1 Gradient and Hamiltonian systems

1.1 Gradient systems

These are quite special systems of ODEs, Hamiltonian ones arising in conservative classical mechanics, and gradient systems, in some ways related to them, arise in a number of applications. They are certainly nongeneric, but in view of their origin, they are common.

A system of the form

$$X' = -\nabla V(X) \tag{1}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is, say, C^∞, is called, for obvious reasons, a gradient system. A critical point of V is a point where $\nabla V = 0$.

These systems have special properties, easy to derive.

Theorem 1. *For the system (1), if V is smooth, we have (i) If c is a regular point of V, then the vector field is perpendicular to the level hypersurface $V^{-1}(c)$ along $V^{-1}(c)$.*

(ii) A point is critical for V iff it is critical for (1).

(iii) At any equilibrium, the eigenvalues of the linearized system are real.

More properties, related to stability, will be discussed in that context.

Proof. □

(i) It is known that the gradient is orthogonal to level surface.

(ii) This is clear essentially by definition.

(iii) The linearization matrix elements are $a_{ij} = -V_{x_i,x_j}$ (the subscript notation of differentiation is used). Since V is smooth, we have $a_{ij} = a_{ji}$, and all eigenvalues are real.

1.2 Hamiltonian systems

If $\mathbf{F}$ is a conservative field, then $\mathbf{F} = -\nabla V$ and the Newtonian equations of motion (the mass is normalized to one) are

$$q' = p \tag{2}$$

$$p' = -\nabla V \tag{3}$$

where $q \in \mathbb{R}^n$ is the position and $p \in \mathbb{R}^n$ is the momentum. That is

$$q' = \frac{\partial H}{\partial p} \tag{4}$$

$$p' = -\frac{\partial H}{\partial q} \tag{5}$$

where

$$H = \frac{p^2}{2} + V(q) \tag{6}$$

is the Hamiltonian. In general, the motion can take place on a manifold, and then, by coordinate changes, H becomes a more general function of q and p. The coordinates q are called generalized positions, and q are the called generalized momenta; they are canonical coordinates on the phase on the cotangent manifold of the given manifold.

An equation of the form (4) is called a Hamiltonian system.

Exercise 1. *Show that a system $x' = F(x)$ is at the same time a Hamiltonian system and a gradient system iff the Hamiltonian H is a harmonic function.*

Proposition 1. *(i) The Hamiltonian is a constant of motion, that is, for any solution $X(t) = (p(t), q(t))$ we have*

$$H(p(t), q(t)) = const \tag{7}$$

where the constant depends on the solution.

(ii) The constant level surfaces of a smooth function $F(p, q)$ are solutions of a Hamiltonian system

$$q' = \frac{\partial F}{\partial p} \tag{8}$$

$$p' = -\frac{\partial F}{\partial q} \tag{9}$$

Proof. (i) We have

$$\frac{dH}{dt} = \nabla_p H \frac{dp}{dt} + \nabla_q \frac{dq}{dt} = -\nabla_p H \nabla_q + \nabla_q H \nabla_p = 0 \tag{10}$$

(ii) This is obtained very similarly. □

1.2.1 Integrability: a few first remarks

Hamiltonian systems (with time-independent Hamiltonian) in one dimension are integrable: the solution can be written in closed form, implicitly, as $H(y(x), x) = c$; in terms of t, once we have $y(x)$ of course we can integrate $x' = G(y(x), x) := f(x)$ by quadratures (using separation of variables). Note that for an equation of the form $y' = G(y, x)$, this is *equivalent* to the system having a *constant of motion*. The latter is defined as a function $K(x, y)$ defined globally in the phase space, (perhaps with the exception of some isolated points where it may have "simple" singularities, such as poles), and with the property that $K(y(x), x) = $ *const* for any given trajectory (the constant can depend on the trajectory, but not on x). Indeed, in this case we have

$$\frac{d}{dx} K(y(x), x) = \frac{\partial K}{\partial y} y' + \frac{\partial K}{\partial x} = 0$$

or

$$y' = -\frac{\partial K}{\partial x} / \frac{\partial K}{\partial y}$$

and the trajectories are the same as those of

$$\dot{x} = \frac{\partial K}{\partial y}; \quad \dot{y} = -\frac{\partial K}{\partial x} \tag{11}$$

which is a Hamiltonian system.

1.2.2 Dependence on initial conditions

Consider the system

$$y' = F(y, x); \; y(0) = y_0; \; y_0 \in \mathbb{R}^n \tag{12}$$

Proposition 2. *If F is smooth, then in a neighborhood of $(0, y_0)$, $y(x; y_0)$ is smooth in both x and y_0.*

Proof. We can prove this by extending the system (12) to include y_0. Maybe more transparently we can use the contraction mapping principle as follows. The proof is standard, so we only sketch it.

We write (12) in integral form,

$$y = y_0 + \int_0^x F(y(s), s)ds = \mathcal{N}(y, x; x_0) \tag{13}$$

and check that for small ε it is a contraction in the sup norm in a ball in $C(\mathbb{D}_\varepsilon \times B)$, the functions continuous in x and y_0, where B is a ball of radius $2\|y_0\|$.

Thus y is continuous in y_0. Now we differentiate formally w.r.t. y_0. Denoting by M the matrix $D_{y_0} y$ we get the matrix equation

$$M' = [D_y F(y(x; y_0))M; \quad M(0) = I \Leftrightarrow M(x) = I + \int_0^x D_y F(y(s; y_0))M(s)ds \tag{14}$$

where $y(x; y_0)$ is taken as a *known function*, which is continuous in x, y_0. This equation is also contractive in the space of matrix valued continuous functions in the sup norm is ε is small. We can continue in this way and see that the derivatives of all order exist and are continuous. It is straightforward to check that $y = \int \frac{\partial y}{\partial \tau} d\tau$ where τ is one of the components of y_0. The existence and continuity of higher order derivatives is checked similarly. $\square$

With lower regularity we can for instance prove the following. Write the differential equation in integral form,

$$x = x_0 + \int_0^t F(x(s))ds = \mathcal{N}(x, x_0) \tag{15}$$

Theorem 2. Assume F is uniformly Lipschitz in x in an open set $\mathcal{O}$ and let K be a compact set contained in $\mathcal{O}$. Then there exists a $T = T(K)$ s.t. for any $x_0 \in K$ the solution $x(t, x_0)$ exists and is in $\mathcal{O}$ for all $t, |t| \leqslant T$ and x is continuous (thus uniformly continuous) in $(x_0, t) \in K \times [-T, T]$.

Proof. Consider the integral equation with initial conditions in a neighborhood of x_0.

$$x = x_0 + \xi + \int_0^t F(x(s))ds = \mathcal{N}(x, x_0) \tag{16}$$

Let κ be the Lipschitz constant of F in $\mathcal{O}$, that is,

$$|F(x) - F(x')| \leqslant \kappa |x - x'|, \ \forall x, x' \in \mathcal{O} \tag{17}$$

Note first that, by the compact covering theorem there is a δ s.t. $\forall x \in K$ and x' s.t. $d(x, x') \leqslant \delta$ we have $x' \in \mathcal{O}$. Define $K' = \{x \in \mathcal{O} | d(x, K) \leqslant \delta\}$ and let $M = \max_{x \in K'} |F|$. Finally, choose T s.t. $MT \leqslant \delta/3$ and $\kappa T \leqslant 1/2$ and $|\xi| < \delta/3$.

Consider the integral equation (16) in the Banach space $\mathcal{B}$ of functions continuous in $|t| \leqslant T$ and in $\xi, \xi + x_0 \in K, |\xi| < \delta/3$, in the sup norm. Take the closed ball $B = \{x \in \mathcal{B} | \|x - x_0\| \leqslant \delta/3\}$. The conditions above ensure that

$$(x, t, x_0 + \xi) \in B \times [-T, T] \times K \Rightarrow \mathcal{N}(x; x_0) \in B \text{ and } \mathcal{N} \text{ is contractive in } B \tag{18}$$

and the result follows. $\square$

1.3 Example

As an example for both systems, we study the following problem: draw the contour plot (constant level curves) of

$$F(x, y) = y^2 + x^2(x - 1)^2 \tag{19}$$

and draw the lines of steepest descent of F.

For the first part we use Proposition 1 above and we write

$$x' = \frac{\partial F}{\partial y} = 2y \tag{20}$$

$$y' = -\frac{\partial F}{\partial x} = -2x(x - 1)(2x - 1) \tag{21}$$

The critical points are $(0, 0), (1/2, 0), (1, 0)$. It is easier to analyze them using the Hamiltonian. Near $(0, 0)$ H is essentially $x^2 + y^2$, that is the origin is a center, and the trajectories are near-circles. We can also note the symmetry $x \to (1 - x)$ so the same conclusion holds for $x = 1$, and the phase portrait is symmetric about $1/2$.

Near $x = 1/2$ we write $x = 1/2 + s$, $H = y^2 + (1/4 - s^2)^2$ and the leading Taylor approximation gives $H \sim y^2 - 1/2s^2$. Then, $1/2$ is a saddle (check). Now we can draw the phase portrait easily, noting that for large x the curves essentially become $x^4 + y^2 = C$ "flattened circles". Clearly, from the interpretation of the problem and the expression of H we see that *all* trajectories are closed.

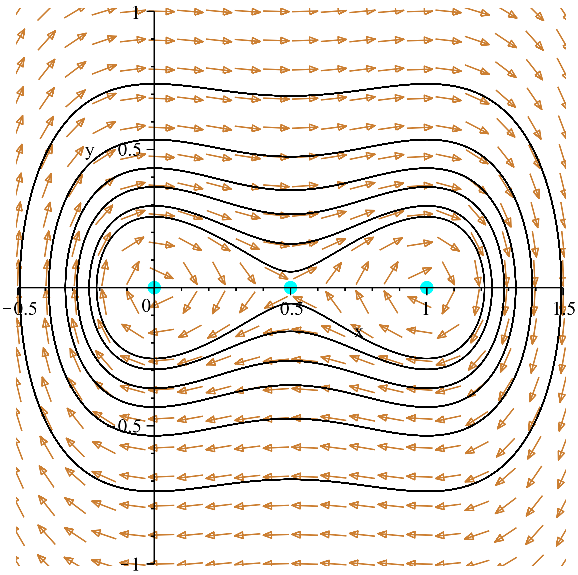

Figure 1:

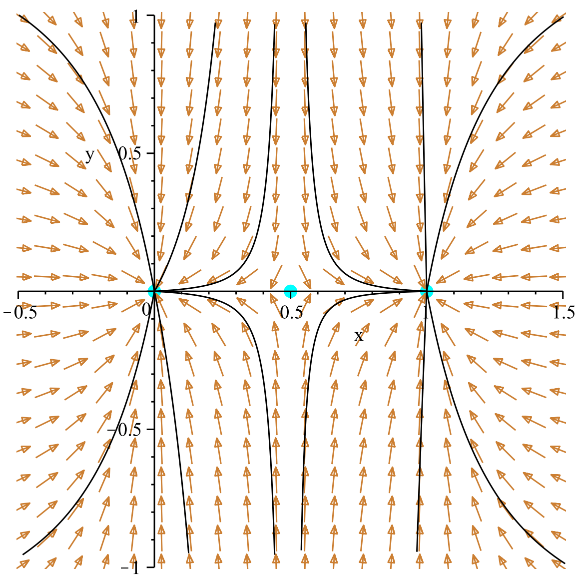

Figure 2:

The perpendicular lines solve the equations

$$x' = -\frac{\partial F}{\partial x} = -2x(x-1)(2x-1) \tag{22}$$

$$y' = -\frac{\partial F}{\partial y} = -2y \tag{23}$$

We note that this equation is separated. In any case, the two equation obviously share the critical points, and the sign diagram can be found immediately from the first figure.

Exercise 2. *Find the phase portrait for this system, and justify rigorously its qualitative features. Find the expression of the trajectories of (22). I found*

$$y = C\left(\frac{1}{(x-1/2)^2} - 4\right)$$

2 Flows, revisited

Often in nonlinear systems, equilibria are of higher order (the linearization has zero eigenvalues). Clearly such points are not hyperbolic and the methods we have seen so far do not apply.

There are no general methods to deal with all cases, but an important one is based on Lyapunov (or Liapunov, or Lyapounov,...) functions.
Definition. A flow is a smooth map

$$(X,t) \to \Phi_t(X)$$

A differential system

$$\dot{x} = F(x);\ \ F \in \mathbb{R}^d \tag{24}$$

generates a flow

$$(X,t) \to x(t;X)$$

where $x(t;X)$ is the solution at time t with initial condition X.

The derivative of a function G along a vector field F is, as usual,

$$D_F(G) = \nabla G \cdot F$$

If we write the differential equation associated to F, (24), then clearly

$$D_F G = \frac{d}{dt} G(x(t))_{|t=0}$$

2.1 Lyapunov stability

Consider the system (24) and assume $x = 0$ is an equilibrium.

Then

1. $x_e = 0$ is Lyapunov stable (or simply stable) if starting with initial conditions near 0 the flow remains in a neighborhood of zero. More precisely, the condition is: for every $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x_0| < \delta$ then $|x(t)| < \varepsilon$ for all $t > 0$.

2. $x_e = 0$ is asymptotically stable if furthermore, trajectories that start close to the equilibrium converge to the equilibrium. That is, the equilibrium x_e is asymptotically stable if it is Lyapunov stable and if there exists $\delta > 0$ so that if $|x_0| < \delta$, then $\lim_{t\to\infty} x(t) = 0$.

2.2 Lyapunov functions

Let X^* be a fixed point of (24). A Lyapunov function for (24) is a function defined in a neighborhood $\mathcal{O}$ of X^* with the following properties

(1) L is differentiable in $\mathcal{O}$.

(2) $L(X^*) = 0$ (this can be arranged by subtracting a constant).

(3) $L(x) > 0$ in $\mathcal{O} \setminus \{X^*\}$.

(4) $D_F L \leqslant 0$ in $\mathcal{O}$.

A strict Lyapunov function is a Lyapunov function for which

(4') $D_F L < 0$ in $\mathcal{O}$.

Finding a Lyapunov function is often nontrivial. In systems coming from physics, the energy is a good candidate. In general systems, one may try to find an exactly integrable equation which is a good approximation for the actual one in a neighborhood of X^* and look at the various constants of motion of the approximation as candidates for Lyapunov functions.

Theorem 3 (Lyapunov stability)**.** *Assume X^* is a fixed point for which there exists a Lyapunov function L. Then*

(i) X^* is stable.

(ii) If L is a strict Lyapunov function then X^* is asymptotically stable.

Proof. (i) Consider a small ball $B \ni X^*$ contained in $\mathcal{O}$. Let α be the minimum of L on ∂B. By the definition of a Lyapunov function, (3), $\alpha > 0$. Consider the following subset:

$$\mathcal{U} = \{x \subset B : L(x) < \alpha\} \tag{25}$$

From the continuity of L, we see that $\mathcal{U}$ is an open set. Clearly, $X^* \subset \mathcal{U}$. Let $X \in \mathcal{U}$. Then $x(t; X)$ is a continuous curve, and it cannot have components outside B without intersecting ∂B. But an intersection is impossible since by monotonicity, $L(x(t)) \leqslant L(X) < \alpha$ for all t. Thus, trajectories starting in $\mathcal{U}$ are confined to B, proving stability.

(ii)

1. Note first that X^* is the only critical point in $\mathcal{O}$ since $\frac{d}{dt}L(x(t; X_1^*)) = 0$ for any fixed point.

2. Note that trajectories $x(t; X)$ with $X \in \mathcal{U}$ are contained in $\overline{B}$, a compact set, and thus they contain limit points, i.e., points x^* s.t. $x(t_n, X) \to x^*$ for some sequence $t_n \uparrow \infty$. Any limit point x^* is strictly inside $\mathcal{U}$ since $L(x^*) < L(x(t); X) < \alpha$.

3. Let x^* be a limit point of a trajectory $x(t; X)$ where $X \in \mathcal{U}$. Then, by 1 and 2, if $x^* \in \mathcal{U} \neq X^*$, then x^* is a regular point of the field.

4. We want to show that $x^* = X^*$. We will do so by contradiction. Assuming $x^* \neq X^*$ we have $L(x^*) =: \lambda > 0$, again by (3) of the definition of L.

