Contents

It is hard to overemphasize the importance of complex analysis in virtually every field of mathematics. Its power stems from a number of sources; one of them is that it is the “ultimate” extension of real numbers: it satisfies all the axioms of the reals except for ordering. Furthermore, the Gelfand-Mazur theorem lists all division algebras. These are: $\mathbb{R}, \mathbb{C}$ and the quaternions (a non-commutative algebra). Relatedly, functions that extend from $\mathbb{R}$ to $\mathbb{C}$, the analytic functions, preserve all their local properties, by the so-called *principle of permanence of relations* that will be discussed later. Hence, to understand the properties of such functions, we'd better look at the entire domain of definition. A popular example is the function $1/(1 + x^2)$ which has a convergent power series at zero, whose radius is 1. There is nothing special about $x = \pm 1$ but as soon as we extend the function to $\mathbb{C}$ we see that $\pm i$ are singular points because of they are roots of the denominator.

More generally, the fundamental theorem of algebra holds in $\mathbb{C}$, not necessarily in $\mathbb{R}$.

To show the exponential decay of Fourier transforms of real-analytic functions, one “needs” complex-analytic arguments.

1. Review: Complex numbers, functions of a complex variable

◦ Complex numbers, $\mathbb{C}$ form a field; addition, multiplication of complex numbers have the same properties as their counterparts in $\mathbb{R}$.

◦ There is no “good” order relation in $\mathbb{C}$. Except for that, we operate with complex numbers in the same way as we operate with real numbers.

◦ A function f of a complex variable is a function defined on some subset of $\mathbb{C}$ with complex values. Alternatively, we can view it as a pair of real valued functions of two real variables. We write $z = x + iy$ with x, y real and $i^2 = -1$ and write $x =\text{Re}(z)$, $y =\text{Im}(z)$. We write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

◦ We note that $i^2 = (-i)^2 = -1$. There is no intrinsic distinction between i and $-i$. This entails a fundamental symmetry of the theory, symmetry with respect to complex conjugation [1].

◦ Based on the basic properties of complex numbers, we can right away define a number of elementary complex functions: z, $1/z$ and

[1] More precisely, $z = x+iy \to \overline{z} = x-iy$ is an involution and a field isomorphism of $\mathbb{C}$

more generally for $m \in \mathbb{Z}$ we easily define z^m and in fact any polynomial $\sum_{m=0}^{K} c_m(z-z_0)^m$.

∘ To be able to define and work with more interesting functions we need to define continuity, derivatives and so on. For this we need to define limits. Seen as a pair of real numbers (x, y), the modulus of z, $|z| = \sqrt{x^2+y^2}$ gives a measure of length and thus of smallness which induces a natural norm which makes $\mathbb{C}$ a **complete metric space**. Convergence then reduces to one of real numbers:

$$z_n \to z \quad \Leftrightarrow |z - z_n| \to 0 \quad \text{as } \ n \to \infty \tag{1.2}$$

The *topology of* $\mathbb{C}$ is the same of that of $\mathbb{R}^2$, if we identify $z = x + iy$ with the point $(x, y) \in \mathbb{R}^2$. Some basic facts in topology are reviewed in Appendix §50.1.

In the sequel, a *domain in* $\mathbb{C}$ is an open connected set in $\mathbb{C}$. Examples are the disks of radius $r \geqslant 0$ centered at some point $z_0 \in \mathbb{C}$:

$$\mathbb{D}(z_0, r) := \{z : |z - z_0| < r\} \tag{1.3}$$

The special cases $r = 0$ (the empty set, $\emptyset$) and $r = \infty$ (the whole of $\mathbb{C}$) are open sets. The unit disk $\mathbb{D}$

$$\mathbb{D} := \mathbb{D}(0, 1)$$

will play a special role as a canonical choice of a disk.

Exercise 1.1. Show that $z_n \to z$ if and only if $\mathrm{Re}\,(z_n) \to \mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z_n) \to \mathrm{Im}\,(z)$. Using completeness of $\mathbb{R}$ show that $\mathbb{C}$ is a complete normed space.

Definition 1.2. *For functions, limits are similarly reduced to the real case:* $\lim_{z\to z_0} f(z) = a$ *iff* $|f(z) - a| \to 0$ *as* $z \to z_0$.

2. Convergent power series

2.1. **Series.** A complex series is written as

$$\sum_{k=0}^{\infty} a_k \tag{2.2}$$

where $a_k, k \geqslant 0$ are complex, and is said to converge if, by definition, the sequence of *partial sums*

$$S_N := \sum_{k=0}^{N} a_k \tag{2.3}$$

converges as $N \to \infty$.

The series is said to converge absolutely if the real-valued series

$$\sum_{k=0}^{\infty} |a_k| \tag{2.4}$$

converges.

Exercise 2.3. Check that a *necessary* condition of convergence is $a_k \to 0$ as $k \to \infty$ and that absolute convergence implies convergence. Verify that the convergence criteria that you know from real analysis: the ratio test, the n-th root test, in fact any test that does not rely on signs carry over to complex series. The proofs over $\mathbb{C}$ require minor, if any, modifications of the standard proofs in $\mathbb{R}$; we will illustrate that shortly.

2.2. **Power series.** A power series centered at z_0 is a series of the form

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{2.5}$$

where c_k, z, z_0 are complex.

Theorem 2.4 (Abel)**.** *If for some $z_1 \neq z_0$ the series*

$$\sum_{k=0}^{\infty} c_k (z_1 - z_0)^k \tag{2.6}$$

converges, then

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{2.7}$$

converges absolutely and uniformly in any disk $\mathbb{D}(z_0, r)$ if $r < |z_1 - z_0|$.

Exercise 2.5. Prove this theorem by reducing it to a familiar property of real series and using the completeness of $\mathbb{C}$.

Abel's theorem tells us that the domain of convergence of a power series is a disk; convergence can extend to parts of the boundary. The largest r for which a series (2.7) converges for all $z \in \mathbb{D}(z_0, r)$ is called the *radius of convergence*. The disk of convergence may be degenerate: in one extreme situation it is a point, $z = z_0$ (zero radius of convergence) in the other, the whole complex domain ("infinite radius of convergence").

3. Continuity and differentiability

Definition. A complex function is continuous at z_0 if $f(z) \to f(z_0)$ as $z \to z_0$.

Exercise 3.6. Show that polynomials are continuous in $\mathbb{C}$.

We can now define differentiability.

Definition. A function f is differentiable at z_0 if, by definition, there is a number, call it $f'(z_0)$ such that

$$\frac{f(z) - f(z_0)}{z - z_0} \to f'(z_0) \quad \text{as} \quad z \to z_0$$

Exercise 3.7. Show that differentiation has the properties we are familiar with from real variables: the sum rule, product rule, chain rule etc. hold for complex differentiation. (Prove this; it amounts to nothing more than mimicking the proofs over the reals.)

Differentiability in $\mathbb{C}$ is far more demanding than differentiability in $\mathbb{R}$. For the same reason, complex differentiable functions are much more regular and have better properties than real-differentiable ones.

We will see that if f is analytic, then its derivative is also analytic, implying that f has continuous derivatives of all orders. This comes from the crucial fact that in complex analysis the derivative has an integral formula.

We will also see later that analyticity in a domain $\mathcal{D}$ is equivalent to the convergence of the Taylor series at all points $z_0 \in \mathcal{D}$.

3.1. **Differentiability of power series.**

Theorem 3.8. *If the power series*

$$S(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{3.2}$$

converges in the open disk $\mathbb{D}(z_0, r)$, $r > 0$ *(see Theorem 2.4), then* $S(z)$ *has derivatives of all orders in* $\mathbb{D}(z_0, r)\}$*. In particular,*

$$S'(z) = \sum_{k=0}^{\infty} kc_k(z - z_0)^{k-1} \tag{3.3}$$

$$S''(z) = \sum_{k=0}^{\infty} k(k-1)c_k(z - z_0)^{k-2} \tag{3.4}$$

$$\ldots\ldots \tag{3.5}$$

$$S^{(p)}(z) = \sum_{k=0}^{\infty} k(k-1)\cdots(k-p+1)c_k(z - z_0)^{k-p} \tag{3.6}$$

$$\ldots\ldots\ldots \tag{3.7}$$

and all these series converge in $\mathbb{D}(z_0, r)$ *to the corresponding derivative of* S*.*

Proof. For the proof we only need to show the result for S': for larger p the proof follows by induction. Furthermore, by taking $z' = z - z_0$ we reduce the problem to the case when $z_0 = 0$. Let $|z| < \rho < r$ and choose h small enough so that $|z| + |h| < \rho$. Note first that, if z and h are in $\mathbb{R}^+$ we have

$$\begin{aligned}(z+h)^n - z^n - nz^{n-1}h &= \frac{n(n-1)}{2}z^{n-2}h^2 + \sum_{k=3}^{n}\binom{n}{k}h^k z^{n-k} \\ &\leqslant \frac{n(n-1)}{2}(z+a)^{n-2}h^2 \leqslant \frac{n(n-1)}{2}\rho^{n-2}h^2\end{aligned} \tag{3.8}$$

where $a \in (0, \delta)$ where we applied *the Taylor remainder theorem.* If z, h are complex the equality above still holds. Hence

$$\begin{aligned}\left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| &= \left|\frac{n(n-1)}{2}z^{n-2}h + \cdots + h^{n-1}\right| \\ \leqslant \frac{n(n-1)}{2}|z|^{n-2}|h| + \sum_{k=3}^{n}\binom{n}{k}|h|^k|z|^{n-k} &\leqslant \frac{n(n-1)}{2}\rho^{n-2}|h|\end{aligned} \tag{3.9}$$

(since in the term after the first inequality $|z|$ and $|h|$ are positive). Thus, for the partial sums $S_N(z) = \sum_{k=0}^{N} c_k(z - z_0)^k$ we have

$$\left|\frac{S_N(z+h) - S_N(z)}{h} - S_N'(z)\right| \leqslant |h| \sum_{k=0}^{N} \frac{k(k-1)}{2}|c_k|\rho^k \leqslant C|h|$$

where $C = \displaystyle\sum_{k=0}^{\infty} \frac{k(k-1)}{2}|c_k|\rho^k$ where the series converges as $\rho < r$. □

Corollary 3.9. *Show that* $S^{(k)}(z_0) = k!c_k$ *and thus* (3.2) *is the convergent Taylor series of* S.

Corollary 3.10. *Assume* $S(z)$ *converges in a disk* $\mathbb{D}(z_0, r)$ *and that there is a sequence* $\{z_n\}_{n\in\mathbb{N}}$ *with an accumulation point at* z_0 *so that* $S(z_n) = 0$ *for all* $n \in \mathbb{N}$. *Then* $S(z)$ *is identically zero.*

Proof. We can assume without loss of generality that $z_0 = 0$. We show that all coefficients of $S(z)$ are zero, and thus $S = 0$. We write

$$S(z) = c_0 + zT(z) \tag{3.10}$$

where T converges in $\mathbb{D}(0, r)$. We have, by assumption

$$S(z_n) = 0 = \lim_{n\to\infty} [c_0 + z_n T(z_n)] \tag{3.11}$$

and thus $c_0 = 0$. From here we proceed by induction, as $S(z)/z$ is a power series with the same properties as S etc. (check!) ▮

3.2. **Some basic functions.**
∘ *The exponential.* We define

$$e^z = \sum_{k=0}^{\infty} \frac{z^n}{n!} \tag{3.12}$$

This series converges for any $z \in \mathbb{C}$ and thus it is differentiable for any z in $\mathbb{C}$ by Theorem 3.8. We have, by (3.3) and (3.12)

$$(e^z)' = e^z \tag{3.13}$$

Thus,

$$(e^z e^{-z})' = 0 \tag{3.14}$$

and thus $e^z e^{-z}$ does not depend on z, and takes the same value everywhere, the value for $z = 0$. But we see immediately that $e^0 = 1$. Thus

$$e^z e^{-z} = 1 \quad \Leftrightarrow e^{-z} = 1/e^z \tag{3.15}$$

In the same way,

$$(e^{z+a}e^{-z})\,' = (e^{z+a})'e^{-z} + e^{z+a}(e^{-z})' = 0$$
$$\Leftrightarrow e^{z+a}e^{-z} = e^a e^0 = e^a \quad \Leftrightarrow e^{z+a} = e^z e^a \tag{3.16}$$

which provides us with the fundamental property of the exponential.

Also, we immediately check Euler's formula: for $\phi \in \mathbb{R}$ we have

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{3.17}$$

Exercise 3.11. *(a) We tacitly used something more in these arguments: what? Fill in the missing details using results we have already proven.*

(c) Define cos and sin in $\mathbb{C}$ by their power series and show that (3.17) *holds for all z in $\mathbb{C}$: $e^{iz} = \cos z + i \sin z$. This leads to the important formulas for* $\sin, \cos$ *in terms of the exponential:*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}; \quad \cos z = \frac{e^{iz} + e^{-iz}}{2i}$$

Use these representations to show that $\sin^2 + \cos^2 = 1$ *throughout* $\mathbb{C}$.

Exercise 3.12 (A first form of permanence of analytic relations)**.** *(a) Let F and f be analytic in a domain $\mathcal{D}$ that contains an interval in $\mathbb{R}$ and assume that $F(f(x)) = 0$ for $x \in \mathbb{R}$. Show that $F(f(z)) = 0$ in $\mathcal{D}$.*

(b) Use (a) to give an alternative proof of the identities of the exponential, based on Corollary 3.10.

(c) Use (a) and the series extension of $\sin, \cos$ *to* $\mathbb{C}$ *to show that* $\sin^2 + \cos^2 = 1$ *in* $\mathbb{C}$.

Exercise 3.13. *Show that*

$$e^s = 1 \quad \Leftrightarrow s = 2N\pi i, \quad N \in \mathbb{Z} \tag{3.18}$$

∘ *The logarithm.* In the complex domain the log is a trickier function. For the moment we look at a simpler question, that of defining $\log(1+z)$ only for $|z| < 1$. This is done via the convergent Taylor series

$$\log(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \cdots \tag{3.19}$$

By (3.3) we get

$$\frac{d}{dz}\log(1+z) = 1 - z + z^2 - z^3 + \cdots = \frac{1}{1+z} \quad \text{if} \quad |z| < 1 \tag{3.20}$$

Exercise 3.14. *Show that if $|s|$ is small we have*

$$\log(e^s) = s; \qquad e^{\log(1+s)} = 1 + s$$

We will return later to the question of defining $\log z$ for more general $z \in \mathbb{C}$, $z \neq 0$ and we will study its properties carefully. It is one of the fundamental "branched" complex functions. Many other branched functions have their branching due to that of the log.

3.3. **Operations with power series.** If S and T are power series convergent in a neighborhood of z_0, then $S+T, S\times T$, S/T if $T(z_0) \neq 0$ and $S(T)$ if $T(z_0) = 0$ (see Exercise 3.15 below) are convergent in some neighborhood of z_0 as well. Formulas for these new series are obtained by noticing that power series are limits of polynomials, and then by finding the formulas for the corresponding polynomials. For instance,

$$ST = s_0t_0 + (s_1t_0 + s_0t_1)(z - z_0) + (s_2t_0 + s_1t_1 + s_0t_2)(z - z_0)^2 + \cdots \tag{3.21}$$

Exercise 3.15. *(a) If S and T are two power series with radius of convergence r, then ST has radius of convergence at least r.*

(b) Write three terms of the series S/T if $T(z_0) \neq 0$.

(c) Under the assumptions above, show that S/T has nonzero radius of convergence.*

(d) Under the assumptions above with the one in (b) replaced by $T(z_0) = 0$, show that $S \circ T$ has nonzero radius of convergence.*

3.4. **The Cauchy-Riemann equations.** Analytic functions can be defined by many equivalent properties, that we will soon explore.

Definition 3.16. *The function f defined in a domain $\mathcal{D}$ is analytic in $\mathcal{D}$ if it is differentiable at all points in $\mathcal{D}$.*

As a first definition equivalent to differentiability, an analytic function is a function which satisfies the Cauchy-Riemann (C-R) equations:

Theorem 3.17 (C-R)**.** *(1) Assume that $f = u + iv$ is analytic in a domain $\mathcal{D}$ in $\mathbb{C}$. Then the Cauchy-Riemann equations hold:*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \tag{3.22}$$

throughout $\mathcal{D}$ and u, v are differentiable in $\mathcal{D}$.

(2) Conversely, if (u, v) are differentiable and satisfy (3.22) in $\mathcal{D}$, then f is differentiable in $\mathcal{D}$.

Proof. (1) Let $f(z) = u(x, y) + iv(x, y)$ and $f'(z_0) = a + ib$. We can again take wlog $z_0 = 0$. We show that the stated equivalence for any $z_0 \in \mathcal{D}$. By a translation, we can arrange that $z_0 = 0$. We have

(3.23)
$$f(z) - f(0) = u(x, y) - u(0, 0) + iv(x, y) - iv(0, 0) = [f'(0) + \varepsilon(z)]z$$
$$= (a+ib)[x-x_0+i(y-y_0)]+\varepsilon(z)z = ax-by+i(ay+bx)+ib(x-x_0)+\varepsilon(z)z$$

where $\varepsilon(z) \to 0$ as $z \to z_0$ implying u_x, u_y, v_x, v_y exist at z_0 and satisfy the C-R equations (see also Exercise 3.18 below).

(ii) Differentiability of u and v at (x_0, y_0) implies

$$f(z) - f(0) = u(x, y) - u(0, 0) + iv(x, y) - iv(0, 0) \tag{3.24}$$
$$= u_x(0, 0)x + u_y(0, 0)y + iv_x(0, 0)x + iv_y(0, 0)y + \varepsilon(x, y)x + i\eta(x, y)y$$

where ε and η go to zero as $z \to z_0$. ∎

Exercise 3.18. Show that (3.23) and (3.24) are compatible if and only if (3.22) hold. (The real and imaginary parts must be equal to each other, and $x - x_0$ and $y - y_0$ are independent quantities.)

3.5. **Analyticity at infinity.** As $|z| \to \infty$, $1/z \to 0$. By definition f is *analytic at infinity* if $f(1/z)$ is analytic at zero.

4. Integrals

Integration plays an important role in complex analysis. As we shall see, the derivative of a function can be written as an integral, and many of the nice properties of analytic functions originate in this fact.

If $f(t) = u(t)+iv(t)$ is a complex-valued function of one real variable t then $\int_a^b f(t)dt$ is defined by

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt \tag{4.2}$$

This reduces the questions of complex integration to the familiar real integration.

Note 4.19. *In the following, unless otherwise specified, we assume that the curves we use are piecewise differentiable.*

Note.

Let $\gamma(t) = x(t) + iy(t)$, $t \in [a, b]$ be a piecewise C^1 parametrized curve. We define

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

essentially in the same way as (4.2): breaking up γ, γ' and f into their real and imaginary part. Do this calculation.

Note that the sign of the integral depends on the *orientation* of γ, specified by stating that t *goes from* a *to* b, rather than only $t \in [a, b]$. It is natural to say that $\gamma(a)$ is the starting point of γ, and $\gamma(b)$ is its final point. The same geometric curve with opposite orientation is denoted by $-\gamma$; a formula is easily found as $(-\gamma)(t) = x(a+b-t)+iy(a+b-t)$, $t \in [a, b]$. We see that

$$\int_{-\gamma} f(z)dz = \int_b^a f(\gamma(t))\gamma'(t)dt = -\int_\gamma f(z)dz \tag{4.3}$$

If $\gamma(a) = \gamma(b)$ the curve is called *closed*. Positive orientation, the counterclockwise one, is assumed (unless otherwise specified), and we denote $\oint_\gamma := \int_\gamma$.

Exercise 4.20. *Show that*

$$\int_\gamma f(z)dz = \int_\gamma \Big(udx - vdy\Big) + i\int_\gamma \Big(udy + vdx\Big) \tag{4.4}$$

Exercise 4.21. *Show that the integral along a curve (as a set) can depend on the parametrization of the curve only through a sign.*

A curve is called *simple* if it has no self-intersections. For example, the circle is simple, but the figure ”8” is not.

A domain $\mathcal{D}$ is called *simply connected* if any simple closed curve γ contained it $\mathcal{D}$ can be deformed to a point continuously through curves completely contained in $\mathcal{D}$. This means:

There is a continuous function of two variables $F(t,s) = x(t,s) + iy(t,s)$ defined on $[a,b]\times[0,1]$ with values in $\mathcal{D}$ such that $F(t,0) = \gamma(t)$ and $F(t,1) = p$, a point in $\mathcal{D}$. Intuitively, a simply connected domain has no holes. For example a disk is a simply connected domain, but a punctured disk (such as $\mathbb{D}\setminus\{0\}$), or an *annulus*: $\{z\in\mathbb{C}|\, r<|z|<R\}$, are not simply connected.

Theorem 4.22 (Cauchy)**.** *Assume $\mathcal{D}$ is a simply connected domain and that f is continuously differentiable in $\mathcal{D}$. If γ is a piecewise differentiable simple closed curve contained in $\mathcal{D}$ then*

$$\oint_\gamma f(z)dz = 0 \tag{4.5}$$

Proof. Start with the decomposition (4.4) and use Green's theorem to write

$$\int_\gamma \Big(udx - vdy\Big) = -\int\int_{\mathrm{Int}(\gamma)} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy = 0 \tag{4.6}$$

which vanishes by (3.22) The second integral in (4.4) is dealt with similarly. □

Cauchy thought that continuity of the derivative was needed. Later on Montel and others weakend this hypothesis imposing only boundedness, L^1, or other conditions. Goursat noticed that mere differentiability suffices.

Theorem 4.23 (Cauchy-Goursat–Goursat (1884))**.** *Assume $\mathcal{D}$ is a simply connected domain and that f is differentiable in $\mathcal{D}$. If γ is a piecewise differentiable simple closed curve contained in $\mathcal{D}$ then*

$$\oint_\gamma f(z)dz = 0 \tag{4.7}$$

Proof. We first note that, due to the continuity of f and the piecewise-differentiability of γ we can approximate with arbitrary accuracy $\int_\gamma f(s)ds$ with integrals along polygonal lines, $\int_P f(s)ds$. Decomposing the polygon into triangles, it is then enough to show that $\int_T f(s)ds = 0$ along any triangle contained in $\mathcal{D}$. This is done in the following steps.

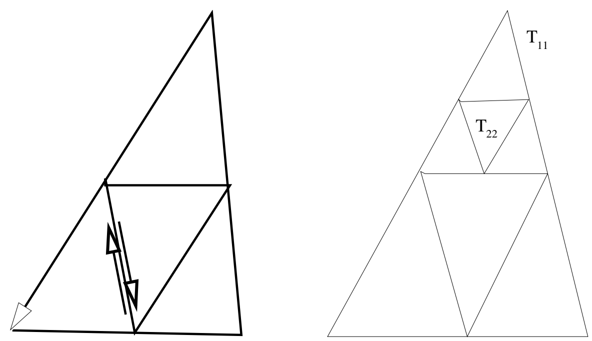


FIGURE 1. Directions of integration after the first partition(left); a few partitions (right).

We split the triangle into 4 smaller triangles and write $\int_T = \sum_1^4 \int_{T_j}$ where the orientation of the inner triangles is chosen so that the integrals along the inner sides cancel. We denote by T_{11} the triangle with the property $\left|\int_{T_{11}} f(s)ds\right| = \max_j \left\{\left|\int_{T_j} f(s)ds\right|\right\}$. We see that $\left|\int_T f(s)ds\right| \leqslant 4\left|\int_{T_{11}} f(s)ds\right|$.

We now split T_{11} into 4 smaller triangles, and proceed in a similar manner to pick T_{22}, and continue inductively in the same fashion.

It follows inductively that for all $n \in \mathbb{N}$, $\left|\int_T f(s)ds\right| \leqslant 4^n \left|\int_{T_{nn}} f(s)ds\right|$. Since the closed triangles T_{nn} are compact and nested there is a $z_0 \in T$ such that $\cap_n T_{nn} = \{z_0\}$.

Now for any fixed n and z on ∂T_{nn} we have $f(z) = f(z_0) + f'(z_0)(z - z_0) + \varepsilon(z)(z - z_0)$ where $\varepsilon(z) \to 0$ as $n \to \infty$ (since $z - z_0 \to 0$ as $n \to \infty$).

We check, using Cauchy's theorem, that $\int_{T_{nn}} f(s)ds = \int_{T_{nn}} \varepsilon(s)(s - z_0)ds$. Note that for some constant C the length of the perimeter of T_{nn} as well as $|z - z_0|$ are bounded by $C2^{-n}$.

It follows that $\left|\int_T f(s)ds\right| \leqslant C^2 4^n 2^{-n} 2^{-n} \sup_{T_{nn}} |\varepsilon(s)| \to 0$ as $n \to \infty$. ∎

It is sometimes useful to integrate analytic functions along the boundary of their analyticity domain. This can be done for instance if f is continuous up to this boundary; Cauchy's theorem still holds:

Exercise 4.24. *Assuming that f is analytic on $\mathcal{D}$, continuous on $\overline{\mathcal{D}}$, and that $\overline{\mathcal{D}}$ is a rectifiable curve of winding number one show that* (4.7) *holds if γ is a simple closed curve in $\overline{\mathcal{D}}$.*

5. Cauchy's formula

5.1. **Homotopic curves.** Let $\mathcal{D}$ be domain in $\mathbb{C}$. Two curves in $\mathcal{D}$ are said to be *homotopic in $\mathcal{D}$* if they can be continuously deformed into each other by a deformation inside $\mathcal{D}$ (see paragraph preceding Theorem 4.23). For example, if $\mathcal{D}$ is simply connected then any simple closed curve is homotopic to a point, see Fig. 2. As another example, if $\mathcal{D}$ is the annulus $\{z \,|1 < |z| < 2\}$ then all circles $|z| = r$ with $1 < r < 2$ are homotopic to each other, but not to a point, while any simple closed curve not going around 0 is homotopic to a point.

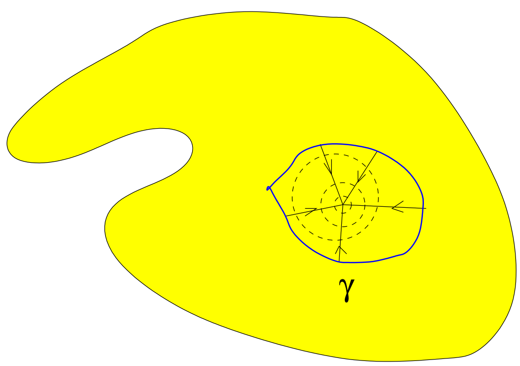


Figure 2. All dotted curves inside γ are homotopic to each other and to the central point.

We will find useful to consider curves $\gamma_{1,2}$ in $\mathcal{D}$, given by two functions $\gamma_{1,2}(t)$ for $t \in [a, b]$, which have the same endpoints, $\gamma_1(a) = \gamma_2(a)$, and $\gamma_1(b) = \gamma_2(b)$. Two such curves are called *homotopic with fixed endpoints* if they can be continuously deformed into each other through a transformation preserving the endpoints with range within $\mathcal{D}$, see Fig. 3.

5.2. **Independence of the integral on the path.** Line integrals are additive w.r.t. the domain of integration: consider an oriented curve γ_1, and then let γ_2 start at the final point of γ_1, say, $\gamma_1(t)$ for t from a to b, $\gamma_2(t)$ for t from b to c, with $\gamma_1(b) = \gamma_2(b)$. We denote for short $\gamma_1 + \gamma_2$ the concatenated curve from t from a to c and we have by definition

$$\int_{\gamma_1 - \gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz \tag{5.2}$$

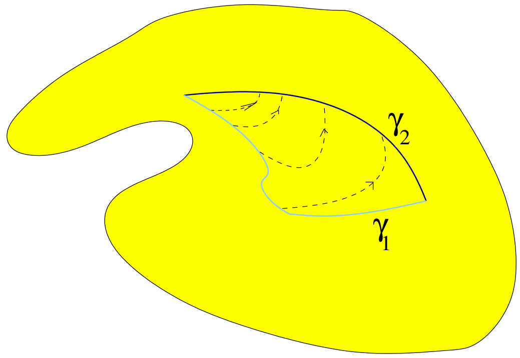


FIGURE 3. Two homotopic curves, γ_1 and γ_2.

(i) Suppose f is analytic in a simply connected domain $\mathcal{D}$, and $\gamma_{1,2}$ are two oriented curves in $\mathcal{D}$ having the same endpoints. Then $\gamma = \gamma_1 - \gamma_2$ is a closed curve. If there is a domain $\mathcal{D}' \subset \mathcal{D}$ containing $\gamma_1 - \gamma_2$ and $\gamma_1 - \gamma_2$ is homotopic to a point in $\mathcal{D}'$, then by (4.7), (5.2), (4.3) we find

$$0 = \int_{\gamma_1 - \gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz - \int_{\gamma_2} f(z)\, dz$$

and therefore *the integral of an analytic function on a simply connected domain is path independent if the paths are homotopic*:

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz \tag{5.3}$$

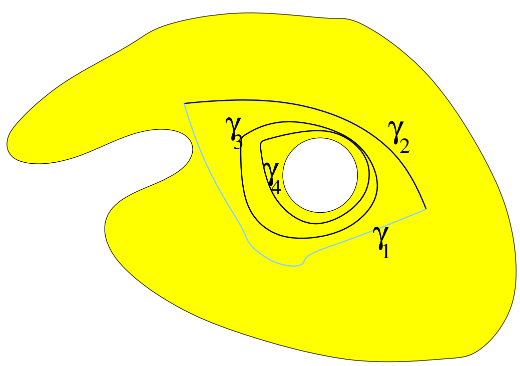


FIGURE 4. γ_1 and γ_2 are not homotopic in the yellow domain while γ_3 and γ_4 are.

