $x \in V$
- *existence of the opposite: for any* $x \in V$ *there is an opposite, denoted by* $-x$, *so that* $x + (-x) = 0$
- *distributivity of scalar multiplication with respect to vector addition:* $c(x+y) = cx + cy$
- *distributivity of scalar multiplication with respect to field addition:* $(c + d)x = cx + dx$
- *compatibility of scalar multiplication with field multiplication:* $c(dx) = (cd)x$
- *identity element of scalar multiplication:* $1x = x$.

Some familiar definitions are reformulated below in a way that allows us to tackle infinite dimensions too.

Definition 4. *A set of vectors* $S \subset V$ *is called* linearly independent *if whenever, for some* $x_1, x_2, \ldots, x_n \in S$ *(for some* n*) there are scalars* $c_1, c_2, \ldots, c_n \in F$ *so that* $c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0$ *then this necessarily implies that all the scalars are zero:* $c_1 = c_2 = \ldots = c_n = 0$.

Note that the zero vector can never belong to in a linearly independent set.

Definition 5. *Given any set of vectors* $S \subset V$ the span of S *is the set*

$$Sp(S) = \{c_1x_1 + c_2x_2 + \ldots + c_nx_n \,|\, x_j \in S,\, c_j \in F,\, n = 1, 2, 3 \ldots\}$$

Note that $Sp(S)$ forms a vector space, included in V; it is a *subspace* of V.

Definition 6. *A set of vectors* $S \subset V$ is a basis of V *if it is linearly independent and its span equals* V.

Theorem 7. *Any vector space has a basis. Moreover, all the basis of* V *have the same cardinality, which is called the dimension of* V.

By the cardinality of a set we usually mean "the number of elements". However, if we allow for infinite dimensions, the notion of "cardinality" helps us distinguish between different types of "infinities". For example, the positive integers form an infinite set, and so do the real numbers; we somehow feel that we should say there are "fewer" positive integers than reals. We call the cardinality of the positive integers *countable*.

Please note that the integers $\mathbb{Z}$ are also countable (we can "count" them: $0, 1, -1, 2, -2, 3, -3, \ldots$), pairs of positive integers $\mathbb{Z}_+^2$ are also countable (count: $(1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (1, 3), \ldots$), and so are $\mathbb{Z}^2$, and the set of rational numbers $\mathbb{Q}$ (rational numbers are ratios of integers m/n). It can be proved that the reals are not countable.

Notations The positive integers are denoted $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ and the natural numbers are denoted $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$. However: some authors do not include 0 in the natural numbers, so when you use a book make sure you know the convention used there.

Examples

1. $\mathbb{R}^n = \{(x_1, \dots x_n) \,|\, x_j \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$ of dimension n, with a basis consisting of the vectors
$e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0 \dots, 0), \dots, e_n = (0, 0, \dots 0, 1)$
Remark: from now on we will prefer to list horizontally the components of vectors.

Any $x = (x_1, \dots x_n)$ can be written as a linear combination of them

$$x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

and there is an inner product:

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$$

and any vector has a norm

$$\|x\|^2 = x_1^1 + \dots + x_n^2$$

2. $\mathbb{R}^{\mathbb{Z}_+} = \{x = (x_1, x_2, x_3, \dots) \,|\, x_j \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$. By analogy with $\mathbb{R}^n$ we can formulate the following *wish list*.

We would like to say that a norm is defined by

$$\|x\|^2 = x_1^1 + \dots + x_n^2 + \dots$$

that an inner product of two sequences is

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n + \dots$$

and that a basis consists of the vectors

$$e_1 = (1, 0, 0, 0, \dots),\ e_2 = (0, 1, 0, 0, 0 \dots),\ e_3 = (0, 0, 1, 0, \dots), \dots \tag{89}$$

since any $x = (x_1, x_2, x_3, \dots)$ can be written as an (infinite) linear combination

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 + \dots = \sum_{n=1}^{\infty} x_n\, e_n \tag{90}$$

But all these are not quite correct, because we have series rather than finite sums. It looks like we should accept series as expansions, and to restrict the sequences we work with in order to get a nice extension to infinite dimensions!

3. Similarly, $\mathbb{C}^n = \{(z_1, \dots z_n) \,|\, z_j \in \mathbb{C}\}$ is a vector space over $\mathbb{C}$ of dimension n, with a basis consisting of the vectors
$(1, 0, \dots, 0)$, $(0, 1, 0 \dots, 0), \dots, (0, 0, \dots 0, 1)$.

4. $\mathbb{C}^{\mathbb{Z}_+} = \{z = (z_1, z_2, \dots z_n, \dots) \,|\, z_j \in \mathbb{C}\}$ is a vector space over $\mathbb{C}$ and any z could be thought as the infinite sum $z_1 e_1 + z_2 e_2 + z_3 e_3 + \dots$ where e_n are given by (88) - but again, this is not a basis in the sense of the definition for vector spaces).

5. The set of polynomials of degree at most n, with coefficients in F (which is $\mathbb{R}$ or $\mathbb{C}$)

$$\mathcal{P}_n = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \,|\, a_j \in F\}$$

is a vector space over the field of scalars F, of dimension $n+1$, and a basis is $1,\, t\,, t^2, \ldots\, t^n$.

6. The set of all polynomials with coefficients in F (which is $\mathbb{R}$ or $\mathbb{C}$)

$$\mathcal{P} = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \,|\, a_j \in F, n \in \mathbb{N}\}$$

is a vector space over the field of scalars F, and has the countable basis $1,\ t,\ t^2,\ t^3 \ldots$.

7. The set of all functions continuous on a closed interval (it could also be open, or extending to ∞):

$$C[a,b] = \{f : [a,b] \to F \,|\, f \text{continuous}\}$$

8. The set of all functions f with absolute value $|f|$ integrable on $[a,b]$:

$$L^1[a,b] = \{f : [a,b] \to F \,|\, |f| \text{integrable}\}$$

Warning: I did not specify what "integrable" means. $L^1[a,b]$ is a bit more complicated, but for practical purposes this is good enough for now. (We will comment more later.)

Note that $\mathcal{P} \subset C[a,b] \subset L^1[a,b]$.

13.2 Inner product

Vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$ have an interesting operation:

Definition 7. *An* **inner product** *on vector space V over F (=$\mathbb{R}$ or $\mathbb{C}$) is an operation which associate to two vectors $x, y \in V$ a scalar $\langle x, y\rangle \in F$ that satisfies the following properties:*
- it is conjugate symmetric: $\langle x, y\rangle = \overline{\langle y, x\rangle}$
- it is linear in the second argument: $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ and $\langle x, cy\rangle = c\langle x, y\rangle$
- its is positive definite: $\langle x, x\rangle \geq 0$ with equality only for $x = 0$.

Note that conjugate symmetry combined with linearity implies that $\langle ., .\rangle$ is conjugate linear in the first variable.

Note that for $F = \mathbb{R}$ an inner product is symmetric and linear in the first argument too.

Please keep in mind that most mathematical books use inner product linear in the first variable, and conjugate linear in the second one. You should make sure you know the convention used by each author.

Definition 8. *A vector space V equipped with an inner product $(V, \langle ., .\rangle)$ is called an* **inner product space***.*

Examples

On $\mathbb{C}^n$ the most used inner product is $\langle x,y\rangle = \sum_{j=1}^{n} \overline{x_j} y_j$.

We may wish to introduce a similar inner product on $\mathbb{C}^{\mathbb{Z}_+}$: $\langle x,y\rangle = \sum_{j=1}^{\infty} \overline{x_j} y_j$. The problem is that the series may not converge!

On the space of polynomials $\mathcal{P}$ or on continuous functions $C[a,b]$ we can introduce the inner product

$$\langle f,g\rangle = \int_a^b \overline{f(x)}\, g(x)\, dx$$

or more generally, using a weight (which is a positive function $w(x)$),

$$\langle f,g\rangle_w = \int_a^b \overline{f(x)}\, g(x)\, w(x) dx$$

We may wish to introduce a similar inner product on $L^1[a,b]$, only the integral may not converge. For example, $f(x) = 1/\sqrt{x}$ is integrable on $[0,1]$, but $f(x)^2$ is not.

13.3 Norm

The inner product defines a length by $\|x\| = \sqrt{\langle x,x\rangle}$. This is a norm, in the following sense:

Definition 9. *Given a vector space V, a* **norm** *is a function on V so that:*
- it is positive definite: $\|x\| \geq 0$ and $\|x\| = 0$ only for $x = 0$
- it is positive homogeneous: $\|cx\| = |c|\,\|x\|$ for all $c \in F$ and $x \in V$
- satisfies the **the triangle inequality** *(i.e. it is sub-additive):*

$$\|x+y\| \leq \|x\| + \|y\|$$

Definition 10. *A vector space V equipped with a norm $(V, \|.\|)$ is called a* **normed space**.