5. By 3 the trajectory $\{x(t; x^*) : t \geqslant 0\}$ is well defined and is contained in $\mathcal{B}$.

6. We then have $L(x(t; x^*)) < \lambda \forall t > 0$.
7. We look at the increasing sequence t_n in 1. For any n, the set

$$\mathcal{V} = \{X : L(x(t_{n+1} - t_n; X))\} < \lambda \tag{26}$$

contains x^* and is open, so

$$L(x(t_{n+1} - t_n; X_1)) < \lambda \tag{27}$$

for all X_1 close enough to x^*.

8. Let n be large enough so that $x(t_m; X) \in \mathcal{V}$ for all $m \geqslant n$.
9. Note that, by existence and uniqueness of solutions at regular points we have

$$x(t_{n+1}; X) = x(t_{n+1} - t_n; x(t_n; X)) \tag{28}$$

10. On the one hand $L(x(t_{n+1})) \downarrow \lambda$ and on the other hand we got $L(x(t_{n+1})) < \lambda$. This is a contradiction.

□

2.3 Examples

Hamiltonian systems, in Cartesian coordinates often assume the form

$$H(q, p) = p^2/2 + V(q) \tag{29}$$

where p is the collection of spatial coordinates and p are the momenta. If this ideal system is subject to external dissipative forces, then the energy cannot increase with time. H is thus a Lyapunov function for the system. If the external force is $F(p, q)$, the new system is generally not Hamiltonian anymore, and the equations of motion become

$$\dot{q} = p \tag{30}$$

$$\dot{p} = -\nabla V + F \tag{31}$$

and thus

$$\frac{dH}{dt} = pF(p, q) \tag{32}$$

which, in a dissipative system should be nonpositive, and typically negative. But, as we see, $dH/dt = 0$ along the curve $p = 0$.

For instance, in the ideal pendulum case with Hamiltonian

$$H = \frac{1}{2}\omega^2 + (1 - \cos\theta) \tag{33}$$

The associated Hamiltonian flow is

$$\theta' = \omega \tag{34}$$

$$\omega' = -\sin\theta \tag{35}$$

Then H is a global Lyapunov function at $(0,0)$ for (36) (in fact, this is true for any system with *nonnegative* Hamiltonian). This is clear from the way Hamiltonian systems are defined.

Then $(0,0)$ is a stable equilibrium. But, clearly, it is not asymptotically stable since $H = const > 0$ on any trajectory not starting at $(0,0)$.

If we add air friction to the system (36), then the equations become

$$\theta' = \omega \tag{36}$$

$$\omega' = -\sin\theta - \kappa\omega \tag{37}$$

where $\kappa > 0$ is the drag coefficient. Note that this time, if we take $L = H$, the same H defined in (33), then

$$\frac{dH}{dt} = -\kappa\omega^2 \tag{38}$$

The function H is a Lyapunov function, but it is not strict, since $H' = 0$ if $\omega = 0$. Thus the system is stable. It is however intuitively clear that furthermore the energy still decreases to zero in the limit, since $\omega = 0$ are isolated points on any trajectory and we expect $(0,0)$ to still be asymptotically stable. In fact, we could adjust the proof of Theorem 3 to show this. However, as we see in (32), this degeneracy is typical and then it is worth having a systematic way to deal with it. This is one application of Lasalle's invariance principle that we will prove next.

3 Some important concepts

We start by introducing some important concepts.

Definition 3.

1. An entire solution $x(t;X)$ is a solution which is defined for all $t \in \mathbb{R}$.
2. A positively invariant set $\mathcal{P}$ is a set such that $x(t,X) \in \mathcal{P}$ for all $t \geqslant 0$. Solutions that start in $\mathcal{P}$ stay in $\mathcal{P}$. Similarly one defines negatively invariant sets, and invariant sets.
3. The basin of attraction of a fixed point X^* is the set of all X such that $x(t;X) \to X^*$ when $t \to \infty$.
4. Given a solution $x(t;X)$, the set of all points x^* such that solution $x(t_n;X) \to x^*$ for some sequence $t_n \to \infty$ is called the set of ω-limit points of $x(t;X)$. At the opposite end, the set of all points x^* such that solution $x(-t_n;X) \to x^*$ for some sequence $t_n \to \infty$ is called the set of α-limit points. These may of course be empty.

   That is,

   $$\omega(X) := \{x : \lim_{n\to\infty} x(t_n, X) = x \text{ for some sequence } t_n \to +\infty\} \tag{39}$$

and, similarly, the α-limit set is defined as

$$\alpha(X) := \{x : \lim_{n\to\infty} x(t_n, X) = x \text{ for some sequence } t_n \to -\infty\}. \tag{40}$$

Proposition 4. Assume X belongs to a closed, positively invariant set $\mathcal{P}$ s.t., with $K = \mathcal{P}$, the hypotheses of Theorem 2 are satisfied. Then, the ω-limit set $\omega(X)$ is a closed invariant set: solutions with initial condition in $\omega(X)$ are **entire**. A similar statement holds for the α-set.

Proof.

1. (Closure) We show the complement of $\omega(X)$ is open. Let $b \in \omega(X)^c$. Then for some $\varepsilon > 0$, $d(x(t, X), b) \geqslant \varepsilon$ for all t. If $|b' - b| < \varepsilon/2$, then by the triangle inequality, $\liminf_{t\to\infty} d(x(t, X), b') > \varepsilon/2 > 0$ for all t.

2. By Theorem 2 the function $x(t, x_0)$ exists for any $x_0 \in \mathcal{P}$, $|t| \leqslant T$ and is uniformly continuous for all $x_0 \in \mathcal{P}$ and $|t| \leqslant T$. Since $\mathcal{P}$ is a compact set in $\mathcal{O}$ and positively invariant, for any $\tau \geqslant 0$ the function $x(\tau, x_0)$ exists for any $x_0 \in \mathcal{P}$ and is uniformly continuous in $x_0 \in \mathcal{P}$ (we write $\tau = nT + T_1, T_1 < T$, use induction in n and the fact that $x(t + t', X) = x(t', x(t, X))$.)

3. Note that for any $|T'| \leqslant T$, the limit $\lim_{n\to\infty} x(t_n + T', X)$ exists and thus belongs to $\omega(X)$. This is the case because $x(t_n + T', X) = x(T'; x(t_n))$ and by uniform continuity of x in the initial condition and in $|t| \leqslant T$. In fact the restriction $|T'| \leqslant T$ is not needed since we can write $T' = nT + T_1$ as in 2 above.

4. As a consequence, note now that for any $|t| \leqslant T$ and $x^* \in \omega(X)$ we have $x(t, x^*) \in \omega(X)$. It follows immediately that $x(t \pm T, x^*) \in \omega(X)$ if $|t| \leqslant T$, writing $t = nT + T_1$ as above, we see that $x(t, x^*) = \lim x(t_n + t, X) \in \omega(X)$.

$\square$

4 Lasalle's invariance principle

Theorem 4. *Let X^* be an equilibrium point for $x' = F(x)$ and let $L : \mathcal{U} \to \mathbb{R}$ ($\mathcal{U}$ open) be a Lyapunov function at X^*. Let $\mathcal{P} \subset \mathcal{U}$ be compact, positively invariant containing X^*. Assume there is no entire trajectory in $\mathcal{P} - \{X^*\}$ along which L is constant. Then X^* is asymptotically stable, and $\mathcal{P}$ is contained in the basin of attraction of X^*.*

Proof. Since $\mathcal{P}$ is compact and positively invariant, then $X \in \mathcal{P} \Rightarrow \omega(X) \subset \mathcal{P}$. If $\omega(X) = \{X^*\}$, $\forall X$, the assumption follows easily (check!). So, we may assume there is an $x^* \neq X^*$ which is also an ω-limit point of some $x(t; X)$. By Proposition 4, the trajectory $x(t; x^*)$ is entire. Since L is nondecreasing along trajectories, we have $L(x(t; X)) \to \alpha = L(x^*)$ as $t \to \infty$. (This is clear for the subsequence t_n, and the rest follows by inequalities: check!) Since $x(t, x^*) = \lim x(t_n + t, X)$, by continuity, $L(x(t, x^*)) = \alpha$, contradiction. $\square$

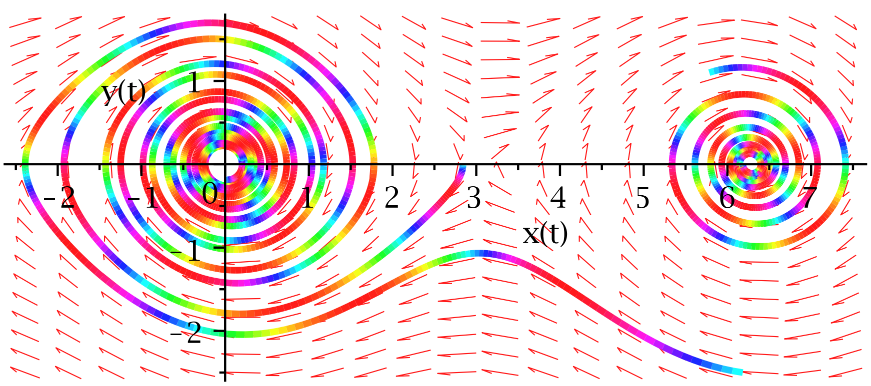

Figure 3:

4.1 Example: analysis of the pendulum with drag

Of course this is a simple example, but the way Lasalle's invariance principle is applied is representative of many other problems.

Intuitively, it is clear that any trajectory that starts with $\omega = 0$ and $\theta \in (-\pi, \pi)$ should asymptotically end up at the equilibrium point $(0, 0)$ (other trajectories, which for the frictionless system would rotate forever, may end up in a different equilibrium, $(2n\pi, 0)$. For zero initial ω, the basin of attraction of $(0, 0)$ should exactly be $(-\pi, \pi)$. In general, the energy should be less than precisely the one in this marginal case, $H = 1 - \cos(\pi) = 2$. Then, the region $\theta_0 \in (-\pi, \pi)$, $H < 1 - \cos(\pi) = 2$ should be the basin of attraction of $(0, 0)$.

So let $c \in (0, 2)$, and let

$$\mathcal{P}_c = \{(\theta, \omega) : H(\theta, \omega) \leqslant c, \text{ and } |\theta| \leqslant \arccos(1 - c) \in (-\pi, \pi)\} \tag{41}$$

In H, θ coordinates, this is simply a closed rectangle and since (H, θ) is a continuous map, its preimage in the (ω, θ) plane is closed too.

Now we show that $\mathcal{P}_c$ is closed and forward invariant. If a trajectory were to exit $\mathcal{P}_c$, it would mean, by continuity, that for some t we have $H = c + \delta$ for a small $\delta > 0$ (ruled out by $\dot{H} \leqslant 0$ along trajectories) or that $|\theta| > \arccos(1 - c)$ for some t which implies, from the formula for H the same thing: $H > c$.

Now there is no nontrivial entire solution (that is, other than $X^* = (0, 0)$) along which $H = const$. Indeed, $H = const$ implies, from (38) that $\omega = 0$ identically along the trajectory. But then, from (35) we see that $\sin\theta = 0$ identically, which, within $\mathcal{P}_c$ simply means $\theta = 0$ identically. Lasalle's theorem applies, and all solutions starting in $\mathcal{P}_c$ approach $(0, 0)$ as $t \to \infty$.

The phase portrait of the damped pendulum is depicted in Fig. 3

5 Gradient systems and Lyapunov functions

Recall that a gradient system is of the form (1), that is

$$X' = -\nabla V(X) \tag{42}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is, say, C^∞ and a critical point of V is a point where $\nabla V = 0$. We have the following result:

Theorem 5. *For the system (1): (i) If c is a regular value of V, then the vector field is orthogonal to the level set of $V^{-1}(c)$.*

(ii) If a critical point X^* is an isolated minimum *of V, $V(X) - V(X^*)$ is a strict Lyapunov function at X^*, and then X^* is asymptotically stable.*

(iii) Any $\alpha-$ limit point of a solution of (1), and any $\omega-$ limit point is an equilibrium.

Note 1. (a) By (iii), any solution of a gradient system tends to a limit point or to infinity.

(b) Thus, descent lines of any smooth manifold have the same property: they link critical points, or they tend to infinity.

(c) We can use some of these properties to determine for instance that a system is not integrable. We write the associated gradient system and determine that it fails one of the properties above, for instance the linearized system at a critical point has an eigenvalue which is not real. Then there cannot exist a smooth H so that $H(x, y(x))$ is constant along trajectories.

Proof. (i) is straightforward.

(ii) If an equilibrium point is isolated, then $\nabla V \neq 0$ in a set of the form $|X - X^*| \in (0, a)$. Then $-|\nabla V|^2 < 0$ in this set. Furthermore, $V(X) - V(X^*) > 0$ for all X with $|X - X^*| \in (0, a)$. Then, in this neighborhood, V is a strict Lyapunov function.

(iii) Note that if x is a point where $\nabla V = 0$, then x is an equilibrium. If Ω is an orbit along which V is constant, then $dV/dt = 0 = -|\nabla V|^2$ at any point along the trajectory, so all points are equilibria. There are no nontrivial limit sets.

□

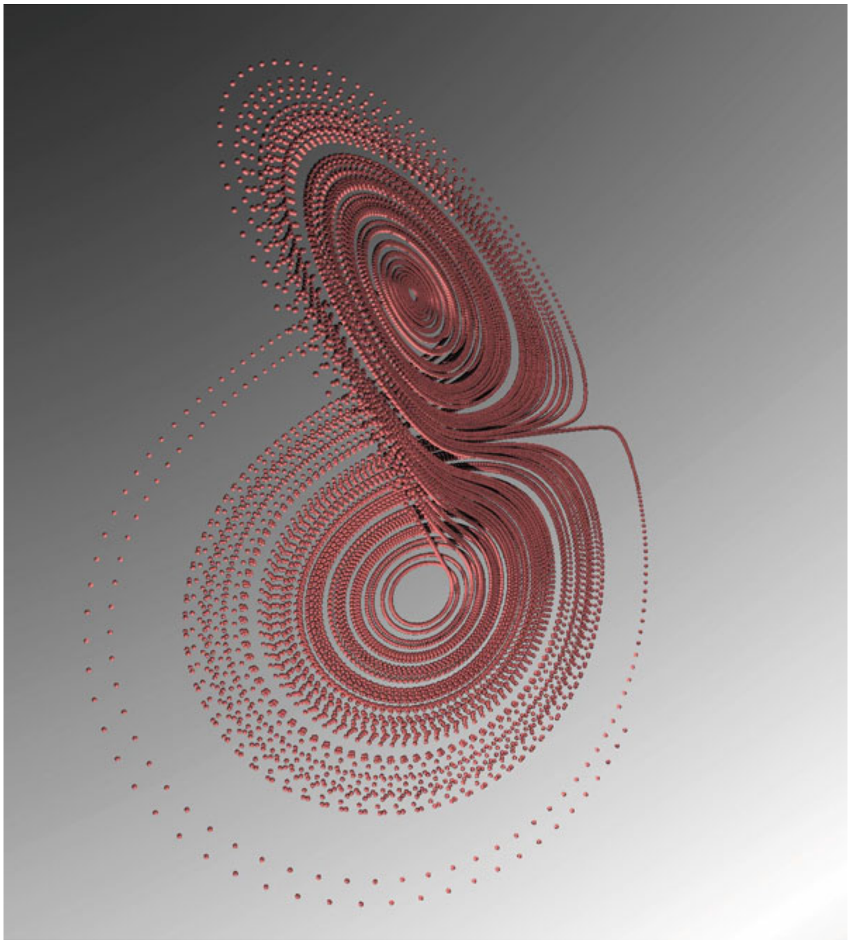

Figure 4: The Lorenz attractor

6 Sections; the flowbox theorem

Consider first a planar system $x' = f(x)$ with f smooth, and a point x_0 such that $f(x_0) \neq 0$. A section through x_0 is a curve which is transversal to the flow, and passes through x_0. To be specific, take a unit vector V_0 at x_0 which is orthogonal to $f(x_0)$, say $(-f_2(x_0), f_1(x_0))/|f(x_0)|$. We draw a line segment in the direction of V_0,

$$\mathcal{S} = \{h(u) := x_0 + uV_0 | u \in (-\varepsilon, \varepsilon)\} \tag{43}$$

Once more, since f is continuous, for small δ there is a small ε so that we have $V_0 \cdot (-f_2(h(u)), f_1(h(u)))/|f(h(u))| \geqslant 1 - \delta$ if $u \in (-\varepsilon, \varepsilon)$. That is, the field is transversal to the section in a small neighborhood of x_0. By the same estimate, $V_0 \cdot (-f_2(h(u)), f_1(h(u)))/|f(h(u))|$ has constant sign along $\mathcal{S}$, which means that the field and the flow cross $\mathcal{S}$ in the same direction throughout $\mathcal{S}$. See the left side of fig. 7.