5.3. **Cauchy's Formula.** Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and $z_0 \in \mathcal{D}$. The functions $(z-z_0)^{-n}$, $n = 1, 2, ...$ are analytic in $\mathcal{D}\setminus\{z_0\}$. Thus, if γ_1 and

γ_2 are two closed curves in $\mathcal{D}$ not passing through z_0, and homotopic to each-other in $\mathcal{D} \setminus \{z_0\}$ then

$$\int_{\gamma_1} (z - z_0)^{-n} dz = \int_{\gamma_2} (z - z_0)^{-n} dz \tag{5.4}$$

Clearly, these integrals are zero if γ_i does not contain z_0 inside. To calculate the integrals on a simple closed curve encircling z_0 it suffices, by (5.4), to do the calculation when the curve is a circle, which can be done explicitly. Indeed, a circle centered at z_0 with radius ρ is parametrized by $z = z_0 + \rho e^{it}$, $t \in [0, 2\pi]$ (where we used Euler's formula), and we get

$$\oint \frac{dz}{(z - z_0)^n} = \frac{i}{\rho^{n-1}} \int_0^{2\pi} e^{-i(n-1)t} dt = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

Definition 5.25. *If γ is a curve and $z_0 \notin \gamma$, then the index of z_0 w.r.t. γ is defined as*

$$\operatorname{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{d\zeta}{\zeta - z_0}$$

and it represents the "number of times γ winds around z_0".

Theorem 5.26 (Cauchy's formula)**.** *If f is analytic in the simply connected domain $\mathcal{D}$ and γ is piecewise a differentiable simple closed curve in $\mathcal{D}$ around z, we have*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(s)}{s - z} ds \tag{5.6}$$

Proof. Note first that $f(s) - f(z) = f'(z)(s - z) + \varepsilon(s)(s - z)$ where $\varepsilon(s) \to 0$ as $s \to z$. Note that ε is continuous in $\mathcal{D} \setminus \{z_0\}$. We choose a small δ and a ρ such that $|\varepsilon(s)| < \delta/(2\pi)$for $s \in \mathbb{D} := \mathbb{D}_\rho(z_0)$. Then,

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(s)}{s - z} ds = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{f(s)}{s - z} ds = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{f(z)}{s - z} ds \tag{5.7}$$
$$+ \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{f(s) - f(z)}{s - z} ds = f(z) \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{ds}{s - z} + f'(z) \oint_{\partial\mathbb{D}} ds + \oint_{\partial\mathbb{D}} \varepsilon(s) ds$$

where the absolute value of the last integral is bounded by δ, and the result follows from (5.5) by letting $\delta \to 0$.

6. Taylor series of analytic functions

Theorem 6.27 (Taylor series; Cauchy's formula for higher derivatives)**.** *If $f(z)$ is continuously differentiable in $\mathcal{D}$ and $z_0 \in \mathcal{D}$ then there* *exists ρ such that, for $z \in \mathbb{D}(z_0;\rho)$ we have*

$$f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k \quad \textit{where} \tag{6.2}$$

$$c_k = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}_\rho(z_0)} \frac{f(s)}{(s-z_0)^{k+1}}ds = \frac{f^{(k)}(z_0)}{k!}$$

Assume f is analytic in $\mathcal{D}$ and let $z_0 \in \mathcal{D}$. Consider the disk $\mathbb{D}_\rho(z_0)$ with ρ small enough so that its closure $\overline{\mathbb{D}_\rho(z_0)}$ is contained in $\mathcal{D}$.

By Theorem 5.26 we have, for $z \in \mathbb{D}_\rho(z_0)$

$$f(z) = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}_\rho(z_0)} \frac{f(s)}{s-z}ds \tag{6.3}$$

We write

$$\frac{1}{s-z} = \frac{1}{s-z_0-(z-z_0)} = \frac{1}{s-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^k \tag{6.4}$$

Note that the geometric series above converges absolutely and the expression above is bounded in absolute value by

$$\frac{1}{|s-z_0|}\sum_{k=0}^{\infty}\left|\frac{z-z_0}{s-z_0}\right|^k$$

Applying the dominated convergence theorem, we see that

$$f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k; \quad c_k = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}_\rho(z_0)} \frac{f(s)}{(s-z_0)^{k+1}}ds \tag{6.5}$$

Exercise 6.28 (Cauchy remainder)**.** *Check that*

$$f(z) = \sum_{k=0}^{n} c_k (z-z_0)^k + E(z,z_0,n) \tag{6.6}$$

where

$$E(z,z_0,n) = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}_\rho(z_0)} \frac{f(s)}{(s-z)}\frac{(z-z_0)^{n+1}}{(s-z_0)^{n+1}} \tag{6.7}$$

Show that, if $\overline{\mathbb{D}}_\rho(z_0) \subset \mathcal{D}$, then

$$\big|E(z,z_0,n)\big| \leqslant \max_{\mathbb{D}_\rho(z_0)} |f| \;\frac{|z-z_0|^{n+1}}{(\rho-|z-z_0|)} \tag{6.8}$$

which in this case is optimal. If $\mathbb{D}_\rho(z_0)$ *is the maximal disk centered at* z_0 *contained in* $\mathcal{D}$*, then the optimal estimate becomes*

$$\big|E(z,z_0,n)\big| \leqslant \inf_{\rho'\leqslant\rho} \sup_{\mathbb{D}_{\rho'}(z_0)} |f| \, \frac{|z-z_0|^{n+1}}{(\rho'-|z-z_0|)} \tag{6.9}$$

From these considerations it follows that

Theorem 6.29. *If* f *is analytic in a domain* $\mathcal{D}$ *and* $z_0 \in \mathcal{D}$*, then* f *has derivatives of any order at* z_0*.*

Therefore if f *is analytic on* $\mathcal{D}$*, so are* f'*,* f''*, etc.*

Furthermore:

Note 6.30. The expression of $f^{(k)}$ as an integral makes differentiation a "smooth" operation on analytic functions, unlike usual differentiation in real analysis.

Remark 6.31. The disk of convergence of the Taylor series of an analytic function cannot, by the estimate (6.9), be zero. We claim that the radius of convergence of the series exactly equals the radius r of the largest disk centered at z_0 where f is analytic ("the distance to the nearest singularity"), see Fig. 5. Indeed, in any smaller disk we can

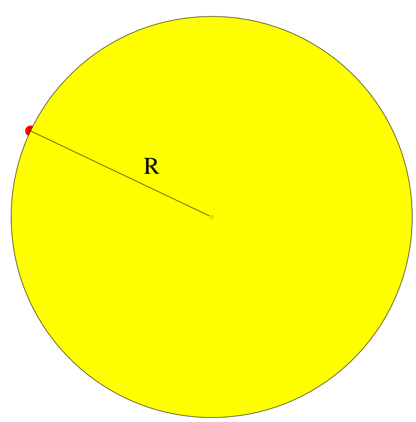


FIGURE 5. Disk of convergence of a Taylor series where the yellow region is a domain of analyticity and the red dot is a singularity.

apply Theorem 6.27 above. If the radius of convergence were larger than r, f would be analytic in a larger domain since convergent power series are analytic.

Example. Consider the function $\frac{1}{1+z^2}$. Its Taylor series at $z = 0$ is

$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty}(-1)^{k+1}z^{2k}, \quad \text{convergent for } |z| < 1$$

and on the boundary of the disk of convergence there are singularities of $\frac{1}{1+z^2}$, namely $z = \pm i$. As a corollary we have

Theorem 6.32 (Liouville's theorem)**.** *A function which is* **entire** *(meaning analytic in all of $\mathbb{C}$) and bounded in $\mathbb{C}$ is constant.*

Proof. Let $M = \sup_{\mathbb{C}} |f|$. We have, by 6.27, for any $\rho > 0$,

$$f'(z) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}_\rho(0)} \frac{f(s)}{(s-z)^2} ds \tag{6.10}$$

and thus

$$|f'(z)| \leqslant \frac{1}{2\pi} M \frac{1}{\rho^2} 2\pi\rho = M/\rho \tag{6.11}$$

Taking $\rho \to \infty$ it follows that $f'(z) \equiv 0$. Then f is a constant. □

Exercise 6.33. ** Show that an entire function other than a polynomial must grow faster than any power of $|z|$ along some path as $z \to \infty$.*

7. The fundamental theorem of algebra

One classical application of Liouville's Theorem is the Fundamental Theorem of Algebra:

Theorem 7.34 (The Fundamental Theorem of Algebra)**.** *A polynomial $P_n(z)$ of degree n has exactly n roots in $\mathbb{C}$, counting multiplicity.*

Proof. It is enough to show the existence of one root when $n \geqslant 1$, since the general form follows inductively by factoring the polynomial.

Exercise 7.35. *Let P be a nonconstant polynomial. Then there exists an R such that $1/P$ is analytic in the domain $\{z : |z| > R\}$ and $1/P(z) \to 0$ as $|z| \to \infty$.*

By this exercise, P_n must have a root, since otherwise $1/P_n(z)$ would be entire and bounded (check). ∎

8. More properties of analytic functions

Assume f is analytic in $\mathbb{D}_\rho(z_0)$ and all derivatives of f are zero at z_0. Then f is zero in the whole of $\mathbb{D}_\rho(z_0)$ (check). More is true.

Proposition 8.36. *Assume f is analytic in a* domain *$\mathcal{D}$ and all derivatives at $z_0 \in \mathcal{D}$ of f are zero. Then f is identically zero in $\mathcal{D}$.*

Proof. For any $z \in \mathcal{D}$ there is a polygonal line P joining z_0 to z: segments $[z_{j-1}, z_j]$, $j = 1, \dots, n$ (with $z_n = z$) and disks $\mathbb{D}_{r_j}(z_j) \subset \mathcal{D}$. (see Proposition 50.19).

Elementary geometry arguments show that we can find some $\rho > 0$ so that any disk centered at a point on P and of radius ρ is contained in $\mathcal{D}$. We cover P by a finite number of disks of radius ρ, centered at spaced points on P, so that the center of each disk is contained in the previous one (for example, their centers are $<\rho/2$ distance apart). The first disk is centered at z_0. Then f is identically zero on the first disk. This means that f and all its derivatives are zero at the center of the second disk, hence f is identically zero on the second disk as well. The argument is continued up to the last disk, showing that $f(z) = 0$.

Theorem 8.37 (Morera)**.** *Let f be continuous in a simply connected domain $\mathcal{D}$ and such that $\oint_\gamma f ds = 0$ for any simple piecewise differentiable closed curve γ contained in $\mathcal{D}$. Then f is analytic in $\mathcal{D}$. The same is true if we restrict the set of curves γ to triangles.*

Proof. Let $z_0 \in \mathcal{D}$ and let $F(z) = \int_{z_0}^{z} f(s)ds$. Here the integral is along any piecewise differentiable path from z_0 to z which is contained in $\mathcal{D}$; note that the value of the integral does not depend on the choice of the path, by the assumption of the theorem. We choose such a path γ.

We take δ small enough, choose a path γ' from z_0 to $z + \delta$ and note that $\int_\gamma f(s)ds - \int_{\gamma'} f(s)ds = \int_z^{z+\delta} f(s)ds$ where the path can be chosen to be a straight line. We write $f(s) = f(z) + \varepsilon(s)$, $\lim_{s\to 0} \varepsilon(s) = 0$ and note that $\int_z^{z+\delta} f(s)ds = f(z)\delta + \varepsilon_1(\delta)\delta$ with $\lim_{\delta\to 0} \varepsilon_1(\delta) = 0$. Hence, F is continuously differentiable in $\mathcal{D}$ and $F' = f$. By Theorem 6.29 f is analytic. The restriction to triangles is left as an exercise, below.
□

Exercise 8.38 (Restriction to triangles)**.** *(a) Find a similar argument in the case when the set of curves γ is restricted to triangles, using polygonal-path connectedness.*

(b) Find an alternative proof based on the piecewise differentiability of the curve and approximations by polygons.

We have now three equivalent views of analytic functions: as differentiable functions of z, as sums of power series, and as continuous functions with zero loop integrals. All these characterizations are quite valuable.

Theorem 8.39 (Weierstrass's theorem)**.** *Assume that f_n are analytic in the domain Ω and converge uniformly on any compact set in Ω to*

f. Then f is analytic in Ω. Furthermore, $f_n' \to f'$ uniformly on any compact set in Ω.

Note 8.40. *Clearly this implies that for any $k \in \mathbb{N}$ $f_n^{(k)} \to f^{(k)}$ uniformly on any compact set in Ω.*

Proof. Let T be a triangle contained in the compact $K \subset \Omega$. Then, by analyticity,

$$\int_T f_n(z)dz = 0 \tag{8.2}$$

Uniform convergence implies that f is continuous and that we can apply dominated convergence which implies

$$\int_T f(z)dz = 0 \tag{8.3}$$

Using Morera's theorem, we see that f is analytic. The properties of the derivatives are immediate, by Cauchy's formula and dominated convergence. ∎

9. Harmonic functions

A real-valued, C^2 function $u(x,y)$ which satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{9.2}$$

in some domain U is called *harmonic* in U.

Theorem 9.41. *Let $\mathcal{D}$ be a simply connected domain in $\mathbb{C}$. A function u is harmonic in $\mathcal{D}$ if and only if u is the real part (or, equivalently, the imaginary part) of an analytic function: $u = Re(f)$ with f analytic in $\mathcal{D}$; f is unique up to an arbitrary imaginary constant.*

In other words, for any harmonic function u, there exists a function v, harmonic on the same domain, and unique up to an additive constant, so that $u + iv$ is analytic. The function v is called *harmonic conjugate of* u.

Proof. If $u = \mathrm{Re}(f)$ then $u \in C^\infty$ (check this, for instance by using the Taylor series of f). Then (9.2) follows immediately from the C-R equations. In the opposite direction, consider the field $\mathbf{E} = (-u_y, u_x)$. We check immediately that this is a potential field and thus $\mathbf{E} = \nabla v$ for some v (unique up to an arbitrary constant). But then, by the C-R theorem, $u + iv$ is analytic in $\mathcal{D}$. □

Lemma 9.42. *Let $f = u + iv$ be analytic near a point z_0 and assume $f'(z_0) \neq 0$. Then the constant level curves $u(x,y) = u(x_0,y_0)$ and*

$v(x,y) = v(x_0,y_0)$ exist near z_0, they are smooth and orthogonal to each-other.

The fact that $f'(z_0) \neq 0$ implies, by C-R that $\nabla u, \nabla v$ are nonzero at x_0, y_0. The rest follows from the implicit function theorem in $\mathbb{R}^2$, and from $\nabla u \cdot \nabla v = 0$, a consequence of C-R.

9.1. **Potential and Hamiltonian flows.** Consider an autonomous system of ODEs in a domain $\mathcal{D}$: the system $\dot{x} = E_1(x,p); \dot{p} = E_2(x,p)$ is a Hamiltonian system if there is an $H \in C^1(\mathcal{D})$ such that $E_1 = \frac{\partial H}{\partial p}$ and $E_2 = -\frac{\partial H}{\partial x}$. It is a potential system if there is a $V \in C^1(\mathcal{D})$ such that $E_1 = \frac{\partial V}{\partial x}$ and $E_2 = \frac{\partial V}{\partial p}$. We see that a system is *both* potential and Hamiltonian if there exist two functions H and V such that $\frac{\partial H}{\partial p} = \frac{\partial V}{\partial x}$ and $\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial p}$. If H and V are smooth enough, then a system is both potential and Hamiltonian **iff** H and V are harmonic functions, the real and imaginary part of an analytic function. In Hamiltonian systems, H is a conserved quantity, meaning: $\frac{d}{dt}H(x(t),y(t)) = \frac{\partial}{\partial H}\dot{x} + \frac{\partial}{\partial H}\dot{p} = 0$ for any solution, as can be easily checked. In gradient systems, $\langle \dot{x}, \dot{p} \rangle$ (when nonzero) clearly gives the direction of steepest ascent of V at the point $\langle x, p \rangle$.

Note 9.43. *The steepest decent/ascent lines of u satisfy the system of ODEs $\dot{x} = \partial v/\partial x$, $\dot{y} = \partial v/\partial y$; the solution is smooth wherever $v_x^2 + v_y^2 \neq 0$. This will be important in understanding the steepest descent method.*

10. The maximum modulus principle

An analytic function in a domain $\mathcal{D}$ can attain its maximum absolute value only on the boundary of $\mathcal{D}$:

Theorem 10.44. *Assume f is analytic and nonconstant in the domain $\mathcal{D}$. Then $|f|$ has no maximum point in $\mathcal{D}$, unless f is a constant.*

Usually the proofs use Cauchy's formula. Look up these other proofs, because they extend to harmonic functions in more than two dimensions.

We will give a proof based on Taylor series.

Proof. Assume that $z_0 \in \mathcal{D}$ is a point of maximum of $|f|$. There is nothing to prove if $f = 0$. Otherwise, replacing f by f/M and z by $z - z_0$ without loss of generality, we can assume that $M = 1$ and $z_0 = 0$. If f is not 1 everywhere, then there exists $k > 0$ so that the Taylor coefficient c_k of f at 0 is nonzero, and in some $\mathbb{D}_\rho(0)$ we have

$$f(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + \cdots = 1 + c_k z^k (1 + d_k z E_1(z))$$

with E_1 analytic at zero. Let z_1 be such that $c_k z_1^k \in \mathbb{R}^+$ and small enough so that $|z_1 d_k E_1(z_1)| < 1$. We see that $|f(z_1)| > 1$, a contradiction. □

Exercise 10.45. *Show that if $|f|$ has a* minimum *in $\mathcal{D}$, then this minimum is zero.*

Exercise 10.46. ** Find the maximum and minimum values of $|\sin z|$ inside the closed unit disk.*

Steepest ascent/descent lines Let f be analytic in a neighborhood of z_0 and $k \in \mathbb{N}$ be the least positive index for which $c_k \neq 0$. A direction $d \in \mathbb{C}$ is a *steepest ascent (descent)* direction at z_0 of $|f|$ if $c_k d^k \in \mathbb{R}^+$ ($c_k d^k \in \mathbb{R}^-$, resp.). A curve γ that follows at each point a steepest ascent/descent direction (is tangent to $d = d(z)$ for all $z \in \gamma$ is a *steepest ascent/descent curve*.

Note 10.47. By note 9.43, the lines of steepest ascent/descent are smooth and unique through any z_0 such that $f(z) = f(z_0) + f'(z_0)(z - z_0) + \ldots$ with $f'(z_0) \neq 0$. A **saddle point** is a point where $f'(z_0) = 0$. Then f is given locally by $z^k + \ldots$ with $k \geqslant 2$. Check that there are exactly k directions of steepest ascent at the saddle point. Hence, lines of steepest descent can be continued, in k possible ways, through a saddle point. These notions are important for understanding the steepest descent method (a.k.a. saddle point method).

Harmonic functions in a domain $\mathcal{D}$ do not have extremum points in $\mathcal{D}$:

Theorem 10.48. *Assume u is harmonic and non-constant in $\mathcal{D}$. Then u has no minimum or maximum in $\mathcal{D}$.*

Proof. Let $u = \mathrm{Re}(f)$ and define $g = e^f$, clearly analytic in $\mathcal{D}$. We have $e^f = e^u e^{iv}$; $|e^f| = e^u$ and then u has a maximum if and only if $|g|$ has a maximum. But this cannot happen strictly inside $\mathcal{D}$. For the minimum, note that $\min(u) = -\max(-u)$ □

10.1. **Application.** The soap film picked up by a thin closed wire has the minimum possible area compatible with the constraint that it is bordered by the wire, since the potential energy is proportional to the surface area. Then, if the wire is close to planar (say the (x, y) plane), the local height $u(x, y)$ of the film satisfies Laplace's equation. (This is not hard to show using some elementary differential geometry.) If the wire is planar, then $u = 0$ on the boundary, and by Theorem 10.48 the minimal surface is flat. This is probably not a surprise. We will however be able to solve Laplace's equation with any boundary constraint, and this will provide us with a lot of insight on minimal surfaces, and conversely, the intuition we have about shapes of soap films gives us an intuition on the solution of Laplace's equation. For instance, it is clear that the shape can have no local extremum, otherwise by flattening it locally would make the surface area smaller.

10.2. **Cauchy principal value integrals (PV).** Suppose that f is analytic in a domain containing the simple closed piecewise differentiable curve C. By Cauchy's theorem we have

$$\frac{1}{2\pi i}\oint_C \frac{f(s)}{s-z}ds = \begin{cases} f(z) \text{ if } z \text{ is inside } C \\ 0 \text{ otherwise} \end{cases} \tag{10.2}$$

What if z lies *on* C? Then the integral is in need of a definition. In one such definition a symmetric segment of the curve centered at z of length ε is cut out and then ε is taken to zero, giving the "Cauchy principal part integral" denoted PV $\oint$ (or P or with a bar through the integral). Another definition is to take the half sum of the integral on a curve circumventing z from the outside and of the integral on a curve circumventing z from the inside.

Exercise 10.49. *Show that if C is a smooth closed curve and f is analytic in a neighborhood of C, then*

$$\frac{1}{2\pi i}\,\mathrm{PV}\oint_C \frac{f(s)}{s-z}ds = \frac{1}{2}f(z) \tag{10.3}$$

and that the two definitions above coincide. Clearly if $f(s)/(s-z)$ extends to an analytic function at z, then the PV integral and the usual one coincide.

Definition 10.50. *[PV on the line] If for any small $\varepsilon > 0$ $f \in L^1(a,-\varepsilon)\cap L^1(\varepsilon,b)$, then PV is defined as*

$$\mathrm{PV}\int_a^b f(s)ds = \lim_{\varepsilon\downarrow 0}\left(\int_a^{L-\varepsilon} f(s)ds + \int_{L+\varepsilon}^b f(s)ds\right) \tag{10.4}$$

if the limit exists.

Example 10.51. The Hilbert transform, an important operator in applications is defined as

$$\mathrm{H}(u)(t) = \frac{1}{\pi}\ \mathrm{PV}\int_{-\infty}^{+\infty}\frac{u(\tau)}{t-\tau}\ \mathrm{d}\tau \tag{10.5}$$

It is is a bounded operator in L^p for $1 < p < \infty$ where the limit exists pointwise almost everywhere, as well as from L^1 to weak L^1. With some Hölder continuity, the limit exists everywhere. The Hilbert transform is crucial in solving Riemann-Hilbert problems.

Exercise 10.52. Show that if $f \in L^1(a,b)$ then

$$\text{PV}\int_a^b f(s)ds = \int_a^b f(s)ds \tag{10.6}$$

11. Automorphisms of the disk

Exercise 11.53. *Let $a \in \mathbb{C}$ with $|a| < 1$ and $\theta \in \mathbb{R}$. Show that*

$$z \mapsto A(z) := e^{i\theta}\frac{a+z}{1+\overline{a}z} \tag{11.2}$$

is a one-to-one transformation of the closed unit disk onto itself. This is of course equivalent to the general family of transformations

$$z \mapsto A(z) := e^{i\theta}\frac{a-z}{1-\overline{a}z} \tag{11.3}$$

a presentation that makes some properties more manifest.

12. Poisson's formula

Proposition 12.54. *Assume u is harmonic in the open unit disk and continuous in the closed unit disk. Then*

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})dt \tag{12.2}$$

Proof. If v is the harmonic conjugate of u then $f := u+iv$ is analytic in the open unit disk, and we have by Cauchy's formula for any $\rho < 1$,

$$\begin{aligned} u(0)+iv(0) = f(0) &= \frac{1}{2\pi i}\oint_{\mathbb{D}_\rho(0)} \frac{f(s)}{s}ds = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it})dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{it})dt + i\frac{1}{2\pi}\int_0^{2\pi} v(\rho e^{it})dt \end{aligned} \tag{12.3}$$

We get (12.2) by taking the real part of (12.3) and passing to the limit $\rho \to 1$. □

Exercise 12.55. ** (i) Let u as in Proposition 12.54 and T as in Exercise 11.53. Show that*

$$U(z) = u(T(z)) \tag{12.4}$$

is harmonic in the open unit disk and continuous in the closed unit disk.

(ii) Show that, if $z_0 = re^{i\theta}$ we have

(12.5)

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta}\frac{r+e^{is}}{1+re^{is}}\right)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(e^{i\theta}\frac{r+e^{is}}{1+re^{is}}\right)ds$$

Proposition 12.56 (Poisson's formula)**.** *(1) Let u be as in Proposition 12.54 and $z_0 = re^{i\theta}$ with $r < 1$. We have*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} u(e^{it})dt \tag{12.6}$$

(2) Conversely, if $f(e^{it}) : [-\pi, \pi)$ is a continuous function, then $u(re^{i\theta})$ defined by (12.5) *where $u(e^{it})$ is replaced by $f(e^{it}$ is harmonic in $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and solves Laplace's equation in $\mathbb{D}$ with $u(e^{it})$ as boundary condition on $\partial\mathbb{D}$.*

The proof of (1) is left any easy exercise:

Exercise 12.57. *Prove* (12.5) *by making the change of variable*

$$e^{i\theta}\frac{r+e^{is}}{1+re^{is}} = e^{it} \tag{12.7}$$

in (12.5).

Proof of (2). By a rotation, we can arrange that $\theta = 0$. Write, for $r < 1$,

$$u(r) = \frac{1}{2\pi}\left(\int_{(-\pi-\delta)\cup(\delta,\pi)} + \int_{-\delta}^{\delta}\right) \frac{1-r^2}{1-2r\cos(t)+r^2} f(e^{it})dt \tag{12.8}$$

and note that by dominated convergence the first integral vanishes as $r \to 1$. For the second integral, we note that $\cos(t) = 1 - t^2\phi(t)/2$ where $\phi(t) \to 1$ as $t \to 0$. The integral becomes

$$\frac{1}{2\pi}\int_{\delta}^{\delta} \frac{(1+r)(1-r)}{(1-r)^2 + rt^2\phi(t)} f(e^{it})dt \tag{12.9}$$

We change variables: $t = (1-r)\tau$ and the second integral becomes

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\frac{\delta}{1-r}}^{\frac{\delta}{1-r}} \frac{(1+r)(1-r)}{(1-r)^2 + (1-r)^2\tau^2\phi((1-r)\tau)} f(e^{i(1-r)\tau})(1-r)d\tau \\ &= \frac{1}{2\pi}\int_{-\frac{\delta}{1-r}}^{\frac{\delta}{1-r}} \frac{1+r}{1+r\tau^2\phi((1-r)\tau)} f(e^{i(1-r)\tau})d\tau \end{aligned} \tag{12.10}$$

We now take the limit $r \to 1$ and get, by dominated convergence

$$\frac{2}{2\pi}f(1)\int_{-\infty}^{\infty}\frac{1}{1+\tau^2}d\tau = f(1)$$

∎

12.1. **The Dirichlet problem for the Laplacian in $\mathbb{D}$.** Formula (12.10) gives the solution of Laplace's equation in two dimensions with Dirichlet boundary conditions, namely with u specified on the boundary , when the domain is $\mathbb{D}$. A simple change of variables adapts this formula to any disk. More generally, we will see that the formula can be adjusted to apply to a general simply connected domain lying in the interior of any simple, closed, piecewise differentiable curve. This is a consequence of the *Riemann mapping theorem*.

12.2. **Application.** The soap film picked up by a thin closed wire has the minimum possible area compatible with the constraint that it is bordered by the wire, since the potential energy is proportional to the surface area. Then, if the wire is close to planar (say in the (x, y) plane), the local height $u(x, y)$ of the film satisfies Laplace's equation. (This is not hard to show using some elementary differential geometry.) If the wire is planar, then $u = 0$ on the boundary, and by Theorem 10.48 the minimal surface is flat. This is probably not a surprise. Formula (12.10) however solves Laplace's equation with any boundary constraint, and this can provide us with a lot of insight on minimal surfaces, and conversely, the intuition we have about shapes of soap films gives us an intuition on the solution of Laplace's equation. For instance, it is clear that the shape can have no local extremum, otherwise by flattening it locally would make the surface area smaller. Before getting to the Riemann mapping theorem we have to restrict ourselves to wires whose projection on a plane is a circle.

Example 12.58. The Faraday cage. "A region surrounded by a conductor does not feel the electrical influence of static outside charges." We show this in two dimensions.

We note first that the electric potential along a conductor, at equilibrium, is zero. For otherwise, there would be a potential difference between two points, thus an electric current $i = V/R$ where R is the resistivity. This would contradict equilibrium. This is of course a uncontroversial physics argument of little mathematical value. We could have modeled the problem mathematically and proved something rigorously, but this would carry us beyond the scope of our course.

Thus we deal with (27.2) with $V = C$ on $\partial\mathcal{D}$. Since $V = C$ is a solution, it is *the* solution. But then $\mathbf{E} = -\nabla V = 0$ which we wanted to prove $\square$.

Problem: (In two dimensions) explain why a region surrounded by a conductor does not feel the electrical influence of static outside charges.

Solution.

Thus we deal with (27.2) with $V = C$ on $\partial\mathcal{D}$. Since $V = C$ is a solution, it is *the* solution. But then $\mathbf{E} = -\nabla V = 0$ which we wanted to prove $\square$.

13. Isolated singularities, Laurent series

Definition 13.59. *f has an* isolated singularity *at z_0 if f is analytic in a* punctured *disk $\mathbb{D}_\rho(z_0) \setminus \{z_0\}$ for some $\rho > 0$.*

More generally, we will analyze functions analytic in annuli $\mathbb{D}_\rho(z_0) \setminus \mathbb{D}_{\rho'}(z_0)$ where $0 < \rho' < \rho$.

For example the functions $e^{1/z}$ and $1/\sin z$ have an isolated singularity at zero, whereas the singularity of $\ln z$ is not isolated (we will see that ln is not well defined in $\mathbb{D} \setminus \{0\}$).