An inner product space is, in particular a normed space (the first two properties of the norm are immediate, the triangle inequality is a geometric property in finite dimensions and requires a proof in infinite dimensions).

There are some very useful normed spaces (of functions), which are not inner product spaces (more about that later).

13.4 The space ℓ^2

Let us consider again $\mathbb{C}^{\mathbb{Z}_+}$, which appears to be the most straightforward way to go from finite dimension to an infinite one. To do linear algebra we need an inner product, or at least a norm. We must then restrict to the "vectors" which do have a norm, in the sense that the series $\|z\|^2 = \sum_{n=1}^{\infty} |z_n|^2$ converges.

Define the vector space ℓ^2 by

$$\ell^2 = \{z = (z_1, z_2, z_3, \ldots) \in \mathbb{C}^{\mathbb{Z}_+} \mid \sum_{n=1}^{\infty} |z_n|^2 < \infty\}$$

On ℓ^2 we therefore have a norm: $\|z\| = \left(\sum_{n=1}^{\infty} |z_n|^2\right)^{1/2}$.

Examples.

1. The constant sequence with $z_n = c$ for all n is not in ℓ^2 unless $c = 0$.
2. The sequences with $z_n = \frac{1}{n^a}$ with $a \in \mathbb{R}$ is in ℓ^2 only for $a > 1/2$ (why?).
3. For complex powers of n, recall that they are defined as

$$n^{a+ib} = e^{(a+ib)\ln n} = e^{a \ln n} e^{ib \ln n} = n^a \, e^{ib \ln n}$$

therefore if $z_n = \frac{1}{n^{a+ib}}$ the sequence is in ℓ^2 only for $a > 1/2$.

Actually, on ℓ^2 the inner product converges as well, due to the following inequality, which is one of the most important and powerful tools in infinite dimensions (the triangle inequality is also fundamental):

Theorem 8. The Cauchy-Schwartz inequality

In an inner product space we have

$$\left| \langle x, y \rangle \right| \leq \|x\| \, \|y\|$$

Therefore, if $\|x\|$ and $\|y\|$ converge, then $\langle x, y \rangle$ converges and moreover, equality holds if and only if x, y are linearly dependent (which means $x = 0$ or $y = 0$ or $x = cy$)

Proof.

Intuitively: if $\|x\|$ and $\|y\|$ converge, then $Sp(x, y)$ is an inner product space which is two-dimensional at most, therefore the Cauchy-Schwartz inequality follows from the one in finite dimensions.

Here are detailed rigorous arguments for the case of ℓ^2, with a review of the main concepts on convergent series.

Recall: a series $\sum_{n=1}^{\infty} a_n$ is said to converge to A, and we write $\sum_{n=1}^{\infty} a_n = A$ if the sequence formed by its partial sums $S_N = \sum_{n=1}^{N} a_n$ converges to A as $N \to \infty$.

Recall: a series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Recall: absolute convergence implies convergence.

Recall: a series with positive terms either converges or has the limit $+\infty$. For such a series, say $\sum_{n=1}^{\infty} |a_n|$, it is customary to write $\sum_{n=1}^{\infty} |a_n| < \infty$ to express that it converges.

Take the partial sums of $\langle x, y \rangle$, use the triangle inequality in dimension N, then Cauchy-Schwartz in dimension N (which follows from $u \cdot v = \|u\| \, \|v\| \cos\theta$):

$$\left| \sum_{n=1}^{N} \overline{x_n} \, y_n \right| \leq \sum_{n=1}^{N} \left| \overline{x_n} \, y_n \right| = \sum_{n=1}^{N} |x_n| \, |y_n| \leq \left(\sum_{n=1}^{N} |x_n|^2 \right)^{1/2} \left(\sum_{n=1}^{N} |y_n|^2 \right)^{1/2}$$

Taking the limit $N \to \infty$ the convergence of the $\langle x, y \rangle$ series follows if $\|x\|$ and $\|y\|$ converge.

The argument showing when equality holds is not given here, as it is in accordance with what happens in finite dimensions. $\square$

We obtained that ℓ^2 is an inner product space, by the Cauchy-Schwartz inequality.

The vectors e_n of (88) do belong to ℓ^2, and they do form an *orthonormal set*: $e_n \perp e_k$ for $n \neq k$ (since $\langle e_n, e_k \rangle = 0$) and $\|e_n\| = 1$).

13.5 Metric Spaces

Now we would like to make sense of the expansion (89). For this, we need to state what we mean by convergence in ℓ^2.

We can do that using the usual definition of convergence (in $\mathbb{R}$) by replacing the distance between two vectors x, y by $d(x, y) = \|x - y\|$. This distance is a metric, in the following sense:

Definition 11. *A distance (or a metric) on a set M is a function $d(x, y)$ for $x, y \in M$ with the following properties:*
it is nonnegative: $d(x, y) \geq 0$
it separates the points: $d(x, y) = 0$ if and only if $x = y$
it is symmetric: $d(x, y) = d(y, x)$
it satisfies the triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

(Two of conditions follow from the other, but it is better to have them all listed.)

Definition 12. *A metric space (M, d) is a set M equipped with a distance d.*

Note that any normed space is a metric space by defining the distance d:

$$d(x, y) = \|x - y\| \tag{91}$$

But there are many interesting metric spaces which are not normed (they may not even be vector spaces!). For example, we can define distances on a sphere (or on the surface of the Earth!) by measuring the (shortest) distance between two points on a large circle joining them.

Once we have a distance, we can define convergence of sequences:

Definition 13. *Consider a metric space (M, d).*
We say that a sequence $s_1, s_2, s_3 \ldots \in M$ converges to $L \in M$ if for any $\varepsilon > 0$ there is an N (N depends on ε) so that $d(s_n, L) < \varepsilon$ for all $n \geq N$.

Note that we can use this definition for convergence in normed spaces using the distance (90).

The series expansion (89) of any $x \in \ell^2$ converges (why?). We now have a satisfactory theory of ℓ^2.

14 Completeness

There is one more property that is essential for calculus: there need to exist enough limits.

The spaces $\mathbb{R}^n$, $\mathbb{C}^n$, ℓ^2 all have this property (inherited from $\mathbb{R}$). However, the space of polynomials $\mathcal{P}$ and $C[a,b]$ as inner product spaces with $\langle f, g\rangle = \int \overline{f} g$ do not have this property.

14.1 Complete spaces

In the case of $\mathbb{R}$ this special property can be intuitively formulated as: if a sequence of real numbers a_n tends to "pile up" then it is convergent. Here is the rigorous formulation:

Definition 14. *The sequence $\{a_n\}_{n\in\mathbb{N}}$ is called a* Cauchy sequence *if for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ (N depending on ε) so that*

$$|a_n - a_m| < \varepsilon \quad \textit{for all } n, m > N$$

Theorem 9. *Any Cauchy sequence of real numbers is convergent.*

(This is a fundamental property of real numbers, at the essence of what they are.)

We can define Cauchy sequences in a metric space very similarly:

Definition 15. *Let (M, d) be a metric space.*

The sequence $\{a_n\}_{n\in\mathbb{N}} \subset M$ is called a Cauchy sequence *if for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ (N depending on ε) so that*

$$d(a_n, a_m) < \varepsilon \quad \textit{for all } n, m > N$$

We would like to work in metric spaces where every Cauchy sequence is convergent. However, this is not always the case:

Definition 16. *A metric space (M, d) is called* complete *if every Cauchy sequence is convergent in M.*

(Note that the limit of the Cauchy sequence *must belong to M*.)

In the particular case of normed spaces (when the distance is given by the norm):

Definition 17. *A normed space* $(V, \|\,.\,\|)$ *which is* complete *is called a Banach*[2] *space.*

In the even more special case of inner product spaces (when the norm is given by an inner product):

Definition 18. *An inner product space* $(V, \langle\,.\,\rangle)$ *which is* complete *is called a Hilbert*[3] *space.*

14.2 Examples

These are very important examples (some proofs are needed but not given here).

1. The usual $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces.

2. The space ℓ^2 of sequences:

$$\ell^2 \equiv \ell^2(\mathbb{Z}_+) = \{x = (x_1, x_2, x_3, \ldots) \,|\, x_n \in \mathbb{C},\ \sum_{n=1}^{\infty} |x_n|^2\ < \infty\}$$

endowed with the $\boldsymbol{\ell^2}$ **inner product**

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \overline{x_n} y_n$$

is a Hilbert space.

For example, the sequence $x = (x_1, x_2, x_3, \ldots)$ with $x_n = 1/n$ belongs to ℓ^2 and so do sequences with $x_n = ca^n$ if $|a| < 1$.