Definition 5. *The segment $\mathcal{S}$ defined above is called local section at x_0.*

6.0.1 The flowbox theorem for planar system; geometric approach

There is a diffeomorphic change of coordinates *in some neighborhood of* x_0, $x \leftrightarrow z$ so that in coordinates z the field is simply $\dot{z} = \mathbf{e}_1 := (1, 0)$.

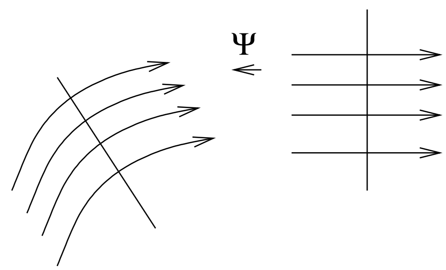

Figure 5: Flowbox and transformation

To straighten the field, we construct the following map, from a neighborhood of x_0 of the form

$$\mathcal{N} = \{\Psi(t,u) := x(t;h(u)) : |t| < \delta, u \in (-\varepsilon, \varepsilon)\}$$

where ε and δ are sufficiently small. Then, $(t,u) \mapsto x(t;h(u))$ is a diffeomorphism since the Jacobian of the transformation at $(0,0)$ is

$$\det \begin{pmatrix} \frac{\partial \Psi_1}{\partial t} & \frac{\partial \Psi_1}{\partial u} \\ \frac{\partial \Psi_2}{\partial t} & \frac{\partial \Psi_2}{\partial u} \end{pmatrix} = \det \begin{pmatrix} f_1 & V_1 \\ f_2 & V_2 \end{pmatrix} = |f(x_0)| \neq 0 \tag{44}$$

Clearly, the inverse image of trajectories through Ψ are straight lines, (t, u_0), as depicted. The associated flow in the set $\Psi^{-1}(\mathcal{N})$ is

$$\frac{dt}{dt} = 1; \quad \frac{du}{dt} = 0 \tag{45}$$

6.0.2 The flowbox theorem, in general

Consider a vector field F at a regular point, say 0, with $F(0) \neq 0$. Without loss of generality we can assume that $F(0) = \alpha e_1$ where e_1 is the first unit vector and by rescaling time we can assume $\alpha = 1$. We seek a local diffeomorphism $x = w + h(w), h = o(w)$ s.t.

$$\dot{w} = e_1 \tag{46}$$

This is the case if

$$\dot{w} + \frac{\partial h}{\partial w}\dot{w} = e_1 + \frac{\partial h}{\partial w} e_1 = \dot{x} = F(w + h(w)) \tag{47}$$

and thus, since $F(0) = e_1$, we have

$$\frac{\partial h}{\partial w} e_1 = F(w + h(w)) - F(0) = g(w + h(w)) \tag{48}$$

which is equivalent to the system

$$\frac{\partial h_j}{\partial w_1} = g_j(w + h(w)) \tag{49}$$

which we write in integral form

$$h_j(w_1, ..., w_n) = \int_0^{w_1} g_j(s + h_1(s, w_2, ..., w_n), ..., w_n + h_n(s, w_2, ..., w_n)))ds \tag{50}$$

which is contractive for small w. We see that $h_j(0, w_2, ..., w_n) = 0$. This is built into the initial conditions in the integral equation, and also natural, since $w_1 = w_1(0) + t; w_i = w_i(0)$.

6.0.3 Local versus global

Given the linearization theorem above, any system would be Hamiltonian, locally, near a regular point of the field. A system is a properly Hamiltonian one if the Hamiltonian is defined in a wide enough region of the phase space.

7 Limit sets, Poincaré maps, the Poincaré Bendixson theorem

In two dimensions, there are typically two types of limit sets: equilibria and periodic orbits (which are thereby limit cycles). Exceptions occur when a limit set contains a number of equilibria, as we will see in examples.

The Poincaré-Bendixson theorem states that if $\omega(X)$ is a nonempty compact limit set of a *planar system of ODEs* containing *no equilibria*, then $\omega(X)$ is a closed orbit. We will return to this important theorem and prove it.

Beyond two dimensions however, the possibilities are far vaster and limit sets can be quite complicated. Fig. 4 depicts a limit set for the Lorenz system, in three dimensions. Note how the trajectories seem to spiral erratically around two points. The limit set here has a fractal structure.

We begin the analysis with the two dimensional case, which plays an important role in applications.

We have already studied the system $r' = 1/2(r - r^3)$ in Cartesian coordinates. There the circle of radius one was a periodic orbit, and a limit cycle. All trajectories, except for the trivial one $(0, 0)$ tended to it as $t \to \infty$.

We have also analyzed many cases of nodes, saddle points etc, where trajectories have equilibria as limit sets, or else they go to infinity.

A rather exceptional situation is that where the limit sets contain equilibria. Here is one example

7.1 Example: equilibria on the limit set

Consider the system

$$x' = \sin x(-\cos x - \cos y) \tag{51}$$
$$y' = \sin y(\cos x - \cos y) \tag{52}$$

The phase portrait is depicted in Fig. 6.

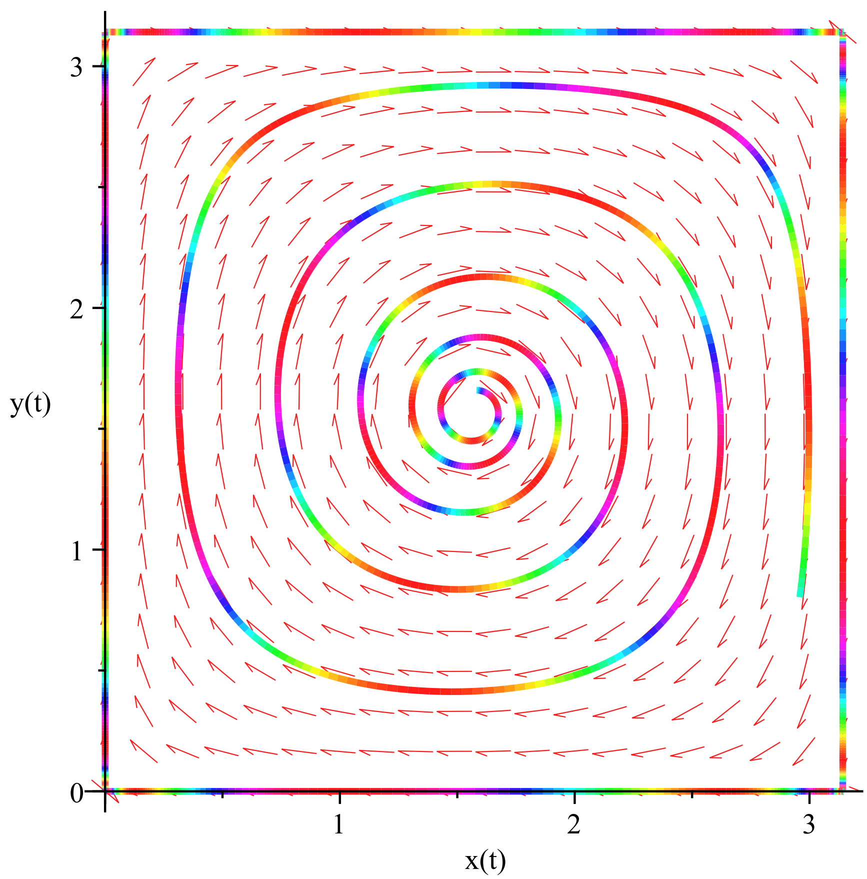

Figure 6: Phase portrait for (51).

Exercise 1. *Find the equilibria of this field and their type. Justify the qualitative elements in Fig. 6.*

In the example above, we see that the limit set is a collection of fixed points and orbits, none of which periodic.

7.1.1 Closed orbits

A closed orbit is a solution whose trajectory is a closed curve (with no equilibria on it). Let $\mathcal{C}$ be such a trajectory.

Note that the trajectory $x(t, X)$ is differentiable, the flow is always in the direction of the field, since

$$\dot{x}(t) = f(x(t))$$

and furthermore, the speed is, as we see from the above

$$|\dot{x}(t)| = |f(x(t))|$$

Since trajectories and f are smooth and there are no equilibria along $\mathcal{C}$, $|\dot{x}(t)| = |f(x)|$ is bounded below, and $\mathcal{C}$ is traversed in finite time. That is, starting at a point $x_1 \in \mathcal{C}$, after a (finite) time T, then, the solution returns to x_1. From that time on, the solution must repeat itself identically, by uniqueness of solutions. It then means that the solution is periodic, and there is a smallest τ so that $\Phi_{t+\tau}(x_1) = \Phi(x_1)$. This τ is called the period of the orbit.

Proposition 6. *(i) If x_1 and x_2 lie on the same solution curve, then $\omega(x_1) = \omega(x_2)$ and $\alpha(x_1) = \alpha(x_2)$.*

(ii) If $\mathcal{P}$ is a closed, positively invariant set and $x_2 \in \mathcal{P}$, then $\omega(x_2) \subset \mathcal{P}$; similarly for negatively invariant sets and $\alpha(x_2)$.

(iii) A closed invariant set, and in particular a limit set, contains the $\alpha-$limit and the $\omega-$limit of every point in it.

Proof. Exercise. □

Exercise 2. *Show that τ is the same for any two points x_1, x_2 on $\mathcal{C}$.*

7.2 Time of arrival

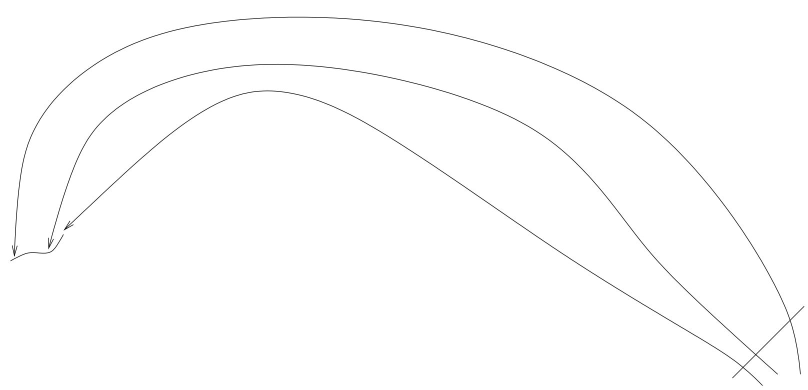

Figure 7: Time of arrival function

We consider all solutions in the domain $\mathcal{O}$ where the field is defined and a section $\mathcal{S}$. Some of the trajectories intersect $\mathcal{S}$. Since the trajectories are continuous, if $x(t, z_0)$ intersects $\mathcal{S}$, then there is a first time of arrival, the smallest t so that $x(t, z_0) \in \mathcal{S}$.

This time of arrival is continuous in z_0, as shown in the next proposition.

Proposition 7. *Let $\mathcal{S}$ be a local section at x_0 and assume $x(t_0; z_0) = x_0$. Let $\mathcal{W}$ be a neighborhood of z_0. Then there is an open set containing z_0, $\mathcal{U} \subset \mathcal{W}$ and a differentiable function $\tau : \mathcal{U} \to \mathbb{R}$ such that $\tau(z_0) = t_0$ and*

$$x(\tau(X), X) \in \mathcal{S} \tag{53}$$

for each $X \in \mathcal{U}$.

Note 2. *In some sense, a small subsegment of the section $\mathcal{S}$ is carried* backwards *smoothly through the field arbitrarily far, assuming that the backward flow exist for a sufficiently long time, and that the subsegment is small enough.*

Proof. A point x_1 belongs to the line ℓ containing $\mathcal{S}$ iff $x_1 = x_0 + uV_0$ for some u. Since V_0 is orthogonal to $f(x_0)$ we see that $x_1 \in \ell$ iff $(x_1 - x_0) \cdot f(x_0) = 0$.

We look now at the more general function

$$G(x, t) = (x(t; X) - x_0) \cdot f(x_0) \tag{54}$$

We have, by assumption

$$G(z_0, t_0) = 0 \tag{55}$$

We want to see whether we can apply the implicit function theorem to

$$G(x, t) = 0 \tag{56}$$

For this we need to check $\frac{\partial}{\partial t} G\big|_{(z_0, t_0)}$. But this equals

$$x'(t; X) \cdot f(x_0)\big|_{(t_0, z_0)} = |f(x_0)|^2 \neq 0 \tag{57}$$

Then, there is a neighborhood of t_0 and a differentiable function $\tau(x)$ so that

$$G(x, \tau(x)) = 0 \tag{58}$$

□

7.3 The Poincaré map

The Poincaré map is a useful tool in determining whether closed trajectories (that is, periodic orbits) are stable or not. This means that taking an initial close enough to the periodic orbit, the trajectory thus obtained would approach the periodic orbit or not.

The basic idea is simple, we look at a section containing a point on the periodic orbit, and then follow the successive re-intersections of the perturbed orbit with the section. Now we are dealing with a discrete map $x_{n+1} = P(x_n)$. If $P(x_n) \to x_0$, the point on the closed orbit, then the orbit is asymptotically stable. See Figure 12.

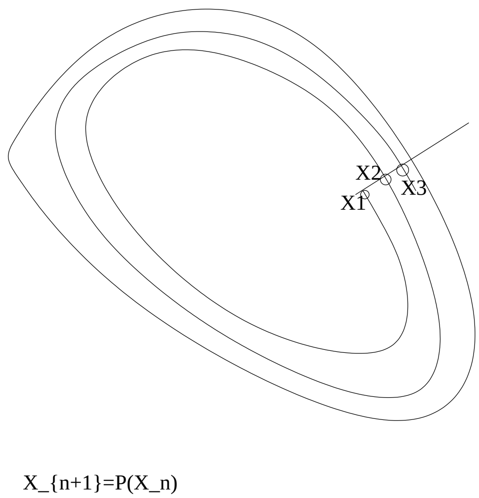

Figure 8: A Poincaré map.

It is often not easy to calculate the Poincaré map; in general it can't quite be easier than calculating the trajectories, but it is a very useful concept, and it has many theoretical applications; furthermore, we often don't need fully explicit knowledge of P.

Let's define the map P rigorously.

Consider a periodic orbit $\mathcal{C}$ and a point $x_0 \in \mathcal{C}$. We have

$$x(\tau; x_0) = x_0 \tag{59}$$

where τ is the period of the orbit. Consider a section $\mathcal{S}$ through x_0. Then according to Proposition 7, there is a neighborhood of $\mathcal{U}$ of x_0 and a continuous function $\tau(x)$ close to the period τ such that $x(\tau(X), X) \in \mathcal{S}$ for all $X \in \mathcal{U}$. Then certainly $\mathcal{S}_1 = \mathcal{U} \cap \mathcal{S}$ is an open set in $\mathcal{S}$ in the induced topology. The return map is thus defined on $\mathcal{S}_1$. It means that for each point in $X \in \mathcal{S}_1$ there is a point $P(X) \in \mathcal{S}$, so that $x(\tau(X); X) = P(X)$ and $\tau(X)$ is the smallest time with this property. Note that now $\tau(x)$ is not a period, though it is "very close to one": the trajectory does not return to the same point.

This is the Poincaré map associated to $\mathcal{C}$ and to its section $\mathcal{S}$.

This can be defined for planar systems as well as for higher dimensional ones, if we now take as a section a subset of a hyperplane through a point $x_0 \in \mathcal{C}$. The statement and proof of Proposition 7 generalize easily to higher dimensions.