Definition 13.60. An isolated singularity z_0 of f is

(1) a **pole of order** M if $a_k = 0$ for all $k < -M$,

(2) a **removable singularity** if f extends to a function $\tilde{f}$ analytic in the whole disk. By slight abuse of notation we typically do not distinguish notationally $\tilde{f}$ from f itself.

(3) an **essential singularity** if the singularity is not of type (1) or (2).

A function all of whose singularities are poles is called **meromorphic**.

In the following we denote the punctured disk $\mathbb{D}_\rho(z_0)\setminus\{z_0\}$ by $\mathbb{D}^*_\rho(z_0)$.

Proposition 13.61 (Laurent series)**.** *A function f analytic in $\mathbb{D}_\rho(z_0) \setminus \mathbb{D}_{\rho'}(z_0)$ where $0 < \rho' < \rho$ has the convergent representation*

$$f(z) = \sum_{k\in\mathbb{Z}} a_k (z - z_0)^k, \qquad \text{convergent for } z \in \mathbb{D}_\rho(z_0) \setminus \mathbb{D}_{\rho'}(z_0) \tag{13.2}$$

where

$$a_k = \frac{1}{2\pi i} \oint \frac{f(s)}{(s - z_0)^{k+1}}\, ds, \qquad k \in \mathbb{Z}$$

with the integral taken on any circle around z_0 of radius less that ρ.

Proof. We take wlog $z_0 = 0$. Consider the annulus between two circles C_o, C_i in $A = \mathbb{D}_\rho \setminus \mathbb{D}_{\rho'}$, as in Fig. 6 ($C_o$ is the outside circle, and C_i is the interior one). Make a cut L in the annulus as shown. Then $A \setminus L$ is simply connected and Cauchy's formula applies there:

$$f(z) = \frac{1}{2\pi i} \oint_{C_o} \frac{f(s)}{s - z} ds - \frac{1}{2\pi i} \oint_{C_i} \frac{f(s)}{s - z} ds \tag{13.3}$$

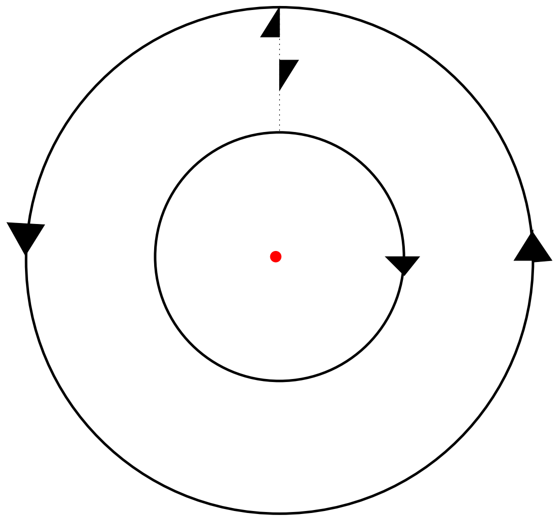


FIGURE 6. Circles of integration and cut in the proof of the Laurent series expansion.

where we used the fact that the boundary of the cut annulus is the closed path composed of C_o, ran counterclockwise from the cut point back to it, followed by the segment ℓ along the cut, then C_i going clockwise, and back to the starting point following $-\ell$. The rest of the proof is similar to Taylor theorem's proof and is left as an exercise below. □

Exercise 13.62. *Complete the proof of formula (13.2) by expanding the integrands in (13.3) in powers of z/s and s/z respectively, and estimating the remainders as we did for obtaining formula (6.3).*

Note. Check that (13.3) gives a decomposition of f into a part f_1 analytic in the disk enclosed by C_o and a function f_2 analytic in $1/z$ in the exterior of the disk bounded by C_i. In [3] this decomposition is used for a nice proof of (13.2). □

For example $e^{1/z}$ has an essential singularity at $z = 0$. Application of (13.2) yields

$$e^{1/z} = \sum_{k=0}^{\infty} z^{-k}/k! \tag{13.4}$$

Note. The part of the Laurent series containing the terms with negative k is called the principal part of the series.

Note. Laurent series are of important theoretical value. However, Laurent are impractical for *calculating* functions in a small neighborhood of an essential singularity. Its convergence gets slower as the singularity is approached.

Theorem 13.63 (Casorati-Weierstrass; also known as the "little Picard theorem"). *Assume f is analytic in $\mathbb{D}_\rho^*(z_0)$ and has an essential singularity at z_0. Then, for any $\rho_1 < \rho$, $f(\mathbb{D}_{\rho_1}^*(z_0))$ is dense in $\mathbb{C}$.*

Proof. Without loss of generality, we may assume $z_0 = 0$. Assume to get a contradiction that there is a $\zeta \in \mathbb{C}$ and a ρ_1 such that $|f(z)-\zeta| > c > 0$ in $\mathbb{D}_{\rho_1}^*(0)$. Then $h(z) = (f(z)-\zeta)^{-1}$ is bounded in $\mathbb{D}_{\rho_1}^*(0)$ and thus 0 is a removable singularity of h (why?). But then $f(z) = \zeta + 1/h$ has a removable singularity or a pole at zero (why?), contradiction. ∎

14. Laurent series and Fourier series

Let f be $2\pi-$periodic and satisfying the following
Assumption The Fourier coefficients of f, $\{c_k\}_k \in \mathbb{Z}\}$ are in $\ell^1(\mathbb{Z})$.

Then,

$$\sum_{k=0}^{\infty} c_k z^k =: g(z)$$

converges absolutely and uniformly in $\overline{\mathbb{D}}$, and thus g is analytic in $\mathbb{D}$ and continuous up to the boundary. We see that for all $k \leqslant -1$ we have

$$\oint_{\partial\mathbb{D}} \frac{g(s)}{s^{k+1}} ds = 0$$

Similarly, define

$$\sum_{k=1}^{\infty} c_k \zeta^k =: H(\zeta)$$

Then, H is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. This means $h(z) = H(1/z)$ is analytic in $\text{ext}(\overline{\mathbb{D}})$ and continuous down to $\partial\mathbb{D}$. You can check that for all $k \geqslant 0$

$$\oint_{\partial\mathbb{D}} \frac{h(s)}{s^{k+1}} ds = 0 \, \forall k \geqslant 0$$

Thus, $f(s) = h(e^{is}) + g(e^{is})$, $s \in \mathbb{R}$. In this sense:

Proposition 14.64. *If f is periodic and its Fourier coefficients satisfy the assumption above, then f is the sum of two functions, one analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, and the other analytic in ext$(\overline{\mathbb{D}})$ and continuous in ext$(\mathbb{D})$.*

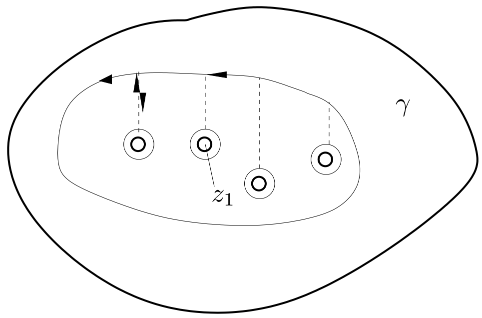


FIGURE 7. Multiply connected domains

15. Calculating the Taylor series of simple functions

One easy way to calculate Taylor series is to use operations with series, see§3.3.

Example. (1) The Taylor series of the function $z^{-1}\sin z$ is

$$\frac{\sin z}{z} = 1 - z^2/6 + z^4/120 + \cdots \tag{15.2}$$

(2) The Taylor series of the function $z/\sin z$ is

(15.3)
$$\frac{z}{\sin z} = \frac{1}{1-(z^2/6 - z^4/120 + \cdots)} = 1 + (z^2/6 - z^4/120 + \cdots) +$$
$$(z^2/6 - z^4/120 + \cdots)^2 + \cdots = 1 + z^2/6 - z^4/120 + z^4/36 + \cdots$$
$$= 1 + z^2/6 + 7z^4/360 + \cdots$$

The first function defined is entire; the second one is not. What is the radius of convergence of the second series?

Exercise 15.65. *Find the integral of* $1/\cos z$ *on a circle of radius* $1/2$ *centered at* $z_0 = \pi/2$.

16. Residues and integrals

Definition 16.66. *A function* f *is meromorphic in a neighborhood of a point* z_0 *if it is analytic in some punctured domain* $\mathcal{D}\setminus\{z_0\}$ *and there is a* $k \geqslant 0$ *so that* $(z-z_0)^k f$ *has a removable singularity at* z_0.

Note 16.67. *Meromorphic functions have* one-sided *convergent Laurent expansions in* $\mathcal{D}\setminus\{z_0\}$*, in the sense that only a finite number of Laurent coefficients of negative order are nonzero.*

Definition 16.68 (Residues)**.** *Let f be meromorphic in $\mathcal{D}\setminus\{z_0\}$. The coefficient c_{-1} in its Laurent expansion at z_0 is called the* **residue** *of f at z_0. You can check that if $\overline{\mathbb{D}}_\rho(z_0)\subset\mathcal{D}$, then $(2\pi i)^{-1}\oint_{\partial\mathbb{D}_\rho(z_0)}=c_{-1}$, hence the name "residue".*

Proposition 16.69. *Let $\mathcal{D}$ be a simply connected domain. Consider a function f which is analytic in the domain $\mathcal{D}\setminus\cup_{k=1}^n\mathbb{D}_k$ where $\mathbb{D}_k$ are disks centered at $z_k\in\mathcal{D}$, and consider a simple closed curve piecewise differentiable γ which encircles each $\mathbb{D}_k$ once (see Definition 18.75 and Figure 9). We have*

$$\oint_\gamma f(s)ds=2\pi i\sum_{k=1}^n \mathrm{Res}(f)_{z=z_k} \tag{16.2}$$

Exercise 16.70. *Prove Proposition 16.69 by deforming and cutting the curve of integration appropriately.*

Example. Calculate

$$\oint\frac{dz}{\sin^3 z}$$

on a circle of radius 1/2 around the origin.
Solution. We have, in $D(0,1/2)$,

(16.3)
$$\frac{1}{\sin^3 z}=\frac{1}{(z-z^3/6+z^5/120\cdots)^3}=\frac{1}{z^3}\frac{1}{(1-z^2/6+z^4/120\cdots)^3}=\frac{1}{z^3}\left(1+z^2/2+17z^4/120\right)+\cdots)$$

and thus the residue of $\sin^{-3}(z)$ at $z=0$ is $1/2$ and the integral equals πi.

Exercise 16.71. *Show that if f has a pole of order m at $z=z_i$ then*

$$\mathrm{Res}f_{z=z_i}=\frac{[(z-z_i)^m f(z)]_{z=z_i}^{(m-1)}}{(m-1)!} \tag{16.4}$$

by applying Laurent's formula near $z=z_i$.

17. Integrals of trigonometric functions

Contour integration is very useful in calculating or estimating Fourier coefficients of periodic functions. Consider the integral

$$I=\int_0^{2\pi}\frac{\cos(nt)}{2+\cos t}dt=\mathrm{Re}\,(J);\;\; J:=\int_0^{2\pi}\frac{e^{int}}{2+\cos t}dt \tag{17.2}$$

Let $z = e^{it}$. Then

$$J = -i \oint_C \frac{z^{n-1}}{2 + (z + 1/z)/2} dz = -2i \oint_C \frac{z^n}{z^2 + 4z + 1} dz \tag{17.3}$$

where C is the unit circle. The roots of $z^2 + 4z + 1$ are $-2 \pm \sqrt{3}$ and only one, $z_0 = -2 + \sqrt{3}$ lies in the unit disk. Thus,

$$J = -2i \cdot 2\pi i \frac{z_0^n}{2z_0 + 4} \ \Rightarrow I = \frac{4\pi z_0^n}{2z_0 + 4} = \frac{2\pi z_0^n}{\sqrt{3}} \tag{17.4}$$

18. Counting zeros and poles

Notations and definitions. (1) Assume f is analytic in a disk $\mathbb{D}_\rho(z_0)$ and $f(z_0) = 0$. Then, in $\mathbb{D}_\rho(z_0)$ we have

$$f(z) = \sum_{k=1}^{\infty} c_k (z - z_0)^k \tag{18.2}$$

If f is not identically zero then there exists some k_0 such that $c_{k_0} \neq 0$ (see Proposition 8.36). The smallest such k_0 is called the *order* (or *multiplicity*) of the zero z_0. For a meromorphic function g (see p. 13.60), the order of a pole at z_0 is the multiplicity of the root of $1/g$ at z_0.

Exercise 18.72. (The zeros of an analytic function are isolated) *Assume $f \not\equiv 0$ is analytic near z_0 and $f(z_0) = 0$. Use Taylor series to show that there is some disk around z_0 where $f(z) = 0 \Rightarrow z = z_0$.*

Assume f is meromorphic in $\mathcal{D}$; let γ be a piecewise differentiable simple closed curve contained in $\mathcal{D}$ together with its interior Γ. Since $\gamma \cup \Gamma$ is a closed subset of $\mathcal{D}$, the region of analyticity of f strictly exceeds Γ. For the purpose of the next proposition, the assumptions can be relaxed, allowing γ to be the boundary of the analyticity domain of f if we impose continuity conditions on f and f'.

Theorem 18.73 (counting zeros and poles)**.** *Let f be as above, and assume f has no zeros on γ. Let N be the total number of zeros of f in Γ counting multiplicities and let P be the number of poles, each pole being counted p times if it has order p. Then*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(s)}{f(s)} ds = N - P \tag{18.3}$$

Proof. The function f'/f is also meromorphic. Check that f'/f has a pole of order 1 and residue n_i at a zero of order n_i of f and a pole of order 1 and residue $-p_i$ at a pole of order p_i of f (check!). The rest follows from the residue theorem, Proposition 16.69.

Note 18.74. We observe that $N - P$ is an integer and, by continuity,

$$g(\zeta) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f'(s)}{f(s) - \zeta} ds = const.$$

in Γ. Hence, in Γ, f takes every value the same number of times. This number is called the order of f in Γ.

Definition 18.75. *The winding number of z_0 with respect to the closed, piecewise C^1 curve $\gamma \not\ni z_0$ is defined as*

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z - z_0} \tag{18.4}$$

Exercise 18.76. *Show that the winding number defined above is always an integer.*

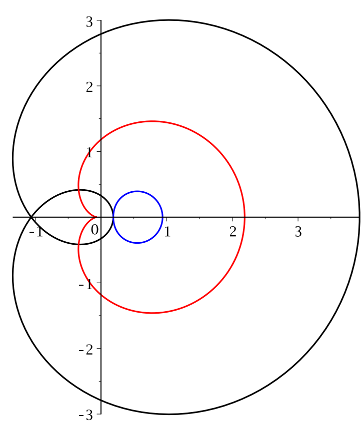


FIGURE 8. The image of the circles of radius 1/4 (innermost curve), of radius 3/4 (middle curve) and of radius 5/4 all centered at zero under the map $(z-1)(z-1/2)$.

Note 18.77 (Some further properties of the log)**.** If γ is a curve without self-intersections that does not pass through zero joining 1 to z, then $\int_1^z ds/s =: \log_\gamma z$, where the integral is taken along γ defines an analytic function in a neighborhood of γ. We can homotopically deform γ and ensure it starts with the interval $[1, 1+\varepsilon]$ for some small ε. Since for positive x we have $e^{\log x} = x$, by permanence of relations, we will have $e^{\log z} = z$ in a neighborhood of γ. If we write the polar representation $z = \rho e^{i\phi}$ and $\log z = u + iv$ we see that $e^u e^{ib} = \rho e^{i\phi}$ whence $\log z = \log|\rho| + i\psi$ where $\psi = \phi + 2n\pi i$ for some $n \in \mathbb{Z}$. The logs defined along different curves may only differ by a multiple of $2\pi i$.

Note 18.78 (The argument principle)**.** If we take, formally for now, $g = \ln f$, then $g' = f'/f$ and then (18.3) shows that the change in $\ln f$ as we traverse positively γ is $N - P$. Another formulation is that if we take the image of a parametrization of γ under f, then, $N - P$ counts the number of times the image turns around zero.

More precisely, since f is not zero on the contour, we can define the log of f on γ as follows. Since the zeros of f' are isolated, there will be only finitely many of them on γ. We break the integral at the points where f' vanishes. Take a $z_0 \in \gamma$ such that $f'(z_0) \neq 0$ and choose any smooth curve γ' joining 1 and $f(z_0)$, which avoids zero. Define $\log f(z_0) = \int_{\gamma'} ds/s$. As discussed, the value of this integral depends on the path through a multiple of $2\pi i$ which will turn out to be immaterial. We take a smal disk $\mathbb{D}_\rho(z_0)$ where $f' \neq 0$, and we note that, for $z \in \mathbb{D}_\rho(z_0)$, $\int_{f(z_0)}^{f(z)} ds/s = \int_{z_0}^{z} f'(u)/f(u)du := \log(f(z)) - \log(f(z_0))$. We note that these integrals are equal piecewise between any two successive points where $f' \neq 0$, thus everywhere alng γ and you can check that this $\log f$ is well defined and analytic in the union of the disks. We then have

$$N - P = \frac{1}{2\pi i}\Delta \log f = \frac{1}{2\pi i}\oint_\gamma \frac{f'(s)}{f(s)}ds = \frac{1}{2\pi i}\oint_{f\circ\gamma}\frac{d\zeta}{\zeta} \tag{18.5}$$

which is the winding number of 0 with respect to $f \circ \gamma$, see Definition 18.75, the number of times, positive or negative, $f \circ \gamma$ winds around 0.

Exercise 18.79. *Let* $f(z) = \exp(1/z) - 1$*; clearly* 0 *is an essential singularity and Proposition 18.73 does not apply. Find however, as a function of* $\varepsilon > 0$*, how many times the curve* $\{f(\varepsilon e^{it}) : t \in [0, 2\pi)\}$ *turns around zero.*

18.1. **Hurwitz's theorem.** This theorem shows that a uniform limit of analytic functions which have no zeros in a domain is either exactly zero, or it is an analytic function which has no zeros in the domain:

Theorem 18.80 (Hurwitz)**.** *If* $\{f_n\}_{n\in\mathbb{N}}$ *are analytic and nonzero in a domain* $\Omega \subset \mathbb{C}$ *and* $\{f_n\}_{n\in\mathbb{N}}$ *converges uniformly on compact sets to* $f \not\equiv 0$*. Then* $f(z)$ *has no zeros on* Ω *either.*

Proof. As we know, the uniform convergence of $\{f_n\}_{n\in\mathbb{N}}$ implies that f is analytic. We do the analysis in the neighborhood of a point z_0; as usual we can take $z_0 = 0$. If $f \not\equiv 0$, then there is a $\mathbb{D}_\delta$ such that $f(z) \neq 0$ in $\partial\mathbb{D}_\delta \setminus \{0\}$ (apply Corollary 3.10). Since $f_n \to f$ uniformly (together with f_n' by Weierstrass's Theorem 8.39), $\{f_n'/f_n\}_{n\in\mathbb{N}}$ to converges f'/f uniformly on the circle $\partial\mathbb{D}_{\delta/2}(0)$. The rest follows from Theorem 18.73. ∎

18.2. **Rouché's Theorem.**

Theorem 18.81 (Rouché)**.** *Assume* f *and* h *are analytic in a domain* $\mathcal{D}$ *containing the piecewise differentiable simple closed curve* γ *and its interior* Γ*. Assume that on* γ *we have* $|h| < |f|$*. Then the number of zeros of* f *and* $f + h$ *in* Γ *is the same (we can think of* $f + h$ *as a "small" perturbation of* f*).*

Proof. Note that all the assumptions hold in a small neighborhood of γ too. We need to all restrict the analysis to such a neighborhood since f might have zeros in Γ. Since $0 \leqslant |h| < |f|$, f can have no zeros in a neighborhood of γ. We have

$$f + h = f \cdot (1 + h/f) =: fQ \tag{18.6}$$

The key remark is that, since we have $|h/f| < 1$, the series

$$q = \sum_{k=1}^{\infty} k^{-1}(-1)^{k+1} g^k; \quad g = h/f$$

converges in a neighborhood of γ and by Theorem 8.39 q is analytic in a neighborhood of γ. We see that

$$q' = g' \sum_{k=0}^{\infty} (-1)^k g^k = \frac{g'}{1+g} = \frac{h'/f - hf'/f^2}{1 + h/f} = \frac{1 + \left(\frac{h}{f}\right)'}{1 + \frac{h}{f}} = \frac{Q'}{Q}$$

This implies

$$\frac{(fQ)'}{fQ} = \frac{f'}{f} + \frac{Q'}{Q}$$

But $\oint \frac{Q'}{Q} = 0$ since, for any $z_0 \in \gamma$, $\oint q' = q(z_0) - q(z_0) = 0$, and the proposition follows (the number of poles of f is by assumption zero).

19. The inverse function theorem

Theorem 19.82. *Assume f is analytic at z_0 and $f'(z_0) = a \neq 0$. Then there exists a disk $\mathbb{D}_\varepsilon(z_0)$ such that f is invertible from $\mathbb{D}_\varepsilon(z_0)$ to $f(\mathbb{D}_\varepsilon(z_0))$ and the inverse is analytic.*

Without loss of generality, we may assume that $z_0 = 0$ and $f(z_0) = 0$. We have $f(z) = az + z^2 g(z)$ where $g(z) \to g(0)$ as $z \to 0$. We want to find a disk of injectivity for f. Note that

$$f(z) - f(z_1) = a(z - z_1) + (z - z_1)^2 g(z) + z_1^2[g(z) - g(z_1)] \tag{19.2}$$
$$= (z - z_1)\Big[a + g(z)(z + z_1) + z_1^2 g'(z)(1 + \varepsilon(z_1)\Big]$$

Take a disk $\mathbb{D}_\rho(z)$ with ρ small enough so that for $z_1 \subset \mathbb{D}_\rho(0)$ we have

$$|(z + z_1)g(z) + z_1^2 g'(z)(1 + \varepsilon_g(z_1))| < |a|$$

Then, in $\mathbb{D}_\rho(z)$, the term on the second line of (19.2) cannot vanish, and it follows that f is injective. The inverse is manifestly continuous, hence the image $f(\mathbb{D}_\rho(z))$ is open. From here, you can show as in calculus that the inverse function defined on $f(\mathbb{D}_\rho(z))$ is differentiable, thus analytic.

Exercise 19.83 (Important generalization)**.** *Assume f is analytic in a neighborhood of z_0, say $z_0 = 0$, that $f(0), f'(0), ..., f^{(k-1)}(0) = 0$ and that $f^{(k)}(0) = a \neq 0$. Show that there is a disk $\mathbb{D}_\varepsilon(f(0))$ such that, for all $t \in \mathbb{D}_\varepsilon(0)$, the equation $f(z) = t$ has exactly k roots. Hint: the proof is a straightforward generalization of the proof of Theorem 19.82*

By definition, a mapping from the domain $\mathcal{D}$ to $\mathbb{C}$ is open if the image of every open set $\mathcal{O} \subset \mathcal{D}$ is open.

Theorem 19.84 (The open mapping theorem)**.** *Let f be analytic and non-constant in the domain $\mathcal{D}$. Then f is open.*

Exercise 19.85. *Use Theorem 19.82 and Exercise 19.83 to prove the open mapping theorem.*

20. Analytic continuation

Assume that f is analytic in $\mathcal{D}$ and f_1 is analytic in $\mathcal{D}_1$, $\mathcal{D}_1 \supsetneq \mathcal{D}$ and $f = f_1$ in $\mathcal{D}$. Then f_1 is an *analytic extension* of f. We also say that f_1 has been obtained from f by *analytic continuation*.

The point of view favored by Weierstrass was to regard analytic functions as properly defined chains of Taylor series, up to a natural equivalence (more about this later), each one of them being the analytic continuation of the adjacent ones. If f is analytic at z_0, then there exists a disk of radius ε centered at z_0 such that f is the sum of this series; we take ε_0 to be the largest ε with this property. If we take a point z_1 inside this disk, f is analytic at z_1 too, and thus near z_1 it is given by a series centered at z_1. The disk of convergence of this series is, as we know, at least equal to the distance $d(z_1, \partial\mathbb{D}_\varepsilon(z_0))$, but might be larger. (Take as an illustration the function $1/(1+z)$ with disks centered at at $z = 0$ and at $z = 1/2$.) In the latter case, we have found a function f_1, piecewise given by the two Taylor series, which is analytic in the union $\mathbb{D}_\varepsilon(z_0) \cup \mathbb{D}_{\varepsilon_1}(z_1)$. In fact, we can continue this process and define chains $z_0, z_1, ...$ such that f_i is analytic in $\mathbb{D}_{\varepsilon_i}(z_i)$. Now, there is a natural equivalence relation between the various fs thus defined: f_α is equivalent to f_β if they are analytic continuations of each other.

Definition 20.86 (Global Analytic Functions in the sense of Weierstrass)**.** *An analytic function in the sense of Weierstrass is an equivalence class with respect to this relation.*

Uniqueness. If there is an analytic continuation in $\mathbb{D}_\varepsilon(z_0) \cup \mathbb{D}_{\varepsilon_1}(z_1)$, then it is unique (use Proposition 8.36 to show this).

This "global analytic function" is not necessarily a function, since the chains may intersect each other while the value of the continuation of f in the overlap region can be different. Indeed the log defined by $\int ds/s$ along any curve avoiding zero has the property that the value at -1 if the curve is an upper semicircle differs from the one on the lower semicircle by $2\pi i$: indeed the residue of $1/s$ at 0 is 1. Now, as discussed, the log is well defined along any curve avoiding 0 from 1 to any point z, and it is analytic in a neighborhood of each curve. But as you see, there is disagreement about the value at -1, and at any other point ultimately: this function is not single-valued.

Cuts. One way to restore single-valuedness is to define log in $\mathbb{C} \setminus (-\infty, 0]$ where $(-\infty, 0]$ is called a **cut** and its function is to prevent curves to circle around 0.

We note that the cut $(-\infty, 0]$ is **highly arbitrary**. It can be replaced by any ray $\{te^{i\phi} : \phi \in [0, 2\pi)\}$ or any curve that has the same functionality, namely to prevent the paths of continuation to encircle 0. This is worth emphasizing this, because many books tend to give the wrong impression that the cut $(-\infty, 0]$, giving the "principal branch of the log", is somehow set in stone. Notably too, the **value of the log may depend on the cut**.

Exercise 20.87. *What is the value of* $\log e^{3\pi i/4}$ *if the cut is* $(-\infty, 0]$ *and what is it if the cut is* $\{it : t \geqslant 0\}$*?*

Natural boundaries. For general analytic functions, it might happen that one encounters lines, or more generally closed regions through which no analytic continuation exists.

Such a boundary is called "natural boundary". It represents a set of "terminus" points for a global analytic function in the sense of Weierstrass.

The standard example of such a function is $f(z) = \sum_{k=0}^{\infty} z^{2^k}$: we have $f(z) \to \infty$ as $z \to 1$, and also as $z \to -1$ to i and $-i$ and more generally as $z \to e^{2\pi i N/2^M}$, $(N, M) \in \mathbb{N}^2$, which form a dense set on the unit circle. This shows that **no point** on the circle is a point of analyticity (why is that?). This is a special case of a lacunary series: if the sequence of natural numbers p_k has the property that $\liminf_{n\to\infty} p_{n+1}/p_n = 1 + \delta$, then the series $\sum_{k=0}^{\infty} c_k z^{p_k}$ is called **lacunary**. There is a whole literature about them. The following theorem is due to Hadamard, and the proof can be found in [7].

Theorem 20.88. *Assume that for $n \in \mathbb{N}$ we have $p_{n+1}/p_n > 1 + 1/n$ and that the series $S(z) \sum_{n=0}^{\infty} c_n z^{p_n}$ has radius of convergence 1. Then $\partial\mathbb{D}$ is a natural boundary for f.*

The same was shown to hold in the more general case $\liminf p_{n+1}/p_n = 1 + \delta$ by Mandelbrojt (1921).

For later: Natural boundaries occur in the *uniformization of nontrivial Riemann surface*. Such is $\mathcal{R}$, the set of curves in $\mathbb{C} \setminus \{0, 1\}$ modulo homotopies. A uniformization ψ map is a bi-analytic bijection between $\mathcal{R}$ and $\mathbb{D}$. Then $\partial\mathbb{D}$ is a natural boundary for ψ^{-1}.

Exercise 20.89 (Example of natural boundary due to Poincaré)**.** *** Consider the rational numbers $r = p/q$ (we assume p and q are relatively prime) and associate to it $N_{pq} = 7^{|p|}5^{|q|}$ (check that this is injective as a function from $\mathbb{Q}^+$ to $\mathbb{N}$). Similarly, you will find an isomorphism between $\mathbb{Q}^-$ and $\mathbb{Z}^-$. Take the function*

$$f(z) = \sum_{N_{pq}}^{\infty} \frac{2^{-N_{pq}}}{z - p/q} \tag{20.2}$$

Show that the series converges for $z \in \mathbb{C} \setminus \mathbb{R}$ and that $\mathbb{R}$ is a natural boundary for f.

How can this example can be modified to obtain an analytic function f in any domain bounded by a simple closed curve γ, and γ is a natural boundary of f?

21. The Schwarz reflection principle

Assume f is analytic in the domains $\mathcal{D}_1, \mathcal{D}_2$ which have a common piece of boundary, a piecewise differentiable curve γ. Assume further that f is continuous across γ. Then, by Morera's theorem, f is analytic in $\mathcal{D}_1 \cup \mathcal{D}_2$ (check this statement). If all is known about a function f is analyticity in $\mathcal{D}_1$, and the fact that a piece of the boundary is an *analytic curve* (meaning, there is a parametrization by a function $\gamma : [0,1] \to \mathbb{C}$ which extends analytically in a complex neighborhood of $[0,1]$) up to which f is continuous, and a further condition on the values of f on γ holds, then it is possible to extend f analytically past γ. We start with the simplest such case.