A variation of this space is that of bilateral sequences:

$$\ell^2(\mathbb{Z}) = \{x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2 \ldots) \,|\, x_n \in \mathbb{C},\ \sum_{n=-\infty}^{\infty} |x_n|^2\ < \infty\}$$

with *the* ℓ^2 *inner product* $\langle x, y\rangle = \sum_{n=-\infty}^{\infty} \overline{x_n} y_n$ is a Hilbert space.

Recall that a bilateral series $\sum_{n=-\infty}^{\infty} a_n$ is called convergent if both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=-\infty}^{0} a_n$ are convergent.

3. The space $C[a, b]$ of continuous function on the interval $[a, b]$, endowed with the $\mathbf{L^2}$ **inner product**

$$\langle f, g\rangle = \int_a^b \overline{f(t)}\, g(t)\, dt$$

[2]Stefan Banach (1892-1945) was a Polish mathematician, founder of modern functional analysis - a domain of mathematics these lectures belong to.

[3]David Hilbert (1862-1943) was a German mathematician, recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries. He discovered and developed a broad range of fundamental ideas in many areas, including the theory of Hilbert spaces, one of the foundations of functional analysis.

is *not* a Hilbert space, since *it is not complete*.

For example, take the following approximations of step functions

$$f_n(t) = \begin{cases} 0 & \text{if } t \in [0,1] \\ n\,(t-1) & \text{if } 1 < t < 1 + \frac{1}{n} \\ 1 & \text{if } t \in [1 + \frac{1}{n}, 2] \end{cases}$$

The f_n are continuous on $[0,2]$ (the middle line in the definition of f_n represents a segment joining the two edges of the step). The sequence $\{f_n\}_n$ is Cauchy in the L^2 norm: (say $n < m$)

$$\|f_n - f_m\|^2 = \int_0^2 |f_n(t) - f_m(t)|^2\, dt \;=\; \int_1^{1+\frac{1}{n}} |f_n(t) - f_m(t)|^2\, dt$$

$$= \int_1^{1+\frac{1}{m}} [n\,(t-1) - m\,(t-1)]^2\; dt \;+\; \int_{1+\frac{1}{m}}^{1+\frac{1}{n}} [n\,(t-1) - 1]^2\; dt \;=$$

$$= 1/3\,\frac{(n-m)^2}{m^3} - 1/3\,\frac{(n-m)^3}{m^3 n} = 1/3\,\frac{(n-m)^2}{m^2 n} < \frac{1}{3n} \to 0$$

However, f_n is not L^2 convergent in $C[0,2]$. Indeed, in fact f_n converges to the step function

$$f(t) = \begin{cases} 0 & \text{if } t \in [0,1) \\ 1 & \text{if } t \in [1,2] \end{cases}$$

because

$$\|f_n - f\|^2 = \int_0^2 |f_n(t) - f(t)|^2\, dt \;=\; \int_1^{1+\frac{1}{n}} [n\,(t-1) - 1]^2\; dt \;=\; \frac{1}{3n} \to 0$$

therefore the L^2 limit of f_n is f, a function that does not belong to $C[0,2]$.

14.3 Completion and closure

The last example suggests that if a metric space of interest is not closed, then one can add to that space all the possible limits of the Cauchy sequences, and then we obtain a closed space.

This procedure is called "closure", and the closure of a metric space M is denoted by $\overline{M}$.

14.4 The Hilbert space $L^2[a,b]$

The closure of $C[a,b]$ *in the* L^2 *norm* (the closure depends on the metric!) is a space denoted by $L^2[a,b]$, *the space of square integrable*[(4)] *functions*:

$$L^2[a,b] = \{ f : [a,b] \to \mathbb{C}(\text{or } \mathbb{R}) \;\big|\; |f|^2 \text{ is integrable on } [a,b] \,\}$$

[(4)] These are Lebesgue integrable functions. They are quite close to the familiar Riemann integrable functions, but the advantage is that the Lebesgue integrability behaves better when taking limits.

or, for short,

$$L^2[a,b] = \{ f : [a,b] \to \mathbb{C} \,\big|\, \int_a^b |f(t)|^2 \, dt \,<\, \infty \,\} \tag{92}$$

This notation is very suggestive: for practical purposes, if you have an f so that you can integrate $|f|^2$ (as a proper or improper integral), and the result is a finite number, then $f \in L^2$.

Examples of functions in L^2: continuous functions, functions with jump discontinuities (a finite number of jumps, or even countably many!), some functions which go to infinity: for example $x^{-1/4} \in L^2[0,1]$ because $\int_0^1 (x^{-1/4})^2 dx = \int_0^1 x^{-1/2} dx = 2x^{1/2}\big|_0^1 = 2 < \infty$.

A wrinkle: in $L^2[a,b]$, if two functions differ by their values at only a finite number of points (or even on a countable set) they are considered equal. For example, the functions f_1 and f_2 below are equal:

$$f_1(t) = \begin{cases} 0 & \text{if } t \in [0,1] \\ 1 & \text{if } t \in (1,2] \end{cases} \qquad f_2(t) = \begin{cases} 0 & \text{if } t \in [0,1) \\ 1 & \text{if } t \in [1,2] \end{cases} \qquad f_1 = f_2 \text{ in } L^2[0,2]$$

as indeed the L^2-norm $\|f_1 - f_2\| = 0$.

Therefore $L^2[a,b] = L^2(a,b)$.

Note the similarity of the definition (91) of L^2 with the definition of ℓ^2.

Note that by the Cauchy-Schwartz inequality, the L^2 inner product of two function in $L^2[a,b]$ is finite. Therefore the space $L^2[a,b]$ is a Hilbert space.

The space L^2 is also used on infinite intervals, like $L^2[a,\infty)$, and $L^2(\mathbb{R})$.

14.4.1 An important variation of the L^2 space

Instead of using the usual element of length dt we can use an element of "weighted" length $w(t)dt$; one physical interpretation is that if $[a,b]$ represents a wire of variable density $w(t)$ then the element of mass on the wire is $w(t)dt$. If $w(t)$ is a positive function define

$$L^2([a,b], w(t)dt) = \{ f : [a,b] \to \mathbb{C} \,\big|\, \int_a^b |f(t)|^2 \, w(t)dt \,<\, \infty \,\}$$

with the inner product given by

$$\langle f, g \rangle_w = \int_a^b \overline{f(t)} \, g(t) \, w(t)dt$$

Note that $f \in L^2([a,b], w(t)dt)$ if and only if $\frac{f}{\sqrt{w}} \in L^2[a,b]$.

14.5 The Banach space $C[a, b]$

Recall: a function continuous on a closed interval does have an absolute maximum and an absolute minimum.

The space of continuous function on $[a, b]$ is closed with respect to **the sup norm** $\|.\|_\infty$:

$$\|f\| = \|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$$

In other words, completeness of $C[a, b]$ in the sup-norm means that: if f_n are continuous on $[a, b]$ and if $f_n \to f$ in the sense that $\lim_{n\to\infty} \sup_{x\in[a,b]} |f_n(x) - f(x)| = 0$ then f is continuous.

14.6 The Banach spaces L^p

The space

$$Ł^p[a, b] = \{f : [a, b] \to \mathbb{C}(\text{or } \mathbb{R}) \,\big|\, \int_a^b |f(t)|^p \, dt \, < \, \infty \}$$

is complete in the L^p norm $\|f\|_p = (\int_a^b |f|^p)^{1/p}$.

L^p are called the Lebesgue[5] spaces (only L^2 is a Hilbert space).

14.7 Closed sets, dense sets

Definition 19. *Let (M, d) be a closed metric space.*

A subset $F \subset M$ is called closed *if it contains the limits of the Cauchy sequences in F: if $f_n \in F$ and $f_n \to f$ then $f \in F$.*

Examples: for $M = \mathbb{R}$ the intervals (a, b), $(a, +\infty)$ are not closed sets, but the intervals $[a, b]$, $[a, +\infty)$ are closed. For $M = \mathbb{R}^2$ the disk $x^2 + y^2 \le 1$ is closed, but the disk $x^2 + y^2 < 1$ is not closed, and neither is the punctured disk $0 < x^2 + y^2 \le 1$.

Definition 20. *Given a closed metric space (M, d) and a set $S \subset M$ we call the set $\overline{S}$* the closure of S in M *if $\overline{S}$ contains all the limits of the Cauchy sequences sequences $(f_n)_n \subset S$.*

Note that $S \subset \overline{S}$ (since if $f \in S$ we can take the trivial Cauchy sequence constantly equal to f, $f_n = f$ for all n.

Examples: for $M = \mathbb{R}$ the closure of the open interval (a, b) is the closed interval $[a, b]$. For $M = \mathbb{R}^2$ the closure of the (open) disk $x^2 + y^2 < 1$, and of the punctured disk $0 < x^2 + y^2 \le 1$ is the closed disk $x^2 + y^2 \le 1$.