In two dimensions, we can identify the segments $\mathcal{S}$ and $\mathcal{S}_1$ with intervals on the real line, $u \in (-a, a)$, and $u \in (-\varepsilon, \varepsilon)$ respectively, see also Definition 5. Then P defines an analogous transformation of the interval $(-\varepsilon, \varepsilon)$, which we still denote by P though this is technically a different function, and we have

$$P(0) = 0$$

$$P(u) \in (-a, a), \quad \forall u \in (-\varepsilon, \varepsilon)$$

We have the following easy result, the proof of which we leave as an exercise.

Proposition 8. *Assume that $x' = f(x)$ is a planar system with a closed orbit $\mathcal{C}$, let $x_0 \in \mathcal{C}$ and $\mathcal{S}$ a section at x_0. Define the Poincaré map P on an interval $(-\varepsilon, \varepsilon)$ as above, by identifying the section with a real interval centered at zero. If $|P'(x_0)| < 1$ then the orbit $\mathcal{C}$ is asymptotically stable.*

Example 3. Consider the planar system

$$r' = r(1-r) \tag{60}$$

$$\theta' = 1 \tag{61}$$

In Cartesian coordinates it has a fixed point, $x = y = 0$ and a closed orbit, $x = \cos t, y = \sin t; x^2 + y^2 = 1$. Any ray originating at $(0, 0)$ is a section of the flow. We choose the positive real axis as $\mathcal{S}$. Let's construct the Poincaré map. Since $\theta' = 1$, the return time is 1, for any $x \in \mathbb{R}^+$ we have $x(2\pi; X) = x(0, X)$. We have $P(1) = 1$ since 1 lies on the unit circle. In this case we can calculate explicitly the solutions, thus the Poincaré map and its derivative.

We have

$$\ln r(t) - \ln(r(t) - 1) = t + C \tag{62}$$

and thus

$$r(t) = \frac{Ce^t}{Ce^t - 1} \tag{63}$$

where we determine C by imposing the initial condition $r(0) = x$: $C = x/(x-1)$. Thus,

$$r(t) = \frac{xe^t}{1 - x + xe^t} \tag{64}$$

and therefore we get the Poincaré map by taking $t = 2\pi$,

$$P(x) = \frac{xe^{2\pi}}{1 - x + xe^{2\pi}} \tag{65}$$

Direct calculation shows that $P'(1) = e^{-2\pi}$, and thus the closed orbit is stable. We could have seen this directly from (65) by taking $t \to \infty$.

Note that here we could calculate the orbits explicitly. Thus we don't quite need the Poincaré map anyway, we could just look at (64). When explicit solutions, or at least an explicit formula for the closed orbit is missing, calculating the Poincaré map can be quite a challenge.

8 Monotone sequences in two dimensions

There are two kinds of monotonicity that we can consider. One is monotonicity along a solution: $x_1, ..., x_n$ is monotone along the solution if $x_n = x(t_n, X)$ and t_n is increasing in n. Or, we can consider monotonicity along a segment, or more generally a piece of a curve. On a piece of a smooth curve, or on an interval we also have a natural order (or two rather), by arclength parameterization of the curve: $\gamma_2 > \gamma_1$ if γ_2 is farther from the chosen endpoint. To avoid this rather trivial distinction (dependence on the choice of endpoint) we say that a sequence $\{\gamma_n\}_n$ is monotone along the curve if γ_n is inbetween γ_{n-1} and γ_{n+1} for all n. Or we could say that a sequence is monotone if it is either increasing or else decreasing.

If we deal with a trajectory crossing a curve, then the two types of monotonicity need not coincide, in general. But for sections, they do.

Proposition 9. *Assume $x(t; X); t \in [0, \tau]$ is a solution of a planar system $x' = f(x)$, s.t. f is regular and nonzero in a sufficiently large region. Let $\mathcal{S}$ be a local section. Then monotonicity along the solution $x(t; X)$ assumed to intersect $\mathcal{S}$ at $x_1, x_2, ...$ (finitely or infinitely many intersection) and along $\mathcal{S}$ coincide.*

Note that all intersections are taken to be with $\mathcal{S}$, along which, by definition, they are always transversal.

Proof. We assume we have three successive distinct intersections with $\mathcal{S}$, x_1, x_2, x_3 (if two of them coincide, then the trajectory is a closed orbit and there is nothing to prove).

We want to show that x_3 is not inside the interval (x_1, x_2) (on the section, or on its image on $\mathbb{R}$). Consider the curve $\mathcal{C}_1 = \{x(t; x_1) : t \in [0, t_2]\}$ where t_2 is the first time of re-intersection of $x(t; x_1)$ with $\mathcal{S}$. By definition $x(t_2 - t_1; x_1) = x_2$. $\mathcal{C}_1$ is a smooth curve, with no self-intersection (since the field is assumed regular along the curve) thus of finite length. If completed with the line segment $\mathcal{J}$ linking x_1 and x_2, $\mathcal{C}_1 \cup \mathcal{J}$ is a closed continuous curve. By Jordan's lemma, we can define the inside int $\mathcal{C}$ and the outside of the curve, D =ext $\mathcal{C}$. Note that the field has a definite direction along $[x_1, x_2]$, by the definition of a section. Note also that it points towards ext $\mathcal{C}$, since $x(t; x_1)$ exits int $\mathcal{C}$ at $t = t_2$. Then, no trajectory can enter int $\mathcal{C}$. Indeed, it should intersect either $x(t; x_1)$ or else $[x_1, x_2]$. The first option is impossible by uniqueness of solutions. The second case is ruled out since the field points outwards from $\mathcal{J}$. Thus $x(t_3, x) = x_3$ must lie in ext $\mathcal{C}$, thus outside $[x_1, x_2]$. □

The next result shows points towards limiting points being special: parts of closed curves, or simply infinity.

Proposition 10. *Consider a planar system and $z \in \omega(x)$ (or $z \in \alpha(x)$), assumed a regular point of the field. Consider a local section $\mathcal{S}$ through a regular point $\tilde{z}$. Then the intersection of $\{\Phi_t(z) : t > 0\} \cap \mathcal{S}$ has at most one point (note that we are dealing with $\Phi_t(z)$ and not $\Phi_t(x)$).*

Proof. Assume there are two distinct intersection points $x(t_1, z) = z_1$ and $x(t_2, z) = z_2$ on $\mathcal{S}$. By Proposition 4, $\{\Phi_t(z) : t > 0\} \subset \omega(x)$; in particular, z_1 and z_2 are also in $\omega(x)$. There are then infinitely many points on $\Phi_t(x)$ arbitrarily close to z_1 and infinitely many others arbitrarily close to z_2, by the definition of $\omega(x)$.

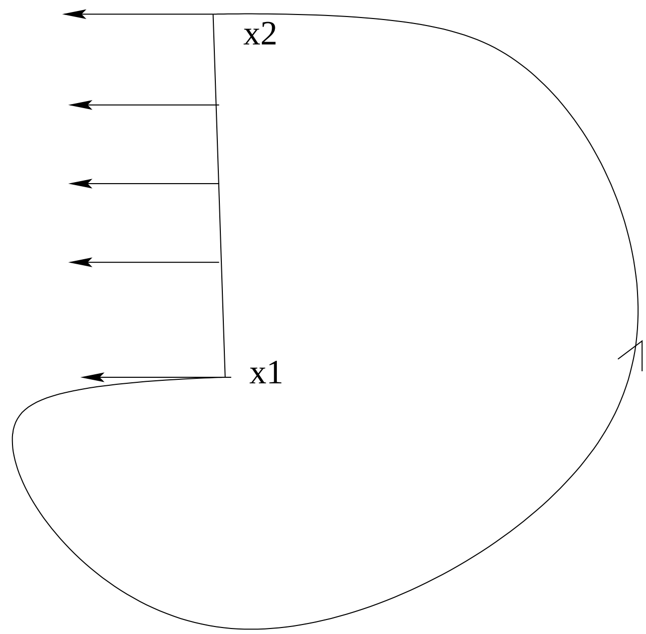

Figure 9: Monotone sequence theorem

We can assume without loss of generality that the points that converge to z_1 lie on $\mathcal{S}$. Indeed, this can be arranged by a small change in t_j as follows:

The first arrival times at $\mathcal{S}$ for the trajectory $\Phi_t(z_1)$ is clearly zero. By the continuity of τ, if j is large enough s.t. $\Phi_{t_j}(x)$ is close to z_1, then $\tau(\Phi_{t_j}(x))$ exists by Proposition 7; $\tau(\Phi_{t_j}(x))$ and is arbitrarily small if j is large enough. Thus, by choosing $t_j + \tau(\Phi_{t_j}(x))$ instead of t_j, we can arrange that $\Phi_{t_j}(x) \in \mathcal{S}$. Similarly, we can arrange that the points converging to z_2 are on $\mathcal{S}$.

Also w.l.o.g. (rotating and translating the figure) we can assume that $\mathcal{S} = (-a, b) \in \mathbb{R}$ and $[z_1, z_2] \subset (-a, b)$. We know that $x(t_j, X)$, where t_j are the increasing times when $x(t_j, X) \in \mathcal{S}$, are monotone in $\mathcal{S} = (-a, b)$. Thus they converge. But then, by definition of convergence, they cannot be arbitrarily close to two distinct points.

$\square$

9 The Poincaré-Bendixson theorem

Theorem 6 (Poincaré-Bendixson)**.** *Let $\Omega = \omega(x)$ be a nonempty compact limit set of a planar system of ODEs, containing no equilibria. Then Ω is a closed orbit.*

Proof. First, recall that Ω is invariant. Let $y \in \Omega$. Then $\Phi_t(y)$ is contained Ω, and then $\Phi_t(y)$ has infinitely many accumulation points in Ω. Let z be one of them and let t_j be s.t. $x(t_j, y) = z + o(1)$. Let $\mathcal{S}$ be a section though z. As in the Proposition 10, we can assume that $x(t_j, y) \in \mathcal{S}$. By Proposition 10, $x(t_j, y) = x(t_{j'}, y) \forall j, j'$, and thus the trajectory is periodic and $t_{j+1} - t_j = T$ is the period.

We have to now show that Ω is a closed orbit (and not a collection of distinct ones).

We take a section through y, and consider the sequence $x(t_j, X)$ of points on $x(t, X)$ approaching y. By the continuity of the Poincaré map and continuity w.r.t. initial conditions, $t_{j+1} - t_j = T + o(1)$ for large j. Thus any $t = t_j + s$ for some j and $s \in (o(1), T + O(1))$, By continuity w.r.t. initial conditions, $x(t_j + s, X) = x(s, y) + o(1)$, thus the distance between $x(s, y)$ and Ω) is zero. □

Exercise 1. *Where have we used the fact that the system is planar? Think how crucial dimensionality is for this proof.*

10 Applications of the Poincaré-Bendixson theorem

Definition. *A limit cycle is a closed orbit γ which is the ω-set, or an $\alpha-$ set of a point $X \notin \gamma$. These are called ω limit cycles or α limit cycles respectively.*

As we see, closed orbits are limit cycles only if *other* trajectories approach them arbitrarily. There are of course closed orbits which are not limit cycles. For instance, the system $x' = -y, y' = x$ with orbits $x^2 + y^2 = C$ for any C clearly has no limit cycles.

$\omega-$ limit cycles have at least one-sided stability.

Corollary 11. *Assume γ is an $\omega-$limit cycle. Then there is a one-sided (or two-sided) neighborhood $\mathcal{N}$ of γ s.t. $X \in \mathcal{N} \Rightarrow \omega(X) = \gamma$.*

Proof. Take a section $\mathcal{S}$ through any point on γ. Similar to the construction for the monotonicity proof, we take the region $\mathcal{R}$ bounded by the trajectory from x_j to x_{j+2}, see Fig. 11. Note that any point X starting on $\mathcal{S}$ in an open neighborhood of some X in (x_j, x_{j+1}) has the property $\omega(X) = \gamma$. Indeed, the blue region in the figure has, by assumption no equilibrium and, because of non-intersection of trajectories and continuity of the return time, if j is large enough, will cross the section $\mathcal{S}$ in a time $t_{j+1} - t_j + o(1)$ somewhere in (x_{j+1}, x_{j+2}), and in general will cross $\mathcal{S}$ in (x_k, x_{k+1}) for all $k > j$. The rest is immediate. □

Corollary 12. *Assume $\omega(X) = \gamma$, $\gamma \not\ni X$ is a limit cycle. Then there exists a neighborhood $\mathcal{O}$ of X s.t. $\forall X' \in \mathcal{O}$ we have $\gamma = \omega(X')$.*

Proof. Let t_0 be large enough so that $\Phi_t(X) \in \mathcal{N}$, the one-sided neighborhood of stability of γ, for all $t \geqslant t_0$. Take any $t_1 > t_0$ and a small enough neighborhood $\mathcal{O}_1$ of $x_1 = \Phi_{t_1}(X)$, so that, in particular, $\mathcal{O}_1 \subset \mathcal{N}$. Clearly, $\Phi_{-t_1}(x_1) = X$. As $\text{diam}(\mathcal{O}_1) \to 0$, we have $\text{diam}(\Phi_{-t_1}(\mathcal{O}_1)) \to 0$ as well, by continuity

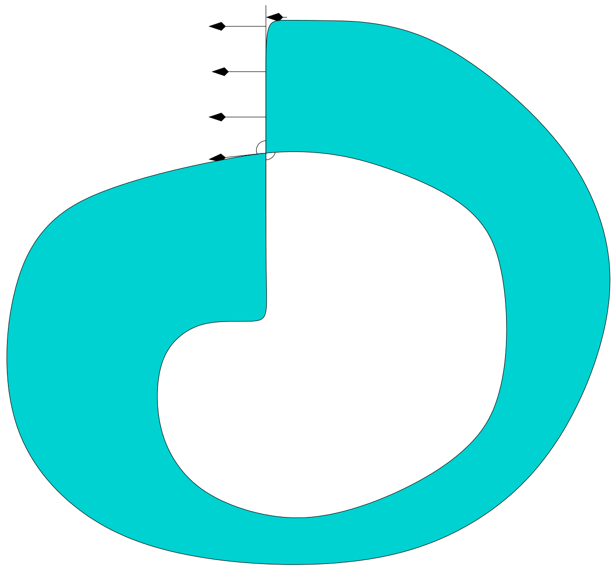

Figure 10: One-sided stability

with respect to initial conditions. Also by continuity of Φ_t, and noting that $\Phi_{-t}(Z) = (\Phi_t)^{-1}(Z)$, we see that $\mathcal{O}_2 := \Phi_{-t_1}(\mathcal{O}_1)$ is an open set, which clearly contains X. By construction, $\omega(X') = \gamma$ for all $X' \in \mathcal{O}_2$. $\square$

Corollary 13. *If a planar system has a first integral J that is not constant in any open set, then it has no limit cycles.*

Proof. Indeed, if $\gamma = \omega(X)$ is a limit cycle for some X, then by Corollary 12 there is a neighborhood $\mathcal{O}_X$ so that $\omega(X') = \gamma$ for all $X' \in \mathcal{O}_X$. We know that J is constant along any trajectory. Let $Y_0 \in \gamma$. By continuity, $J(X') = J(Y_0)$ for any $X' \in \mathcal{O}_X$. $\square$

Corollary 14. *Let $\mathcal{P}$ be a* compact*, simply connected, positively invariant set. Then $\mathcal{P}$ contains at least a limit cycle or an equilibrium.*

Proof. Assume to get a contradiction that there were no equilibria or limit cycles in $\mathcal{P}$. By invariance, $\mathcal{P}$ must contain the ω limit set Ω of any $X \in \mathcal{P}$. By Poincaré-Bendixson, $\omega(X)$ is a closed curve which is not a limit cycle, and thus $X \in \omega(X)$, and $\omega(X)$ is a closed orbit. Take now X_1 in $\text{int}(\omega(X))$; then similarly $\omega(X_1)$ is a closed orbit and $X_1 \in \omega(X_1) \subsetneq \omega(X)$. We can form, by induction, a nested sequence of closed orbits $\omega(X_j)$, each of them strictly contained in the interior of the previous one. We consider now the set of all such nested sequences and let ν be the inf of the areas of the regions inside these $\omega(X_j)$. If $\nu \neq 0$, then we take a nested sequence of closed orbits whose areas converge to ν and let $\mathcal{P}$ be the intersection of all $\omega(X_j) \cup \text{int}(\omega(X_j))$. This is a compact, simply connected invariant set $\mathcal{P}$. If $\mathcal{P}$ has nonempty interior, then for any point X

in int($\mathcal{P}$), $\omega(X)$ is a closed trajectory also contained in int($\mathcal{P}$) (why?) and this $\omega(X)$ necessarily has area $<$ area($\mathcal{P}$) $< \nu$, contradiction. If instead $\mathcal{P}$ has empty interior, then for any $X \in \mathcal{P}$, $\omega(X)$ cannot be a curve, as smooth curves have nonempty interior. Then $\omega(X)$ is an equilibrium, contradiction. □

Corollary 15. *Let γ be a closed orbit and $\mathcal{U}$ its interior. Then $\mathcal{U}$ contains at least an equilibrium.*

Proof of the Corollary. We first show that if there is no equilibrium in $\mathcal{U}$ then there are infinitely many limit cycles. Indeed, $\mathcal{P} = \mathcal{U} \cup \gamma$ is positively invariant and then it must contain a limit cycle. If γ itself is the only limit cycle or equilibrium in $\mathcal{P}$, then, since $\mathcal{P}$ is also negatively invariant, γ is also the $\alpha-$limit set of any point in $\mathcal{P}$, but this would violate monotonicity along sections (check!). If there were finitely many limit cycles in $\mathcal{P}$ then there would be one of minimal area, impossible by the arguments in Corollary 14.