Theorem 21.90 (The Schwarz reflection principle)**.** *Assume f is analytic in a domain $\mathcal{D}$ in the upper half plane (UHP, also denoted in these notes by $\mathbb{H}^u$) whose boundary contains an interval $I \subset \mathbb{R}$ and assume f is continuous on $\mathcal{D} \cup I$ and real valued on I. Then f has analytic continuation across I, in a domain $\mathcal{D} \cup \mathcal{D}^*$ where $\mathcal{D}^* = \{\overline{z} : z \in \mathcal{D}\}$.*

Note 21.91. see §32.4 for a generalization of this result.

Proof. Consider the function F, given by $F(z) = f(z)$ in $\mathcal{D} \cup I$ and equal to $\overline{f(\overline{z})}$ in $\mathcal{D}^* \cup I$. This function is continuous in $\mathcal{D} \cup I \cup \mathcal{D}^*$ (explain this continuity). It is also analytic in $\mathcal{D}^*$ as it can be immediately seen using a local Taylor series argument. Now Morera's theorem applies: integrals along closed curves completely contained in $\mathcal{D}$ or $\mathcal{D}^*$ are evidently zero, whereas since a closed curve crossing I can be split into two integrals, with I as the splitting, traversed twice, in opposite directions (where is the fact that f is real on I used?). Check the details.

Note. When we learn more about conformal mappings, we shall see that much more generally, a function admits a continuation across a curve γ if the curve is an analytic arc (we will define this precisely) and $f(\gamma)$ is an analytic arc as well.

Example 21.92. The square root function defined by $\sqrt{z} = \rho^{1/2} e^{i\phi/2}$ if $z = \rho e^{i\phi}, \phi \in [0, \pi)$ is analytic in the upper half-plane and continuous down to $[0, \infty)$ and real-valued there, and thus can be continued analytically in the lower half plane by Schwarz reflection. What is the continuation? Let $z = \rho e^{-i\theta} \in \mathbb{H}_l$ $(\theta \in (0, \pi))$. Then $\overline{z} = \rho e^{i\theta} \in \mathbb{H}^u$ where $\surd$ was defined: $\sqrt{\overline{z}} = \rho^{1/2} e^{i\theta/2}$. Then $\overline{\sqrt{\overline{z}}} = \rho^{1/2} e^{-i\theta/2}$.

22. Multi-valued functions

As we discussed, as a result of analytic continuation in the complex plane we may get a *global analytic function* which is not necessarily a function on $\mathbb{C}$ since the definition is path–dependent; the function is thus defined on a space of paths or curves, modulo homotopies.

As long as the domain of continuation is simply connected, we still get a function in the usual sense:

Exercise 22.93 (The **monodromy theorem**). * Assume that f is analytic in $\mathbb{D}_\varepsilon(z_0)$ and that we have and two piecewise differentiable curves γ_1 and γ_2 joining z_0 to z which can be continuously deformed into each-other and furthermore analytic continuation exists along each intermediate curve:

That is, there is a smooth map $\gamma : [0,1]^2 \mapsto \mathbb{C}$ such that $\gamma(s,0) = z_0 \; \forall s \in [0,1]$ and $\gamma(s,1) = z_1 \forall s \in [0,1]$ and furthermore f admits analytic continuation from z_0 to z_1 along $t \mapsto \gamma(s,t), t \in [0,1]$ for any $s \in [0,1]$.

Assume that $\mathcal{D} = \gamma((0,1)^2)$ is simply connected. Show that there is an analytic function F in $\mathcal{D}$ which coincides with f in $\mathbb{D}_\varepsilon(z_0)$. As we know, this continuation is then unique. *(Rough sketch: consider the first curve which by compactness is covered by a finite number of disks of analytic continuation. Choose an intermediate curve close enough so that it is well covered by the same disks. From this point, it should be straightforward.)*

This means in a nutshell: **if f has analytic continuation along any path in the domain $\mathcal{D}$, then it is analytic in $\mathcal{D}$.** Some continuations must explicitly fail to prevent analyticity.

22.1. Generalization: log of a function.

If g is a function defined in a region in $\mathbb{C}$ we can define $\ln g$ by

$$\ln g = \int_a^z \frac{g'(s)}{g(s)} ds \tag{22.2}$$

Now, depending on the properties of g, the homotopy classes will be in general more complicated than those of $\log z$.

If, for instance, g is a meromorphic function, then all the zeros and poles $S = \{z_i, p_j\}$ of g_1 are points where the integral is undefined. $\log g$ is defined on homotopy classes of curves over $\mathbb{C} \setminus S$.

It is convenient to define a branch of $\ln g$ by cutting the plane along rays originating at the points in S. Check that such cuts exist for any such g.

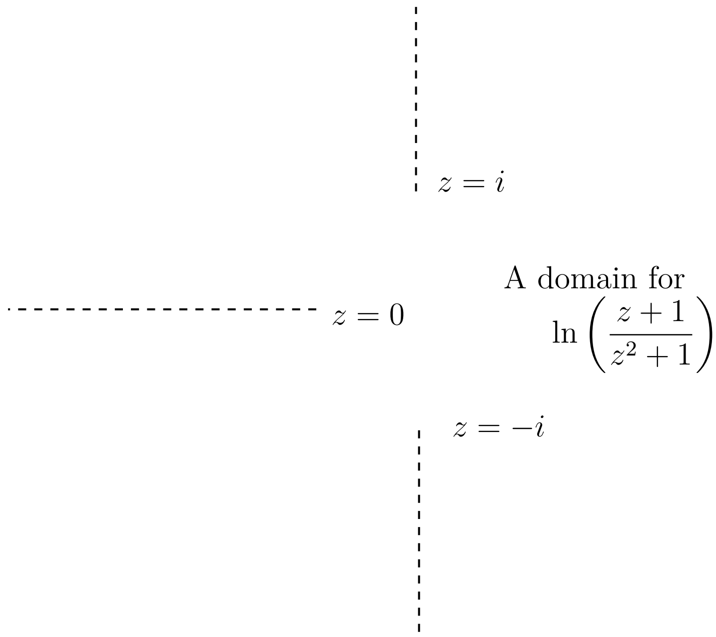


FIGURE 9. Cuts defining the log of a rational function.

22.2. **General powers of z.** Once we have defined the log, it is natural to define

$$z^\alpha = e^{\alpha \ln z} \tag{22.3}$$

Since $\ln z$ is defined on the homotopy classes of curves over $\mathbb{C} \setminus \{0\}$, so is z^α. For a general $\alpha \in \mathbb{C}$, the multivaluedness of z^α is inherited from the multivaluedness of the log: $e^{\alpha(\ln z + 2N\pi I)} = e^{\alpha \ln z} e^{2N\pi i \alpha}$. Note however that if $p \in \mathbb{Z}$ then the value does not depend on the homotopy class and the definition (22.3) defines a function in $\mathbb{C} \setminus \{0\}$, with a pole at zero if $p < 0$ and a removable singularity if $p \geqslant 0$. Check that this definition coincides with the usual power on $\mathbb{R}^+$, hence in $\mathbb{C}$.

Another special case is that when $\alpha = p/q$, p, q relatively prime integers. In this case $e^{2(p/q)\pi i}$ only takes q distinct values.

Note. Beware of possible pitfalls.

$$e^{\ln z_1 + \ln z_2} = e^{\ln z_1} e^{\ln z_2} = z_1 z_2 \tag{22.4}$$

However, this does not mean $\ln z_1 + \ln z_2 = \ln z_1 z_2$, but just that

$$\ln z_1 + \ln z_2 = \ln z_1 z_2 + 2N\pi i \tag{22.5}$$

For the same reason, $z^{\alpha_1} z^{\alpha_2}$ is not necessarily $z^{\alpha_1+\alpha_2}$. Note the fallacious calculation[2]

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1 \quad (?!) \tag{22.6}$$

23. Riemann surfaces: a first view

I will start with two definitions of a Riemann surface adapted from Wikipedia [3]:

Definition 23.94. A Riemann surface Ω is a connected complex manifold of complex dimension one. This means that Ω is a connected Hausdorff space that is endowed with an atlas of charts to $\mathbb{D}$: for every point $x \in \Omega$ there is a neighborhood of x that is homeomorphic to $\mathbb{D}$, and the transition maps between two overlapping charts are required to be analytic.

Definition 23.95. A Riemann surface Ω is an oriented manifold of (real) dimension two – a two-sided surface – together with a conformal structure. Again, manifold means that locally at any point x of Ω, the space is homeomorphic to a subset of the real plane. The supplement "Riemann" signifies that Ω is endowed with an additional structure which allows angle measurement on the manifold, namely an equivalence class of so-called Riemannian metrics. Two such metrics are considered equivalent if the angles they measure are the same. Choosing an equivalence class of metrics on Ω is the additional datum of the conformal structure.

Clearly, $\mathbb{C}$ itself is a Riemann surface. So are domains $\mathcal{D} \subset \mathbb{C}$.

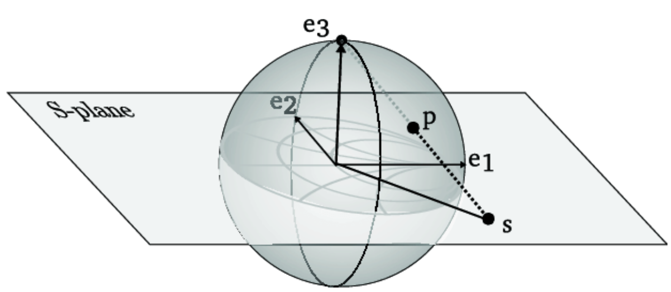


Figure 10. The Riemann sphere (taken from Google images)

[2] This calculation was done as presented by an early version of a computer algebra program.

[3] https://en.wikipedia.org/wiki/Riemann_surface

An important surface in complex analysis is the Riemann sphere $S^2 = \hat{\mathbb{C}}$. The complex plane contains the equator of a sphere $\hat{\mathbb{C}}$, represented by $\partial\mathbb{D} \subset \mathbb{C}$. For any point z in the plane, one draws a line segment from the north pole of the sphere through the sphere, cutting the sphere at some point $p(z)$. Clearly, to any point z in the plane, there corresponds a unique $p(z)$ on $\hat{\mathbb{C}}$. $0 \in \mathbb{C}$ corresponds to the south pole in $\hat{\mathbb{C}}$. There is no point in $\mathbb{C}$ corresponding to the north pole on $\hat{\mathbb{C}}$; this is "the point at infinity". Points in the southern hemisphere map to points in $\mathbb{D}$ while those in the northern hemisphere are in $\overline{\mathbb{D}}^c$.

We will later study the properties of bianalytic transformations, and will see that, in the limit where the size of a triangle goes to zero, a bianalytic (conformal) map takes such a triangle into a similar triangle: the angles are preserved. The triangle is simply rotated and rescaled, in this limit.

Bianalytic transformations are the natural isomorphisms of domains, and more generally of Riemann surfaces in complex analysis. The very important and deep *uniformization theorem* states that, up to such isomorphisms, there are only 3 distinct *simply connected* Riemann surfaces: $\hat{\mathbb{C}}, \mathbb{C}$ and $\mathbb{D}$. Riemann surfaces that are not "very simple" are conformally equivalent to $\mathbb{D}$.

Typically, we cannot embed usefully Riemann surfaces in $\mathbb{R}^3$ and they are not easy to depict. In common cases, a useful way to handle Riemann surfaces is to think of them as spaces of curves, closed or not, modulo homotopies. The endpoints of these curves are the points on the Riemann surface. Thus, the log that we have defined by integrating ds/s over various paths avoiding zero is naturally generating such a set of equivalence classes.

An important object is **the fundamental group** $\pi(\Omega, p)$ generated by the homotopy classes of *closed curves* joining p to p under composition. Here, the product is concatenation of curves, the inverse of γ, γ^{-1} is $t \mapsto \gamma(1-t)$ and $1 = \gamma\gamma^{-1}$.

Exercise 23.96. *Check that the fundamental group of* $\mathbb{C} \setminus \{0\}$ *is the free group generated by one element, an abelian group.*

Let us uniformize Ω_0, the Riemann surface of the log. That means finding a bianalytic bijection from Ω_0 to one of $\hat{\mathbb{C}}, \mathbb{C}, \mathbb{D}$.

The log is well defined by $\log z = \int_1^z ds/s$ on curves from 1 to z modulo homotopies in $\hat{\mathbb{C}} \setminus \{0\}$, and is manifestly analytic in z. The fact that it is injective follows from the calculation $\log z_1 = \log z_2 \Rightarrow \exp\log z_1 = \exp\log z_2 \Rightarrow z_1 = z_2$. It is also onto since any $z \neq 0$ can be written as $\log x + i\phi + 2N\pi i$ for some $x \in \mathbb{R}^+$, $\phi \in [0, 2\pi)$ and $N \in \mathbb{Z}$.

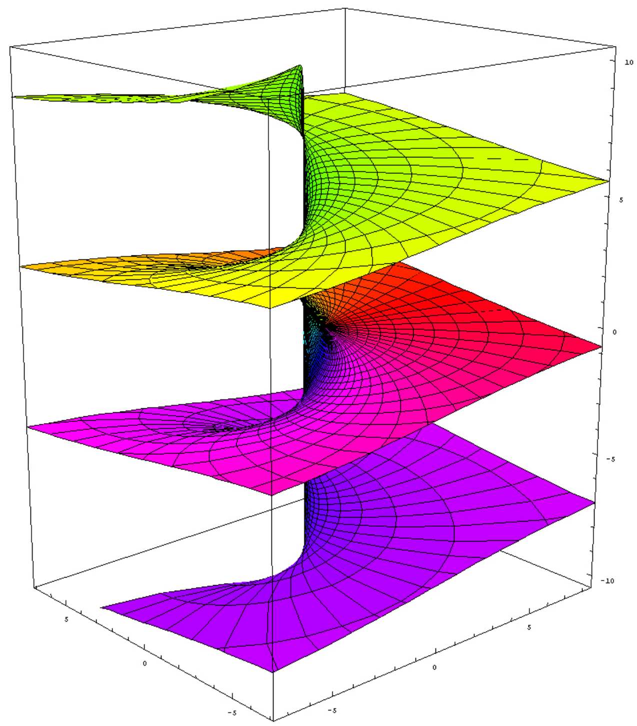


FIGURE 11. The Riemann surface of the log (from https://en.wikipedia.org/wiki/Complex_logarithm)

The Riemann surface of the log is uniformized by the log onto $\mathbb{C}$.

In the opposite direction, let us look again at the action of the exponential on $\mathbb{C}$. As we have seen, exp is one-to-one bi-analytic from any vertical strip

$$S_n\{z = u + iv : u \in \mathbb{R}, v \in [2n\pi i, 2(n+1)\pi i)\} \to \mathbb{C} \setminus \{0\}$$

(think again of the polar representation of complex numbers). If we take exp to be defined on $\mathbb{C}$, it is not invertible because it is many-to-one. It is not univalent. But in each strip it is one-to-one and we can take as its inverse $\log|z| + i\phi + 2n\pi i$. By taking $\cup_n S_n$ we cover all the possible values of log. We see that, if we define exp from $\mathbb{C}$ to the Riemann surface of the log, then it is one-to-one, onto and bi-analytic.

Covering maps. A covering map (also called a covering or projection) is a surjective open map $f : X \to Y$ that is locally a homeomorphism. In a covering map, the preimages $f^{-1}(y)$ are discrete sets in X whose cardinality (finite or infinte) do not depend on y.

Universal covers The universal cover of a connected topological space X is a simply connected space Y with a map $f : Y \to X$ that is a covering map. If X is simply connected, i.e., has a trivial fundamental group, then it is its own universal cover. For instance, the sphere $S^2 = \hat{\mathbb{C}}$ is its own universal cover. The universal cover is always unique and, under very mild assumptions, always exists. In fact, the universal cover of a topological space X exists iff the space X is connected, locally pathwise-connected, and semilocally simply connected.

We see that the exponential is the covering map of $\mathbb{C} \setminus \{0\}$ and $\mathbb{C}$ is its covering space. Note that, of course, $\mathbb{C} \setminus \{0\}$ is not simply connected while the covering space, $\mathbb{C}$, is. If we look at the image of $f(z) = e^z = ue^{iv}$ where u is fixed and $v \in (a, b) \subset \mathbb{R}$ where $b \leqslant \infty$ as a parametrized curve, this curves circles, infinitely many times as $b \to \infty$. around 0. The set of all these curves are the natural curves on which the log is analytic. Here $f^{-1}(y) = \{a + 2n\pi i : e^a = y\}$.

As another example, $f = z \mapsto z^2$ as a map from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$ is a covering map in which f^{-1} always consists of two points. The exponential is a universal cover of $\mathbb{C} \setminus \{0\}$ where the inverse image of a point is an infinite set.

As another example, consider the equivalence relation $x + iy \sim x + m + i(y + n)$ iff $(m, n) \in \mathbb{Z}^2$ and the quotient map $\pi : \mathbb{C} :\to \mathbb{C}/\sim$. Then π is the universal cover of the torus $\mathbb{T}$.

The Riemann surface of the square root As the log, the square root, definable in terms of the log is analytic on the Riemann surface of the log. We note however, from its log-based definition, that closed curves starting at some $z_0 \neq 0$, say $z_0 = 1$ that encircle 0 twice yield the same value of the square root, here 1. What is different from the case of the log is that the fundamental group has a relation: $a^2 = 1$. The Riemann surface of the square root is different from that of the log: the second floor of the "infinite parking lot" is glued to the first. It is not possible to embed this surface in $\mathbb{R}^3$, but we can draw a more or less suggestive picture, below.

24. Evaluation of definite integrals

Contour integrals can be evaluated using the residue theorem. Many definite integrals for which the endpoints are at infinity can also be evaluated using the residue theorem; we can think of them as closed contour integrals on the Riemann sphere, although infinity is often a singular point for the functions of interest. Similarly, a number of integrals whose endpoints are singular points can also be evaluated in closed form. Here is a first simple case.

Proposition 24.97. *Let R be a rational function, continuous on $\mathbb{R}$ and such for some $C > 0$ and all $z \in \mathbb{C}$ we have $|R(z)| \leqslant C|z|^{-2}$ in $\mathbb{C}$.*

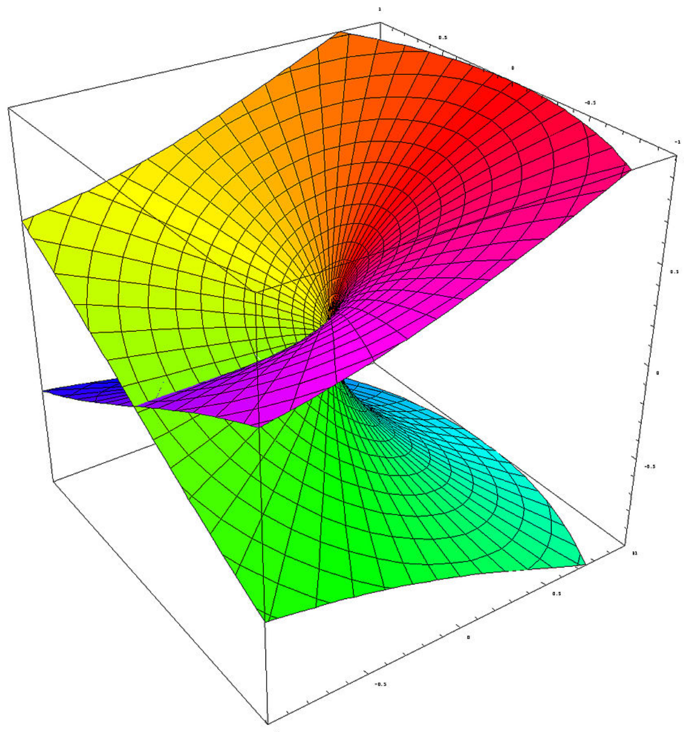


FIGURE 12. An approximate depiction of the Riemann surface of the square root (from https://commons.wikimedia.org/wiki/)

(This happens if the numerator has degree lower by at least two than the denominator.) Then

(24.2)
$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{z_i \in UHP} Res(R; z = z_i) = 2\pi i \sum_{z_i \in LHP} Res(R; z = z_i)$$

where z_i *are poles of* R.

Here and in the sequel "UHP" and "LHP" denote the upper (lower, resp.) half planes; alternative notations are $\mathbb{H}^u, \mathbb{H}_l$ resp. Check that if all the poles of R are in the upper, or in the lower half planes, then the integral vanishes. *Proof.* Under the given assumptions, we take as

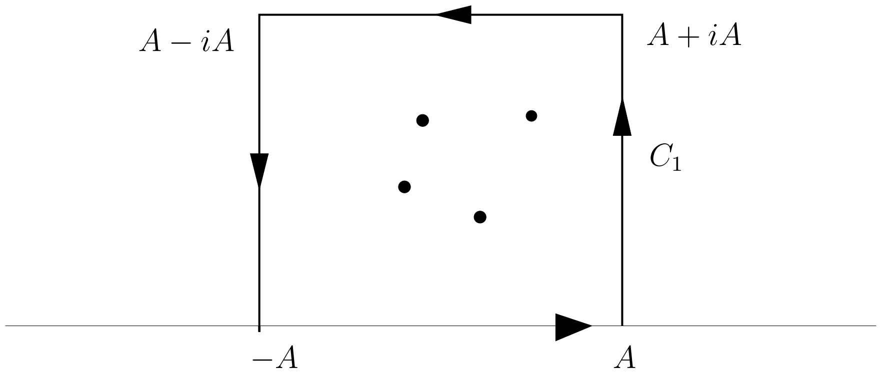


FIGURE 13.

a contour the square in the figure below and write

$$(24.3)\quad \int_{-\infty}^{\infty} R(x)dx = \lim_{A\to\infty}\int_{-A}^{A} R(x)dx$$
$$= \lim_{A\to\infty}\oint_{[-A,A]\cup C_1} R(z)dz - \lim_{A\to\infty}\int_{C_1} R(z)dz$$
$$= 2\pi i \sum_{z_i\in\mathbb{H}^u} \mathrm{Res}(R; z=z_i) - \lim_{A\to\infty}\int_{C_1} R(z)dz = 2\pi i \sum_{z_i\in\mathbb{H}^u} \mathrm{Res}(R; z=z_i)$$

since

$$(24.4)\qquad \left|\int_{C_1} R(z)dz\right| \leqslant const A^{-2}(3A) = 3A^{-1} \to 0 \text{ as } A\to\infty$$

Note It is useful to interpret what we have done as taking the initial conour of integration $\mathbb{R}^+$ and "pushing" or "deforming" it towards $+i\infty$. Every time a pole is crossed, a residue is collected. Since there are only finitely many poles, from a certain "height" on the contour can be pushed all the way to infinity, and that integral vanishes since the integrand vanishes at a sufficient rate.

Example. Find

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$$

Solution The singularities of R in the upper half plane are at $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$ with residues $1/[(1+x^4)']_{z=z_i}$. The result is $I = \pi/\sqrt{2}$.

25. Certain integrals with rational and trigonometric functions

We focus on integrals often occurring in Fourier transforms, of a type which can be reduced to

$$\int_{-\infty}^{\infty} e^{iax} Q(x)dx \tag{25.2}$$

where Q has appropriate decay so that the integral makes sense. Here we take $a > 0$ for convenience, as in the inverse Fourier transform. We would like to push the contour, as above, towards $+i\infty$ since the exponential goes to zero in the process. We need Q to satisfy decay and analyticity assumptions too, for this process to be possible. Jordan's lemma provides such a result suitable for applications.

Lemma 25.98 (Jordan)**.** *Assume $a > 0$ and that Q is analytic in the domain $\mathcal{D} = \{z : \mathrm{Im}\,(z) \geqslant 0, |z| > c\}$ and that γ in the UHP is a semicircle of radius $\rho > c$ centered at zero. Assume furthermore that for large $|z|$ we have $|Q(z)| \leqslant M$. Then,*

$$\left| \int_\gamma e^{iaz} Q(z)dz \right| \leqslant \frac{M\pi}{a} \tag{25.3}$$

In particular, if $Q(z) \to 0$ as $z \to \infty$, then

$$\int_\gamma e^{iaz} Q(z)dz \to 0 \quad \textit{as} \quad \rho \to \infty \tag{25.4}$$

Proof. Let ρ_0 be such that $|Q(z)| \leqslant M$ for all z with $|z| > \rho_0$. Then, for $\rho > \rho_0$ and γ as above we have

$$\left| \int_\gamma e^{iaz} Q(z)dz \right| = \left| \int_0^\pi e^{ia\rho e^{i\phi}} Q(\rho e^{i\phi}) \rho i e^{i\phi} d\phi \right| \tag{25.5}$$
$$\leqslant M \int_0^\pi \rho e^{-\rho a \sin\phi} d\phi = 2M \int_0^{\frac{\pi}{2}} \rho e^{-\rho a \sin\phi} d\phi$$

(By the symmetry $\sin t = \sin(\pi - t)$ the integral is twice the one on $[0, \pi/2]$.) To calculate the last integral we bound below $\sin\theta$ by $b\theta$ for some $b > 0$. By an elementary calculation we see that $t^{-1}\sin t$ is decreasing on $[0, \pi/2]$ and thus $\sin\theta \geqslant 2\theta/\pi$ for θ in $[0, \pi/2]$ and we get

$$\left| \int_\gamma e^{iaz} Q(z)dz \right| \leqslant M \int_0^{\pi/2} 2\rho e^{-2\rho a\phi/\pi} d\phi \leqslant \frac{M\pi}{a} \tag{25.6}$$

and the result follows. The second statement is immediate. ∎

Proposition 25.99. *Assume $a > 0$ and Q is a rational function continuous on $\mathbb{R}$ and vanishing as $|z| \to \infty$ (that is, the degree of the denominator exceeds the degree of the numerator). Then*

$$\int_{-\infty}^{\infty} Q(x)e^{iax}dx = 2\pi i \sum_{z_i \in \mathbb{H}^u} \operatorname{Res}(Q(z)e^{iaz}; z = z_i) \tag{25.7}$$

The proof is left as an exercise: it is a simple combination of Jordan's lemma and of the arguments in Proposition 24.97.

The fact that we are dealing with a rational function is crucial; simply $1/z$-like decay would not ensure the existence of the improper integrals involved. Note that the improper integral $\int_1^\infty z^{-1}e^{iz}$ exists, and that for large $|z|$ and some constant $c \in \mathbb{C}$, $|Q(z) - a/z| \leqslant const(|z|+1)^{-2}$ and that $(|z|+1)^{-2} \in L^1$.

Example Let $\tau > 0$ and find

$$I = \int_0^\infty \frac{\cos \tau x}{x^2+1}dx \tag{25.8}$$

Solution. The function is even; thus we have

$$2I = \int_{-\infty}^{\infty} \frac{\cos \tau x}{x^2+1}dx = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{i\tau x}}{x^2+1}dx \tag{25.9}$$

which is of the form in Proposition 25.99 and thus a little algebra shows

$$I = \frac{\pi}{2}e^{-\tau}$$

Note that we have calculated the cos Fourier transform of a function which is analytic in a neighborhood of the real line, and the transform is exponentially small as $\tau \to \infty$. This is not by accident: formulate and prove a result of this type for cosine transforms rational functions with no poles on $\mathbb{R}$.

Example([13] p. 116) Assume $\operatorname{Re} z > 0$. Show that

$$I(z) = \int_0^\infty t^{-1}(e^{-t} - e^{-tz})dt = \log z \tag{25.10}$$

Solution (for another solution look at the reference cited) Note that the integrand is continuous at zero and the integral is well defined. Furthermore, it depends analytically on z and by dominated convergence we get

$$I'(z) = \int_0^\infty e^{-tz}dt = z^{-1} \quad \Leftrightarrow I(z) = \log z + C \tag{25.11}$$

Check that the constant C is zero.

Example: A common definite integral. Show that

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2} \tag{25.12}$$

Solution This brings something new, since a naive attempt to write

$$\int_{-\infty}^{\infty} \frac{\sin t}{t} dt = \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{it}}{t} dt \quad (??) \tag{25.13}$$

cannot work as such, since the rhs is ill–defined. But we can still apply the ideas of the residue calculations in these lectures. Here is how.

(1) Use the box argument (see figure below) and $A = (2N+1)\pi/2, N \in \mathbb{Z}$ to show that

$$\int_{-\infty}^{\infty} \frac{\sin t}{t} dt = \int_{-\infty+i}^{\infty+i} \frac{\sin t}{t} dt = \int_{-\infty}^{\infty} \frac{\sin(t+i)}{t+i} dt$$

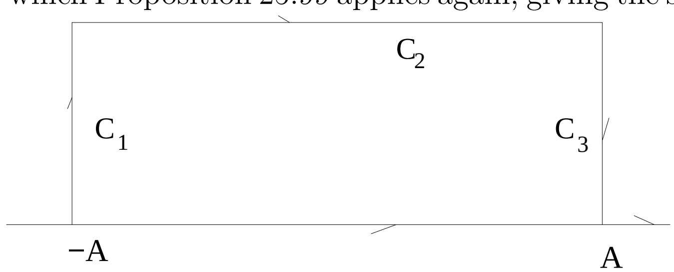
(2) Now we can write

$$\int_{-\infty}^{\infty} \frac{\sin(t+i)}{t+i} dt = \int_{-\infty}^{\infty} \frac{e^{i(t+i)} - e^{-i(t+i)}}{2i(t+i)} dt = \int_{-\infty}^{\infty} \frac{e^{i(t+i)}}{2i(t+i)} dt - \int_{-\infty}^{\infty} \frac{e^{-i(t+i)}}{2i(t+i)} dt$$

The first integral is zero, by Proposition 25.99. The last term equals

$$\overline{\int_{-\infty}^{\infty} \frac{-e^{i(t-i)}}{-2i(t-i)} dt}$$

to which Proposition 25.99 applies again, giving the stated result (check!)