[5] Henri Lebesgue (1875-1941) was a French mathematician most famous for Lebesgue's theory of integration.

Definition 21. *Given a closed metric space (M, d) and a set $S \subset M$ we say that* a set S is dense in M if $\overline{S} = M$.

It is worth repeating: S is dense in M means that any $f \in M$ can be approximated by elements of S: there exists $f_n \in S$ so that $\lim_{n\to\infty} f_n = f$.

Examples: for $M = \mathbb{R}$ the rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$ (why?). For $M = \ell^2$ the sequences that terminate[(6)] are dense in ℓ^2.

Dense sets have important practical consequences: when we need to establish a property of M that is preserved when taking limits (e.g. equalities or inequalities), then we can prove the property on S and then, by taking limits, we find it on M.

14.8 Sets dense in the Hilbert space L^2

$C[a, b]$ is dense (in the L^2 norm) in $L^2[a, b]$ (by our construction of L^2).

In fact, we can assume functions as smooth as we wish, and still obtain dense spaces: the smaller space $C^1[a, b]$, of functions which have a continuous derivative, is also dense in $L^2[a, b]$, and so is $C^r[a, b]$, functions with r continuous derivatives, for any r (including $r = \infty$).

Even the smaller space $\mathcal{P}$ (of polynomials) is dense in $L^2[a, b]$.

Another type of (inner product sub)spaces dense in L^2 are those consisting of functions satisfying zero boundary conditions, for example

$$\{f \in C[a, b] \,|\, f(a) = 0\} \tag{93}$$

(Why: any function f in (92) can be approximated in the L^2-norm by functions in (92): consider the sequence of functions f_n which equal f for $x \geq a + \frac{1}{n}$ and whose graph is the segment joining $(a, 0)$ to $(a + \frac{1}{n}, f(a + \frac{1}{n}))$ for $a \leq x < a + \frac{1}{n}$. Then the L^2 norm of $f_n - f$ converges to zero, much like in the Example 3. of §14.2.)

Similarly,

$$C_0[a, b] = \{f \in C[a, b] \,|\, f(a) = f(b) = 0\}$$

is dense in $L^2[a, b]$, and so is the following very useful space

$$C_0^1[a, b] = \{f \,|\, f' \text{ continuous on } [a, b],\ f(a) = f(b) = 0,\ f'(a) = f'(b) = 0\}$$

14.9 Polynomials are dense in the Banach space $C[a, b]$

The space of polynomials is dense in $C[a, b]$ in the sup norm. (This is the Weierstrass' Theorem: any continuous function f on $[a, b]$ can be uniformly approximated by polynomials, in the sense that there is a sequence of polynomials p_n so that $\lim_{n\to\infty} \sup_{[a,b]} |f - p_n| = 0$.)

Please note that closure and density are relative to a norm. For example, $C[a, b]$ is not closed in the L^2 norm, but it is closed in the sup-norm.

(6) These are the sequences $x = (x_1, x_2, \dots)$ so that $x_n = 0$ for all n large enough.

15 Hilbert Spaces

In a Hilbert space we can do linear algebra (since it is a vector space), geometry (since we have lengths and angles) and calculus (since it is complete). And they are all combined when we write series expansions.

Recall:

Definition 22. *A Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.*

Recall the two fundamental examples: the space of sequences ℓ^2, and the space of square integrable function L^2.

15.1 When does a norm come from an inner product?

In every Hilbert space the **parallelogram identity** holds: for any $f, g \in H$

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2) \tag{94}$$

(in a parallelogram the sum of the squares of the sides equals the sum of the squares of the diagonals).

Relation (93) is proved by a direct calculation (expanding the norms in terms of the inner products and collecting the terms).

The remarkable fact is that, if the parallelogram identity holds in a Banach space, then its norm actually comes from an inner product, and can recover the inner product only in terms of the norm by
the polarization identity:
for complex spaces

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right)$$

and for real spaces

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right)$$

15.2 The inner product is continuous

This means that we can take limits inside the inner product:

Theorem 10. *If $f_n \in H$, with $f_n \to f$ then $\langle f_n, g\rangle \to \langle f, g\rangle$.*

Indeed: $|\langle f_n, g\rangle - \langle f, g\rangle| = |\langle f_n - f, g\rangle| \leq$ (by Cauchy-Schwartz) $\|f_n - f\|\,\|g\| \to 0$. (Recall: $f_n \to f$ means that $\|f_n - f\| \to 0$.)

15.3 Orthonormal bases

Consider a Hilbert space H.
Just like in linear algebra we define:

Definition 23. *If $\langle f, g\rangle = 0$ then $f, g \in H$ are called orthogonal ($f \perp g$).*

and

Definition 24. *A set $B \subset H$ is called orthonormal if all $f, g \in B$ are orthogonal ($f \perp g$) and unitary ($\|f\| = 1$).*

Note that as in linear algebra, an orthonormal set is a linearly independent set (why?).
And departing from linear algebra:

Definition 25. *A set $B \subset H$ is called an* **orthonormal basis** *for H if it is an orthonormal set and it is complete in the sense that the span of B is* dense *in H: $\overline{Sp(B)} = H$.*

Note that in the above B is not a basis in the sense of linear algebra (unless H is finite-dimensional). But like in linear algebra:

Theorem 11. *Any Hilbert space admits an orthonormal basis; furthermore, any two orthonormal bases of the same space have the same cardinality, called the Hilbert dimension of the space.*

From here on we will only consider Hilbert spaces which admit a (finite or) countable orthonormal basis.
These are the Hilbert spaces encountered in mechanics and electricity.
It can be proved that this condition is equivalent to the existence of a countable set S dense in H. This condition is often easier to check, and the property is called "H is separable". Application: $L^2[a, b]$ is separable (why?) therefore $L^2[a, b]$ has a countable orthonormal basis. Many physical problems are solved by finding special orthonormal basis of $L^2[a, b]$!
[For your amusement: it is quite easy to construct a Hilbert space with an uncountable basis, e.g. just like we took $\ell^2(\mathbb{Z}_+)$ we could take $\ell^2(\mathbb{R})$.]

We assume from now on that our Hilbert spaces H do have a countable orthonormal basis.
The following theorem shows that many of the properties of inner product vector spaces which are finite dimensional (think $\mathbb{R}^n$ or $\mathbb{C}^n$ with the standard basis) are very similar for Hilbert spaces; the main difference is that in infinite dimensions instead of sums we have series - which means sums followed by limits, see for example the infinite linear combination (94).

Theorem 12. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space with a countable basis.*
Let $u_1, \ldots, u_n, ..$ be an orthonormal basis.

The following hold.

(i) Let $c_1, \ldots, c_n, ..$ be scalars so that $(c_1, \ldots, c_n, ..) \in \ell^2$. Then the series

$$f = \sum_{n=1}^{\infty} c_n u_n \tag{95}$$

converges and its sum $f \in \mathcal{H}$.

Moreover

$$\|f\|^2 = \sum_{n=1}^{\infty} |c_n|^2$$

(ii) Conversely, every $f \in \mathcal{H}$ has an expansion (94). The scalars c_n satisfy $c_n = \langle u_n, f\rangle$ and are called **generalized Fourier coefficients** *of f.*

Therefore any $f \in \mathcal{H}$ can be written as a **generalized Fourier series***:*

$$f = \sum_{n=1}^{\infty} \langle u_n, f\rangle\, u_n \tag{96}$$

and **Parseval's identity** *holds:*

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle u_n, f\rangle|^2 \tag{97}$$

As a consequence, **Bessel's inequality** *holds:*

$$\|f\|^2 \geq \sum_{n \in J} |f_n|^2 \qquad \text{for any } J \subset \mathbb{Z}_+ \tag{98}$$

(iii) If $f, g \in \mathcal{H}$ then

$$\langle f, g\rangle = \sum_{n=1}^{\infty} \langle f, u_n\rangle \langle u_n, g\rangle \tag{99}$$

Note that (separable) Hilbert spaces are essentially ℓ^2, since given an orthonormal basis $u_1, u_2, \ldots$, the elements $f \in H$ can be identified with the sequence of their generalized Fourier coefficients $(c_1, c_2, c_3, \ldots)$ which, by (96), belongs to ℓ^2.

Remark. If $v_1, \ldots, v_n, \ldots$ form an orthogonal basis, but not an orthonormal basis (some v_n are not unit vectors) then one can produce an orthonormal basis by setting $u_n = v_n/\|v_n\|$, which used in formula (95) gives the expansion of f in terms of v_n as

$$f = \sum_{n=1}^{\infty} \frac{\langle v_n, f\rangle}{\|v_n\|^2}\, v_n \tag{100}$$

15.4 Generalized Fourier series in L^2

If $H = L^2[a,b]$ then (94) means that

$$\lim_{N\to\infty} \int_a^b \left|f(x) - \sum_{n=1}^{N} c_n u_n(x)\right|^2 dx = 0 \tag{101}$$

(we took the square of the norm rather than the norm).