Thus, that there are infinitely many limit cycles γ_n in U. We can furthermore assume they are contained in each other, since each limit cycle contains an equilibrium or yet another limit cycle (strict inclusion). Now we can repeat the last part of the proof of Corollary 14, since a limit cycle is, in particular, a closed orbit. (At the end of that proof, $\omega(X)$ cannot be a closed orbit, otherwise, once more, it would contain an even smaller one.) □

Corollary 16. *If K is positively (or negatively) invariant, then it contains an equilibrium.*

Proof. Combine Corollaries 14 and 15. □

11 The Painlevé property

As mentioned on p.2, Hamiltonian systems (with time-independent Hamiltonian) in one dimension are integrable: the solution can be written in closed form, implicitly, as $H(y(x), x) = c$; in terms of t, once we have $y(x)$ of course we can integrate $x' = G(y(x), x) := f(x)$ in closed form, by separation of variables. The classification of equations into integrable and nonintegrable, and in the latter case finding out whether the behavior is chaotic plays a major role in the study of dynamical systems.

As usual, for an $n-$th order differential equation $x' = f(x)$, a constant of motion is a function $K(u_1, ..., u_n, t)$ with a predefined degree of smoothness (analytic, meromorphic, C^r etc.) and with the property that for any solution $y(t)$ we have

$$\frac{d}{dt}K(y(t), y'(t), ..., y^{(n-1)}(t), t) = 0$$

There are multiple precise definitions of integrability, and no one perhaps is comprehensive enough to be widely accepted. For us, let us think of a system as being integrable, relative to a certain regularity class of first integrals, if there are sufficiently many global constants of motion so that a particular solution can be found by knowledge of the values of the constants of motion.

If f is analytic, it is usually required that K is analytic too, except perhaps for *isolated* singularities (in particular, single-valued; e.g., the log does not have an isolated singularity at zero, whereas $e^{1/x}$ does).

We note once more that an integral of motion needs to be defined in a wide region. The existence of local constants along trajectories follows immediately either from the flowbox theorem, or from the implicit function theorem: indeed, if $x' = f(x)$ is a system of equations near a regular point, x_0, then evidently there exists a local solution $x(t; x_0) = \varphi_t(x_0)$. It is easy to check that $D_{x_0} x_{|t=0} = I$, so we can write, near $x_0, t = 0$, $x_0 = K(x,t) = \Phi_{-t}(x)$. Clearly K is constant along trajectories. Not a very explicit function, admittedly, but smooth, at least locally. K is thus obtained by integrating the equation backwards in time. This is not a very useful constant of motion however, since in general it is only defined for small t: typically for larger t singularities will arise.

Assume now that f is an analytic function, so that it makes sense to extend the equation to $\mathbb{C}$.

If t is in the complex domain, we can in principle circumvent possible singularities, and define K by analytic continuation around singularities. When is this possible? If the singularities are always isolated, and in particular solutions are single valued, it does not matter which way we go. But if these are, say, square root branch points, if we avoid the singularity on one side we get $+\surd$ and on the other $-\surd$. There is no obvious way to prescribe a systematic path of analytic continuation since the Riemann surface is solution-dependent. We will see in the next section that this is typically not a mere failure to find a systematic prescription.

On the other hand, if we impose the condition that the equation have only isolated singularities (at least, those depending on the initial condition, or *movable*, then we have a single valued global constant of motion, take away some lower dimensional singular manifolds in $\mathbb{C}^2$.

Such equations are said to have the Painlevé property (PP) and are integrable, at least in the sense above. But it turns out, in those considered so far in applications, that more is true: they were all ultimately reduced to linear equations.

Failure of the Painlvé property and nonintegrability

In the case the movable singularities of solutions of a meromorphic equation are branch points we don't expect simple, closed form solutions which are single-valued in $\mathbb{C}$. Indeed, assume that the solution $y(z)$ is given by $\Phi(z, y(z)) = 0$ where Φ is nontrivial, and analytic (or meromorphic) in $\mathbb{C}^2$. Then Φ should be constant along the trajectory. We follow the solution y on a Riemann surface avoiding the singularities. Since the solution is not single valued, after surrounding one singularity we end up with $y_1(z)$, a solution of the same ODE, but *a different one*. The expectation is that by wandering "randomly" around branch points we generate a family of solutions dense in the space of all solutions (*dense branching*). This is because of the huge amount of freedom we have in choosing the continuation path. In case of dense branching– the typical situation in

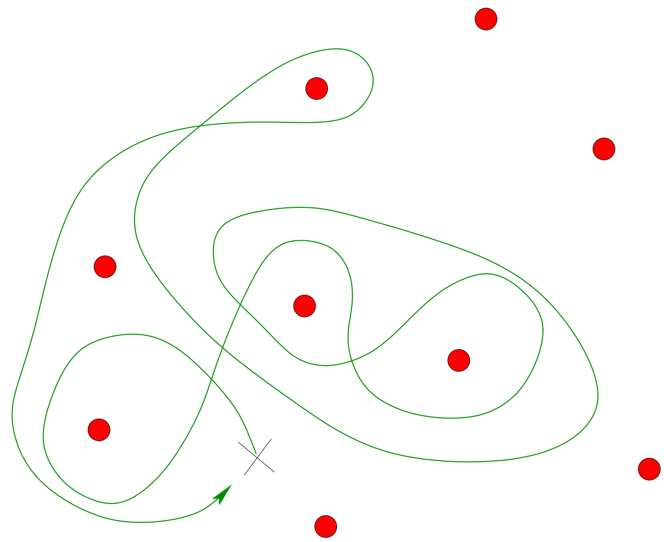

Figure 11: A continuation path for a solution with movable singularities

fact– then $\Phi(z, y)$ takes the same value on a dense set of y and thus it does not depend on y; of course it cannot depend only on x and thus Φ is a number, contradiction. One of course has to check whether in a particular ODE dense branching occurs, but this is generically the case and failure of the PP is a "red flag" when trying to solve equations in any explicit way.

11.1 The Painlevé equations

11.2 Spontaneous singularities: The Painlevé's equation $\mathbf{P}_{\mathrm{I}}$

Let us analyze local singularities of the Painlevé equation P_{I},

$$y'' = y^2 + x \tag{66}$$

The standard existence and uniqueness theorem guarantees that there is a unique solution in any region where y is bounded, and this solution is analytic.

In a neighborhood of a point where y is large, keeping only the largest terms in the equation (*dominant balance*) we get $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we may search for a power-like behavior

$$y \sim A(x - x_0)^p$$

where $p < 0$ obtaining, to leading order, the equation $Ap(p-1)x^{p-2} = A^2(x - x_0)^2$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). Let's look for a power series solution, starting with $6(x - x_0)^{-2}$:

$y = 6(x-x_0)^{-2} + c_{-1}(x-x_0)^{-1} + c_0 + \cdots$. We get: $c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = -x_0/10, c_3 = -1/6$ and c_4 is undetermined, thus free. Choosing a c_4, all others are uniquely determined.

Series solutions at movable singularities for neighboring equations

It is convenient to make the substitutions $y(x) = 6(x-x_0)^{-2} + \delta(x)$ where for consistency we should have $\delta(x) = o((x-x_0)^{-2})$ and taking $x = x_0 + z$ we get the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{67}$$

In this form, the singular point of y is now placed at $z = 0$ and zero is a singularity of the equation.

Note that with the standard substitution that we used for Frobenius systems, $u_1 = \delta, u_2 = z\delta'$ we get the system

$$\begin{aligned} u_1' &= z^{-1}u_2 \\ u_2' &= 12z^{-1}u_1 + z^{-1}u_2 + zu_1^2 + z^2 + zx_0 \end{aligned} \tag{68}$$

which can be extended to an autonomous system by adding $\dot{z} = z$, and P1 becomes equivalent to

$$\begin{aligned} \dot{u_1} &= u_2 \\ \dot{u_2} &= 12u_1 + u_2 + z^2u_1^2 + z^3 + z^2x_0 \\ \dot{z} &= z \end{aligned} \tag{69}$$

where 0 is a critical point of the field. The linearized matrix at zero M of this system and its corresponding diagonal form D are given by

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 12 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{70}$$

we see that the system is resonant, with eigenvalues in the Siegel domain.

Typically therefore we do not expect local analytic linearization. Typical would mean here that we allow for generic nonlinear monomials instead of the specific ones.

However, for P1, by "accident" the solutions of (67) are locally analytic.

Substituting

$$\delta(z) = c_0 + c_1 z + ...$$

in (67) we get

$$-12c_0z^{-2} - 12c_1z^{-1} + ... = 0$$

forcing $c_0 = c_1 = 0$. Thus a power series would have the form

$$\delta(z) = \sum_{k=2}^{\infty} c_k z^k \tag{71}$$

which, used in (67), gives

$$[-10c_2 - x_0] + (-6c_3 - 1)\,z + 8c_5 z^3 + \left(18c_6 - {c_2}^2\right) z^4 + (30c_7 - 2c_2c_3)\,z^5 \\ + \left(-2c_2c_4 - {c_3}^2 + 44c_8\right) z^6 + \cdots = 0 \quad (72)$$

We note that the coefficient of z^2 is zero, and c_4 is undetermined. For any value of c_4, the recurrence for c_k determines uniquely these coefficients.

The fact that c_4 is not determined is due to 4 being an eigenvalue of the linear part, and that *there are no obstructing monomials*. If we take a modification of (67), for instance

$$\delta'' = \frac{12}{z^2}\delta + z + az^2 + x_0 + \delta^2 \quad (73)$$

we get as before $c_0 = 0, c_1 = 0$ and

$$[-10c_2 - x_0] + (-6c_3 - 1)\,z + az^2 + 8c_5 z^3 + \left(18c_6 - c_2^2\right) z^4 + \cdots = 0 \quad (74)$$

Now an equation for c_4 is still missing, and the term z^2 cannot be eliminated (unless $a = 0$, which is the original P_1.) As in the linear case, we expect $\log z$ to appear in the expansion. Indeed, substituting

$$\delta(z) = \sum_{k=2}^{6} c_k z^k + Az^4 \ln z \quad (75)$$

in (73) we get

$$-10c_2 - x_0 + (-6c_3 - 1)\,z + (7A + a)\,z^2 + 8c_5 z^3 + \left(18c_6 - c_2^2\right) z^4 + \cdots = 0 \quad (76)$$

which is now solvable (at least to order 6).

Existence of a convergent power series for δ in (67)

To show that there indeed is a convergent such power series solution we substitute Note now that our assumption $\delta = o(z^{-2})$ makes $\delta^2/(\delta/z^2) = z^2\delta = o(1)$ and thus the nonlinear term in (67) is *relatively* small. Thus, *to leading order*, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximately by a linear equation. It is then natural to separate out the large terms from the small terms and write a fixed point equation for the solution based on this separation. We write (67) in the form

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \quad (77)$$

and integrate as if the right side were known. This leads to an equivalent integral equation. Since all unknown terms on the right side are chosen to be *relatively smaller*, by construction this integral equation is expected to be contractive.

Click here for Maple file of the formal calculation ($y'' = y^2 + x$)

The indicial equation for the Euler equation corresponding to the left side of (77) is $r^2 - r - 12 = 0$ with solutions $4, -3$ (same as the eigenvalues of the linearized matrix, of course). By the method of variation of parameters we thus get

$$\delta = \frac{D}{z^3} - \frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta^2(s)ds + \frac{z^4}{7}\int_0^z s^{-3}\delta^2(s)ds$$
$$= -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \quad (78)$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$; C is arbitrary. To find δ formally, we would simply iterate (78) in the following way: We take $r := \delta^2 = 0$ first and obtain $\delta_0 = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4$. Then we take $r = \delta_0^2$ and compute δ_1 from (78) and so on. This yields:

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + \ldots \quad (79)$$

This series is actually convergent. To see that, we scale out the leading power of z in δ, z^2 and write $\delta = z^2 u$. The equation for u is

$$u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z s u^2(s)ds$$
$$= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \quad (80)$$

It is straightforward to check that, given C_1 large enough (compared to $x_0/10$ etc.) there is an ε such that this is a contractive equation for u in the ball $\|u\|_\infty < C_1$ in the space of analytic functions in the disk $|z| < \varepsilon$. We conclude that δ is analytic and that y is meromorphic near $x = x_0$. **Note.** The analysis above does *not* prove that the solutions are meromorphic functions in $\mathbb{C}$ (why not?).

Exercise 1. Show that $y'' = y^2 + P(x)$ where P is a polynomial has the Painlevé property **iff** $P(x) = ax + b$ where, modulo elementary changes of variables, $a = 1, b = 0$.

Click here for Maple file of the formal calculation, for $y'' = y^2 + x^2$

11.2.1 The six Painlevé equations

This is the list of the six Painlevé equations:

$$
\begin{aligned}
P_1 \quad w'' &= 6w^2 + z \\
P_2 \quad w'' &= 2w^3 + zw + \alpha \\
P_3 \quad w'' &= \frac{1}{w}(w')^2 - \frac{1}{z}w' + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \\
P_4 \quad w'' &= \frac{1}{2w}(w')^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \\
P_5 \quad w'' &= \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{1}{z}\frac{dw}{dz} \\
&\quad + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \\
P_6 \quad w'' &= \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{dw}{dz} \\
&\quad + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right) \qquad (81)
\end{aligned}
$$

A nonintegrable example

The following is a nonintegrable case of the Abel class of ODEs:

$$y' = y^3 + x \tag{82}$$

We claim that all singularities are branch points. First, we note that the standard existence and uniqueness theorem guarantees that the solution of (82) is analytic in any region where y is bounded; for a point x_0 to be singular we need that $y \to \infty$ as $x \to x_0$. Near the singular point we must have $y' \sim y^3$ which by direct integration gives $y \sim \pm(2x)^{-1/2}$.

To show that this is indeed the behavior near a point z_0 where y blows up, we take $y = 1/u, x = z_0 + z$ and get

$$\frac{dz}{du} = -\frac{u}{1 + u^3 z_0 + u^3 z} \tag{83}$$

in this presentation $u = 0, z = 0$ is a regular point of the ODE, and the solution is analytic. It is clear that $z = -\frac{u^2}{2}(1 + h(u))$ where h is analytic and $h(0) = 0$. We leave it as a straightforward exercise to check that this implies the existence of two solutions of (82) in the form $y = \pm(x - x_0)^{-1/2} H(\sqrt{x - x_0})$ where H is analytic.

Exercise 2. Show that an equation of the form

$$y' = P(y) + Q(x) \tag{84}$$

where P and Q are polynomials has the Painlvé property iff P is quadratic, in which case the equation is Riccati, thus integrable.

12 Asymptotics of ODEs: first examples

Asymptotic behavior typically refers to behavior near an irregular singular point.