Exercise 25.100. *** Find*

$$\int_0^\infty \frac{\sin^4 t}{t^4} dt$$

26. Integrals of branched functions

We now show that, for $\alpha \in (0,1)$ we have

$$\int_0^\infty \frac{t^{-\alpha}}{t+1} dt = \frac{\pi}{\sin \pi\alpha} \tag{26.2}$$

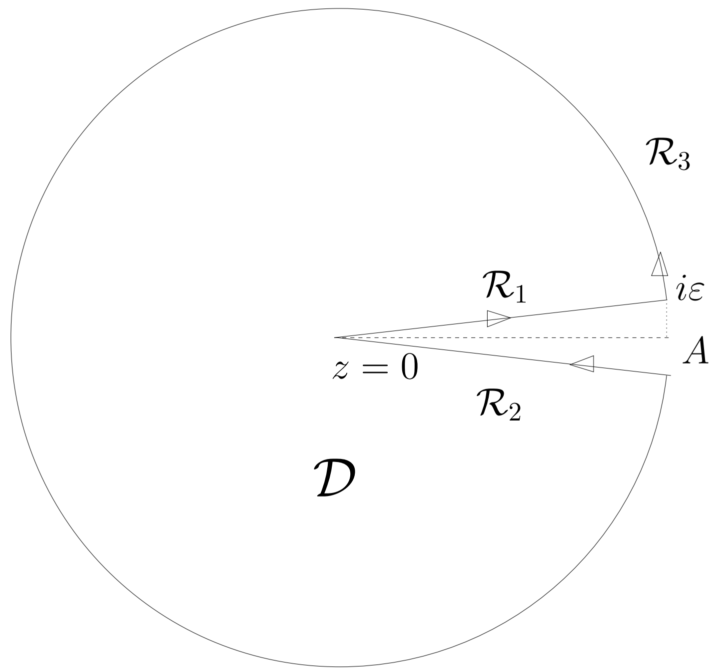


FIGURE 14.

Note that

$$\int_0^\infty \frac{t^{-\alpha}}{t+1}dt = \lim_{\varepsilon\to 0, A\to\infty} \int_\varepsilon^A \frac{t^{-\alpha}}{t+1}dt \tag{26.3}$$

Note that the integrand has an integrable singularity at $t = 0$ and decays like $t^{-\alpha-1}$ for large t, thus the integral is well defined. The integral is performed along $\mathbb{R}^+$ so we know what $t^{-\alpha}$ means. We extend $t^{-\alpha}$ to a global analytic function; it has a branch point at $t = 0$ and no other singularities. Consider the region in the figure below. $t^{-\alpha}$ is analytic in $\mathbb{C} \setminus [0, \infty)$. Note first that the integral along any ray $\rho e^{it}, \rho \in [0, \infty]$ equals the limit when $0 < \varepsilon \to 0$ of the integral along $\rho e^{it}, \rho \in [\varepsilon, \infty]$. Thus

$$\oint_{\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3} \frac{t^{-\alpha}}{t+1}dt = 2\pi i \text{Res}\left(\frac{t^{-\alpha}}{t+1}; z = -1\right) = 2\pi i e^{-\pi i\alpha} \tag{26.4}$$

In the limit $\varepsilon \to 0, A \to \infty$ $\int_{\mathcal{R}_3} \to 0$ and $\int_0^A$ converges to $\int_0^\infty$ and we get

$$\int_{\mathbb{R}^+} \frac{t^{-\alpha} - t^{-\alpha}e^{-2\pi i\alpha}}{t+1}dt = (1 - e^{-2\pi i\alpha}) \int_{\mathbb{R}^+} \frac{t^{-\alpha}}{t+1}dt = 2\pi i e^{-\pi i\alpha} \tag{26.5}$$

The rest is straightforward.

More generally, we have the following result.

Proposition 26.101. *Assume* $\operatorname{Re} a \in (0,1)$ *and* Q *is a rational function which is continuous on* $\mathbb{R}^+$ *and is such that* $x^a Q(x) \to 0$ *as* $x \to 0$ and *as* $x \to \infty$. *Then*

$$\int_0^\infty x^{a-1} Q(x)dx = -\frac{\pi e^{-\pi i a}}{\sin a\pi} \sum Res(z^{a-1}Q(z); z_i) \tag{26.6}$$

where z_i *are the poles of* Q.

Exercise 26.102. *Prove Proposition 26.101.*

Exercise 26.103. ** Let* $a \in (0,1)$. *Calculate*

$$P \int_0^\infty \frac{x^{a-1}}{1-x} dx$$

where P *denotes the Cauchy principal part, as defined before.*

Exercise 26.104.

$$\int_0^\infty \frac{x^{-1/2} \ln x}{x+1} dx$$

(There is a simple way, using the previous results.)

27. Conformal Mapping

Laplace's equation in two dimensions

$$\Delta f = f_{xx} + f_{yy} = 0 \tag{27.2}$$

describes a number of problems in physics; it describes the flow of an incompressible fluid. It also describes the space dependence of the electric potential in a region where the density of charges, ρ, is zero and the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ fields are time-independent. For the latter problem, Maxwell's equations are $\nabla \cdot \mathbf{E} = \varepsilon_0^{-1}\rho = 0$ and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = 0$. The second equation implies $\mathbf{E} = -\nabla V$, for some V (called potential) and the first equation gives $\Delta V = 0$. Since the electric field is produced by charges, the boundary conditions are expected physically to determine the solution. A typical problem would be to solve eq. (27.2) with $u = V$ in $\mathcal{D}$ with V given on $\partial\mathcal{D}$ (Dirichlet problem).

In the case of two-dimensional incompressible fluid flow, let $\langle v, u\rangle$ be the velocity field. Incompressibility translates into

$$\operatorname{div}\langle v, u\rangle = v_x + u_y = 0 \tag{27.3}$$

while the fact that the flow is irrotational implies

$$\nabla \times \mathbf{V} = 0 \Rightarrow u_x - v_y = 0 \tag{27.4}$$

(27.3) and (27.4) imply that $\langle v, u\rangle$ are harmonic conjugates. In any simply connected domain, (27.4) implies $\mathbf{V} = \nabla\varphi$ for some φ called *velocity potential*. We can check that $\Delta\varphi = 0$, thus φ is harmonic. Its harmonic conjugate ψ is called *the stream function*. In the physical applications above, the ODE system associated with **V** and **E** are *both* potential and gradient. In the case of fluid flow, the lines of constancy of ψ are parallel to the flow, see §9.1. If the fluid flows in some domain $\mathcal{D}$, a natural boundary condition (no, not a natural boundary!) is that the fluid does not flow through $\partial\mathcal{D}$, that is $\langle v, u\rangle \cdot \langle n_1, n_2\rangle = 0$ where $\langle n_1, n_2\rangle$ is the normal direction to the boundary; this is also known as a no-penetration condition. Laplace's equation where the normal derivative is given on $\partial\mathcal{D}$ is called a Neumann problem.

We already know the general solution of the Dirichlet problem when $\mathcal{D}$ is a disk, (12.7). The solution of the Dirichlet problem exists and is unique in any connected domain $\mathcal{D}$ with smooth enough boundary and continuous data on the boundary.

27.1. **Uniqueness.** We showed the existence of a of the Dirichlet problem, given in terms of a Poisson integral. We show uniqueness here, which is very easy: if we had two solutions u_1, u_2 then $u = u_1 - u_2$ would satisfy (27.2) with $u = 0$ on ∂D. But a harmonic function reaches both its maximum and minimum on the boundary. Thus $u \equiv 0$. A similar argument shows that in the Neumann problem, u is determined up to an arbitrary constant.

27.2. **Existence.** We have the uniques solution of the Dirichlet problem in $\mathbb{D}$. What about other domains?

It is often the case in PDEs that a symmetry group exist and then it is very useful in solving the equation and/or determining its properties.

It turns out that (27.2) has a *huge* symmetry group: the equation is *conformally invariant*. This means the following.

Proposition 27.105. *If u solves (27.2) in $\mathcal{D}$, a simply connected domain, and $f = f_1 + if_2 : \mathcal{D}_1 \to \mathcal{D}$ is analytic, then $u(f_1(s,t), f_2(s,t))$ is a solution of (27.2) in $\mathcal{D}_1$.*

Proof. We know that u has a harmonic conjugate v determined up to an additive constant. Let $g = u + iv$. Then g is analytic in $\mathcal{D}$. Then the composite function $g(f)$ is analytic in $\mathcal{D}_1$, and in particular $u(f_1(s,t), f_2(s,t))$ and $v(f_1(s,t), f_2(s,t))$ satisfy the C-R equations in $\mathcal{D}_1$. But then $u(f_1(s,t), f_2(s,t))$ is harmonic in $\mathcal{D}_1$. $\square$.

We will be mostly interested in analytic *homeomorphisms* which have many nice properties. Two regions that are analytically homeomorphic to each-other are called *conformally equivalent*.

The Riemann mapping theorem, which we will prove later, states that any simply connected domain other that $\mathbb{C}$ itself is conformally

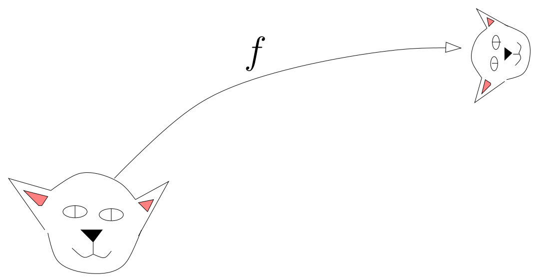


FIGURE 15. To be made rigorous in the sequel.

equivalent to the unit disk. The boundary of the region is then mapped onto the unit circle. The "orbit" of the disk under the group of conformal homeomorphisms group contains every simply connected region other that $\mathbb{C}$ itself.

The conformal group is large enough so that by its action we can solve Laplace's equation in any simply connected domain (the boundary has to be smooth enough for the boundary condition to make sense; $C^{1,\alpha}$ is sufficient.)

This is one of many motivations for a careful study of conformal maps.

27.3. **Preservation of angles and small shapes: heuristics.** Let f be analytic at z_0, $f'(z_0) = a \neq 0$ (w.l.o.g. $z_0 = 0$, $f(0) = 0$) and consider a tiny neighborhood $\mathcal{N}$ of zero. If z_β are points in $\mathcal{N}$ then

$$f(z_\beta) \approx az_\beta \tag{27.5}$$

All these points get multiplied by the *same* number a. Multiplication by a complex number rescales it by $|a|$ and rotates it by $\arg a$. If we think of z_β as describing a figure, then $f(z_\beta)$ describes the same figure, rotated and rescaled. The shape (form) of the figure is thus preserved and the transformation is *conformal*.

Since a tiny square of side ε becomes a square of side $|a\varepsilon|$ areas are changed by a factor of $|a^2|$.

We make this rigorous in what follows.

27.4. **Preservation of angles.** Additivity of arguments holds up to a multiple of 2π. In the calculations below, this multiple will be immaterial. Assume f is analytic in a disk D, $z_0 \in D$ and that $f'(z_0) \neq 0$. The angle between two smooth curves $\gamma(t)$ and $\Gamma(t)$ which cross at a point $z = \gamma(t_0) = \Gamma(t_1)$ (w.l.o.g. we can take $t_0 = t_1 = 0$) is by definition the angle between their tangent vectors, that is $\arg \gamma'(0) - \arg \Gamma'(0)$, assuming of course that these derivatives don't vanish.

The angle between the images of these curves is given by

(27.6)

$$\arg[f(\gamma)'(0)] - \arg[f(\Gamma)'(0)] = \arg[f'(\gamma(0))\gamma'(0)] - \arg[f'(\Gamma(0))\Gamma'(0)]$$

$$= \arg f'(\gamma(0)) + \arg \gamma'(0) - \Big(\arg f'(\Gamma(0)) + \arg \Gamma'(0) \Big) = \arg \gamma'(0) - \arg \Gamma'(0)$$

That is to say the image of two curves intersecting at an angle α is a pair of curves intersecting at the same angle α. Preservation of angles means that a small enough domain is transformed into a similar one, only rotated and rescaled.

Exercise 27.106. *More precisely, consider a sequence of similar triangles whose sides go to zero and with a common point z_0, with $f'(z_0) \neq 0$. Consider their images through f. For each triangle, rescale both the triangle by the size L of one of its sides, and rescale their images through f by $L|f'(z_0)|$. Show that in the limit of vanishing sides, the images of these triangle converge to triangles congruent to them. Then, polygons are preserved in the same sense, and by approximation by polygons any "small" smooth shapes are also preserved.*

27.5. **Rescaling of arc length.** The arc length along a curve $\gamma(t)$ is given by

$$L(\gamma) = \int_a^b |\gamma'(t)| dt =: \int_\gamma d|z| \tag{27.7}$$

If f is analytic, then

(27.8)

$$L(f(\gamma)) = \int_a^b |f(\gamma)'(t)| dt = \int_a^b |f'(\gamma(t))| \, |\gamma'(t)| dt = \int_\gamma |f'(z)| d|z|$$

and thus the arc length is locally stretched by $|f'(z)|$, as seen in Exercise 27.106.

27.6. **Transformation of areas.** The area of a domain A is its Lebesgue measure,

$$\iint_A dx dy \tag{27.9}$$

Let $u + iv = f : B \to A$ be injective and analytic. After the transformation $(x, y) \mapsto (u(x, y), v(x, y))$ the area becomes

$$\iint_{f^{-1}(A)} |J| du dv \tag{27.10}$$

where the Jacobian J is, using the C-R equations, $|f'|^2$ (check!). Tis also follows from Exercise 27.106 if you think how small squares are transformed.

Note 27.107. It is interesting to remark that it is enough that (u, v) is a smooth transformation that preserves angles for $u+iv$ to be analytic. It is also enough that it rescales any figure by the same amount for it to be analytic or anti-analytic ($\overline{f}$ is analytic). This is not difficult to show; see [3], p 74. This gives a very nice characterization of analytic functions: they are those which are "locally Euclidian".

Note 27.108. Observe that we did not require f to be globally one-to-one. The simple fact that f is analytic with nonzero derivative makes it conformal.

Note 27.109 (Note on bijectivity)**.** *In* $\mathbb{R}$ *it suffices that* $f' \neq 0$ *everywhere for* f *to be bijective. This is* **not** *the case in* $\mathbb{C}$*. Clearly* $\exp' = \exp \neq 0$ *in* $\mathbb{C}$ *but* exp *is not injective.*

We need to impose bijectivity for two regions to be conformally *equivalent*. On the other hand, if f is an analytic homeomorphism between $\mathcal{D}_1$ and $\mathcal{D}_2$ then f is conformal (that is, $f' \neq 0$ in $\mathcal{D}$). This also follows from the following proposition.

Proposition 27.110 (Ramified expansions: Puiseux series)**.** Assume that $f : \mathcal{D}_1 \mapsto \mathcal{D}_2$ is analytic, that for some $z_0 \in \mathcal{D}_1$ we have $f^{(j)}(z_0) = 0$ if $j = 1, ..., m-1$ and $f^{(m)}(z_0) = a \neq 0$. (Take $z_0 = 0, f(z_0) = 0$.) Let $\{\omega_k\}_{1 \leqslant k \leqslant m}$ be the mth roots of unity. Then, there exists an analytic function $\mathcal{A}$ in a neighborhood of 0, $\mathcal{A}(0) = 0$ such that, given a choice of $t^{1/m}$, all the solutions of the equation $f(z) = t$ near zero are given by $z_k = \mathcal{A}(\omega_k t^{1/m}), 1 \leqslant k \leqslant m$.

Furthermore, $\mathcal{A}'(0) \neq 0$ and hence $\mathcal{A}$ is a local bianalytic bijection.

Proof. In a neighborhood of 0 we have, with $a \neq 0$

$$f(z) = z^m(a + b_1 z + b_2 z^2 + \cdots) = z^m g(z)$$

and $g(z)$ is analytic, $g(0) = a$ and thus $g \neq 0$ in some disk $\mathbb{D}_{\varepsilon_1}(0)$. Choose a branch $h = g^{1/m}$ and of $t^{1/m}$. For a given ζ, our equation becomes $(zh(z))^m = t$, equivalent to m equations, $zh(z) = \omega_k t^{1/m}$. The function zh is analytic at zero and $(zh)'(0) = h(0) \neq 0$. By the inverse function theorem, the equation $zh = \omega_k t^{1/m}$ has exactly one solution, $\mathcal{A}(\omega_k t^{1/m})$, where $\mathcal{A}$ is a locally analytic bijection. □

Note 27.111. Writing $\mathcal{A}(s) = \sum_{j \geqslant 0} a_j s^j$ we see that $z(t) = \sum_{j \geqslant 0} a_j \omega_k^j t^{j/m}$ a *ramified* expansion, a special case of a Puiseux series.

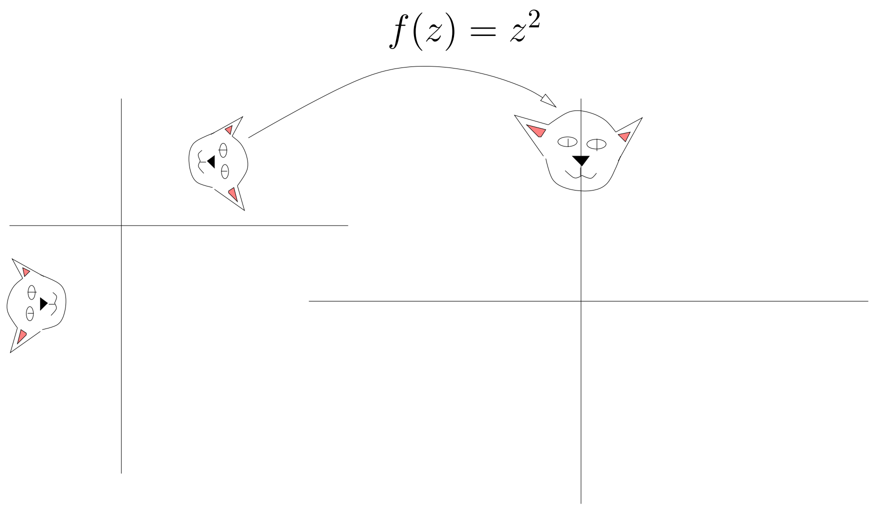


FIGURE 16. To be made rigorous in the sequel.

This gives us another proof of the open mapping theorem,

Theorem 27.112. *Let $\mathcal{D}$ be a domain and $f : \mathcal{D} \to \mathbb{C}$ be a nonconstant analytic function. Then f is open, that is, the images of open sets are open.*

Proof. This follows immediately from the fact that $\mathcal{A}$ is a bijection and is left as an exercise.

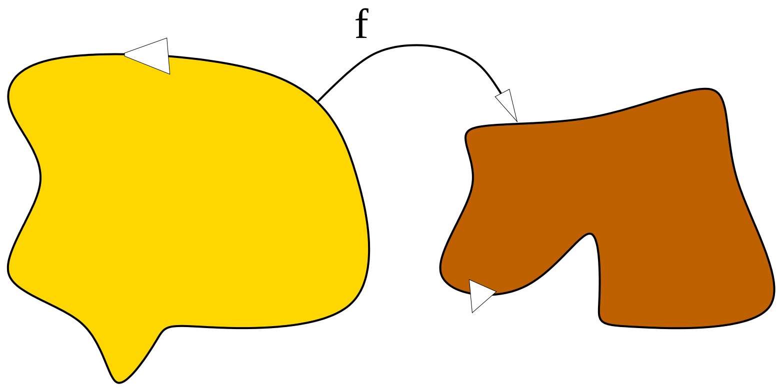
∎

27.6.1. *Conformally equivalent domains.*

Definition 27.113. *Two domains $\mathcal{D}_1$ and $\mathcal{D}_2$ are* **conformally equivalent** *if there is a biholomorphic bijection $f : \mathcal{D}_1 \to \mathcal{D}_2$. In this case, we say that f* **maps conformally** *$\mathcal{D}_1$ to $\mathcal{D}_2$.*

The boundary behavior of analytic maps is a very important tool to determine their conformal mapping properties.

Recall that a curve is traversed in anticlockwise direction if the parameterization is such that the interior is to the left of the curve as the parameter increases.



Proposition 27.114. *Assume that $f : \mathcal{D} \mapsto \mathcal{D}_1$ is analytic and γ is a simple piecewise differentiable* closed *curve contained in $\mathcal{D}$ together with its interior.*

If f is one-to-one from γ to $f(\gamma)$, then f maps one-to-one conformally Int*(γ) onto $f($*Int*$(\gamma))$ and preserves the orientation of the curve.*

Proof. It follows easily from the assumptions that $f(\gamma)$ is also a simple curve. Let $w_0 \in$Int$(f(\gamma))$. Cauchy's formula implies

$$\frac{1}{2\pi i}\int_{f(\gamma)}\frac{dw}{w-w_0}=1 \tag{27.11}$$

On the other hand by assumption f is one-to-one on γ and we can change variables $w = f(z)$, $z \in \gamma$, and we get

$$1=\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)dz}{f(z)-w_0}=\frac{1}{2\pi i}\int_{\gamma}\frac{(f(z)-w_0)'dz}{f(z)-w_0} \tag{27.12}$$

and by Proposition 18.73 (and since f is analytic) this shows that $f(z)-w_0$ has exactly one zero in Int(γ), or there is exactly one z_0 such that $f(z_0)=w_0$. Then f is conformal, one-to-one onto between Int(γ) and $f(\text{Int}(\gamma))$. This also shows that f preserves orientation, otherwise the integral would be -1.

27.7. **Automorphisms of the plane.**

Theorem 27.115. *The group of automorphisms of the plane is exactly the Euclidean group, $z \mapsto az+b, a \neq 0$.*

Proof. In one direction it is clear: each Euclidean transformation is in $Aut(\mathbb{C})$.

If $f \in Aut(\mathbb{C})$ then by a translation we can arrange that $f(0)=0$. Since f is bijective, in particular bijective from $\mathbb{D}$ to $f(\mathbb{D})$, there is a $c>0$ s.t.

$$|f(z)|>c \text{ for } |z|\geqslant 1 \tag{27.13}$$

We want to show that f does not have an essential singularity at infinity, meaning that 0 is not an essential singularity for $h(z) = f(1/z)$. This is clearly so, or otherwise, by Casorati-Weierstrass we would contradict (27.13). Hence $f = \sum_{k=-\infty}^{0} c_k z^{-k} + P(z)$ in a neighborhood of infinity and thus f is entire and polynomially bounded in $\mathbb{C}$, and thus it is a polynomial by Exercise 6.33. But, if $\deg(P) = n > 1$, then the equation $P(1/z) = w$ would have $n > 1$ roots for large z, by Proposition 27.110. ∎

27.8. **Automorphisms of the Riemann sphere.** A linear fractional transformation (LFT), or Möbius map is a map of the form

$$S(z) = \frac{az+b}{cz+d} \tag{27.14}$$

where $ad - bc \neq 0$. Clearly, S is meromorphic, with only one pole at $z = -d/c$.

Exercise 27.116. Associate to a LFT the coefficients matrix

$$\hat{M} := \frac{az+b}{cz+d} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{27.15}$$

If T_1 and T_2 are LFTs, then show that

$$\hat{M}(T_1 \circ T_2) = \hat{M}(T_1)\hat{M}(T_2) \tag{27.16}$$

where the product on the right side of (27.16) is the usual matrix product. Hence LFT with composition are isomorphic to $SL(2,\mathbb{C})$ with multiplication (some authors prefer $\mathrm{PSL}(2,\mathbb{C})$ to emphasize the projective invariance, and in some sense the arbitrariness of the normalization $ac - bd = 1$. It is a normalization that we do not wish to ensure every time a LFT is constructed). In particular, they form a group.

Theorem 27.117. *$F : \hat{\mathbb{C}} \to \mathbb{C}$ is analytic iff F is a rational function.*

Analytic functions are defined of course through charts.

Proof. Constants are of course analytic; assume F is nonconstant. Since f is analytic, it cannot have essential singularities in $\hat{\mathbb{C}}$ (but it can have poles since ∞ is a regular point in $\hat{\mathbb{C}}$, the North pole). Hence, F is meromorphic in $\mathbb{C}$. Since no value of F can accumulate anywhere in $\hat{\mathbb{C}}$ and since $\hat{\mathbb{C}}$ is compact, F has finitely many poles. Therefore, for a polynomial $P, PF = Q$ is analytic in $\mathbb{C}$. At ∞ the Laurent series of P has to be of the form $P_1(z) + A(1/z)$ where A is analytic at zero. But then Q is analytic in $\mathbb{C}$ and polynomially bounded hence it is a polynomial. ∎

Theorem 27.118. *The automorphism group of* $\hat{\mathbb{C}}$ *is the Möbius group.*

Proof. It is straightforward to check that Möbius maps are automorphisms of $\hat{\mathbb{C}}$. If f is an automorphism of $\hat{\mathbb{C}}$ then of course it is analytic and thus, by Theorem 27.117, it is of the form P/Q, P, Q polynomials. Since in $\mathbb{C}$ the value ∞ is taken at most once with multiplicity one, we must have $Q(z) = cz + d$. Now, $z \mapsto 1/z$ is an automorphism of $\hat{\mathbb{C}}$, and this implies that Q/P is also an automorphism, hence $P(z) = az + b$. We must have $ac - bd = 0$ or else the fraction simplifies to a constant. ∎

28. Linear fractional transformations (Möbius transformations)

If $c = 0$ we have a linear function. If $c \neq 0$ we write

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c^2(z+d/c)} \tag{28.1}$$

It is clear from (28.1) that $S(z_1) = S(z_2)$ iff $z_1 = z_2$ and in particular $S'(z) \neq 0$: If a LFT F is defined in the domain $\mathcal{D}$ (that is, $-d/c \notin \mathcal{D}$), then $\mathcal{D}$ and $F(\mathcal{D})$ are conformally equivalent. It also follows from the decomposition (28.1) that

Proposition 28.1. *The Möbius group is generated by* $z \mapsto z + a$, $z \mapsto az$ *and* $z \mapsto 1/z$. *This implies that* $PSL(2, \mathbb{C})$ *is generated by the matrices of the form*

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{28.2}$$

(we took a negative sign in the last matrix to have determinant one.)

Hence, the automorphisms of $\hat{\mathbb{C}}$ are represented by $\mathrm{PSL}(2, \mathbb{C})$ while those of $\mathbb{C}$ are given by the subgroup generated by the matrices $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

Exercise 28.2. *Show that* $z \mapsto 1/z$ *maps a line or a circle into a line or a circle. Hint: Show first that the equation*

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$$

where $\alpha, \gamma \in \mathbb{R}$ *and* $\beta \in \mathbb{C}$ *and* $|\beta|^2 > \alpha\gamma$ *is the most general equation of a line or a circle. Then apply the transformation to the equation.*

As a result we have an important property of LFTs

Proposition 28.3. *A LFT maps a line or a circle into a line or a circle.*

Proof. By Exercise 28.2 inversion has the property above, and clearly, so do Euclidean transformations.□

28.1. **Finding specific LFTs.** A line is determined by two of its points and a circle is determined by three. We now show that for any two circles/lines there is a LFT mapping one into the other, and find the formula for it. Let z_1, z_2, z_3 be three points in $\mathbb{C}$. Then the transformation

$$S = \frac{z_1 - z_3}{z_1 - z_2}\,\frac{z - z_2}{z - z_3} \tag{28.3}$$

maps z_1, z_2, z_3 into $1, 0, \infty$ in this order; this is easy to check. If one of z_1, z_2, z_3 is ∞, we pass the transformation to the limit. The transformation taking (∞, z_1, z_2) to $(1, 0, \infty)$ is is

$$\frac{z - z_1}{z - z_2} \tag{28.4}$$

To find a transformation that maps z_1, z_2, z_3 into $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3$ in this order, we can write it as $\tilde{\tilde{S}} := \tilde{S}^{-1}S$. By Exercise 28.4 this transformation is unique.

Exercise 28.4. *Check that a LFT that takes* $(1, 0, \infty)$ *into itself is the identity.*

28.1.1. *Cross ratio.* If $z_i, i = 1...4$ are four distinct points and $w_i = S(z_i)$ then (check!)

$$\frac{w_1 - w_2}{w_1 - w_3}\,\frac{w_3 - w_4}{w_2 - w_4} = \frac{z_1 - z_2}{z_1 - z_3}\,\frac{z_3 - z_4}{z_2 - z_4}$$

This is often a handy way to determine the image of a fourth point when the transformation is calculated using three points.

28.2. **Mappings of regions.** We know that LFTs are *conformal and one-to-one* and transform circles/lines onto circles/lines. Proposition 27.114 shows that LFTs map disks/half planes to disks/half planes. To see the latter, note that on the Riemann sphere LFTs map circles to circles, hence disks to disks. This argument requires some tidying up, left as an easy exercise.

Exercise 28.5. (i) Show that the *Cayley transform* $f(z) = (z-i)/(z+i)$ maps conformally the UHP to $\mathbb{D}$. Its inverse, $i(1+w)/(1-w)$, of course maps conformally $\mathbb{D}$ to the UHP.

(ii) Find a LFTs that maps the disk $(x-1)^2+(y-2)^2=4$ onto the unit circle and the center is mapped to $i/2$.

(iii)(*) Find the most general LFT that maps the unit disk onto itself.

28.3. As usual, we let $\mathbb{D}$ be the open unit disk.

Theorem 28.6 (Schwarz lemma)**.** *Let $f:\mathbb{D}\to\mathbb{D}$ be analytic and such that $f(0)=0$. Then*

(i)

$$|f(z)| \leqslant |z| \tag{28.5}$$

for all $z\in\mathbb{D}$.