Formula (100) is often expressed as

$$f \text{ equals } \sum_{n=1}^{\infty} c_n u_n \text{ in the least square sense, or } \textit{in mean square} \tag{102}$$

see also §15.8.

Only if $f(x)$ and $u_n(x)$ are smooth enough it is true that the series $\sum c_n u_n$ is point-wise convergent, meaning that

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x) \tag{103}$$

(precise conditions can be given) But this is **not** the case for every $f \in L^2$. In general, the series on the right side of (102) may not converge for all x, and even if it converges, it may not equal $f(x)$.

15.5 Example of orthonormal basis in $L^2[a,b]$: Fourier series

The functions $\sin x$, $\cos x$ have period 2π. The same is true for any linear combination of $\sin nx$, $\cos nx$: any **trigonometric polynomial**(7)

$$T(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(nx) + b_n \sin(nx)\right] \tag{104}$$

is a periodic function, period 2π.

What if instead of a finite sum in (103) we consider a series? Periodicity should survive taking limits (if the sequence of functions is not oscillating too wildly).

Exercise. Check that the functions $1, \sin(nx), \cos(nx)$, $(n = 1, 2, 3\ldots)$ form an orthogonal system in $L^2[-\pi,\pi]$. Then normalize these functions, to obtain an orthonormal system.

Moreover

(7)It is convenient to write the constant term as $a_0/2$ rather than a_0 so that formula (105) applies for all $n \geq 0$, and not need a separate formula for a_0.

Theorem 13. *The functions* $1, \sin(nx), \cos(nx), \ (n = 1, 2, 3 \dots)$ *form a basis for the real-valued Hilbert space* $L^2([-\pi, \pi], \mathbb{R})$

The proof of this important and deep theorem is not included here.

Is is clear that if we allow the scalars a_n, b_n to be complex numbers, then the same trigonometric monomials form a basis for the complex-valued Hilbert space $L^2([-\pi, \pi], \mathbb{C})$.

It follows that if $f(x) \in L^2[-\pi, \pi]$ then $f(x)$ can be approximated (in mean square) by the partial sums of a Fourier series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{105}$$

Exercise. Assuming that $f \in L^2[-\pi, \pi]$ has its Fourier series expansion (104) verify that the Fourier coefficients a_n and b_n are given by the formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \cos(nx) \, dx \ , \ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \sin(nx) \, dx \tag{106}$$

A more concise and compact formula can be written using Euler's formula $e^{inx} = \cos(nx) + i \sin(nx)$.

Exercise. Check that the functions $e^{inx}, \ n \in \mathbb{Z}$ form an orthogonal system in the complex valued functions $L^2[-\pi, \pi]$. Normalize them to obtain an orthonormal system.

An orthonormal basis for the complex Hilbert space$L^2[-\pi, \pi]$ is thus found, and any complex valued function in $L^2[-\pi, \pi]$ can be expanded in a Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}$$

with the Fourier coefficients given by

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

(why?) The series converges in square mean (i.e. in the L^2 norm), and, moreover, pointwise if $f(x)$ is "smooth enough" (precise mathematical formulations form the topic of one domain of mathematics, harmonic analysis).

Exercise. Show that the Fourier coefficients $a_n, b_n, \hat{f}_n$ are related by the formulas

$$a_n = \hat{f}_n + \hat{f}_{-n} \text{ for } n = 0, 1, 2, \dots, \ \ b_n = i(\hat{f}_n - \hat{f}_{-n}) \text{ for } n = 1, 2, \dots$$

Find the conditions on $\hat{f}_n$ equivalent to the fact that function $f(x)$ is real valued.

On a general interval $[a, b]$ the Fourier series is generated by expansion in terms of $exp(2\pi inx/(b-a))$ (with $n \in \mathbb{Z}$) as it is easily seen by a rescaling of x.

Exercise. Use a linear change of the variable x (i.e. setting $x = cx + d$ for suitable scalars c, d) to show that the Fourier series of a function $g \in L^2[a, b]$ has the form

$$g = \sum_{n=-\infty}^{\infty} \hat{g}_n e^{2\pi inx/(b-a)}$$

and find the formula that expresses $\hat{g}_n$ in terms of $g(x)$.

15.6 Other bases for L^2: orthogonal polynomials

Recall that polynomials are dense in $L^2[a, b]$. The fact that every function in L^2 can be approximated by polynomials is extremely useful in applications. It would be even better to have an orthonormal basis of polynomials.

The Legendre orthogonal polynomials are polynomials which form a basis for the vector space $\mathcal{P}$ and orthogonal in $L^2[-1, 1]$.

Exercise. Use a Gram-Schmidt process on the polynomials $1, x, x^2, x^3$ to obtain an orthonormal set; these are the first four Legendre polynomials.

More generally, consider weighted L^2 spaces, $L^2([a, b], w(x)dx)$. As we noted, the polynomials form a dense set, and they are spanned by $1, x, x^2, \ldots$ The Gram-Schmidt process (with respect to the weighted inner product) produces a sequence of orthonormal polynomials which span a dense set in the weighted L^2, hence it forms an orthonormal basis for $L^2([a, b], w(x)dx)$.

Orthogonal polynomials (with respect to a given weight, on a given interval) have been playing a fundamental role in many areas of mathematics and its applications, and are invaluable in approximations.

Exercise. The Laguerre orthogonal polynomials are orthogonal in the weighted $L^2([0, +\infty), e^{-x}dx)$. Use a Gram-Schmidt process on the polynomials $1, x, x^2, x^3$ to obtain an orthonormal set; these are the first four Laguerre polynomials.

15.7 Orthogonal complements, The Projection Theorem

The constructions are quite similar to the finite-dimensional case. One important difference is that we often need to assume that subspaces are closed, or otherwise take their closure.

Definition 26. *If S is a subset of a Hilbert space H its orthogonal is*

$$S^\perp = \{f \in H \mid \langle s, f\rangle = 0, \text{ for all } s \in S\}$$

Remark that $S^\perp$ is a **closed subspace**, therefore it is a Hilbert space itself. Why: as in linear algebra, $S^\perp$ is a vector subspace; to see that it is closed take a sequence $f_n \in S^\perp$ so that $f_n \to f$, and show that $f \in S^\perp$. Indeed, for any $s \in S$ we have $0 = \langle f_n, s\rangle \to \langle f, s\rangle$ (by Theorem 15) so $f \in S^\perp$. $\square$

Definition 27. *If V is a* closed subspace *of H, then $V^\perp$ is called the orthogonal complement of V.*

Note that if V is a vector subspace (not necessarily closed) then $(V^\perp)^\perp = \overline{V}$ (another point to be careful about in infinite dimensions).

15.7.1 Orthogonal Projections

Like in finite dimensions, in Hilbert spaces the projection f_V of f onto a subspace V is the vector that minimizes the distance between f and V. Note however: when we try to upgrade a statement from finite to infinite dimensions we often need to assume that our subspaces are closed, and if they are not, to replace them by their closure.

Theorem 14. *Given V a* closed subspace *of H and $f \in H$ there exists a unique $f_V \in V$ which minimizes the distance from f to V:*

$$\|f - f_V\| = \min\{\|f-g\| \,|\, g \in V\}$$

f_V is called the **orthogonal projection of f onto V**.

The outline of the proof is as follows: let $g_n \in V$ so that $\|f - g_n\| \to \inf\{\|f-g\| \,|\, g \in V\}$. A calculation (expanding $\|g_n - g_m\|^2$ and using Cauchy-Schwartz) yields that the sequence $(g_n)_n$ is Cauchy. Since V was assumed closed, then g_n converges, and f_V is its limit. $\square$

The linear operator $P_V : H \to H$ which maps f to f_V is called the **orthogonal projection onto V**.

Like in finite dimensions, H decomposes as a direct sum between a subspace V and its orthogonal complement: but V must to be **closed**:

Theorem 15. *(The Projection Theorem)*

Let V be a **closed subspace** *of H.*

Every $f \in H$ can be written uniquely as $f = f_V + f_\perp$, with $f_V \in V$ and $f_\perp \in V^\perp$.

In other words, $H = V \oplus V^\perp$.

The Projection Theorem follows by proving that $f - f_V$ is orthogonal to V (which follows after calculations not included here). $\square$

15.8 Least squares approximation via subspaces

Consider an orthonormal basis of H: $u_1, u_2, u_3, \ldots$. Any given $f \in H$ can be written as a (convergent) series

$$f = \sum_{n=1}^{\infty} \hat{f}_n u_n, \quad \text{where } \hat{f}_n = \langle u_n, f\rangle$$

We can approximate f by finite sums, with control of the errors, in the following way.