Remember from Frobenius theory that regular singular points (say $z = 0$) of an nth order ODE are characterized by the order of the poles relative to the order of differentiation. Homogeneous equations with regular singularities are of the form

$$y^{(n)} + \frac{A_1(z)}{z} y^{(n-1)} + \cdots \frac{A_j(z)}{z^j} y^{(n-j)} + \cdots \frac{A_n(z)}{z^n} y = 0 \tag{85}$$

where $A_j(z)$ are analytic at zero.

A singularity is regular **iff** there is a fundamental system of solutions in the form of a finite combination of terms of the form $z^a, \mathcal{A}(z), \ln^j z$ where a may be complex, $j \leqslant n-1$, $\mathcal{A}(z)$ analytic.

Thus the general solution at an irregular singular point is *not* given by a convergent power series. Two things can happen:

- Solutions do not have power-like behavior (usually this means exponential behavior).
- Series exist but are divergent.

Consider first the very simple ODE

$$y' = Ay/z^p; \quad p > 1 \tag{86}$$

near $z = 0$. The general solution is

$$y = C \exp(-Az^{-p+1}/(p-1)) \tag{87}$$

Note that this function has no power series at $z = 0$ (in $\mathbb{C}$); the behavior is exponential.

Most often, irregular singularities are placed at infinity (to characterize a singularity at infinity, make the substitution $z = 1/x$). Then, in first order equations with coefficients behaving polynomially, infinity is an irregular singular point if the equation is of the form $y' = Ax^q(1 + o(1))y,\ q > -1$. Equation (86), after the transformation $z = 1/x$ becomes

$$y' = ax^q y;\ a = -A, q = p - 2 \tag{88}$$

and infinity is an irregular singular point if $q > -1$, and the solution is given by

$$y = C \exp\left(\frac{ax^{q+1}}{q+1}\right) \tag{89}$$

For the second new phenomenon, consider the equation

$$y' = -y + 1/x; \quad y \to \infty \tag{90}$$

We can make it homogeneous by multiplying by x and differentiating once more. By taking $z = 1/x$ you convince yourself that the resulting equation is second order with a fourth order pole at zero.

Eq. (90) has a power series solution. Indeed, inserting

$$y = \sum_{k=0}^{\infty} c_k / x^k \tag{91}$$

in (90) we get $c_k = (k-1)c_{k-1}; \quad c_1 = 1 \;\Rightarrow c_k = (k-1)!$ and thus

$$y = \sum_{k=0}^{\infty} k!/x^{k+1} \tag{92}$$

The domain of convergence of this expansion in *empty.*

Many equations for special functions have an irregular singularity at infinity. Typical equations

1. Bessel:
$$y'' + x^{-1}y' + (1 - \alpha^2/x^2)y = 0 \tag{93}$$

2. Parabolic cylinder functions
$$y'' + (\nu + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0 \tag{94}$$

3. Airy functions
$$y'' = xy \tag{95}$$
as well as many nonlinear ones

4. Elliptic functions
$$y'' = y^2 + 1 \tag{96}$$

5. Painlevé P1
$$y'' = 6y^2 + x \tag{97}$$
etc.

It is important to understand the behavior of irregular singularities. Start again from the example (86). It is clear that the singularity remains irregular if z^{-p} is replaced by $z^{-p} + \cdots$ where $\cdots$ are terms with *higher* powers of z.

The WKB method

Given the exponential behavior at an irregular singular point, it is natural to make an *exponential substitution* $y = e^w$. Of course, at the end, we re-obtain the solution we had before. Only, the equation for w' will admit power-like, instead of exponential behavior.

This substitution works in much more generality, and it is behind what is known as the WKB method.

Let's illustrate this on second order equations with rational coefficients:

$$y'' + R_1 y' + R_2 y = 0; \quad R_1, R_2 \text{ rational} \tag{98}$$

A Liouville transformation $y = \exp(-\frac{1}{2}\int R_1)$ transforms (100) into

$$u'' + (R_2 - \frac{1}{2}R_1 - \frac{1}{4}R_1^2)u = 0 \tag{99}$$

and thus we can assume without loss of generality that the equation was of the form

$$y'' - Ry = 0; \quad R = P/Q \ \ P, Q \text{ polynomials} \tag{100}$$

to start with. The singularity at infinity is irregular iff $\deg P \geqslant \deg Q - 1$. The substitution suggested by the previous discussion is $y = e^W$. This gives

$$W'^2 + W'' = R \tag{101}$$

It is easy to see theat the dominant balance is $W'^2 \sim R$. Then, $W \sim x^a$, $a = \deg P - \deg Q$. Since the differential equation can be written in integral form, the asymptotic behavior is differentiable. This means

$$W'^2 \sim x^{2a-2} \gg x^{a-2} = W'' \tag{102}$$

The balance $W'^2 \gg W''$ is quite universal in WKB-like problems. Then we write the equation as

$$f = \pm\sqrt{R - f'}; \quad f = W' \tag{103}$$

and iterate under the assumption (102). This implies $f' \ll R$, and with the plus sign we get

$$f^{[n+1]} = \sqrt{R - f'^{[n]}}; \quad f^{[0]} = 0 \tag{104}$$

We get

$$\begin{aligned}
f^{[1]} &= \sqrt{R} + \cdots \\
f^{[2]} &= \sqrt{R} - \frac{1}{4}\frac{R'}{R} + \cdots \\
f^{[3]} &= \sqrt{R} - \frac{1}{4}\frac{R'}{R} - \frac{5R'^2 - 4RR''}{R^{5/2}} + \cdots
\end{aligned} \tag{105}$$

We see that, if $R \sim x^a$, then

$$R'/R \sim x^{-1}; \quad \frac{5R'^2 - 4RR''}{R^{5/2}} \sim x^{-2-a/2} \tag{106}$$

confirming the asymptotic nature of the expansion: the successive corrections have more and more negative powers of x. By integration

$$W=\int_a^x\sqrt{R(s)}ds-\frac{1}{4}\ln R+\frac{1}{8}\frac{R'}{R^{\frac{3}{2}}}+\frac{1}{32}\int_a^x\frac{R(s)'^2}{R(s)^{\frac{5}{2}}}ds \tag{107}$$

giving

$$y\sim R^{-\frac{1}{4}}e^{\int_a^x\sqrt{R(s)ds}}\left(1+\frac{1}{8}\frac{R'}{R^{\frac{3}{2}}}+\frac{1}{32}\int_a^x\frac{R(s)'^2}{R(s)^{\frac{5}{2}}}ds+\cdots\right) \tag{108}$$

Similarly, the minus sign results in

$$y\sim R^{-\frac{1}{4}}e^{-\int_a^x\sqrt{R(s)ds}}\left(1-\frac{1}{8}\frac{R'}{R^{\frac{3}{2}}}-\frac{1}{32}\int_a^x\frac{R(s)'^2}{R(s)^{\frac{5}{2}}}ds+\cdots\right) \tag{109}$$

12.1 Example: the Airy equation (95)

Substituting $y=e^w$ in (95) we get

$$w''+{w'}^2=x \tag{110}$$

or, choosing the plus sign,

$$f=\sqrt{x-f'}=\sqrt{x}-\frac{f'}{2\sqrt{x}}-\frac{{f'}^2}{8x^{3/2}}+\cdots \tag{111}$$

The sequence of iterations (105) gives

$$\begin{aligned} f^{[0]}&=\sqrt{x}\\ f^{[1]}&=\sqrt{x}-\frac{1}{4x}\\ f^{[2]}&=\sqrt{x}-\frac{1}{4x}-\frac{5}{32}x^{-5/2} \end{aligned} \tag{112}$$

etc. In terms of w, we get

$$w=C_1+\frac{2}{3}x^{3/2}-\frac{1}{4}\ln x+\frac{5}{48}x^{-3/2} \tag{113}$$

and thus

$$y\sim Ce^{\frac{2}{3}x^{\frac{3}{2}}}x^{-1/4}\left(1+\frac{5}{48}x^{-3/2}+\cdots\right) \tag{114}$$

We justify this asymptotic expansion next.

12.1.1 Rigorous justification of the asymptotics for (95)

Theorem 7. *There exist two linearly independent solutions of* (95) *with the (two) asymptotic behaviors (corresponding to different choices of sign)*

$$y_\pm \sim e^{\pm\frac{2}{3}x^{\frac{3}{2}}} x^{-1/4}(1+o(1)) \text{ as } x \to +\infty \tag{115}$$

A similar analysis can be performed for $x \to -\infty$.

Proof. It is enough to show that $w_\pm - [\pm\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\ln x] \to 0$ as $x \to \infty$. We choose the sign $+$, for which the analysis is slightly more involved. Define $w = \sqrt{x} + g$ and consider the equation for g,

$$g' + 2\sqrt{x}g = -\frac{1}{2\sqrt{x}} - g^2 \tag{116}$$

or, more generally

$$g' + 2\sqrt{x}g = H(x) \tag{117}$$

The differential equation (117) with initial condition $g(x_0) = 0$ (chosen for simplicity) where $x_0 > 0$ will be chosen large, is equivalent to

$$g(x) = \mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} H(s)\, \mathrm{e}^{4/3\, s^{3/2}} ds \tag{118}$$

In our specific case, we have

$$H(x) := -\frac{1}{2\sqrt{x}} - g(x)^2 \tag{119}$$

and thus

$$g(x) = -\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds - \mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} g^2(s) \mathrm{e}^{4/3\, s^{3/2}} ds \tag{120}$$

What is the expected behavior of the first integral? We can see this by L'Hospital (which, you can check, applies). We have

$$\frac{\left(\displaystyle\int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds\right)'}{\left(\dfrac{\mathrm{e}^{4/3\, x^{3/2}}}{4x}\right)'} = \frac{1}{1 - 2x^{-3/2}} \to 1 \ (x \to +\infty) \tag{121}$$

and thus

$$= -\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds \sim -\frac{1}{4x} \quad x \to +\infty \tag{122}$$

Let's more generally, look at the behavior of

$$\int_{x_0}^{x} s^n e^{As^m} ds \tag{123}$$

where $m > 0$ We look at the value of l for which, by L'Hospital, we would get

$$\frac{\left(\int_{x_0}^{x} x^n e^{Ax^m}\right)'}{\left(x^l e^{Ax^m}\right)'} = \frac{x^{n+1-l-m}}{Am + lx^{-m}} \to C \tag{124}$$

where $C \neq 0$ is some constant. We need $l = n - m + 1$ and $C = (Am)^{-1}$. In particular,

$$\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} s^{-a} \mathrm{e}^{4/3\,s^{3/2}} ds \sim \frac{1}{2} x^{-a-1/2} \;\; x \to +\infty \tag{125}$$

Since the behavior of the first term of (118) is $-1/(4x)$, consistent with our formal WKB analysis and thus g should be $O(1/x)$, this suggests we write $g = u/x$. We get

$$\begin{aligned} &u\,(x) \\ &= -x\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\,s^{3/2}} ds - x\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} u^2(s) s^{-2} \mathrm{e}^{4/3\,s^{3/2}} ds =: \mathcal{N}u \end{aligned} \tag{126}$$

We analyze this equation in $L^\infty[x_0, \infty)$. We first need bounds on the main ingredients of (126), that is on integrals of the form

$$\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} s^{-a} \mathrm{e}^{4/3\,s^{3/2}} ds \tag{127}$$

which are valid on $[x_0, \infty)$ and not merely as $x \to \infty$. The asymptotic information (125) is helpful, but concrete bounds would provide more information (though this is not necessary if we merely want to prove an expansion as $x \to \infty$). From the limiting information, it follows that for any $A > 1$, if x_0 is large enough, we have

$$\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} s^{-a} \mathrm{e}^{4/3\,s^{3/2}} ds < A \frac{1}{2x^{a+1/2}}, \tag{128}$$

To find a specific x_0, we look at

$$f(x) = \int_{x_0}^{x} s^{-a} \mathrm{e}^{4/3\,s^{3/2}} ds - A\mathrm{e}^{4/3\,x^{3/2}} \frac{1}{2x^{a+1/2}} \tag{129}$$

We note that $f(x_0) < 0$. Calculating f', we get

$$\begin{aligned} f'(x) &= x^{-a}\mathrm{e}^{4/3\,x^{3/2}} - Ax^{-a}\mathrm{e}^{4/3\,x^{3/2}} \left(1 - \frac{a+1/2}{2x^{3/2}}\right) \\ &= -x^{-a}\mathrm{e}^{4/3\,x^{3/2}} \left[A - 1 - \frac{A(a+1/2)}{2x^{3/2}}\right] \end{aligned} \tag{130}$$

It is clear that $f' < 0$ if $x > x(A)$ where

$$x(A)^{3/2} = \frac{A(1+2a)}{4(A-1)} \tag{131}$$

Thus we proved

Lemma 17. *If $A > 1$, with $x(A)$ as given in (131) we have*

$$\int_{x(A)}^{x} s^{-a} e^{4/3\, s^{3/2}} ds < A e^{4/3\, x^{3/2}} \frac{1}{2x^{a+1/2}} \tag{132}$$

for all $x > x(A)$.

We will now write (126) in contractive form in a suitable ball in $L^\infty[x_0, \infty)$. We will make some choices of A, x_0 etc, to write down something specific. The proof of the theorem is completed by the following result.

Lemma 18. *Let $A = 2$ and $x_0 \geqslant x(A), 2\}$. Consider the ball*

$$B = \{u : \sup_{x \geqslant x_0} |u(x)| \leqslant 1\} \tag{133}$$

Then $\mathcal{N}$ is contractive in B, and thus (126) has a unique solution u_0 there.

Proof of the lemma. It is straightforward to check that $\mathcal{N}B \subset B$. We have

$$|\mathcal{N}(u_2 - u_1)| = \left| x e^{-4/3\, x^{3/2}} \int_{x_0}^{x} (u_2 - u_1)(u_2 + u_1) s^{-2} e^{4/3\, s^{3/2}} ds \right|$$
$$\leqslant \|u_2 - u_1\| \frac{2|x|}{|x|^{5/2}} \leqslant \frac{2}{|x_0|^{3/2}} \|u_2 - u_1\| \leqslant 2^{-1/2} \|u_2 - u_1\| \tag{134}$$

□

On the other hand, as $x \to \infty$, using (125) and the fact that $\|u_0\| < 1$, we have

$$g = -\frac{1}{4x} + o(1/x) \quad \text{as} \quad x \to \infty \tag{135}$$

□

Nonlinear ODEs

Some nonlinear ODEs admit special solution that have asymptotic expansions at infinity. Such is the case of many Painlevé equations, the Abel equation discussed before, and more generally ODEs having that can be brought to the form

$$y' = \Lambda y + x^{-1} B y + F(1/x, y); \quad F(z, y) \text{ analytic near } (0,0),\ F = o(x^{-m}, y^2) \tag{136}$$

where m is large enough and Λ and B are constant matrices.

Consider Abel's equation (82) in the limit $x \to +\infty$. We first find the asymptotic behavior of solutions formally, and then justify the argument. We use the method of *dominant balance* that we will discuss in detail later. As x becomes large, y, y', or both need to become large if the equation (82) is to hold. Assume first that the balance is between y' and x and that $y \ll x$. If $y' \sim x$ then we have $y \sim x^2/2$ and $y^3 \sim x^6/8$, and this is inconsistent since it would imply $x^8/8 = O(x)$. Now, if we assume $x \ll y^3$ then the balance would be $y' \approx y^3$, implying $y \sim -\frac{1}{2}(x - x_0)^{-2}$; but this is small for $x - x_0 \gg 1$, which conflicts with what we assumed, $x \ll y^3$. We have one possibility left: $y = \omega x^{1/3}(1 + o(1))$, where $\omega^3 = -1$, which assuming differentiability implies $y' = O(x^{-2/3})$ which is now consistent. We substitute

$$y = \omega x^{1/3}(1 + v(x)) \tag{137}$$

in (82); for definiteness, we choose $\omega = e^{i\pi/3}$, though any cube root of -1 would work. We get

$$\omega x^{1/3}v' + 3xv + 3xv^2 + xv^3 + \frac{\omega}{3}x^{-2/3} + \frac{\omega}{3}x^{-2/3}v = 0 \tag{138}$$

Now a consistent balance is between $3xv$ and $-\frac{\omega}{3}x^{-2/3}$ meaning that $v = O(x^{-5/3})$.