(ii) If there is some nonzero $z_0\in\mathbb{D}$ such that for $z=z_0$ we have equality in (28.5) then $f(z)=e^{i\phi}z$ for some $\phi\in\mathbb{R}$.

(iii) $|f'(0)|\leqslant 1$ and if equality holds then again $f(z)=e^{i\phi}z$ for some $\phi\in\mathbb{R}$.

Proof. (i) Since $f(0)=0$, the function $f(z)/z$ extends analytically in $\mathbb{D}$. By the maximum modulus principle,

$$\left|\frac{f(z)}{z}\right| \leqslant \lim_{r\uparrow 1}\max_{|z|=r}\left|\frac{f(z)}{z}\right| \leqslant 1$$

(ii) If z_0 is such that equality in (28.5) holds, then z_0 is a point of maximum of $|f(z)/z|$, which cannot happen unless $f(z)/z=C=f(z_0)/z_0$.

(iii) The inequality follows immediately from (28.5). Assume $f'(0)=e^{i\phi}$, $\phi\in\mathbb{R}$. If $f(z)\not\equiv e^{i\phi}z$, then we can write

$$f(z)/z = e^{i\phi}(1+z^m e^{i\psi}h(z))$$

where h is analytic and $h(0)\in\mathbb{R}^+$. If we then take $z=\varepsilon\exp(-i\psi/m)$ with ε small enough we contradict (i). ☐

Corollary 28.7. *If h is an* automorphism *of the unit disk and $h(0)=0$ then $h(z)=e^{i\phi}z$ for some $\phi\in\mathbb{R}$.*

Proof. We must have, by Theorem 28.6 $|h(z)|\leqslant|z|$. But the inverse function h^{-1} is also an automorphism of the unit disk and $h^{-1}(0)=0$. Thus $|h^{-1}(z)|\leqslant|z|$ for all z, in particular $|z|=|h^{-1}(h(z))|\leqslant|h(z)|$ or $|z|\leqslant|h(z)|$. Thus $|h(z)|=|z|$ for all z and the result follows from Theorem 28.6 (ii).☐

28.4. **Automorphisms of the unit disk.** We have seen that the subgroup of LFTs:

$$S(z) = e^{i\phi}\frac{z-\alpha}{1-\overline{\alpha}z} \quad \text{with } \phi \in \mathbb{R} \text{ and } |\alpha| < 1 \tag{28.6}$$

maps the unit disk one-to-one onto itself.

The converse is also true:

Theorem 28.8. *Any automorphism f of $\mathbb{D}$ into itself is of the form (28.6) with $\alpha = f^{-1}(0)$.*

Proof. The function $h = S \circ f^{-1}$ is an automorphism of the unit disk and $h(0) = S(\alpha) = 0$. But then Corollary 28.7 applies and the result follows. □.

Exercise 28.9. Show that the automorphisms of the upper half plane are of the form $\frac{az+b}{cz+d}$ where a, b, c, d are real and $ad - bc > 0$ (see also Exercise 27.116). The automorphism is unique, the identity, if $\varphi(0) = 0$ and $\varphi(1) = 1$.

As we have seen, this subgroup of the LFTs is isomorphic to $PSL(2, \mathbb{R})$.

The automorphisms of the canonical forms of simply connected Riemann surfaces, $\hat{\mathbb{C}}, \mathbb{C}, \mathbb{D}$, are therefore subgroups of $PSL(2, \mathbb{C})$.

Of course, knowing the generators of a group can be very useful, for instance when we need to prove a property of the whole group.

29. The modular group

In preparation for elliptic function theory, we now look into the properties of an important subgroup of $PSL(2, \mathbb{R})$ namely the **modular group** $\Gamma = PSL(2, \mathbb{Z})$, course isomorphic to the subgroup of LFT

$$z \mapsto \frac{az+b}{cz+d},$$

$a, b, c, d \in \mathbb{Z}$, $|ad - bc| = 1$, with M and $-M$ identified.

Note 29.10. *The fact that $ad - bc = \pm 1$ implies that the fractions $a/b, a/c, c/d, b/d$ are all irreducible. Check this.*

Definition 29.11 (Lattices)**.** *A two-dimensional lattice generated by the complex numbers, or* periods*, ω_1 and ω_2 with $\omega_1/\omega_2 \notin \mathbb{R}$ is the set*

$$\Lambda_{\omega_1,\omega_2} := \Lambda = \{z \in \mathbb{C} : z = m_1\omega_1 + m_2\omega_2;\ m_1, m_2 \in \mathbb{Z}\} \tag{29.7}$$

Any such Λ is a **module** over $\mathbb{Z}$. Geometrically, a lattice provides a tesselation of the plane by identical parallelograms:

Definition 29.12. *A tessellation, or tiling of a region is a family of disjoint open sets whose union of closures cover the region.*

29.1. **Bases of lattices.** Lattices are clearly discrete sets. Hence there is a ω_1 of minimal absolute value. Choose any $\omega \in \Lambda$ which is not of the form $n\omega_1$ and among these ω choose one of minimal absolute value; call it ω_2. We must have $\omega_2/\omega_1 \notin \mathbb{R}$, or else if n is s.t. $n < \omega_2/\omega_1 < n+1$, then $|n\omega_1 - \omega_2| < |\omega_1|$, a contradiction.

Proposition 29.13. *$PSL(2,\mathbb{Z})$ is a symmetry of any lattice Λ. In other words $PSL(2,\mathbb{Z})\Lambda = \Lambda$.*

Proof. (Simplified, from [3].) If

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \tag{29.8}$$

with $a, b, c, d \in \mathbb{Z}, ad - bc = 1$, and $\Lambda = \Lambda_{\omega_1,\omega_2}$ with $\omega_2/\omega_1 \notin \mathbb{R}$, we want to show that $M\Lambda = \Lambda$. Let

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{29.9}$$

Note that (ω_1', ω_2') and (ω_1, ω_2) play a symmetric role, for

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = M^{-1} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} \tag{29.10}$$

Eq. (29.10) shows in particular that $\omega_2'/\omega_1' \notin \mathbb{R}$. Check that, for instance by assuming the contrary and arriving at a contradiction. Now, the pair (29.9) and (29.10) shows that $\Lambda_{\omega_1,\omega_2} \subset \Lambda_{\omega_1',\omega_2'}$ and $\Lambda_{\omega_1',\omega_2'} \subset \Lambda_{\omega_1,\omega_2}$. ∎

Corollary 29.14.
- *Any two bases of a lattice Λ are related by a modular transformation* (29.9)*.*
- *$PSL(2,\mathbb{Z})$ are automorphisms of lattices, and hence of the 2d torus $\mathbb{T}$ (in fact, these are all).*

29.2. **The fundamental region of $PSL(2,\mathbb{Z})$.** There is a special basis of importance for PSL$(2,\mathbb{Z})$ and for the theory of elliptic functions:

Theorem 29.15. *There exists a basis (ω_1, ω_2) such that the ratio $\tau = \omega_2/\omega_1$ satisfies the following conditions:*

(1) $\operatorname{Im}\tau > 0$
(2) $-\frac{1}{2} < \operatorname{Re}\tau \leqslant \frac{1}{2}$
(3) $|\tau| \geqslant 1$
(4) $\operatorname{Re}\tau \geqslant 0$ *if* $|\tau| = 1$.

The ratio τ is uniquely determined by these conditions, and there is a choice of two, four, or six corresponding bases. The region $\mathcal{R}$ is shown in Fig. 29.2.

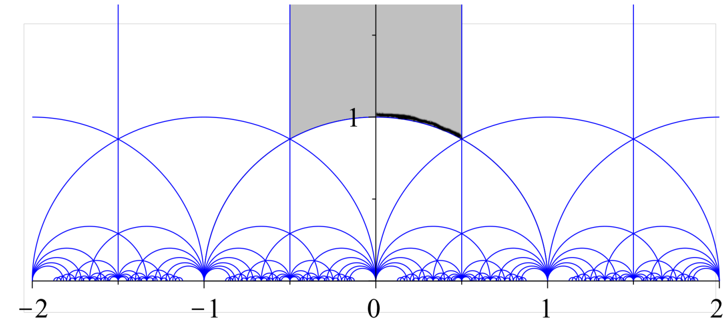


FIGURE 17. The fundamental region $\mathcal{R}$ (the grey region together with the right part of the circle, that is, starting at i) and the action of $\mathrm{PSL}(2,\mathbb{Z})$ on it, figure from Wiki: [14].

Proof. (Slightly simplified version of the proof in [3].) Choose ω_1 and ω_2 as in the beginning of §29.1. Then, $|\omega_1| \leqslant |\omega_2|$, $|\omega_2| \leqslant |\omega_1 + \omega_2|$ and $|\omega_2| \leqslant |\omega_1 - \omega_2|$. These translate in $|\tau| \geqslant 1$ and $|\mathrm{Re}\,\tau| \geqslant \frac{1}{2}$. If $\mathrm{Im}\,\tau < 0$, replacing (ω_1, ω_2) by $(-\omega_1, \omega_2)$ makes $\mathrm{Im}\,\tau > 0$ while preserving $|\mathrm{Re}\,\tau| \geqslant \frac{1}{2}$. Finally, if it happens that $\mathrm{Re}\,\tau = -\frac{1}{2}$ we pass to $\omega_1, \omega_1 + \omega_2$. If $|\tau| = 1$ and $\mathrm{Re}\,\tau < 0$ we replace τ by $-\tau^{-1}$, i.e. (ω_1, ω_2) by $(-\omega_2, \omega_1)$ finishing the required normalization. If τ' corresponding to another basis is also in $\mathcal{R}$, then a calculation based on Corollary 29.14 shows that

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad \mathrm{Im}\,\tau' = \frac{\mathrm{sign}(ad - bc)\mathrm{Im}\,\tau}{|c\tau + d|^2} = \frac{\mathrm{Im}\,\tau}{|c\tau + d|^2} \tag{29.11}$$

since $\tau' \in \mathcal{R}$; this also implies $ad - bc = 1$. We want to show $\tau' = \tau$ and in this τ and τ' play symmetric roles and we are free to assume $\mathrm{Im}\,\tau' \geqslant \mathrm{Im}\,\tau$. By (29.11) this gives

$$|c\tau + d| \leqslant 1 \tag{29.12}$$

Writing $\tau = \alpha + i\beta$ we have $\alpha \in (-1/2, 1/2]$, $\beta^2 + \alpha^2 \geqslant 1$. Eq. (29.12) gives $(\alpha c + d)^2 + \beta^2 c^2 \leqslant 1$. We must have $|c| \leqslant 1$ since $|c| \geqslant 2$ gives $|\beta| \leqslant \frac{1}{2}$ forcing $|\tau| < 1$.

If $c = 0$, $ad - bc = 1$ implies $ad = 1$, hence $a = d = 1$ or $a = d = -1$. From the first equality in (29.11) we then have $\tau' = \tau \pm b$, hence $\mathrm{Im}\,\tau = \mathrm{Im}\,\tau'$, $|b| = |\mathrm{Re}\,\tau - \mathrm{Re}\,\tau'| < 1$, and from Assumption 2, $b = 0$ implying $\tau = \tau'$.

If $|c| = 1$, then $|\tau + d| \leqslant 1$ or $|\tau - d| \leqslant 1$ implies $d < 2$, hence $d = 0$ or $d = \pm 1$. If $d = 0$, since $|\tau| \geqslant 1$ we must have $|\tau| = 1$ and now $bc = -1$ gives $b/c = -1$, hence $\tau' = \pm a - 1/\tau = \pm a - \overline{\tau}$ implying

$1 > \mathrm{Re}\,(\tau+\tau') = \pm a$, Hence $a=0$ and $\tau' = -1/\tau$ which is only possible if $\tau = i$ and $\tau' = i$.

If $d = \pm 1$ then $|\tau \pm 1| \leqslant 1$. The closest point to -1 in $\overline{\mathcal{R}}$ is $(-\frac{1}{2}, \frac{\sqrt{3}}{2}) \notin \mathcal{R}$, hence $d=-1$ and $|\tau - 1| \leqslant 1$. Proceeding as before, the closest point to $\overline{\mathcal{R}}$ is $e^{i\pi/3} = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \in \mathcal{R}$, a point of minimal imaginary part, and where $|\tau - 1| = 1$. But by (29.11) $\mathrm{Im}\,\tau' = \mathrm{Im}\,\tau$, hence $\tau' = \tau$.

To classify all possible choices of bases we have to find the fixed points in $\mathcal{R}$ of $\mathrm{PSL}(2,\mathbb{Z})$. The matrices -1 are identified and result in $\tau \to \tau$, but $(-\omega_1, -\omega_2)$ is a different basis. The analysis above shows that the only possible fixed points are $\tau = i$ and $\tau = e^{\pi i/3}$. These are fixed points of $\tau \mapsto 1/\tau$ and $\tau \mapsto (\tau-1)/\tau$. ∎

Exercise 29.16. *Find all $z \in UHP$ for which there is some $M \in PSL(2,\mathbb{Z})$ s.t. $Mz = z$*

We denote the interior of a set A by A°.

Corollary 29.17. *If $M \in$PSL$(2,\mathbb{Z})$, then $(M\mathcal{R})^\circ \cap \mathcal{R}^\circ \neq \emptyset$ iff $M = 1$, or equivalently, if $A = M\mathcal{R}$ and $A' = M'\mathcal{R}$ then $A^\circ \cap (A')^\circ \neq \emptyset$ iff $M = M'$.*

Proof. If $\tau \in (\mathcal{R})^\circ \cap (M\mathcal{R})^\circ$ and $M \neq 1$, then $\tau = M\tau'$, hence τ and τ' belong to the same lattice. By Theorem 29.15, we must have $\tau = M\tau$. Hence, if $M \neq 1$ then $\tau \in (\mathcal{R})^\circ$, contradiction. ∎

Definition 29.18. *A group G acts on a topological space X in a properly discontinuous way if for all $x \in X$ and $g \in G$ there is a neighborhood $\mathcal{N}$ such that $g(\mathcal{N}) \cap \mathcal{N} \neq \emptyset \Leftrightarrow g = 1$.*

Exercise 29.19. *Does PSL$(2,\mathbb{Z})$ act in a properly discontinuous way on the upper half plane?*

29.3. **The generators of the modular group.**

Proposition 29.20. *The modular group is generated by $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, or in terms of LFTs, $\sigma = z \mapsto -1/z$ and $\tau = z \mapsto z+1$.*

Proof. [4] Let $F(z) = \frac{az+b}{cz+d}$. We distinguish three cases.

(1) $a = 0$. Then $bc = -1$ which can only happen if $b = -c = \pm 1$. In this case, $F(z) = \pm 1/(\pm z + d) = \sigma\tau^{\pm d}(z)$.

[4] Adapted and simplified from a post by rmdmc89

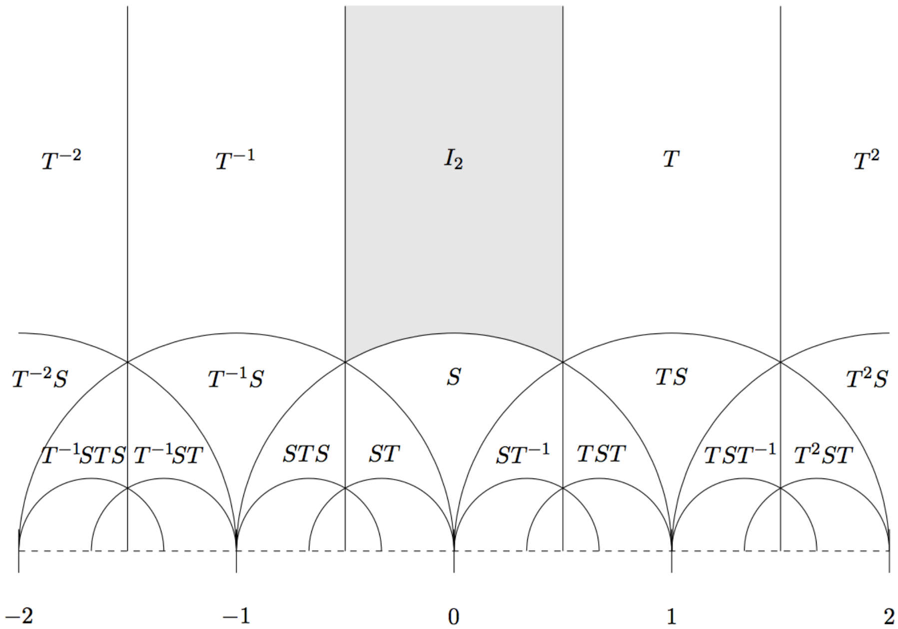


FIGURE 18. Marked action of S,T

(2) $|a| = 1$. By projective equivalence, we may assume $a = 1$, and then $d - bc = 1$. We bring this transformation back to case 1 by τ, σ:
$$\tau^c \sigma \left(\frac{z+b}{cz+d} \right) = \tau^c \frac{-cz-d}{z+b} = \frac{-cz-d}{z+b} + c = -\frac{1}{z+b}$$
indeed of type (1).

(3) $|a| > 1$ (it can't be < 1!). If we show that we can strictly lower $|a|$. Since $a \in \mathbb{Z}$, it means we can eventually bring ourselves to case (2). By Note 29.10 (a and c are coprime) we have $|a| \neq |c|$. By applying σ if necessary we can arrange $|a| > |c|$. We can assume $c \neq 0$ or else σ brings us to (1). Then

$$\tau^{\pm 1} F = \frac{(a \pm c)z + (b-d)}{cz+d} \tag{29.13}$$

∎

Exercise 29.21. * 1. Show that the modular group induces a tessellation of the UHP, namely $\{(M\mathcal{R})^\circ : M \in PSL(2,\mathbb{Z})\}$. One way is to apply Corollary 29.17 and note that for any $z \in \mathcal{R}$ and $x \in \mathbb{R}^+$, xz is in the closure of the tessellation by taking $r = p/q \in \mathbb{Q}^+$ and taking

LFTs of the form $(npz+b)/(cz+nq)$, $n \in \mathbb{N}, n \to \infty$. This shows that the strip $\{|\mathrm{Re}\, z| \leqslant 1/2, \mathrm{Im}\, z \geqslant 0\}$ is in the closure of the tesselation.

2. Find the tesselations of the unit disk induced by conjugation of $\mathrm{PSL}(2,\mathbb{Z})$ with the Cayley transform. Find the Poincaré metric on $\mathbb{D}$ using the Cayley transform.

29.4. **The hyperbolic plane.** On the UHP define the metric tensor

$$ds^2 = \frac{dx^2+dy^2}{y^2} = \frac{dz\overline{dz}}{y^2} \tag{29.14}$$

The metric is given by

$$\rho(z_1,z_2) = 2\tanh^{-1}\left|\frac{z_1-z_2}{z_1-\overline{z_2}}\right| = \log\frac{|z_1-\overline{z_2}|+|z_1-z_2|}{|z_1-\overline{z_2}|-|z_1-z_2|} \tag{29.15}$$

Exercise 29.22. *Check that the metric is invariant under LFT's, hence under $SL(2,\mathbb{R})$. Check that the geodesics for this metric are circular arcs $\perp \mathbb{R}$ and vertical lines ending on $\mathbb{R}$.*

The hyperbolic plane has constant negative curvature.

29.5. **Schwarz reflection of domains about a circle.** We have seen in Theorem 21.90 that we can analytically continue functions from the upper half plane into a symmetric domain in the lower half plane provided they are continuous up to the real line and real-valued there. Consider a domain $\mathcal{D}$ bounded by a curve and an arccircle C, and let F be analytic in $\mathcal{D}$ and continuous up to C. At any point $z \in C$ we can decompose $f(z) = f_1 + f_2$ where the direction of f_1 is tangent to the circle and f_2 is perpendicular to it. Assume that $f_2 = 0$. Let M be a Möbius transformation that takes C to a real interval J and $\mathcal{D}$ to a domain $\mathcal{D}'$ in the UHP. Then $f \circ M^{-1}$ is analytic in $\mathcal{D}'$, continuous down to J and real valued on J. Then there is a function analytic F in the domain $\mathcal{D}' \cup \mathcal{D}'' \cup J$ where $\mathcal{D}''$ is the reflection of $\mathcal{D}$ across J s.t. $F|_{\mathcal{D}} = f \circ M^{-1}$ and $F \circ M^{-1}$ provides analytic continuation of f to $\mathcal{D} \cup R\mathcal{D} \cup C$ where $R\mathcal{D}$ is the reflection of $\mathcal{D}$ across the arccircle C.

Exercise 29.23. *Find the formula of the reflection of a point across an circle, defined as in the construction above.*

30. Some special biholomorphic transformations

. Note. We usually call biholomorphic transformations of domains **conformal maps.**

Many elementary conformal maps are composed of Möbius maps, ramifications: $z \mapsto z^\alpha$, exponential and trig maps and some other simple maps such as the Joukowski map. We illustrate below a number of useful transformations; see the references, esp. [1], [11] for more examples. A good number of interesting domains can be mapped to the unit disk using combinations of these transformations. Note that by Proposition 27.114 it suffices to examine carefully the way the boundaries are mapped to understand the action of a map on a whole domain.

1. Möbius transformations. We have discussed these in some detail already. A very useful special case is the Cayley transform $\dfrac{z-i}{z+i}$ that maps conformally the UHP to $\mathbb{D}$.

2. The map $z \mapsto z^{1/\alpha}$. See Fig. 30.

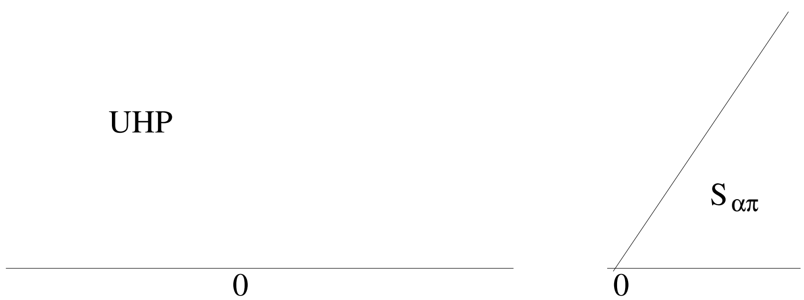


FIGURE 19. The map $z \mapsto z^{1/\alpha}$ maps the open sector of opening $\pi\alpha$ on the right bianalytically to the open upper half plane and the boundary point zero to zero: the upper boundary of the sector is rotated to $-\mathbb{R}$.

3. Maps of the cut plane.

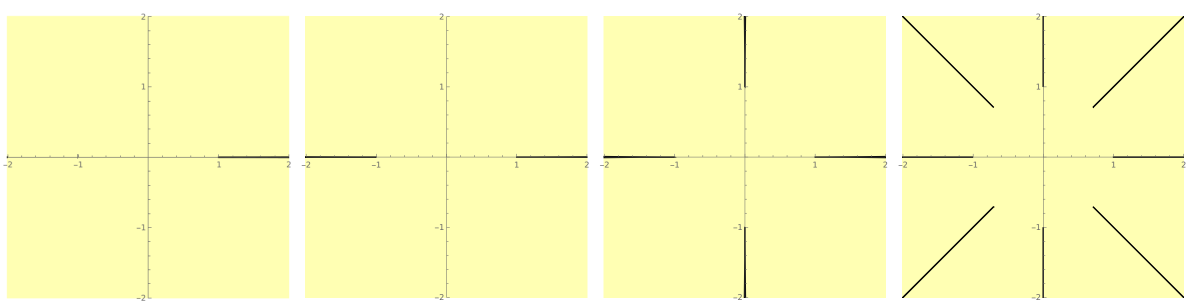
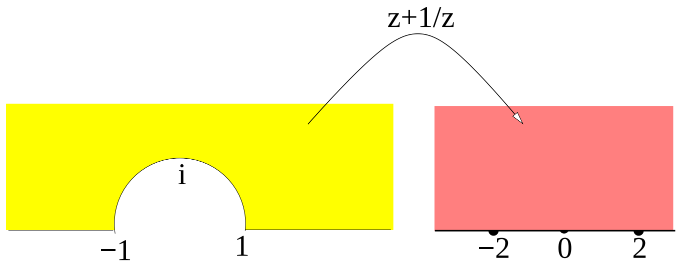


FIGURE 20. $\mathbb{C}$ with one, two, four and eight symmetric cuts. More generally, the directions of the cuts could be along the roots of unity, $\omega_1, ..., \omega_k, k \in \mathbb{N}$.

Exercise 30.24. (1) Show that $\phi_1(z) = \frac{4z}{(1+z)^2}$ maps conformally the unit disk to $\mathbb{C} \setminus [1, \infty)$. Using Proposition 27.114 and this map, verify that the upper and lower sides in $\mathbb{C}$ of $(1, \infty)$ are to be treated as separate parts of the boundary. We have used this point already in the construction of Laurent series.

(2) Show that for any $n \in \mathbb{N}$ $\phi_n(z) = [\phi_1(z^n)]^{1/n}$ maps conformally $\mathbb{D}$ to $\mathbb{C}$ with symmetric cuts at $\omega_k, k = 1, ..., n$.



4. The Joukovski transformation.

This is an interesting map which straightens the region in the upper half plane above the unit circle (of course, by slight modifications, you can choose other radii or centers along $\mathbb{R}^+$) to the upper half plane. It is given by

$$J(z) = z + \frac{1}{z} \tag{30.16}$$

Exercise 30.25. Check that J indeed maps conformally the region in the figure to the UHP (hint: follow the boundary). What is the effect of J on the reflection across $\mathbb{R}$ of yellow region?

5. The exponential maps the region $\{z | 0 < \operatorname{Im} z < \pi\}$ conformally to the domain $\mathbb{C} \setminus (-\infty, 0]$.

Exercise 30.26. *Use some of the maps discussed above to conformally map $\mathbb{D}$ to this strip and the UHP to the strip above.*

5. The cosine maps the region $\{z | \operatorname{Im} z > 0, 0 < x < \pi\}$ conformally to the UHP.

Exercise 30.27. *Note that the cosine and the exponential are linked via a Joukovski map. Can you explain the mapping of* cos *based on* **5.** *and J?*

Note 30.28. *Keep in mind that the conformal maps above* **are not tied to the domains presented***, except loosely through the fact that these domains are often chosen to be some maximal region of conformality. Often subdomains of a maximal domain are interesting in applications.*

Application to fluid flow. Recall our notations: $\langle v, u \rangle = \nabla \phi$, where ϕ is the velocity potential and is a harmonic function, while v, u are the vertical and horizontal velocities of the fluid particles. The complex potential is $\phi + i\psi$ where ψ is the stream function.

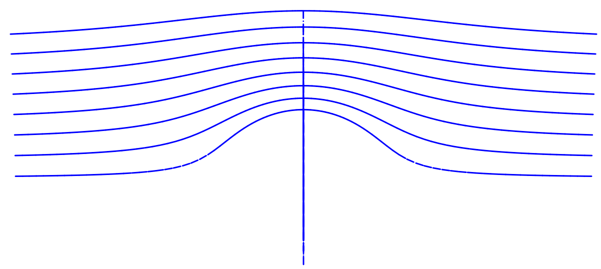


FIGURE 21. Velocity field lines in Joukovski's domain

As a nice application of the Joukovski map, we can find the flow lines of a river passing above a cylindrical obstacle. The problem becomes 2d, by symmetry, and we are dealing with the flow impeded by a circle. Again by symmetry, we only need to understand the flow in the UHP, impeded by a half-circle: exactly the Joukowski domain.

We map conformally the Joukowski domain to the UHP and solve the problem there. Say the complex solution is $F(z)$. Then our solution, in the Joukowski domain, is $F(J(z))$.

In the UHP, the flow of a fluid would be horizontal with constant horizontal velocity (say 1) $\langle v, u\rangle = \langle 0, 1\rangle$. This gives $\phi(x, y) = y + C$ where we take $C = 0$. Up to an irrelevant constant, $\Psi = -x$, and thus the relevant complexified solution is $F(z) = y - ix = -iz$. (In particular, we have the *no-penetration condition* $v = 0$ along the boudary, $\mathbb{R}$.) Then $-iJ(z)$ is the solution with the given boundary conditions in the Joukowski domain.

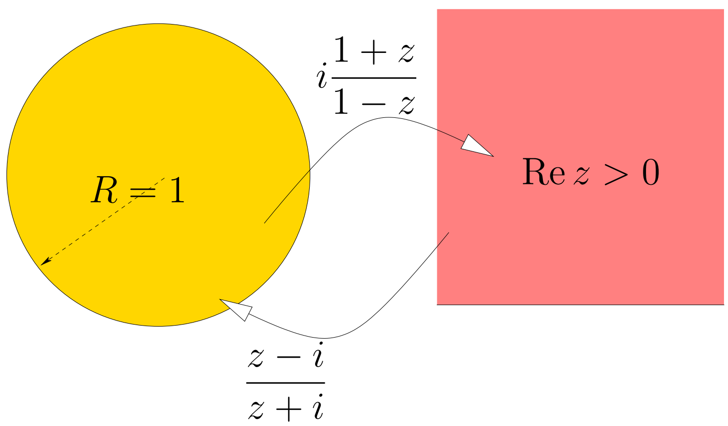
The fluid flow lines are then given by $J_2(x_1, y_1) = const.$ plotted with Maple in Fig 30.

Exercise 30.29. *

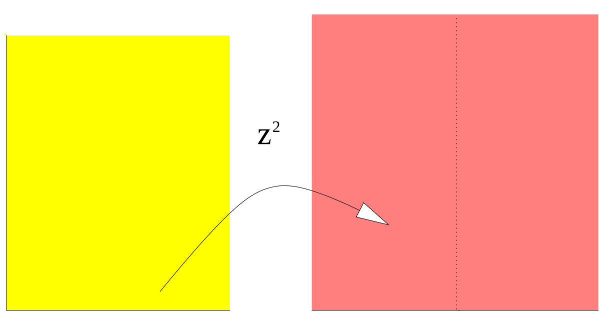
Find an explicit formula for flow lines in the previous example.