It is clear that the N^{th} partial Fourier sum:

$$f^{[N]} = \sum_{n=1}^{N} \hat{f}_n u_n$$

represents the orthogonal projection of f onto $V_N = \text{Sp}(u_1, u_2, \dots, u_N)$ (why?). Since V_N is closed (it is finite-dimensional, hence it is essentially $\mathbb{R}^N$ or $\mathbb{C}^N$) then by Theorem 14 we have that

$$\|f - f^{[N]}\| = \min\{\|f - g\| \mid g \in V_N\} = \min\{ E_N(c_1, \dots, c_N) \big| \, c_k \in \mathbb{C}\}$$

$$\text{where} \quad E_N(c_1, \dots, c_N) = \|f - \sum_{n=1}^{N} c_n u_n\|$$

The function $E_N^2(c_1, \dots, c_N)$ is called *Gauss' mean squared error*. For example, for $H = L^2[a, b]$ we have

$$E_N^2(c_1, \dots, c_N) = \int_a^b \left|f(x) - \sum_{n=1}^{N} c_n u_n(x)\right|^2 dx$$

Note that $\left|f(x) - \sum_{n=1}^{N} c_n u_n(x)\right|^2 \equiv s(x)$ is the squared error at x, and then $\frac{1}{b-a}\int_a^b s(x)dx$ is its mean.

Assume the Hilbert space is over the reals. To minimize the squared error (solving the least squares problem) we look for stationary points, hence we solve the system

$$\frac{\partial E_N^2}{\partial c_k} = 0, \ \ k = 1, \dots, N$$

whose solution is (try it!) $c_k = \langle u_k, f\rangle = \hat{f}_k$ (for $k = 1, \dots, N$).

The error in approximating f by $f^{[N]}$ is $E_N(\hat{f}_1, \dots, \hat{f}_N)$ therefore

$$E_N^2(\hat{f}_1, \dots, \hat{f}_N) = \|f - f^{[N]}\|^2 = \| \sum_{n=N+1}^{\infty} \hat{f}_n u_n\|^2 = \sum_{n=N+1}^{\infty} |\hat{f}_n|^2 \leq \|f\|^2$$

16 Linear operators in Hilbert spaces

Let H be a Hilbert space over the scalar field $F = \mathbb{R}$ or $\mathbb{C}$, with a countable orthonormal basis $u_1, u_2, u_3, \dots$.

In order to extend the concept of a matrix to infinite dimensions, it is preferable to look at the linear operator associated to it.

Definition 28. *An operator $T : H \to H$ is called linear if $T(f + g) = Tf + Tg$ and $T(cf) = cTf$ for all $f, g \in H$ and $c \in F$.*

The linearity conditions are often written more compactly as

$$T(cf + dg) = cTf + dTg \quad \text{for all } f, g \in H, \text{ and } c, d \in F$$

The adjoint T^* of an operator T is defined as in the finite-dimensional case, by $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all f, g.

We will see that it is sometimes convenient to consider operators T which are defined on a domain $D(T) \subset H$ which is smaller than the whole Hilbert space H. In such cases T^* is (usually) also defined on a domain $D(T^*) \subset H$ and we will call T^* the *adjoint* of T.

We will call the operator T *formally self-adjoint* (or symmetric) if

$$\langle Tf, g\rangle = \langle f, Tg\rangle \quad \text{for all } f \in D(T), \ g \in D(T^*)$$

We call T *self-adjoint* if it is formally self-adjoint and $D(T) = D(T^*)$.

16.1 Shift operators on ℓ^2

The examples below illustrate a very important distinction of infinite dimension: for a linear operator to be invertible we need to check both that its kernel is null (meaning that the operator is one-to-one) and that its range is the whole space (the operator is onto). We will see that the two conditions are not equivalent in infinite dimensions, and that an isometry, while it clearly is one-to-one, need not be onto.

1. Consider the left shift operator $S : \ell^2 \to \ell^2$:

$$S(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, \ldots)$$

which is clearly linear.

Obviously $\mathrm{Ker}(S) = Sp(e_1)$ (S is not one-to-one) and $\mathrm{Ran}(S) = \ell^2$ (S is onto).

2. Consider the right shift operator R on ℓ^2:

$$R(x_1, x_2, x_3, x_4, \ldots) = (0, x_1, x_2, x_3, \ldots)$$

The operator R is clearly linear. Note that R is an isometry, since

$$\|Rx\|^2 = \sum_{n=1}^{\infty} |x_n|^2 = \|x\|^2$$

thus we have $\mathrm{Ker}(R) = \{0\}$ (R is one-to-one). But R is not onto since $\mathrm{Ran}(R) = \{x \in \ell^2 \,|\, x_1 = 0\}$.

16.2 Unitary operators. Isomorphic Hilbert spaces

Definition 29. *A linear operator $U : H \to H$ is called* unitary *if $UU^* = U^*U = I$ (i.e. its inverse equals it adjoint).*

This definition is similar to the finite dimensional case, and, it follows that if U is unitary, then U is an isometry. Example 2 in §16.1 shows that the converse is not true in infinite dimensions (an isometry may not be onto). It can be shown that

Theorem 16. *If a linear operator is an isometry, and is onto, then it is unitary.*

We noted that if H is a Hilbert space with a countable orthonormal basis then H is essentially ℓ^2. Mathematically, this is stated as follows:

Theorem 17. *Let H be a Hilbert space with a countable orthonormal basis. Then H* is isomorphic *to ℓ^2 in the sense that there exists a unitary operator $U : \ell^2 \to H$.*

To be precise, if H is a complex Hilbert space, then we consider the complex ℓ^2 (sequences of complex numbers), while if H is a real Hilbert space, then we consider the real ℓ^2 (sequences of real numbers.)

The proof of Theorem 17 is immediate: the operator U is constructed in the obvious way. Denoting by $u_1, u_2, \ldots$ an orthonormal basis of H we define

$$U(a_1, a_2, \ldots) = a_1u_1 + a_2u_2 + \ldots = \sum_{n=1}^{\infty} a_nu_n \quad \text{for each } (a_1, a_2, \ldots) \in \ell^2$$

We need to show that $\sum_{n=1}^{\infty} a_nu_n \in H$, that U is one-to-one (clearly true by the definition of the Hilbert space basis) and onto (intuitively clear but requires an argument). Finally, by the Parseval's identity U is an isometry, and by Theorem 16 it is unitary. $\square$

Other examples of unitary operators.

It can be shown that the Fourier transform is a unitary operator on $L^2(\mathbb{R})$.

16.3 Integral operators

16.3.1 Illustration - solutions of a first order differential equation

The simplest integral operator takes f to $\int_a^x f(s)\,ds$ (it is clearly linear!). More general integral operators have the form

$$Kf(x) = \int_a^b G(x,s)f(s)\,ds \tag{107}$$

where $G(x, s)$ (called "integral kernel") is continuous, or has jump discontinuities.

(Note the similarity of (106) with the action of a matrix on a vector: if $G = (G_{xs})_{x,s}$ is a matrix, and $f = (f_s)_s$ is a vector, the $(Gf)_x = \sum_s G_{xs} f_s$.)

Integral operators appear as solutions of nonhomogenous differential equations; then G is called the Green's functions.

As a first example, consider the differential problem

$$\frac{dy}{dx} + y = f(x), \quad y(0) = 0 \tag{108}$$

where we look for solutions $y(x)$ for x in a finite interval, say $x \in [0, 1]$.

It is well known that the general solution of (107) is obtained by multiplying by the integrating factor (exp of the integral of the coefficient of y) giving

$$\frac{d}{dx}\left(e^x y\right) = e^x f(x) \quad \text{therefore} \quad y(x) = Ce^{-x} + e^{-x} \int e^x f(x)\, dx$$

After fitting in the initial condition $y(0) = 0$ we obtain the solution

$$y(x) = e^{-x} \int_0^x e^s f(s)\, ds \tag{109}$$

which has the form (106) for the (Green's function of the problem)

$$G(x, s) = e^{-x}\, e^s\, \chi_{[0,x]}(s) \tag{110}$$

where $\chi_{[0,x]}$ is the characteristic function of the interval $[0, x]$, defined as

$$\chi_{[0,x]}(s) = \begin{cases} 1 & \text{if } s \in [0, x] \\ 0 & \text{if } s \notin [0, x] \end{cases} \tag{111}$$

In many applications it is important to understand how solutions of (107) depend on the nonhomogeneous term f, for example, if a small change of f produces only a small change of the solution $y = Kf$. This is affirmative if we have the following estimate:

$$\text{there is a constant } C \text{ so that} \quad \|Kf\| \le C\,\|f\| \quad \text{for all } f \tag{112}$$

If an operator K satisfies (111) then K is called *bounded*. In fact, condition (111) is equivalent to the fact that the linear operator K is *continuous*.