To determine the power series formally, we would keep $3xv$ on the left side, place all other terms on the right and iterate, starting with $v^{[0]} = 0$.

For the purpose of justifying the analysis we place the formally largest term(s) containing v *and* v' on the left side and the smaller terms as well as the terms not depending on v on the right side:

$$\omega x^{1/3}v' + 3xv = h(x, v(x));\ \ -h(x, v(x)) := 3xv^2 + xv^3 + \frac{\omega}{3}x^{-2/3} + \frac{\omega}{3}x^{-2/3}v \tag{139}$$

We treat (139) as a linear inhomogeneous equation, and solve it thinking for the moment that h is given.

This leads to

$$v = \mathcal{N}(v);$$
$$\mathcal{N}(v) := Ce^{-\frac{9}{5\omega}x^{5/3}} + \frac{1}{\omega}e^{-\frac{9}{5\omega}x^{5/3}}\int_{x_0}^{x} e^{\frac{9}{5\omega}s^{5/3}}s^{-1/3}h(s, v(s))ds \tag{140}$$

We chose the limits of integration in such a way that the integrand is maximal when $s = x$: if $x \to +\infty$, then $x^{-1/3}e^{\frac{9}{5\omega}x^{5/3}} \to \infty$, and our choice corresponds indeed to this prescription.

The largest of the terms not containing v on the right side of (140) comes from the term $\frac{\omega}{3}x^{-2/3}$ in h, and is of the order $\frac{1}{3}x^{-5/3}(1 + o(1))$. Indeed, (125) gives

$$\frac{\int_a^x e^{bs^m}/s^n ds}{e^{bx^m}/x^n} \sim b^{-1}m^{-1}x^{1-m};\ \ x \to +\infty \tag{141}$$

Again by dominant balance, we expect $v = O(x^{-5/3})$. Thus, it is natural to choose x_0 large enough and introduce the Banach space

$$\{f : \|f\| := \sup_{x > x_0} |x^{5/3} f(x)| < \infty\} \tag{142}$$

or the region $|x| > x_0$ in a sector $\mathcal{S}$ in the complex domain where $\mathrm{Re}\left(\frac{1}{\omega} x^{5/3}\right) > 0 :\ \arg x \in (-\frac{\pi}{10}, \frac{\pi}{2})$:

$$\mathcal{B} = \{f : \|f\| := \sup_{x \in \mathcal{S}} |x^{5/3} f(x)| < \infty\} \tag{143}$$

and within this space a ball of size large enough $-\frac{2}{3}-$ to accommodate for the largest term on the right side, $\frac{\omega}{3} x^{-2/3}$:

$$B_1 := \{f \in \mathcal{B} : \|f\| \leqslant \frac{2}{3}\} \tag{144}$$

Lemma 19. *For given C, if x_0 is large enough, then the operator $\mathcal{N}$ is contractive in B_1 and thus* (140) *(as well as* (139)*) has a unique solution there.*

Proof. We first check that $\mathcal{N}(B_1) \subset B_1$, by estimating each term in $\mathcal{N}$. By (141) we have for large enough x_0, $|\mathcal{N} x^{-m}| = \frac{1}{3}|x|^{-m-1}(1+o(1))$. In particular, $|\mathcal{N}\frac{\omega}{3} x^{-2/3}| \leqslant \frac{\omega}{9}|x|^{-5/3}(1+o(1))$. The contribution of the other terms are much smaller. For instance, $|xv^2| < Cx^{1-5/2}\|v\|$ we have $|\mathcal{N}(xv^2)| = C|x|^{-5/2}(1 + o(1))$.

To show contractivity, we note that, for $k > 1$,

$$|\mathcal{N}(v_2^k - v_1^k)| \leqslant k\|v_2 - v_1\||\mathcal{N}\left[x^{-5/3} 2(2/3)^{k-1} x^{-5(k-1)/3}\right]$$

□

Note 4. We see that, in the way above, we *cannot*, in principle solve the given ODE for any IC, that is for any C and x_0: for a given C we need a large enough x_0, but this allows for "small" IC only.

13 Elements of eigenfunction theory–material complementary to Coddington-Levinson

13.1 Properties of the Wronskian of a system

Lemma 20. *Let A be a matrix on $\mathbb{C}^n$. We have*

$$\det(I + \varepsilon A) = 1 + \varepsilon \mathrm{Tr}\ A + O(\varepsilon^2) \quad as \quad \varepsilon \to 0 \tag{145}$$

Proof 1. The property is obvious for

$$\begin{pmatrix} 1 + \varepsilon a_{11} & \varepsilon a_{12} \\ \varepsilon a_{21} & 1 + \varepsilon a_{22} \end{pmatrix} \tag{146}$$

For the general case, use induction and row expansion. □

Proof 2. Note that $\det B = \prod_j (1 + b_j)$, where b_j are the eigenvalues of B (repeated if the multiplicity is not one). If $(I + \varepsilon A)v = \mu v$ then $\varepsilon A v = (\mu - 1)v$ that is, $v = v_j$ is an eigenvector of A: $Av_j = a_j v$. Thus $(1+\varepsilon a_j)v_j = (I+\varepsilon A)v_j = \mu v_j \Rightarrow \mu = (1 + \varepsilon a_j)$. The property now follows. $\square$

13.1.1 The Wronskian

The definition is

$$W[f_1, ..., f_n] = \begin{vmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix} \tag{147}$$

Lemma 21. *Let*

$$M' = AM \tag{148}$$

be a matrix equation in $\mathbb{C}^n$. We have

$$\det M(t) = \det M(0) \exp\left(\int_0^t \operatorname{Tr} A(s) ds \right) \tag{149}$$

Proof. We have (just by differentiability)

$$M(t + \varepsilon) - M(t) = A(t)M(t)\varepsilon + o(\varepsilon) \tag{150}$$

and thus

$$\begin{aligned} M^{-1}(t)M(t+\varepsilon) &= I + M^{-1}(t)A(t)M(t)\varepsilon + o(\varepsilon) \\ \Rightarrow \det\,\big(M^{-1}(t)M(t+\varepsilon)\big) &= \det\,\big(I + M^{-1}(t)A(t)M(t)\varepsilon + o(\varepsilon)\big) \\ &= 1 + \operatorname{Tr}(A)\varepsilon + o(\varepsilon) \end{aligned} \tag{151}$$

and thus

$$\begin{aligned} \frac{\det M(t+\varepsilon)}{\det M(t)} = 1 + \operatorname{Tr}(A)\varepsilon + o(\varepsilon) \Rightarrow \frac{\det M(t+\varepsilon) - \det M(t)}{\varepsilon} \\ = \det M(t)\operatorname{Tr}(A(t)) + o(1) \Rightarrow (\det M(t))' = \det M(t)\operatorname{Tr}(A(t)) \end{aligned} \tag{152}$$

and the result follows by integration. $\square$

Note that an equation of the kind we are considering,

$$Lf = p_0(t)f^{(n)} + p_1(t)f^{(n-1)} + \cdots + p_n(t)f = \lambda f \tag{153}$$

has the matrix equation counterpart

$$M' = AM \tag{154}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \dots & \dots & \dots & \dots & \\ -\frac{p_n}{p_0} & -\frac{p_{n-1}}{p_0} & -\frac{p_{n-2}}{p_0} & ... & -\frac{p_1}{p_0} \end{pmatrix} \tag{155}$$

and

$$M = \begin{pmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ \dots & \dots & \dots \\ f_1^{(n-1)} & ... & f_n^{(n-1)} \end{pmatrix} \tag{156}$$

Clearly, $\mathrm{Tr}\, A = -p_1/p_0$. Thus we have

Corollary 22. *The Wronskian W of a fundamental system for* (153) *satisfies*

$$W(t) = W(0) \exp\left(-\int_0^t \frac{p_1(s)}{p_0(s)} ds\right) \tag{157}$$

14 Discrete dynamical systems

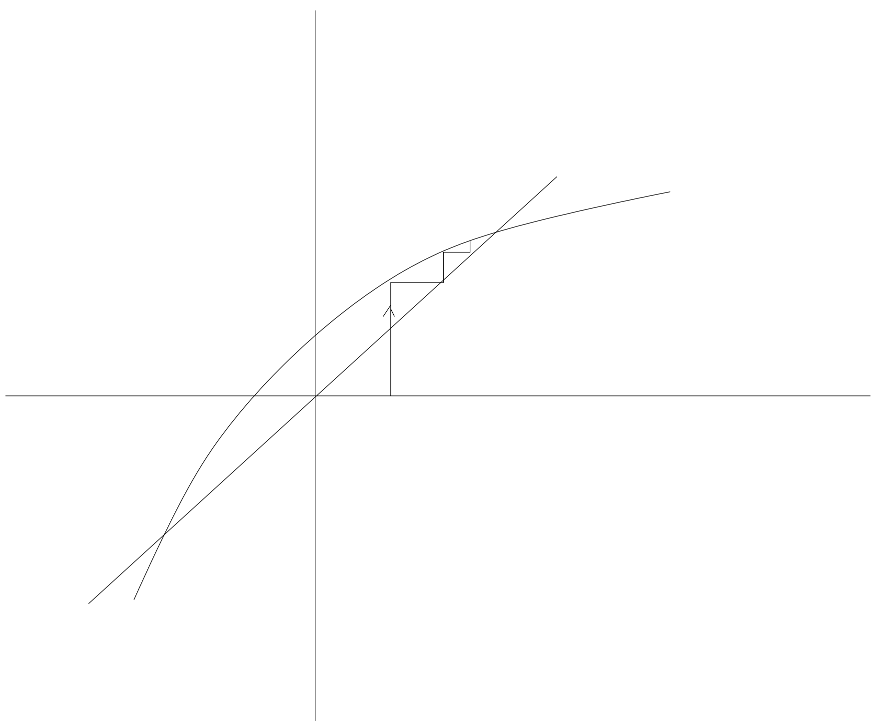

Figure 12:

The study of the Poincaré map leads naturally to the study of discrete dynamics. In this case we have closed trajectory, x_0 a point on it, $\mathcal{S}$ a section through x_0 and we take a point x_1 near x_0, on the section. If x_1 is sufficiently close to x, it must cross again the section, at x_1', still close to x_1, after the *return time* which is then close to the period of the orbit. The application $x_1 \to x_1'$ defines the Poincaré map, which is smooth on the manifold near x_0.

The study of the behavior of differential systems is near closed orbits is often more easily understood by looking at the properties of the Poincaré map.

In one dimension first, we are dealing with a smooth function f, where the iterates of f are what we want to understand.

We write $f^n(x) = f(f(...(f(x))))$ n times. The *orbit* of a point x_0 is the sequence $\{f^n(x_0)\}_{n\in\mathbb{N}}$, assuming that $f^n(x_0)$ is defined for all n. In particular, we may assume that $f : J \to J$, where $J \subset \mathbb{R}$ is an interval, possibly the whole line.

The effects of the iteration are often easy to see on the graph of the iteration, in which we use the bisector $y = x$ to conveniently determine the new point. We have $(x_0, 0) \to (x_0, f(x_0)) \to (f(x_0), f(x_0)) \to (f(x_0), f(f(x_0)))$, where the two-dimensionality and the "intermediate" step helps in fact drawing the iteration faster: we go from x_0 up to the graph, horizontally to the bisector, vertically back to the graph, and repeat this sequence.

There are simple iterations, for which the result is simple to understand globally, such as

$$f(x) = x^2$$

where it is clear that $x = 1$ is a fixed point, if $|x_0| < 1$ the iteration goes to zero, and it goes to infinity if $|x_0| > 1$.

Local behavior near a fixed point is also, usually, not difficult to understand, analytically and geometrically.

Theorem 8. *(a) Assume f is smooth, $f(x_0) = x_0$ and $|f'(x_0)| < 1$. Then x_0 is a sink, that is, for x_1 in a neighborhood of x_0 we have $f^n(x_1) \to x_0$.*

(b) If instead we have $|f'(x_0)| > 1$, then x_0 is a source, that is, for x_1 in a small neighborhood $\mathcal{O}$ of x_0 we have $f^n(x_1) \notin \mathcal{O}$ for some n (this does not mean that $f^m(x_1)$ cannot return "later" to $\mathcal{O}$, it just means that points very nearby are repelled, in the short run.)

Proof. We show (a), (b) being very similar. Without loss of generality, we take $x_0 = 0$. There is a $\lambda < 1$ and ε small enough so that $|f'(x)| < \lambda$ for $|x| < \varepsilon$. If we take x_1 with $|x_1| < \varepsilon$, we have $|f(x_1)| = |f'(c)||x_1| < \lambda|x_1|(< \varepsilon)$, so the inequality remains true for $f(x_1) : |f(f(x_1))| < \lambda|f(x_1)| < \lambda^2|x_1|$ and in general $f^n(x_1) = O(\lambda^n) \to 0$ as $n \to \infty$.

In fact, it is not hard to show that, for smooth f, the evolution is essentially geometric decay.

□

When the derivative is one, in absolute value, the fixed point is called neutral or indifferent. It does not mean that it can't still be a sink or a source, just that we cannot resort to an argument based on the derivative, as above.

Example 5. *We can examine the following three cases:*
(a) $f(x) = x + x^3$.
(b) $f(x) = x - x^3$
(c) $f(x) = x + x^2$.

It is clear that in the first case, any positive initial condition is driven to $+\infty$. Indeed, the sequence $f^n(x_1)$ is increasing, and it either goes to infinity or else it has a limit. But the latter case cannot happen, because the limit should satisfy $l = l + l^3$, that is $l = 0$, whereas the sequence was increasing.

The other cases are analyzed similarly: in (a), if $x_0 < 0$ then the sequence still diverges. Case (c) is more interesting, since the sequence converges to zero if $x_1 < 0$ is small enough and to ∞ for all $x_1 > 0$. We leave the details to the reader.

It is useful to see what the behavior of such sequences is, in more detail.

Let's take the case (c), where $x_1 < 0$. We have

$$x_{n+1} = x_n + x_n^2$$

where we expect the evolution to be slow, since the relative change is vanishingly small. We then approximate the true evolution by a differential equation

$$(d/dn)x = x^2$$

giving

$$x_n = (C - n)^{-1}$$

We can show rigorously that this is the behavior, by taking $x_n = -1/(n+c_0)+\delta$, $\delta_{n_0} = 0$ and we get

$$\delta_{n+1} - \delta_n = \frac{1}{n^2(n+1)} - \frac{2}{n}\delta_n + \delta_n^2 \tag{158}$$

and thus

$$\delta_n = \sum_{j=n_0}^{n} \left(\frac{1}{j^2(j+1)} - \frac{2}{j}\delta_j + \delta_j^2 \right) \tag{159}$$

Exercise 1. *Show that (159) defines a contraction in the space of sequences with the property* $|\delta_n| < C/n^2$*, where you choose* C *carefully.*

Exercise 2. *Find the behavior for small positive* x_1 *in (b), and then prove rigorously what you found.*

14.1 Bifurcations

The local number of fixed points can only change when $f'(x_0) = 1$. As before, we can assume without loss of generality that $x_0 = 0$.

We have

Theorem 9. *Assume $f(x, \lambda)$ is a smooth family of maps, that $f(0, 0) = 0$ and that $f_x(0, 0) \neq 1$. Then, for small enough λ there exists a smooth function $\varphi(\lambda)$, also small, so that $f(\varphi(\lambda), \lambda) = \varphi(\lambda)$, and the character of the fixed point (source or sink) is the same as that for $\lambda = 0$.*

Exercise 3. *Prove the theorem, using the implicit function theorem.*

The logistic map

This is the iteration of the simple quadratic polynomial

$$f(x) = \lambda x(1 - x)$$

or, the recurrence

$$x_{n+1} = \lambda x_n(1 - x_n) \tag{160}$$

14.1.1 Iteration on $[0, 1]$

We can check that for $\lambda \in [0, 1)$ $x = 0$ is a stable fixed point while for $x \in (1, 3)$ the nonzero root of $\lambda x(1 - x) - x = 0$ is stable.

For higher values of λ, up to about $\lambda = 3.8$ there are *stable periodic orbits*. A periodic orbit of a map f of period k is a fixed point of f^k.