Other common conformal maps are depicted in the following figures. The Cayley transform $\frac{z-i}{z+i}$ maps conformally the UHP to $\mathbb{D}$. It plays an important role in the theory of unbounded operators. Any other

$\mathbb{C} \setminus \overline{\mathbb{R}^-}$



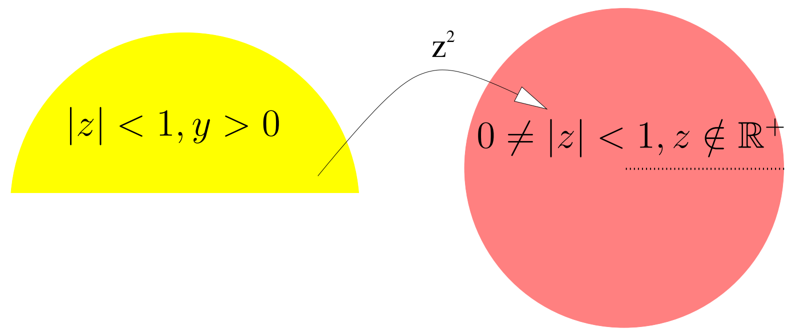
function f that maps conformally the UHP to $\mathbb{D}$ must be the Cayley transform composed to the left with an automorphism of he disk. Write the general form of such a map.

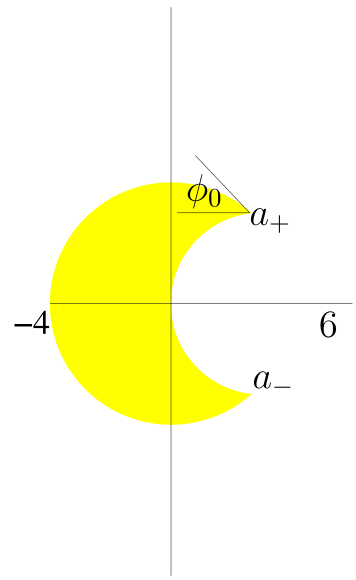


Exercise 30.30. *** (i) Check the transformation in the figures above.*

(ii) Draw a similar picture for the mapping $\sin z$ *from the upper half strip bordered by the half-lines* $x = \pm\pi/2, y > 0$.

(iii) Find a conformal map of the quarter disk $|z| < 1, \arg(z) \in (0, \pi/2)$ *onto the upper half plane.*





(iv) Find a conformal map of the half disk $|z| < 1, \arg(z) \in (0, \pi)$ onto the half strip $x < 0, y \in (0, \pi)$.

(iv) Find a conformal map of the right half plane (RHP) with a cut along $[0, 1]$ in the RHP.

Example 1 It is useful to remark that we can find linear fractional transformations which map conformally a region between two circles into a half plane (or disk) using very simple transformations. Let us map the "moon crescent" M below into a half plane.

The equations of the circles are $x^2 + y^2 = 16$ and $(x - 3)^2 + y^2 = 9$. Solving these equations for the intersection points $a_\pm$ we get $a_\pm = 8/3 \pm 4i\sqrt{5}/3$. The angle between the circles equals the angle between $a_+ - 3$ and a_+ that is $\arg[(a_+ - 3)/a_+] = \arctan(\sqrt{5}/2)$. The idea is that if we map by an LFT one of the intersection points to infinity, the arccircles become lines for which question is easier.

If we map $a_- \mapsto 0$, $0 \mapsto 1$ and $a_+ \mapsto \infty$ by a LFT, concretely

$$\frac{a_+}{a_-}\frac{z-a_-}{z-a_+} \tag{30.17}$$

then both arccircles become *rays* (since they end at ∞). The small arc becomes $\mathbb{R}^+$ and the larger one a ray of angle ϕ_0 (by conformality at a_-: check these statements).

To transform this sector of opening π_0 to the upper half plane we simply use a ramified transformation $z \mapsto z^{\pi/\phi_0}$.

Example 2 Solve $\Delta u = 0$ in the region $|z| < 1, \arg(z) \in (0, \pi/2)$ such that on the boundary we have: $u = 1$ on the arc and $u = 0$ otherwise.

Solution. Strategy: We find conformal map of this region into the strip $\{z = x + iy : y \in (0,1)\}$ such that the arc goes into $y = 1$ and the segments into $y = 0$. The solution of the problem in this region is clear: $u = \operatorname{Im} z$. Then we map back this function through the transformations made.

How to find the transformation? We are dealing with circles, strips, etc so it is hopeful we can get the job done by composing elementary transformations. There is no unique way to achieve that, but the end result must be the same.

(1) The transformation $z \mapsto z^2$ opens up the quarter disk into a half disk. On the boundary we still have: $u = 1$ on the arc and $u = 0$ otherwise.

(2) We can now open the half disk into a quarter plane, by sending the point $z = 1$ to infinity, as in Example 1, by a linear fractional transformation. We need to place a pole at $z = 1$ and a zero at $z = -1$. Thus the second transformation is $z \mapsto i\frac{1+z}{1-z}$. The segment starting at -1 ending at 1 is transformed in a line too, and the line is clearly $\mathbb{R}^+$ since the application is real and positive on $[0, 1)$ and 1 is a pole. What about the half circle? It must become a ray since the image starts at $z = 0$ and ends at infinity. Which line? The image of $z = i$ is $w = i$. Now we deal with the first quadrant with boundary condition $u = 1$ on $i\mathbb{R}^+$ and $u = 0$ on $\mathbb{R}^+$.

(3) We open up the quadrant onto the upper half plane by $z \mapsto z^2$.

(4) We now use a rescaled log to complete the transformation. The composite transformation is

$$\frac{2}{\pi}\ln\left(\frac{1+z^2}{1-z^2}\right)$$

Exercise** The temperature distribution also satisfies Laplace's equation. (1) Map M onto a strip as in Example 2. What is the distribution of temperature in the domain M if the temperature on the larger arc is 1 and 0 on the smaller one? What shape do the lines of constant temperature have?

(2) What is the distribution of temperature in the domain and with the boundary conditions described in example 2? Draw an approximate picture of the lines of

Example. (These calculations need to be checked). In a region free of charge, the electrostatic potential V is harmonic, and hence it equals $\operatorname{Re} F$ where $F = V + iW$ and W is a harmonic conjugate of V. In electrostatics, the electric field is given by $E = -\nabla V$.

We approximate the electrically charged mast of a ship by a sector S as in Fig. 30 with very small α and the other in the second quadrant, and the air outside the mast is then a sector S^c of angle $\pi\beta$, $\beta \approx 2$, as in Fig. 30 with the upper side rotated almost all the way around. The map of S^c to the UHP is $z^{1/\beta}$. This means the mast is like a **knife's edge, rather than a pin**. There is another unrealistic assumption, that **the density of charge is constant.** In reality, charges accumulate at the tip, making the field stronger.

Now the LHP is electrically charged, and by symmetry, the charge density is constant. Still by symmetry (or, more rigorously by Gauss' law) the electric field in the UHP is constant and vertical. This gives $\tilde{V}(x,y) = cy$ (let's take $c = 1$), and, as in the fluid flow problem, $\tilde{V} + i\tilde{W} = -iz$. Then, we have $V + iW = -iz^{1/\beta} \approx f(z) = -iz^{1/2}$. Using the Cauchy-Riemann equations, we have $E_x(z) = \frac{\partial V}{\partial x} = \operatorname{Re}(f'(z))$ and $E_y(z) = -\frac{\partial V}{\partial y} = -\operatorname{Im}(f'(z))$. We deduce that $|E(z)| = |f'(z)| \approx \frac{1}{2}|z|^{-1/2}$. The electric field blows up in this way when we approach the tip of the mast. In practice E grows until it reaches the ionization threshold of the air around it, upon which it keeps discharging in the air. The current through air creates plasma, resulting in Saint Elmo's fire. (A local electric field of about 100 kV/m is required to begin a discharge in moist air.)

31. The Riemann Mapping Theorem

Using elementary transformations we can conformally map a limited family of domains onto $\mathbb{D}$; with the maps we used in the previous section, the boundary is always very simple. The Schwarz-Christoffel formulas in §32, generally nonelementary, provide conformal maps between any polygons and the upper half plane (which, as we saw can be mapped onto the unit disk). For polygons with more than 4 sides, some numerical calculations are needed. Conformally mapping more general domains, even in a computer assisted fashion, is a difficult problem.

In principle, however, **any simply connected domain other than** $\mathbb{C}$ can be mapped onto $\mathbb{D}$:

Theorem 31.1 (Riemann Mapping theorem)**.** *Given any simply connected domain* $\mathcal{D}$ other than $\mathbb{C}$ *there is biholomorphic map* f *between* $\mathcal{D}$ *and* $\mathbb{D}$.

Normalizing f *by* $f(z_0) = 0$ *and* $f'(z_0) \in \mathbb{R}^+$*, for some* $z_0 \in \mathcal{D}$*, such a map is unique.*

The Riemann mapping theorem was stated by Riemann in 1851 for domains with piecewise regular boundary, and he provided a proof based on solving a Dirichlet problem for the Laplacian. The method used, the *Dirichlet's principle* was not quite right. Weierstrass found that indeed this *can* happen for the Dirichlet functional.

The next two pages are taken from an article (see top of the pages) describing Riemann's proof in detail. The first rigorous proof along these lines, in full generality, was given by Caratheodory in 1912.

holomorphic function F on the set with $F' = f$. We shall say then that U is *holomorphically simply connected*.

It is easy to show that topological simple connectivity as defined implies the holomorphic antiderivative property just described. The theorem itself in the following form will show among other things the converse, namely, that the holomorphic simple connectivity implies topological simple connectivity.

Theorem 1.1 (Riemann) *Suppose that U is a connected open subset of $\mathbb{C}$ with $U \neq \mathbb{C}$. If U is holomorphically simply connected, then U is biholomorphic to the unit disc, i.e., there is a one-to-one holomorphic function from U onto the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.*

In the proof of this result, it will be useful to be able to assume that U is bounded. For this, we recall the familiar fact that such a U as in the theorem is always biholomorphic to a bounded open set. The proof of this in summary form goes like this. Since $U \neq \mathbb{C}$, we can replace U by a translate to suppose that $0 \notin U$. The function $1/z$ is then holomorphic on U and hence has an antiderivative $L(z)$, say. Changing L by an additive constant will arrange that $\exp(L(z)) = z$ (this is the usual process for finding complex logarithms). Then $\exp(L(z)/2)$ is one-to-one on U. Choose an open disc in the image of $\exp(L(z)/2)$. The negative of this disc is disjoint from the image of $\exp(L(z)/2)$. So the image of $\exp(L(z)/2)$, which is biholomorphic to U, is itself biholomorphic to a bounded open subset of $\mathbb{C}$, via a linear fractional transformation.

Note that it is not clear by definition that the holomorphic simple connectivity is preserved by a biholomorphic mapping since the meaning of taking the derivative is different when the coordinates change; but by the complex chain rule this is a matter of a holomorphic factor which can be assimilated into the original function. Checking the details of this is left to the reader as an exercise.

So now we can assume without loss of generality that the open set U is bounded. And by translation we can now assume $0 \in U$. We shall look for a biholomorphic mapping from U to the unit disc D which takes 0 to 0. Of course, if there is a biholomorphic map from U to the unit disc at all, there is one that takes 0 to 0 since a linear fractional transformation taking the unit disc D to itself will take any given point to the origin, and in particular the image of 0 to begin with can be moved to the origin.

Now if $H: U \to D$ is biholomorphic and has $H(0) = 0$, then $H(z)/z$ has a removable singularity at 0. Hence, H can be written as $zh(z)$, where h is holomorphic on U and $h(0) \neq 0$. That $h(0) \neq 0$ follows because H is supposed to be one-to-one and hence must have derivative vanishing nowhere. Of course, $h(z)$ is also nonzero for every other $z \in U$ because 0 is the only point of U with $H(z) = 0$.

Now there is an antiderivative L of h'/h on U. The product $h(z)\exp(-L(z))$ is constant since it has derivative identically equal to 0 and U is connected. Changing L by an additive constant, we can assume $h(z)\exp(-L(z)) = 1$ for all $z \in U$. (This familiar argument will occur several times here).

The essential point of Riemann's method was to consider the harmonic function $\mathrm{Re}\, L(z)$. This is of course equal to $\ln|h(z)|$. Since the "boundary values" of $|H(z)|$ have to be 1, it must be that the boundary value of $|h|$ at a boundary point z_0 of U has to be

$1/|z_0|$. In particular, the harmonic function $\ln|h|$ has to have boundary value at z_0 equal to $-\ln|z_0|$.

At this point, Riemann appealed to what he referred to as the Dirichlet Principle. The Euler–Lagrange equation for the variational problem of minimizing the so-called Dirichlet (energy) integral for a real-valued function $f(x, y)$, namely minimizing this integral

$$\int_U \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \mathrm{d}x\mathrm{d}y$$

under the condition that $f = g$ on the boundary ∂U of U, is easily computed to satisfy $\Delta f = 0$ (§18 of [10, 11]).

So Riemann proposed that the harmonic function with the boundary values $-\ln|z_0|$ at each boundary point z_0 could be found by minimization of the Dirichlet integral. And Riemann was well aware of how to construct h and hence H from knowing $\ln|h(z)|$. Riemann actually expressed this all in terms of $\ln|H|$ and the idea of Green's function, a function with boundary value 0 and a specified singularity at (in our case) the point 0, namely the function had to be of the form $\ln|z| + u(z)$ with u harmonic near the point 0. This is equivalent for open sets in $\mathbb{C}$ to our discussion, though the Green's function notion is useful when one tries to extend the Riemann Mapping Theorem to the uniformization problem where there is no *a priori* global z-coordinate.

The main difficulty is that there is no particular reason to suppose that there is in fact any minimum for the Dirichlet integral in this situation. There is also a less serious difficulty of explaining why the resulting function is one-to-one and onto—intuitively this is just a matter of winding numbers if one can approximate U from inside by domains with smooth or piecewise-smooth closed curve boundaries. One supposes that Riemann may have taken this part for obvious, though it is actually quite subtle if one does not appeal to any pre-existing topological intuitions. We shall give a precise argument later on. Riemann apparently considers only domains, the boundary of which is smooth in some sense. Osgood made the major forward step treating simply connected open sets in general, thus proving what we call today the Riemann Mapping Theorem. The Osgood proof is acknowledged directly by Carathéodory [2] where the ideas involved in the usual proof of today, via normal families, are presented. See the footnote (**) of page 108 of [2]. But, for some reason, Osgood's proof fell from favor or even recognition for the history of the Theorem in [9]; there is a reference to Osgood's paper but no comment on it, no acknowledgment that this is in fact the reasonably complete first proof of the general result.

2 The application of the Perron method

Even simply connected bounded open sets in $\mathbb{C}$ can have complicated boundaries. The boundary of the Koch snowflake, for example, has Hausdorff dimension greater than 1 [15]. And Osgood [6] already gave an example with boundary having positive (2-dimensional) measure. Thus, it is appropriate to introduce carefully what is to be meant by finding functions with specified boundary values. For this purpose, let U be a bounded open set in $\mathbb{C}$ and ∂U be its boundary, that is the complement of U within the closure $\mathrm{cl}(U)$ of U in $\mathbb{C}$, or equivalently the intersection of $\mathrm{cl}(U)$ with $\mathbb{C} - U$. Suppose that $b{:}\partial U \to \mathbb{R}$ is a continuous function. Then we say that a harmonic function $h{:}U \to \mathbb{R}$ is

*

The usual proof of this major theorem involves concepts and results that are very important and useful of their own. We will study these in detail.

31.1. **Equicontinuity.** We look at functions $f : M \mapsto M'$ where M, M' are metric spaces. We recall that if the metrics are d and d', a function is uniformly continuous if

$$\forall\delta\ \exists\varepsilon\ \Big(\forall(z, z_0) \in M^2,\ d(z, z_0) < \varepsilon \Rightarrow d'(f(z), f(z_0)) < \delta\Big) \tag{31.1}$$

We can assume that the metric d' is a **bounded function**, for we can always replace it by $d'' = d'/(1 + d')$ (check that d'' is a metric, topologically equivalent to d' [5]).

Definition. An equicontinuous family $\mathcal{F}$ is a collection of continuous functions with the same continuity parameters at every point:

(31.2)

$$\forall x \exists\delta(x) s.t. \forall\varepsilon > 0, \forall y \& \forall f \in \mathcal{F}, d(x, y) < \delta \Rightarrow d'(f(x), f(y) < \varepsilon$$

Definition 31.2 (**Normal Families**). *Let M, M' be a complete metric spaces. Then a collection of continuous functions $\mathcal{F}$ from M to M' is a* **normal family** *if it is pre-compact in the topology of uniform convergence.*

Exhaustion by compact sets. We note that if the metric space M is $\sigma-$compact, then, by definition we can cover M by a countable nested family of compact sets. This is the case of is $\mathbb{C}$, $\mathbb{R}^n$ where

$$M = \bigcup_{n\in\mathbb{N}} K_n, \quad K_n = \{x \in M : |x| \leqslant n\} \tag{31.3}$$

Metrizability of the topology of uniform convergence on compact sets On each K_n in (31.3) we define the distance between two functions f and g in a manner analogous to the L^∞ distance:

$$\delta_n(f, g) = \sup_{x\in K_n} d''(f(x), g(x)) \tag{31.4}$$

and we create a distance on the whole of M which takes advantage of the compact sets exhaustion:

$$\rho(f, g) = \sum_{n=1}^{\infty} 2^{-n}\delta_n(f, g) \tag{31.5}$$

(recall that $d'' \leqslant 1$).

[5] In separable spaces, i.e. ones which contain a countable dense set, equivalence holds iff convergence in one metric is equivalent to convergence in the other.

Exercise 31.3. *Let M be a $\sigma-$ compact metric space. Check that ρ in* (31.5) *is a metric on $C(M)$. Check that convergence with respect to ρ is equivalent to uniform convergence on compact sets. Check that $\mathcal{F}$ is a complete metric space if M' is a complete metric space.*

The following is immediate.

Proposition 31.4. *A family $\mathcal{F}$ is normal iff its completion $\overline{\mathcal{F}}$ with respect to ρ is compact.*

31.1.1. *The Ascoli-Arzelà Theorem.*

Theorem 31.5 (Ascoli-Arzelà)**.** *A family $\mathcal{F}$ of continuous functions in the region $\Omega \subset \mathbb{C}$ with values in a metric space M' is* **normal** *in Ω iff the following conditions are both satisfied:*

(i) $\mathcal{F}$ is equicontinuous at any $x \in \Omega$.

(ii) $\mathcal{F}$ is equibounded: $\forall z \in \Omega$ there is a compact set M' $K_1 = K_1(z) \subset M'$ such that $\forall f \in \mathcal{F}, f(z) \in K_1$.

This is a standard theorem. We leave the proof for the Appendix, §50.3.

Proposition 31.6. *Let now $M' \subset \mathbb{C}$ and $\mathcal{F}$ be a normal family from Ω to M'. Let $K \subset \Omega$ be compact. Then $\mathcal{F}$ is bounded on K:*

$$\sup_{z\in K, f\in\mathcal{F}} |f(z)| = m < \infty \tag{31.6}$$

Proof. Since $\mathcal{F}$ is a normal family, by Theorem 31.5, using equicontinuity, it follows that for any point a we can find $\delta(a)$ such that

$$\forall z, \forall f \in \mathcal{F},\ |a - b| < \delta(a) \Rightarrow |f(a) - f(z)| < 1 \tag{31.7}$$

Extract a finite covering of K from the balls $\mathbb{D}_{\delta(a)}(a)$, let a_j be the centers of the balls and δ_0 be the smallest δ in the finite cover. We denote $m_j = \sup\{|f(a_j)| : f \in \mathcal{F}\}$ and $m = 1 + \max_j\{m_j\}$. Then, for any $x \in K$ there is an a_j such that $|x - a_j| < \delta_0$. Therefore, by the choice of m and a_j (31.7) we have, for any $f \in \mathcal{F}$,

$$|f(x)| \leqslant |f(x) - f(a_j)| + |f(a_j)| = 1 + m_j \leqslant m \quad \square \tag{31.8}$$

Theorem 31.7 (Montel)**.** *Consider a domain $\mathcal{D} \subset \mathbb{C}$ and assume $\mathcal{F}$ is a family of analytic functions $\mathcal{D}$ such that for every compact set $K \subset \mathcal{D}$ we have* $\sup\{|f(z)| : z \in K, f \in \mathcal{F}\} = m(K) < \infty$. *Then the family is normal.*

Proof. The family is clearly equibounded. We show that the derivatives are also equibounded from which equicontinuity is immediate.

Let K be a compact set in $\mathcal{D} \subset \mathbb{C}$, take an $r < \text{dist}(K, \partial\mathcal{D})$ and a finite cover of K, $\mathcal{O} = \cup_{j=1}^{m} \mathbb{D}_r(a_j)$ and let $r' = \text{dist}(\partial O, K)$. We have $\overline{O} \subset \mathcal{D}$ and

$$|f'(z)| \leqslant \frac{1}{2\pi}\left|\oint_{\partial O} \frac{f(s)}{(s-z)^2} ds\right| \leqslant \frac{2m}{r'} \tag{31.9}$$

proving the result. ▮

31.2. **The Riemann Mapping Theorem.**

Definition 31.8. (i) $\mathcal{D}$ and $\mathcal{D}'$ will be called **conformally equivalent** if there is a map which is analytic together with its inverse (biholomorphism) from $\mathcal{D}$ onto $\mathcal{D}'$. (ii) An injective analytic function is called univalent (or *schlicht*).

Theorem 31.9 (Riemann mapping theorem)**.** *Given any nonempty simply connected domain $\Omega \subset \mathbb{C}$ other than $\mathbb{C}$ itself, a point $z_0 \in \Omega$ and the normalization conditions $\varphi(z_0) = 0, \varphi'(z_0) \in \mathbb{R}^+$ there exists a unique biholomorphism $\varphi(z)$ between Ω and $\mathbb{D}$.*

Note 31.10. The fact that $\mathbb{C}$ must be an exception follows from the fact that an entire bounded function is constant.

Note 31.11. *Equivalently, any two nonempty simply connected domains different from $\mathbb{C}$ are conformally equivalent. (cf. Exercise 28.9).*

Proof of the Riemann Mapping Theorem
Uniqueness. This part is easier. If φ_1 and φ_2 are two functions with the stated properties, then $S := \varphi_1(\varphi_2)^{-1}$ is an automorphism of $\mathbb{D}$ and $S(0) = 0$. By Schwarz's lemma $S(z) = ze^{i\theta}$ for some θ. $S'(0) > 0$ implies $\theta = 0$.
Existence.

Note 31.12. Clearly, in the proof we can replace the arbitrary domain $\mathcal{D}$ with any set conformally equivalent to it. Therefore, we first simplify the domain by elementary transformations.

Note 31.13. By assumption there is a point in $\mathbb{C} \setminus \Omega$, which, by translation if needed, we can assume to be 0.

Note 31.14. (1) We can always replace Ω by a conformally equivalent Ω' which is bounded. Indeed, since $0 \notin \Omega$, the map $f = z \mapsto 1/z$ is bounded on Ω. $f(\Omega)$ is conformally equivalent to Ω and it is bounded. Indeed:

Lemma 31.15. *Assume $\Omega \subset \mathbb{D}$ is nonempty and $0 \notin \Omega$. Then,* *(i) There is a biholomorphic branch of the log* ($\log z = \int_1^z s^{-1} ds$) *between* Ω *and* $L = \log(\Omega)$ *(with* exp *as its inverse, of course).*

(ii) $L \cap (L + 2\pi i) = \emptyset$

(iii) There is a branch of the square root such that $\sqrt{\Omega} \cap (-\sqrt{\Omega}) = \emptyset$.

(iv) L *omits an open set.*

Proof. (i) In one direction, log is analytic in Ω by the way we normalized the domain. Now, if $\log z_1 = \log z_2$, by taking the exponential we see that $z_1 = z_2$.

(ii) If w and $w + 2\pi i$ were both in L, then $w = \log z_1$ and $w + 2\pi i = \log z_2$. But then $|z_1| = |z_2|$ in the defining integral and $\arg z_1 = \arg z_2 + 2\pi i$ which means the path of integration between z_1 and z_2 is closed and surrounds 0. Then this closed curve is not contractible to a point ($0 \notin \Omega$), violating simple-connectedness.

(iii) follows from (i) and (ii).

(iv) If $w_0 \in L$ together with a disk $\mathbb{D}_\varepsilon(w_0)$, then (recall the open mapping theorem), then $\mathbb{D}_\varepsilon(w_0) \cap (2\pi i + \mathbb{D}_\varepsilon(w_0)) = \emptyset$. $2\pi i + \mathbb{D}_\varepsilon(w_0)$ is the ommitted disk ∎

If w_1 is the center of an omitted disk, then the map $w \mapsto \frac{1}{z - w_1}$ is bounded in L.

(2) We can assume that $\Omega \subset \mathbb{D}$. Indeed, this is simply achieved by rescaling.

Lemma 31.16. *Assume* $\Omega \subset \mathbb{D}$ *is nonempty and* $0 \notin \Omega$. *Then,*

(i) $\mathbb{C} \setminus \log(\Omega)$ *contains an open disk.*

(ii) Let $z_0 \in \Omega$. *There is a biholomorphism* g *between* Ω *and* $g(\Omega) \subset \mathbb{D}$ *with* $g(z_0) = 0, g'(z_0) > 0$.

Proof. (i) L is also bounded.

(ii) As seen in Note 31.14 2 we can assume that $\Omega \subset \mathbb{D}$. An automorphism ϕ of the disk maps Ω to $\Omega' \subset \mathbb{D}$ and z_0 to zero. ϕ' can be changed by multiplication by $e^{i\theta}$ to make $\phi'(z_0) = 0$.

∎

We can thus assume wlog that $\Omega \subset \mathbb{D}$ and $z_0 = 0$.

Preview of the rest of the proof.

Note 31.17. (1) Let ϕ be any bijection between Ω and $S \subset \mathbb{D}$. We want to see how $\phi'(z_0)$ is correlated to the size of S, in the sense of set inclusion. Assume for a moment we knew the Riemann mapping theorem is true. Let $\phi_1 : \Omega \to S_1$ and $\phi_2 : \Omega \to S_2 \supsetneq$

S_1. Assume for simplicity that $z_0 = 0$. Consider the chains below where all the maps are biholomorphisms.

$$\mathbb{D} \xrightarrow{\psi} \Omega \xrightarrow{\phi_2} S_2 \xrightarrow{\chi} \mathbb{D} \tag{31.10}$$

We can normalize them so that $\chi \circ \phi_2 \circ \psi$ vanishes at zero and has positive derivative there. It is an automorphism of the disk, and the normalization above makes it into the identity.

$$\mathbb{D} \xrightarrow{\psi} \Omega \xrightarrow{\phi_1} S_2 \xrightarrow{\chi} \tilde{S}_1 \subsetneq \mathbb{D} \tag{31.11}$$

The strict inclusion is due to the fact that $S_1 \subsetneq S_2$. By the chain rule, we get that $\phi_1'(0) < \phi_2'(0)$.

Hence, returning to our problem, we are aiming at maximizing $\phi'(z_0)$.

(2) We take the sup M of these $\phi'(z_0)$ and find a subsequence $\phi_n \to \phi$ which converges and $\phi'(z_0) = M$. We then show that $\phi(\Omega) = \mathbb{D}$: otherwise we can find a ϕ_1 with $\phi_1'(z_0) > M$ by taking appropriate square roots, a contradiction.

Since, in $\mathbb{D}$, $0 < |x| < 1$ we have $1 > \sqrt{|x|} > |x|$: $\sqrt{\ }$ is an *expansive map*. If $\Omega \neq \mathbb{D}$, we should be able to select a point not in $\phi(\Omega)$, define a branch of the $\sqrt{\ }$ and expand $\phi(\Omega)$ by taking an appropriate square root.

Definition 31.18. *Let $\mathcal{F}$ be the set of biholomorphic maps between Ω and a subset of $\mathbb{D}$, which vanish at zero and have positive derivative there.*

Proposition 31.19. *(i) $M := \sup\{f'(z_0) | f \in \mathcal{F}\}$ is attained by an F in $\mathcal{F}$.*

(ii) If $F(\Omega) \neq \mathbb{D}$, then there is an $F_1 \in \mathcal{F}$ with $F_1'(z_0) > M$ (a contradiction which finishes the proof of the theorem).

Proof. (i) For any $f \in \mathcal{F}$, by assumption, $|f(z)| < 1 \; \forall z \in \Omega$. By Montel's Theorem 31.6 $\mathcal{F}$ is a normal family, thus if $f_n \in \mathcal{F}$ and $f_n'(z_0) = m_n \to M$ then $\{f_n\}_{n\in\mathbb{N}}$ has a convergent subsequence to a function F. By Weierstrass's theorem (8.39) $f'(z_0) = M$. By Hurwitz's theorem (18.80) is a biholomorphism. To see that, note that the limit F cannot be a constant since $F'(z_0) = M \neq 0$. Take any $z_1 \in \Omega$ and consider the region $\Omega^* = \Omega \setminus \{z_1\}$. Since f_n are bijections, $f_n(z) - f_n(z_1) \neq 0$ in Ω^*. Hence $F(z) - F(z_1) \neq 0$ in Ω^*.