It is not hard to see that the integral operator K given by (106), (109) is bounded on $L^2[0, 1]$; this means that the mean squared average of Kf does not exceed a constant times the mean squared average of f (the estimates are shown in §16.3.2 below).

We can find better estimates, point-wise rather than in average, since the linear operator K is also bounded as an operator on the Banach space $C[0, 1]$, and this means that the maximum of $|Kf(x)|$ does not exceed C times the maximum of $|f|$ (see the proof of this estimate below in §16.3.3).

While the estimate in $C[0, 1]$ (in the sup-norm) is stronger and more informative than the estimate in $L^2[0, 1]$ (in square-average), we have to pay a price: we can only obtain them for continuous forcing terms f.

General principle: **integral operators do offer good estimates**.

16.3.2 Proof that K is bounded on $L^2[0,1]$ (Optional)

Let us denote the norm in $L^2[0,1]$ by $\|\cdot\|_{L^2}$, and the inner product by $\langle\cdot,\cdot\rangle_{L^2}$ to avoid possible confusions.

To see that K is bounded on $Ł^2[0,1]$ note that $Kf(x)$ is, for each fixed x, an L^2 inner product, and using the Cauchy-Schwartz inequality

$$|Kf(x)| = |\langle G(x,\cdot), f(\cdot)\rangle_{L^2}| \leq \|G(x,\cdot)\|_{L^2}\,\|f\|_{L^2} \tag{113}$$

We can calculate using (109)

$$\|G(x,\cdot)\|_{L^2}^2 = \int_0^1 G(x,s)^2\,ds = \int_0^x e^{2s-2x}ds = \frac{1}{2}\left(1-e^{-2x}\right) < \frac{1}{2}$$

which used in (112) gives

$$|Kf(x)| \leq \frac{1}{2}\,\|f\|_{L^2} \tag{114}$$

for any $f \in L^2[0,1]$. Then from (113) we obtain

$$\|Kf\|_{L^2}^2 = \int_0^1 |Kf(x)|^2 dx \leq \frac{1}{4}\,\|f\|_{L^2}^2 \tag{115}$$

so K satisfies (111) in the L^2-norm for $C = 1/2$. $\Box$

16.3.3 Proof that K is bounded on $C[0,1]$ (Optional)

Recall that $C[0,1]$ is a Banach space with the sup-norm $\|\cdot\|_\infty$. Note that we obviously have $|f(x)| \leq \|f\|_\infty$ for any continuous f.

We need to use the following (very useful) inequality:

$$\Big|\int_a^b F(s)\,ds\Big| \leq \int_a^b |F(s)|\,ds$$

which is intuitively clear if we think of the definite integral of a function as the signed area between the graph of the function and the x-axis.

We then have, for any $f \in C[0,1]$,

$$|Kf(x)| = \left|e^{-x}\int_0^x e^s f(s)\,ds\right| \leq e^{-x}\int_0^x e^s\,|f(s)|\,ds$$

$$\leq \|f\|_\infty\, e^{-x}\int_0^x e^s\,ds = \|f\|_\infty\,(1-e^{-x}) \leq \|f\|_\infty$$

hence K satisfies (111) in the sup-norm for $C = 1$. $\Box$

16.3.4 Remarks about the δ function.

An alternative method for finding the Green's function of (107) is by solving $\frac{dy}{dx} + y = \delta(x-s)$ where δ is Dirac delta function. Its solution turns out to be $G(x,s)$ in (109). Then, using the fundamental property of the delta function that

$$f(x) = \int \delta(x-s) f(s) ds \tag{116}$$

by superposition of solutions we obtain (108).

It should be noted that Dirac's delta function does not belong to L^2: since it equals zero everywhere but a single point then in L^2 it must coincide with the zero function. An alternative view of this fact is that the delta function cannot be obtained as the L^2-limit of a sequence of continuous functions.

What is the δ function? Note that it is mainly used via its action on functions, like (115). As a precise mathematical object, then δ is an object that acts on functions - it is called a distribution. The study of distributions and of their applications form a separate chapter in mathematics.

16.4 Differential operators in $L^2[a,b]$

The simplest differential operator is $\frac{d}{dx}$, which takes f to $\frac{df}{dx}$. It is clearly linear.

There are two fundamental differences between the differential operator $\frac{d}{dx}$ and the integral operators (106):

1) we cannot define the derivative on all the functions in L^2, and

2) $\frac{d}{dx}$ is not a bounded operator: we cannot estimate the derivative of a function by only knowing the magnitude of the function (not in the sup-norm, and not even in average). Indeed, you can easily imagine a smooth function, whose values are close to zero and never exceed 1, but which has a very narrow and sudden spike. The narrower the spike, the higher the derivative, even if the average of the function remains small.

16.4.1 Illustration on a first order problem.

Consider again the problem (107). It can be written as

$$Ly = f, \quad y(0) = 0$$

where L is the differential operator

$$L = \frac{d}{dx} + 1, \quad \text{therefore } Ly = \left(\frac{d}{dx} + 1\right) y = \frac{dy}{dx} + y$$

If we are interested to allow the nonhomogeneous term f to have jump discontinuities, since $y' = f - y$ then y' will have jump discontinuities (at such a point y' may not be defined). An example of a such function is:

$$\frac{d}{dx}|x| = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

A good domain for L is then

$$D = \{y \in L^2[0,1] \,|\, y' \in L^2[0,1],\ y(0) = 0\}$$

Note that the initial condition was incorporated in D. Note also that D is dense in $L^2[0,1]$.

Note that the integral operator K found in §16.3.1 is the inverse of the differential operator L (defined on D).

16.5 A second order boundary value problem

Quite often separable partial differential equations lead to second order boundary value problems, illustrated here on a simple example.

Problem: Find the values of the constant λ for which the equation

$$y'' + \lambda y = 0 \tag{117}$$

has non-identically zero solutions for $x \in [0,\pi]$ satisfying the boundary conditions

$$y(0) = 0,\ \ y(\pi) = 0 \tag{118}$$

Equation (116) models the simple vibrating string: $x \in [0,1]$ represents the position on the string, and $y(x)$ is the displacement at position x. The boundary conditions (117) mean that the endpoints $x = 0$ and $x = 1$ of the string are kept fixed.

16.5.1 Formulation of the problem using a differential operator

Equation (116) can be written as

$$Ly = \lambda y, \quad \text{where } L = -\frac{d^2}{dx^2} \tag{119}$$

with the domain (dense in $L^2[0,\pi]$)

$$D = \{y \in L^2[0,\pi] \,|\, y'' \in L^2[0,\pi],\ \ y(0) = 0,\ \ y(\pi) = 0\} \tag{120}$$

(it must be noted that D needs to be more precisely specified, stating conditions on y and y').

A solution of (116), (117) is a solution of (118) in the domain (119). If this solution is not (identically) zero, then it is an eigenfunction of L, corresponding to the eigenvalue λ.

Note that L is formally self-adjoint on D since for $y, g \in D$ we have

$$\langle Ly, g\rangle = \int_0^{\pi} \overline{Ly(x)}g(x)\,dx = -\int_0^1 \overline{y''(x)}g(x)\,dx$$

and integration by parts gives (since $g(0) = 0 = g(\pi)$)

$$= -\overline{y'(x)}g(x)\big|_0^\pi + \int_0^\pi \overline{y'(x)}g'(x)\,dx = \int_0^\pi \overline{y'(x)}g'(x)\,dx$$

and integrating by parts again (and using $y(0) = 0 = y(\pi)$)

$$= \overline{y(x)}g(x)\big|_0^\pi - \int_0^\pi \overline{y(x)}g''(x)\,dx = \langle y, Lg\rangle$$

(Note that the boundary terms vanish by virtue of the boundary conditions.)

It can be shown, just like in the finite-dimensional case, that since L is self-adjoint then its eigenvalues are real and the eigenfunctions of L corresponding to different eigenvalues are orthogonal (see §16.5.2).

Remark that $L = -\frac{d^2}{dx^2}$ is positive definite, motivating the choice of a negative sign in front of the second derivative. Indeed, for $y \neq 0$

$$\langle Ly, y\rangle = -\int_0^1 \overline{y''(x)}y(x)\,dx = \int_0^\pi \overline{y'(x)}y'(x)\,dx = \int_0^\pi |y'(x)|^2\,dx > 0$$

where the last inequality is strict because $\int_0^\pi |y'(x)|^2\,dx = 0$ implies $y' = 0$, therefore y is constant, and due to the zero boundary conditions then $y = 0$.