For $\lambda = 3.5$ successive iterations with $x_0 = 0.5$ yield

$$0.5, 0.875, \underbrace{0.382812, 0.826935, 0.500898, 0.874997}_{period}, 0.382820, 0.826941, 0.500884,$$

The *eventual period* is the same for any initial condition.

14.1.2 Iterating the map $4x(1 - x)$ on $[0, 1]$

For $\lambda = 4$, using (166) we see what substitution simplifies the recurrence: if $x_n = \sin^2(2\pi t)$ then $x_{n+1} = \sin^2(4\pi t)$, and thus $x_n = \sin^2(2^{n+1}\pi t_0)$.

The map is semi-conjugated (the map is not invertible) to $t \to 2t \mod 1$.

14.2 The logistic map in $\mathbb{C}$; conjugation with the doubling map

The *Julia set* J, *the filled-in Julia set* and the *Fatou set* are defined as follows.

Taken as an iteration in $\mathbb{C}$, if f is a rational function and $x_n = f^n(x)$ where f^n is the $n - th$ iteration of f we have the following equivalent characterizations of J.

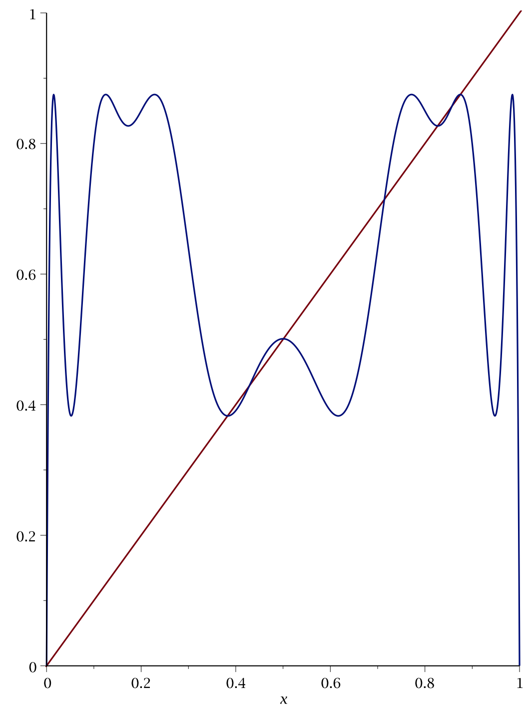

Figure 13: Plot of f^4 for $\lambda = 3.5$.

- J is the closure of *repelling periodic points*.
- J is the *set of limit points of* $\cup_n f^{-n}(z)$
- Importantly for us: If f is entire, e.g. a polynomial, then J is the boundary of the set of points s.t. $f^n(z) \to \infty$ as $n \to \infty$, and also
- If f is a polynomial, then J is the boundary of the filled Julia set; that is, the boundary of the set $\{z : \sup_n |f^n(z)| < \infty\}$.

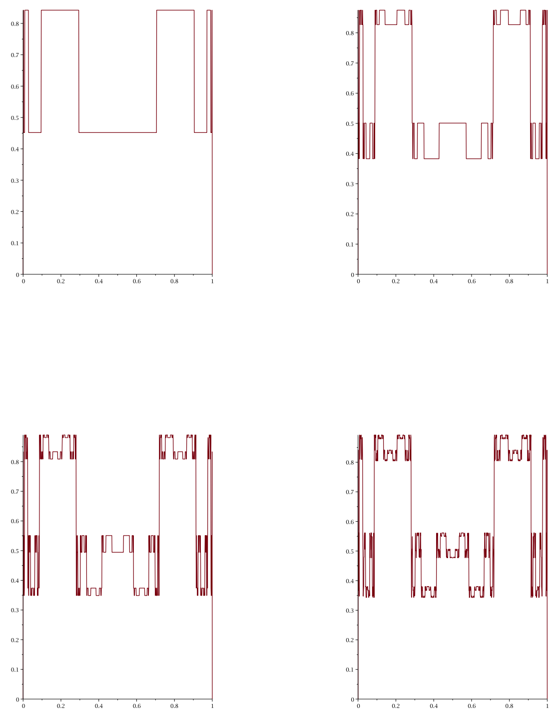

Figure 14: Plot of f^{1000}, for $\lambda = 3.4, 3.5, 3.56, 3.569$.

Exercise 4. Show that the Julia set coincides with the interval $[0, 1]$ if $\lambda = 4$ (without using the explicit formulas for G given below). Fig. 14.1.1 should give you a hint.

The change of variable

$$x = -\frac{z}{\lambda} + \frac{1}{2}$$

transforms (160) into the $c-$parameter form

$$z_{n+1} = z_n^2 + c, \ \ c = \frac{\lambda}{2} - \frac{\lambda^2}{4} \tag{161}$$

Evidently, the Julia and Fatou sets of (160) and of (161) are the same.

The *Mandelbrot set* $\mathcal{M}$ is the set of values of c for which z_n in (161) starting with the *fixed initial condition* $z_0 = 0$ are bounded.

The main cardioid in the Mandelbrot set is the set of values of c in (161) for

$$|\lambda| < 1$$

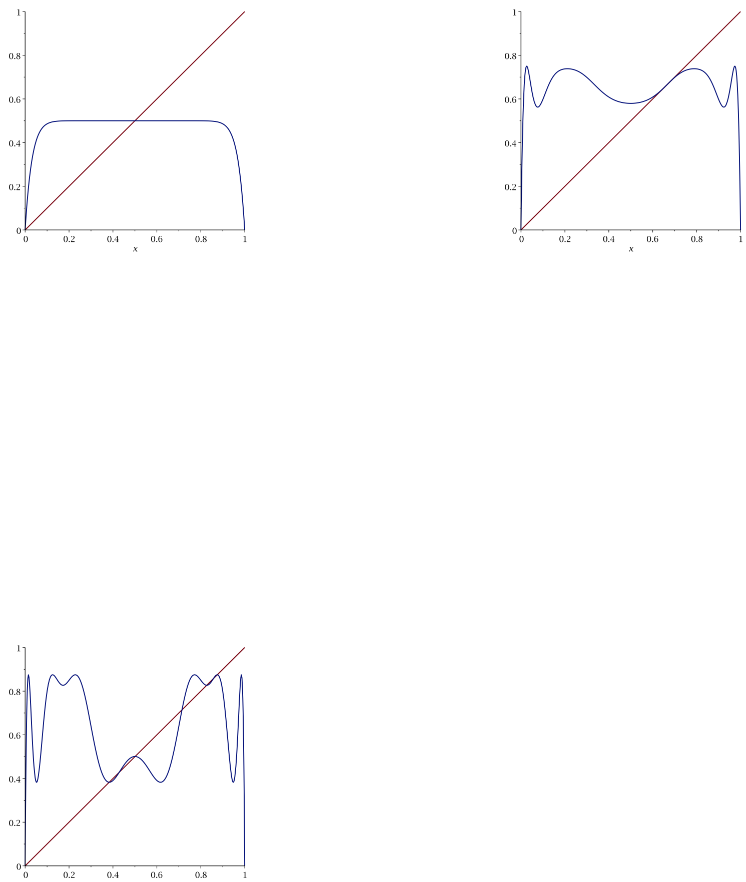

Figure 15: The map f^4 where $f(x) = \lambda x(1-x),\ \lambda = 2, 3, 3.5$.

Since the transformation is not one-to-one, for these c, the corresponding λ's

Figure 16: The map f^4 where $f(x) = \lambda x(1-x)$, $\lambda = 3.8, 4$.

are given by

$$\lambda = \{\rho e^{it}, 2 - \rho e^{it} : \rho \in [0,1), t \in [0, 2\pi)\} \tag{162}$$

and this is exactly the set of λ for which there is an attracting fixed point (check!). The bulbs correspond to values of λ for which there exist attracting periodic points.

After the substitution $x = -y^{-1}$,

$$y_{n+1} = \frac{y_n^2}{\lambda(1+y_n)} = f(y_n) \tag{163}$$

If $c \in \mathcal{M}$ there exists a unique map F, analytic near zero, with $F(0) = 0$, $F'(0) = \lambda^{-1}$ so that $(F \circ f \circ F^{-1})(z) = z^2$. Its inverse, G, used in [3], conjugates (163) to the canonical map $z_{n+1} = z_n^2$, and it can be checked that

$$G(z)^2 = \lambda G(z^2)(1 + G(z)); \quad G(0) = 0,\ G'(0) = \lambda \tag{164}$$

Equivalently (with $\varphi = 1/G$) there exists a unique map φ analytic in $\mathbb{D} \setminus \{0\}$,

$$f(\varphi(z)) = \varphi(z^2); \quad z \in \mathbb{D} \setminus \{0\}; \quad \lim_{z\to 0} z\varphi(z) = 1/\lambda \tag{165}$$

It follows that $G(\partial\mathbb{D}) = J$. For a short proof see [2].

In any hyperbolic component of $\mathcal{M}$ (components of $\mathcal{M}$ corresponding to (unique) attracting cycles), the points $z \in \text{fix } f^n$ on the corresponding Julia set have the property $|f_n'(z)| > 1$ and φ is continuous in $\overline{\mathbb{D}}$.

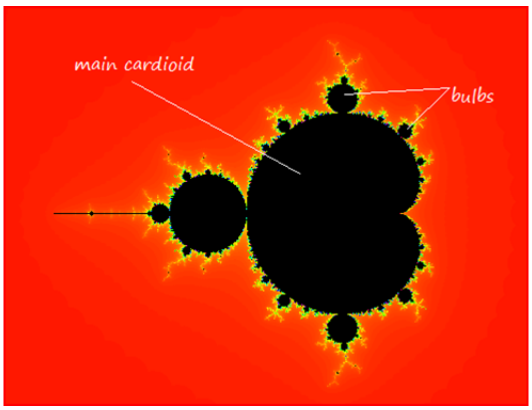

Figure 17: The Mandelbrot set [6]

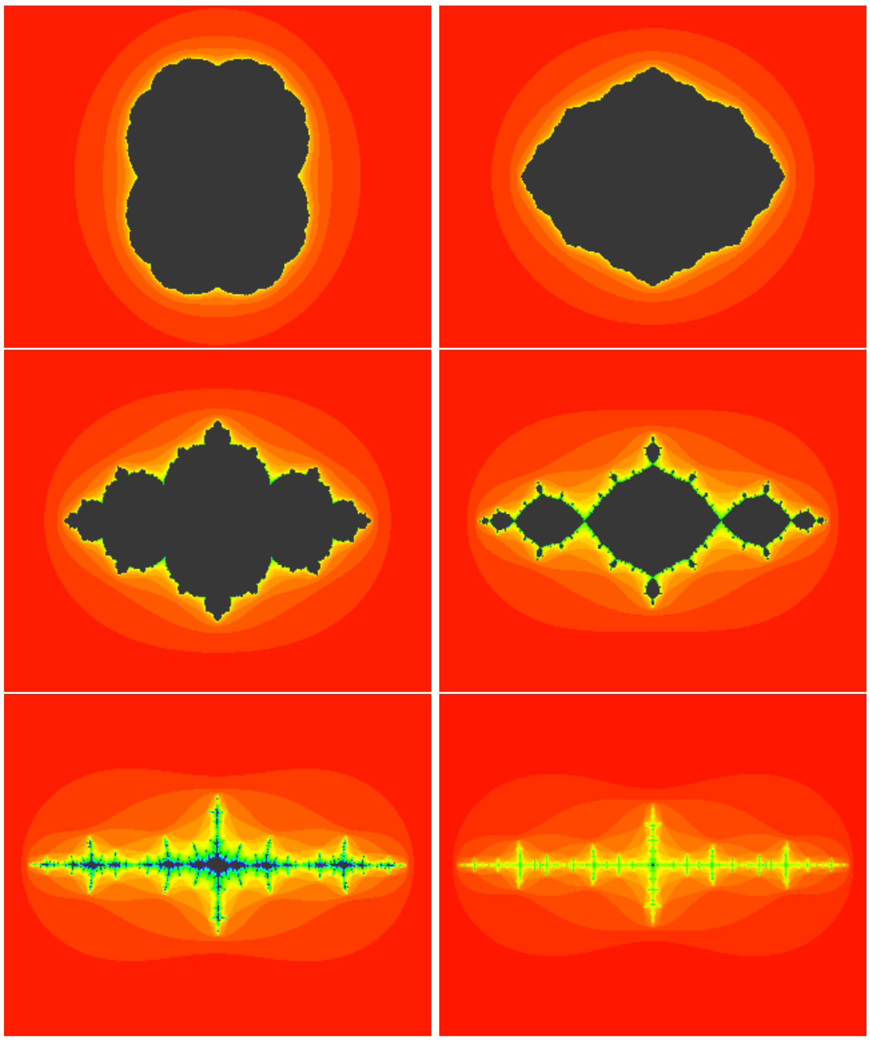

Figure 18: The filled in Julia set for $\lambda = 0.504, 2.48, 2.89, 3.28, 3.56, 3.68$ [7].

Check that for $\lambda = 4$, G has the explicit form

$$G(z) = \frac{4z}{(1-z)^2} \tag{166}$$

Check that this transformation maps conformally the interior of the unit disk onto $\mathbb{C} \setminus (-\infty, -1]$.

The other explicitly solvable cases are $\lambda = -2, 0, 2$. For $\lambda = -2$,

$$G(z) = \frac{-2z}{z^2 + z + 1} \tag{167}$$

while for $\lambda = 2$,

$$G(z) = \frac{2z}{1-z} \tag{168}$$

For $\lambda = 2$, G maps the disk $|z| \leqslant 1$ conformally onto the right half plane and is one-to-one on the boundary. This corresponds to the disk $\{x : |x - \frac{1}{2}| \leqslant \frac{1}{2}\}$. Equivalently, the map

$$H = \frac{x-1}{2x}$$

solves the equation

$$H(x^2) = 2H(x)(1 - H(x))$$

14.2.1 Chaos

A map from, say $I = [0, 1]$ into itself is chaotic if [4]

1. Periodic points are dense in I.

2. f is transitive on I. That is, given any two subintervals of I, U_1 and U_2, there is a point $x_0 \in U_1$ and an $n > 0$ s.t. $f^n(x_0) \in U_2$.

3. f has sensitive dependence on parameters: there is a sensitivity constant $\beta > 0$ s.t., for any $x_0 \in I$ and any open interval U about x_0, there is some other point $y_0 \in U$ s.t.

   $$|f^n(x_0) - f^n(y_0)| > \beta$$

   Surprisingly, the third condition (which depends on a metric) follows from the second one (which is purely topological).

   Clearly, the doubling map extracts one by one the binary digits of a number. The sequence of digits is the simplest example of symbolic dynamics: they describe whether the n-th iterate of the initial condition falls in $[0, \frac{1}{2})$ or in $[\frac{1}{2}, 1]$. In general, a symbolic dynamic description of a map consists of a partition $\{A_1, A_2, ..., A_n\}$ of the phase space and of a sequence of numbers j where $\sigma_n(x_0) = j$ if $f^n(x_0) \in A_j$.

   A nice application of symbolic dynamics is Sharkovskii's theorem, which implies the following:

Theorem 10. *If a discrete dynamical system on the real line has a periodic point of least period 3, then it must have periodic points of every other period.*

More precisely, the following holds: Consider the following ordering of the naturals:

$$\begin{aligned} 3 \prec 5 \prec 7 \prec 9 \prec ... \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec 2\cdot 9 \prec \\ ... \prec 2^2\cdot 3 \prec 2^2\cdot 5 \prec 2^2\cdot 7 \prec 2^2\cdot 9 \prec \\ ... \prec 2^3\cdot 3 \prec 2^3\cdot 5 \prec 2^3\cdot 7 \prec 2^3\cdot 9 \\ ... \prec 2^5 \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1. \end{aligned} \quad (169)$$

Theorem 11 (Sharkovskii)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be continuous function and suppose $p \prec q$ in the above ordering. Then if F has a point of least period p, then F also has a point of least period q.*

In the case of the logistic map, *period doubling* starts at $\lambda \approx 3.5699$: the period of the stable periodic fixed point keeps doubling as λ is increased. More precisely,

$$\frac{\lambda_{n-1} - \lambda_{n-2}}{\lambda_n - \lambda_{n-1}} = c_n \to c = 4.6692\cdots$$

where c is the Feigenbaum constant.

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