Then the eigenvalues of L are positive, like in the finite-dimensional case. Indeed, if $Lf = \lambda f$ ($f \neq 0$) then $\langle f, Lf\rangle = \langle f, \lambda f\rangle$ hence

$$-\int_0^1 \overline{f(x)}\,f''(x)\,dx = \lambda\int_0^1 |f(x)|^2\,dx$$

then integrating by parts and using the fact that $f(0) = f(1) = 0$ we obtain

$$\int_0^1 |f'(x)|^2\,dx = \lambda\int_0^1 |f(x)|^2\,dx$$

which implies $\lambda > 0$.

Let us find the eigenvalues and eigenfunctions of L. Since $\lambda \neq 0$ the general solution of (116) is $y = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$ (note that $\sqrt{-\lambda} = i\sqrt{\lambda}$ since $\lambda > 0$). Since $y(0) = 0$ it follows that $C_2 = -C_1$, so, choosing $C_1 = 1$, $y(x) = e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x}$. The condition $y(\pi) = 0$ implies that $e^{2i\sqrt{\lambda}\pi} = 1$ therefore $\sqrt{\lambda} \in \mathbb{Z}$ so $\lambda = \lambda_n = n^2$, $n = 1, 2, \ldots$ and the corresponding eigenfunctions are $y_n(x) = e^{inx} - e^{-inx} = 2i\sin(nx)$.

A proof is needed to show that the eigenfunctions are complete, i.e. they form a basis for the Hilbert space $L^2[0, \pi]$. The question of finding the eigenvalues, eigenfunctions, proving their completeness and finding their properties, is the subject of study in Sturm-Liouville theory.

16.5.2 Review

Let A be a self-adjoint operator.

1) If λ is an eigenvalue of A, then $\lambda \in \mathbb{R}$.

Indeed, $Av = \lambda v$ for some $v \neq 0$. Then on one hand we have

$$\langle v, Av\rangle = \langle v, \lambda v\rangle = \lambda\langle v, v\rangle = \lambda\|v\|^2$$

and on the other hand

$$\langle v, Av\rangle = \langle Av, v\rangle = \langle \lambda v, v\rangle = \overline{\lambda}\langle v, v\rangle = \overline{\lambda}\|v\|^2$$

therefore $\lambda\|v\|^2 = \overline{\lambda}\|v\|^2$ so $(\lambda - \overline{\lambda})\|v\|^2 = 0$ therefore $\lambda = \overline{\lambda}$ so $\lambda \in \mathbb{R}$.

2) If $\lambda \neq \mu$ are to eigenvalues of A, $Av = \lambda v$ $(v \neq 0)$, and $Au = \mu u$ $(u \neq 0)$ then $u \perp v$.

Indeed, on one hand

$$\langle u, Av\rangle = \langle u, \lambda v\rangle = \lambda\langle u, v\rangle$$

and on the other hand

$$\langle u, Av\rangle = \langle Au, v\rangle = \langle \mu u, v\rangle = \mu\langle u, v\rangle$$

therefore $\lambda\langle u, v\rangle = \mu\langle u, v\rangle$ so $(\lambda - \mu)\langle u, v\rangle = 0$ therefore $\langle u, v\rangle = 0$.

16.5.3 Solution of a boundary value problem using an integral operator

Let us consider again equation (118) on the domain (119) and rewrite it as $(L - \lambda)y = 0$. We have non-zero solutions y (eigenfunctions of L) only when $Ker(L - \lambda) \neq \{0\}$.

Let us consider the totally opposite case, and look at complex numbers z for which $L - z$ is invertible. (Recall that, in general, the condition that the kernel be zero does not guarantee invertibility of an operator in infinite dimensions.)

Let us invert $L - z$; this means that for any f we solve $(L - z)y = f$ giving $y = (L - z)^{-1}f$. The operator $(L - z)^{-1}$ is called *the resolvent* of L.

To find y we solve the differential equation

$$y'' + zy = -f \tag{121}$$

with the boundary conditions

$$y(0) = 0,\ y(\pi) = 0 \tag{122}$$

Recall that the general solution of (120) is given by

$$C_1 y_1 + C_2 y_2 - y_1 \int \frac{y_2}{W} f + y_2 \int \frac{y_1}{W} f$$

where y_1, y_2 are two linearly independent solutions of the homogeneous equation $y'' + zy = 0$ and $W = W[y_1, y_2] = y_1' y_2 - y_1 y_2'$ is their Wronskian.

We need to distinguish the cases $z = 0$ and $z \neq 0$.

I. If $z = 0$, then $y_1 = 1, y_2 = x$ and we can easily solve the problem (120), (121).

II. If $z \neq 0$, then $y_1 = \exp(kx)$ and $y_2 = \exp(-kx)$ where $k = \sqrt{-z}$ (note that k may be a complex number; in particular, if $z > 0$ then $k = i\sqrt{z}$). Their Wronskian is $W = 2k$, so the general solution of (120) has the form

$$C_1 e^{kx} + C_2 e^{-kx} - \frac{e^{kx}}{2k} \int_0^x e^{-ks} f(s)\, ds + \frac{e^{-kx}}{2k} \int_0^x e^{ks} f(s)\, ds$$

The boundary condition $y(0) = 0$ implies $C_2 = -C_1$ therefore

$$y(x) = C \left(e^{kx} - e^{-kx}\right) - \frac{e^{kx}}{2k} \int_0^x e^{-ks} f(s)\, ds + \frac{e^{-kx}}{2k} \int_0^x e^{ks} f(s)\, ds$$

$$\equiv C \left(e^{kx} - e^{-kx}\right) + \int_0^\pi g(x,s)\, f(s)\, ds \tag{123}$$

where

$$g(x,s) = \frac{-1}{2k} \left(e^{kx-ks} - e^{-kx+ks}\right) \chi_{[0,x]}(s)$$

Imposing the boundary condition $y(\pi) = 0$ we obtain that C must satisfy

$$C \left(e^{k\pi} - e^{-k\pi}\right) + \int_0^\pi g(\pi,s)\, f(s)\, ds = 0$$

Solving for C and substituting in (122) we obtain that the solution of the problem (120), (121) has the form (106)

$$y(x) = \int_0^\pi G(x,s)\, f(s)\, ds \, = \, (L - z)^{-1} f$$

where $G(x,s)$ is the Green function of the problem

$$G(x,s) = g(x,s) \, - \, \frac{e^{kx} - e^{-kx}}{e^{k\pi} - e^{-k\pi}}\, g(\pi,s) \tag{124}$$

Note that G is not defined if $e^{k\pi} - e^{-k\pi} = 0$ which means for $ik \in \mathbb{Z}$. Since $k = \sqrt{-z}$ ($z \neq 0$), this means that for $z = n^2$ ($n = 1, 2, \ldots$) the Green function is undefined (for these values the denominator of G vanishes): the resolvent $(L - z)^{-1}$ does not exist for $z = \lambda_n = n^2$.

Note also that $G(x,s)$ is continuous (the discontinuity of $\chi_{[0,x]}(s)$ at $s = x$ does not result in a discontinuity of $G(x,s)$ at $s = x$ because $g(x,x) = 0$).

It is clear that if z is real then $(L - z)^{-1}$ is formally self-adjoint because L is formally self-adjoint.

The study of operators on Hilbert spaces is the topic of Functional Analysis. In one of its chapters it is proved that integral operators (106) (and other similar operators, called *compact operators*) which are self-adjoint are very much like self-adjoint matrices, in that they have real eigenvalues μ_n and the corresponding eigenfunctions u_n form an orthonormal basis for the Hilbert space.

The infinite dimensionality of the Hilbert space implies that there are infinitely many eigenvalues: a countable set, which, moreover, tend to zero: $\mu_n \to 0$. (Zero may also be an eigenvalue.)

Let us see how the eigenvalues μ_n and eigenfunctions of the resolvent $(L - z)^{-1}$ are related to the eigenvalues λ_n and eigenfunctions y_n of L.

We have $Ly_n = \lambda_n y_n$ hence $(L - z)y_n = (\lambda_n - z)y_n$ for any number z. If $L - z$ is invertible (we saw that this is the case for $z \neq \lambda_k$ for all k) then $y_n = (\lambda_n - z)(L - z)^{-1} y_n$ so y_n is an eigenfunctions of the resolvent $(L - z)^{-1}$ corresponding to the eigenvalue $\mu_n = (\lambda_n - z)^{-1}$.

Note that $\lambda_n \to \infty$ (since $\mu_n \to 0$).

Note the following quite general facts:

$\circ$ Remark the functional calculus aspect: if λ_n are the eigenvalues of L then $(\lambda_n - z)^{-1}$ are the eigenvalues of $(L - z)^{-1}$ and they correspond to the same eigenvectors.

$\circ$ Note again that the eigenvalues of L appear as values of z for which the Green function has zero denominators.