(ii) Assume there is an $a \in \mathbb{D} \setminus F(\Omega)$. First we use an automorphism of $\mathbb{D}$ to map Ω to a set in $\mathbb{D} \setminus \{0\}$:

$$f_1(z) = \frac{F(z) - a}{1 - \overline{a}F(z)} \tag{31.12}$$

Since $f_1(\Omega) \subset \mathbb{D} \setminus \{0\}$, there is a branch of the square root in $f_1(\Omega)$. Now we can define

$$f_2(z) = \sqrt{f_1(z)}, \ \forall z \in \Omega \tag{31.13}$$

Note that, by Lemma 31.15, f_2 is a biholomorphism and that $f_2(\Omega) \subset \mathbb{D}$. We see that $f_2(z_0) = \sqrt{-a} := b \neq 0$. We now move b to zero and change the phase of the derivative to zero through another automorphism of the disk:

$$F_1(z) = \frac{f_2(z) - b}{1 - \bar{b} f_2(z)} \frac{|f_2'(z_0)|}{f_2'(z_0)}, \ \forall z \in \Omega \tag{31.14}$$

A straightforward calculation shows that

$$F_1'(z_0) = \frac{1 + |a|}{2\sqrt{|a|}} F'(z_0) > F'(z_0) = M \tag{31.15}$$

∎

This is not a constructive approach to the actual conformal map. There is no prescription on how to choose the convergent sequence whose derivatives at zero approach M. We could attempt to apply the square root trick in (ii), but the function sequence may not converge, nor is it guaranteed that the derivatives at zero (clearly increasing) would approach M.

32. Boundary behavior.

It is known, but would take us some time to prove, that *if $\mathcal{D}$ and $\mathcal{D}'$ are Jordan regions (domains bounded by Jordan curves), then the conformal map between $\mathcal{D}$ and $\mathcal{D}'$ extends to a homeomorphism between $\overline{D}$ and $\overline{D}'$.*

32.1. **Behavior at the boundary of biholomorphisms: a general but weaker result.** We derive an easy but useful result [3]: if $\mathcal{D}$ is a simply connected domain and φ maps it conformally onto $\mathbb{D}$, then $\varphi(z)$ approaches $\partial\mathbb{D}$ as z approaches $\partial\mathcal{D}$, in a sense defined below (which does not necessarily imply that $\varphi(z)$ converges).

Let $\mathcal{D}$ be a domain. Informally, a sequence or an arc approaches the boundary if eventually recedes away from any point in the region. The precise definition is:

Definition 32.1. *A sequence $z_n \to \partial\mathcal{D}$ as $n \to \infty$ if for any compact set $K \subset \mathcal{D}$ there exists n_0 such that for all $n > n_0$ we have $z_n \notin K$. Similarly, for an arc $\gamma : [0,1] \to \mathcal{D}$, $\gamma(t) \to \partial\mathcal{D}$ if $\forall K \, \exists t_0 \in (0,1)$ s.t. $\gamma(t) \notin K$ if $t > t_0$.*

Theorem 32.2. *If $\varphi : \mathcal{D} \to \mathcal{D}'$ is a domain biholomorphism and $z_n \to \partial\mathcal{D}$, as $n \to \infty$ then $\varphi(z_n) \to \partial\mathcal{D}'$ as $n \to \infty$.*

Proof. We prove the statement for sequences; the one for arcs is almost identical. Since φ is biholomorphic, any compact covering of $\mathcal{D}$ generates a compact covering of $\mathcal{D}'$ and vice-versa. Let $z_n \to \partial\mathcal{D}$ and let $K' \subset \mathcal{D}'$ be any compact set and $K = \phi^{-1}(K')$. By definition,

$$z_n \to \partial\mathcal{D} \ \Rightarrow \exists n_0 \ s.t. \forall n > n_0 \ z_n \notin K$$

Since φ is one to one, for any $n > n_0$, $\varphi(z_n) \notin K'$ either.

Corollary 32.3. *If $\varphi : \mathcal{D} \to \mathbb{D}$ is a biholomorphism, then $|\varphi(z)| \to 1$ as $z \to \partial\mathcal{D}$.*

32.2. **A reflection principle for harmonic functions.** Let $\mathcal{D}^u$ be a domain in the UHP such that $\partial\mathcal{D}^u \supset I := [a, b] \in \mathbb{R}$. We denote by $\mathcal{D}_l$ the reflection of $\mathcal{D}^u$ across I.

Theorem 32.4. *Assume v is harmonic in $\mathcal{D}^u$ and continuous in $\mathcal{D}^u \cup I$ and $v = 0$ on I. Then v extends to a harmonic function on $\mathcal{D} = \mathcal{D}^u \cup I \cup \mathcal{D}_l$.*

Note 32.5. Note that we cannot simply use the Schwarz reflection principle to prove this theorem, which is stronger. Nothing is know apriori about the continuity of u the harmonic conjugate of v. In fact, the Schwarz reflection principle follows from Theorem 32.4, see Exercise 32.6.

Proof. [9] As in the Schwarz reflection principle, the extension of v is defined through $V(z) = v(z)$ for $z \in \mathcal{D}^u$

$$V(\overline{z}) := -v(z) \ \forall \overline{z} \in \mathcal{D}_l \tag{32.1}$$

and $V(z) = 0$ on I. The property of a function being harmonic is a local one, so it is enough to check that V is harmonic in any disk in $\mathcal{D} = \mathcal{D}^u \cup I \cup \mathcal{D}_l$ for some family of disks covering $\mathcal{D}$. For disks in $\mathcal{D}^u$ this is so by assumption while for disks $\subset \mathcal{D}_l$ it follows directly from (32.1). Consider now a disk $\mathbb{D}_\varepsilon$ containing part of I as its diameter. Laplace's equation $\Delta v = 0$ with continuous boundary condition $V(z), z \in \partial\mathbb{D}_\varepsilon$ has a unique solution v_1. We have $V(\overline{z}) = -V(z)$ on $\partial\mathbb{D}_\varepsilon$, and thus $v_2(z) = -v_1(\overline{z})$ is another solution of Laplace's equation with the same boundary condition, hence $v_1 = v_2$. It follows in particular that $v_1 = 0$ on I. But now V and v_1 satisfy Laplace's equation with the same boundary condition on $\overline{\mathbb{D}_\varepsilon \cap \mathcal{D}^u}$ and on $\overline{\mathbb{D}_\varepsilon \cap \mathcal{D}_l}$, hence they are equal in $[\mathbb{D}_\varepsilon \cap \mathcal{D}^u] \cup [\mathbb{D}_\varepsilon \cap \mathcal{D}_l] \cup I$ (the latter because they both vanish on I. ∎

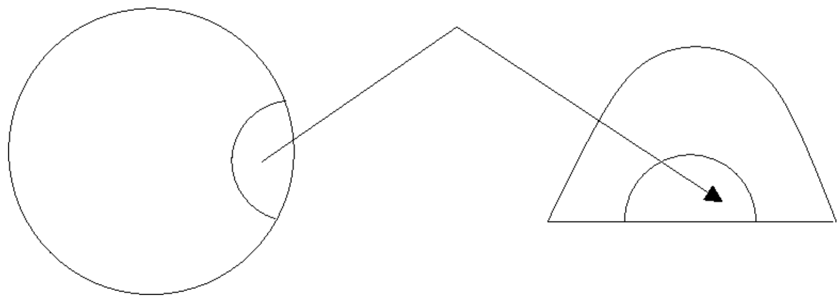


FIGURE 22. Boundary behavior across line segments.

Exercise 32.6. Show that if $f = u + iv$ is analytic in a domain $\mathcal{D}^u$ as in the theorem and v is zero on I, then u is continuous at I. In particular, for the Schwarz reflection principle for analytic functions f in a domain $\mathcal{D}$ in the UHP whose closure contains a real segment I, we only need to show that $\operatorname{Im} f$ is continuous and vanishes on I.

Exercise 32.7. Assume that $\mathcal{D}$ is a simply connected domain in the UHP such that $\overline{\mathcal{D}} \supset I \subset \mathbb{R}$. Let ϕ be the conformal map from $\mathbb{D}$ to $\mathcal{D}$. Consider a half disk strictly contained in $\mathcal{D}$ with diameter on I as in the figure. The arccircle is the image of an analytic curve in $\mathbb{D}$ bounding a domain $\mathcal{D}'$ in $\mathbb{D}$. Show that $\operatorname{Im} \phi(z) \to 0$ as $z \to \partial\mathbb{D} \cap \overline{\mathcal{D}'}$ and apply Theorem 32.4 and Exercise 32.6 to show that ϕ has analytic continuation to the reflection of $\mathcal{D}$ through I. In particular ϕ^{-1} extends continuously to I, and ϕ maps part of $\partial\mathbb{D}$ to I.

Note 32.8. The following interesting construction is given in [3], p.27. If f is an analytic function, then so is f^*, where $f^*(z) = \overline{f(\overline{z})}$. We have

$$2u(x, y) = f(x + iy) + \overline{f(x - iy)} = f(z) + f^*(z) \tag{32.2}$$

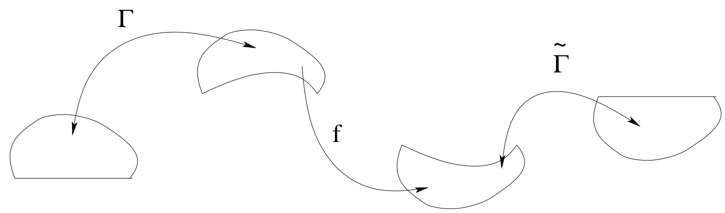
for all x, y for which the rhs makes sense. Assuming also that the expression $u(z/2, z/2i)$ makes sense, we have (check!):

(32.3)

$$2u(z/2, z/2i) = f(z) + f^*(0) = f(z) + C \Rightarrow f(z) = 2u(z/2, z/2i) - C$$
$$(\Rightarrow v(z) = \operatorname{Im}(2u(z/2, z/2i) - C))$$

In particular, no integration is needed to get f or v from u. This certainly works by the principle of permanence of relations for rational functions, or other simple functions. Can you make this work in general?

32.3. **Analytic arcs.**



Definition 32.9. *Let $I = [0, 1]$. A proper analytic arc is the image of I under a function γ which is analytic and injective in a neighborhood of I.*

An equivalent definition of a proper analytic arc is that γ extends to an isomorphism between some neighborhood of I and its image.

32.4. **An extension of the Schwarz reflection principle.** *Informally* this states that if f is analytic in a domain $\mathcal{D}$ which contains an analytic arc γ, if f is continuous up to γ and if the image of γ is an analytic arc, then f extends analytically beyond γ.

Definition 32.10 (Various definitions)**.** Let γ be a proper analytic arc bounding a domain $\mathcal{D}$. Then $\gamma : I \to \mathbb{C}$, $I = [a, b]$ is a *one-sided boundary* of $\mathcal{D}$ if for any $z_0 \in \gamma(I)$ there is a disk $\mathbb{D}_\varepsilon(z_0)$ such that $\gamma^{-1}(\mathbb{D}_\varepsilon(z_0) \cap \mathcal{D})$ is either completely contained in the UHP or else completely contained in the LHP independently of z_0 [6] In this case, we also say that $\mathcal{D}$ lies *on one side* of γ. As usual, by analytic continuation of a function f across a curve, we mean that there is an analytic function $\hat{f}$ in a neighborhood of the curve which coincides with f wherever they are both defined. Note that the analytic continuation is unique.

Definitions The theorem below is the "conformally mapped" Schwarz reflection principle.

Theorem 32.11 (Analytic reflection across arcs)**.** *Let $\mathcal{D}$ be a domain and assume that $\mathcal{D}$ lies on one side of the proper analytic arc γ. Let f be analytic in $\mathcal{D}$ and continuous on $\mathcal{D} \cup \gamma$, assume that $f(\gamma) \subset \tilde{\gamma}$ where $\tilde{\gamma}$ is a proper analytic arc, and finally that $f(\mathcal{D})$ lies on one side of $\tilde{\gamma}$. Then f extends analytically across γ.*

Proof. By assumption γ and $\tilde{\gamma}$ are images of the closed intervals I and $\tilde{I}$ under isomorphisms, $\Gamma, \tilde{\Gamma}$ respectively, f is analytic and continuous up to γ and $\Gamma, \tilde{\Gamma}$ are biholomorphic in a neighborhood of $I, \tilde{I}$ resp. We have $f(\Gamma(I)) = \tilde{\Gamma}(\tilde{I})$, and we can assume w.l.o.g. that I is approached from above. This means $\psi := \tilde{\Gamma}^{-1} \circ f \circ \Gamma$ is analytic in a domain in the

[6] Recall that γ extends to an isomorphism in a neighborhood of I.

UHP, continuous down to I and real valued on I; the usual Schwarz reflection principle applies to ψ. The desired analytic continuation of f is $\Gamma^{-1}\circ\psi\circ\tilde{\Gamma}^{-1}$ where we still denoted by ψ the analytic continuation of ψ. ∎

Exercise 32.12. Let $\mathcal{D}$ be a simply connected nonempty domain with nonempty complement and let ϕ be the conformal map of $\mathbb{D}$ to $\mathcal{D}$ assumed continuous up to the boundary. Show that ϕ has analytic continuation through an arc of $\partial\mathbb{D}$ iff the image of the arc is an analytic arc in $\partial\mathcal{D}$.

Theorem 32.13. *Let v be harmonic in a domain $\mathcal{D}$, and let $\gamma\subset\partial\mathcal{D}$ be a proper analytic arc such that $\mathcal{D}$ lies on one side of γ. Assume that v is continuous up to γ and vanishes on γ. Then v extends to a harmonic function in an open set containing $\mathcal{D}\cup\gamma$.*

Proof. Exercise: use Theorem 32.4 and the strategy in Theorem 32.11. ∎

Theorem 32.14. *Let $\mathcal{D}\subsetneq\mathbb{C},\mathcal{D}\neq\emptyset$ be a simply connected domain and φ a conformal map between $\mathcal{D}$ and $\mathbb{D}$. If $\partial\mathcal{D}$ contains a proper analytic arc γ and lies on one side of γ, then*

(i) φ extends analytically across γ.

(ii) if $z_n\to z_0\in\gamma$, then $\varphi(z_n)\to\varphi(z_0)\in\gamma$.

(iii) Furthermore, this extension is one-to-one on γ, thus in a neighborhood of $\mathcal{D}\cup(\mathcal{D}\cap\gamma)$.

Note 32.15. Applying a linear fractional transformation we see that a similar statement holds if $\mathbb{D}$ is replaced by a half-plane.

In particular, if the boundary of $\mathcal{D}$ is piecewise analytic then φ extends analytically to a domain $\mathcal{D}'\supset\mathcal{D}$. (It is also biholomorphic in some domain $\mathcal{D}''\supset\mathcal{D}$.)

Proof. (i) Follows from Theorem 32.11.

(ii) follows immediately from (i).

(iii) Let $v=\log|\varphi|$; by (ii) $|\varphi|\to 1$ as $\partial\mathcal{D}$ is approached, v is well defined close to $\partial\mathcal{D}$ and

$$\lim_{z\to\partial\mathcal{D}} v(z)=0$$

By Theorem 32.13 v extends to a harmonic function on an open set $\mathcal{D}'$ containing $\gamma\cup\mathcal{D}$. Since v is harmonic, there is an analytic function g s.t. $v=\operatorname{Re} g$. In $\mathcal{D}$, this g coincides up to a constant with φ, and choosing the constant to be zero, we get $g=\log\varphi$; hence $\log\varphi$ has analytic continuation across γ and in particular it is continuous up to

γ. Hence $\varphi = \exp(\log \varphi)$ also extends analytically across γ and maps γ into an arccircle. If φ were not injective on an open subsegment of γ, then that part of γ would be mapped to a closed curve, violating the continuity of φ (why? think of the behavior near the endpoints of that segment of γ).

∎

32.4.1. *Discussion: Tilings with parallelograms. An application of the Schwarz reflection principle to conformal mapping.* Consider a conformal map $\varphi : P \to UHP$ where P is one of the rectangles in the figure (properly normalized, $\varphi =$sn, the elliptic sine function, one of the Jacobi elliptic functions). Theorem 32.11 shows that φ admits analytic continuation through any side of the rectangle, to the adjacent one. We can check that any point in the plane can be reached from one rectangle by successive reflections, and, by an easy count that, while the same point can be reached by different reflections, the number of reflections is either odd or even, regardless of the reflections' paths. We achieved a single-valued continuation of φ to the whole of $\mathbb{C}$ except at the corners. It is *a meromorphic continuation* since the point at infinity must be mapped to a point on the boundary of the rectangle. Since two successive reflections across $\mathbb{R}$ return the point to its original value, the function thus obtained is doubly periodic. At the corners, since they have to be straightened, the function is of the form $a+(z-z_0)^2(1+o(1))$ and since the function is single-valued in a neighborhood of the corners and bounded at the corners, it is analytic there. We'll prove later that, more precisely, for $k \in [0,1]$, $\mathrm{sn}(z,k)$ gives a conformal map of the closed rectangle $[-K,K]\times[0,K']$ onto the UHP, with $0, \pm K, \pm K+iK', iK'$ mapping to $0, \pm 1, \pm k^{-2}, \infty$ respectively; iK' is a simple pole.

32.5. **Behavior at the boundary, a stronger result.**

Definition 32.16. A point $\zeta \in \partial\mathbb{D}$ is called accessible if there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$ s.t. $\lim_{n\to\infty} z_n = \zeta$ and a continuous function $\gamma : [0,1] \to \mathcal{D}$ which passes through all z_n: $t_0 = 0 < t_1 < \cdots < t_n < 1$, $\lim_{n\to\infty} t_n = 1$, $\gamma(t_n) = z_n$ and $\gamma(1) = \zeta$.

Theorem 32.17 (Boundary behavior)**.** *Let $\mathcal{D}$ be a bounded domain. (i) If z_0 is an accessible point of $\partial\mathcal{D}$ then the biholomorphism φ with the unit disk has a limit (call it $\varphi(z_0)$) as $z \to z_0$, $z \in \mathcal{D}$ and $\varphi(z_0) \in \partial\mathbb{D}$.*

(ii) If z_1 and z_2 are accessible points of $\partial\mathcal{D}$ then $\varphi(z_1) \neq \varphi(z_2)$.

We recall that a Jordan curve in $\mathbb{C}$ is a *continuous map* γ defined (say) on $[0,1]$ with values in $\mathbb{C}$ which is injective, that is $\gamma(t_1) = \gamma(t_2)$

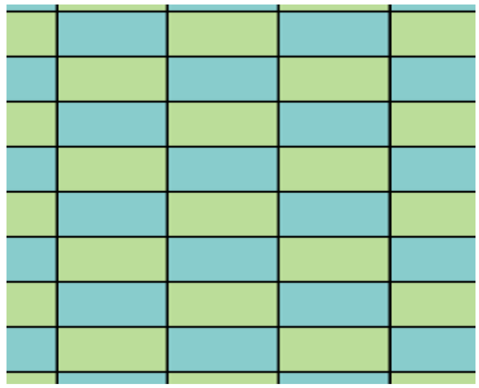


FIGURE 23. Tilings with rectangles
https://upload.wikimedia.org/wikipedia/commons/3/33/Stacked_bond.png

only if $t_1 = t_2$ or $t_1 = 0$ and $t_2 = 1$ where in the latter case it is a closed Jordan curve. We also recall that a closed Jordan curve divides $\mathbb{C}$ into exactly two regions, one bounded and one unbounded. The bounded region is called the interior of the curve. A Jordan domain is the interior of a Jordan curve. It follows from Theorem 32.17 above that if $\mathcal{D}$ is the interior of a simple, closed Jordan curve, then the map φ extends to a continuous injective function in $\overline{\mathcal{D}}$.

For space limitations we do not prove this interesting result; a proof, essentially based on the first proofs by Lindelöf and Koebe, is found in [9]. We shall not use conformal maps of this generality. In §33 we will find that for polygons, φ can be expressed by quadratures.

32.5.1. *A negative result.* The following is a standard example of an inaccessible point [11] shows that, in full generality, continuity of the conformal mapping cannot be expected. Such is the case of the *inaccessible point* below. Take $\mathcal{D}$ to be an open horizontal rectangle with a vertex at zero from which vertical line segments of length, say, 1/2 have been removed, see Fig. 24 (check that 0 is inaccessible!). The image of 0 on $\partial\mathbb{D}$ (which wlog can be taken as $z = 1$) must be a point of discontinuity of the conformal map φ into the unit disk, for $\varphi^{-1}(1) = 0$ and in any neighborhood of 1 there are infinitely many points where φ^{-1} is 1/2.

Illustration of nonanalytic behavior at all points the conformal image of the unit disk. Figure 25 shows the conformal image of the unit disk under the map F defined by the functional equation $G(z^2) = \lambda^{-1}\frac{G(z)^2}{1-G(z)}, \forall z \in \mathbb{D},\ G'(0) = 1$ for $\lambda = 0.5i$. The interior J of

FIGURE 24. The origin is inaccessible

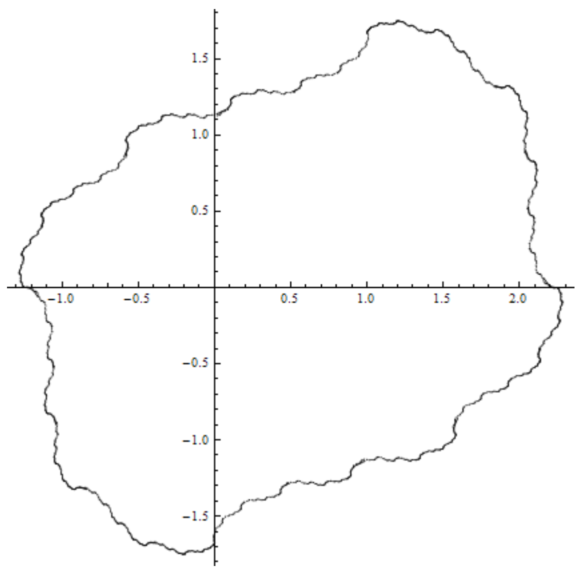


FIGURE 25. The conformal map of the unit disk through F

the curve corresponds to the points in $z \in \mathbb{C}$ for which the solution of the one step recurrence $x_{n+1} = \lambda x_n(1 - x_n)$; $x_0 = z$ converges to zero. J is the *Julia set* for the iteration of the quadratic map $z \mapsto \lambda z(1 - z)$. The unit circle is a natural boundary of G. An (incomplete) argument goes as follows. By continuity, $G(1) = \frac{\lambda}{1+\lambda}$. Differentiating we see that, if $G'(1) = 0$, then $G' = 0$ at all binary rational angles. If it is not zero we can divide by it and get $G(1) = 1 \pm (1 + 2\lambda)^{-1/2}$ which is inconsistent. Hence $G' = 0$ at all binary rationals, or it does not exist at binary rationals. See [6].

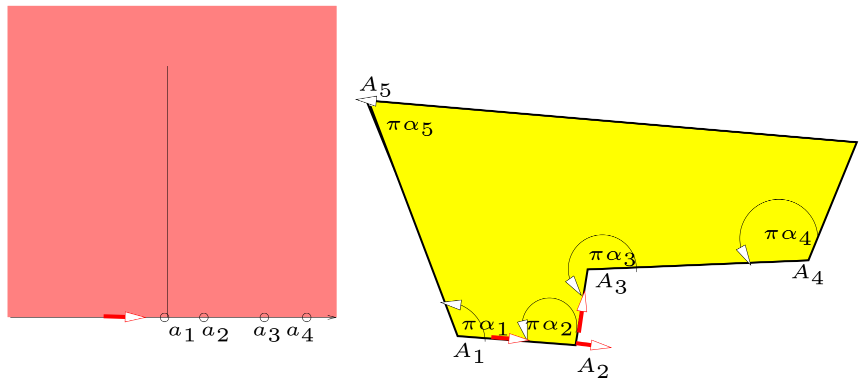


FIGURE 26. Schwarz-Christoffel transformations: a closed polygon.

33. Conformal mappings of polygons and the Schwarz-Christoffel formulas

For polygonal regions, the conformal map to the unit circle (or to $\mathbb{H}^u$ obviously) can be done by quadratures. The transformation is still usually nonelementary, but the integral representation gives us enough control to describe the transformation quite well.

33.1. **Heuristics.** If ψ is biholomorphic at z_0, the angle between the tangent of a curve γ through z_0 and the tangent to its image through ψ is $\arg\psi'(z_0)$; we write this in differential form, $dw = \psi'(z)dz$. We want to map $\mathbb{H}^u$ to the interior of a polygon. We then choose the positive orientation when traversing $\partial\mathbb{H}^u$ (which leaves the domain to its left): this means traversing the boundary from $\mathbb{R}^-$ to $\mathbb{R}^+$. For later convenience we denote by $\pi\alpha_i$ the interior angles of the polygon.

We place without loss of generality a vertex at zero, and rotate the polygon so that one side is in $\mathbb{R}^+$. The red arrow indicates the positive orientation of the polygon. Suppose that we want to map 0 to 0 and the segment in blue on $\partial\mathbb{H}^u$ to the blue segment on the polygon, see Fig. 27. We see that, to the left of $z = 0$, dw is rotated by $-\pi(1-\alpha)$ with respect to dz, while to the right of $z = 0$ (red arrow) dz and dw are parallel. A transformation that behaves like this on the boundary is $\psi'(z) = z^{-(1-\alpha)}$. We see that indeed the argument of ψ' ($\operatorname{Im}\ln\psi'$ which exists locally for $z \neq 0$ since $\psi' \neq 0$) does not change except at the singularity, $z = 0$:

$$(\ln\psi')' = \frac{\alpha-1}{z} \in \mathbb{R} \text{ (since } z \in \mathbb{R}) \Rightarrow d\arg\psi' = 0 \text{ for } z \neq 0 \tag{33.1}$$

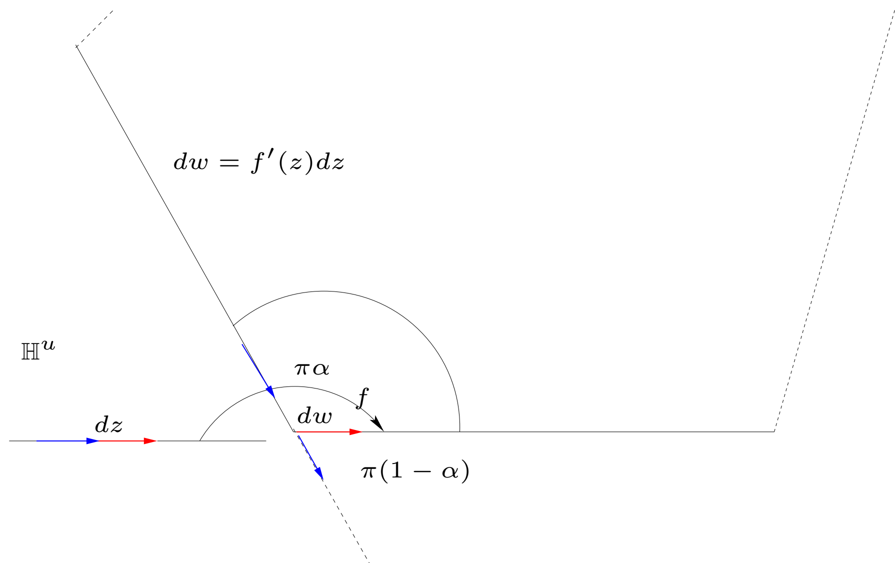


FIGURE 27. Schwarz-Christoffel transformations: two adjacent sides of the polygon.

Also, note that by Schwarz reflection, ψ is analytic at points z which are *not* mapped to vertices of the polygon.

Proposition 33.1. *Any transformation ψ of a one-sided neighborhood $\mathcal{N}$ in $\mathbb{H}^u$ of a segment $I = [-a, b]$; $a, b > 0$ which maps 0 to the vertex $w = 0$ of the polygon and is continuous up to the boundary has the property $\psi(z) = z^{\alpha-1}h(z)$ where h is holomorphic in $\mathcal{N} \cup I \cup \overline{\mathcal{N}}$. In particular, $\psi'(z) = z^{\alpha-1}\sum_{k=0}^{\infty} h_k z^k$ where $h(0) \neq 0$ and the sum is convergent.*

Definition 33.2. *We will call functions which are (locally) of the form $(z-z_0)^a H(z)$ with H holomorphic,* **ramified-analytic**.

Proof. Let $H = \int z^{-(\alpha-1)}\psi'$. Note that $dH = z^{-(\alpha-1)}\psi' dz$ maps the blue vector on the left side of the polygon $\mathcal{N}$ into a vector parallel to the red one, is continuous down to I. The image of I through $H = \int z^{-(\alpha-1)}\psi'$ will, by construction, be an interval in $\mathbb{R}$. Schwarz's reflection principle and the continuity of H ensure that H has analytic continuation in $\mathcal{N} \cup I \cup \overline{\mathcal{N}}$. Thus H' is analytic too. Since the transformation h is one-to one ($H(z)$ is strictly monotonic in $z \in I$, implying $H' = h \neq 0$. ∎

More generally, a transformation of the form

$$\psi' = \prod_{i=1}^{n-1} (z - a_i)^{\alpha_i - 1} \tag{33.2}$$

satisfies

$$(\ln \psi')' = \sum_{i=1}^{n-1} \frac{\alpha_i - 1}{z - a_i} \in \mathbb{R} \ \Rightarrow d \arg \psi' = 0 \text{ for } z \neq a_i \tag{33.3}$$

and $\arg \psi'$ changes by $\pi(1 - \alpha_i)$ upon traversing a_i. The points $A_i := \psi(a_i)$ on the polygon are the only ones where $\arg \psi'$ changes, and it changes by $+\pi(1 - a_i)$ (check the signs!). If the polygon P is a closed curve, then infinity must be mapped into one of the vertices of P, and, since ψ is conformal (to be proven later) it cannot be one of the $\{A_j, j = 1..n-1\}$ so it must be A_n. Recall that the sum of exterior angles of a closed polygon is 2π, and thus $B = \sum_1^{n-1} \beta_i = 1 + \alpha_n$, $\beta_i := a_i - 1$. Note also that $1 - a_i/z$ is analytic at infinity: as $z \to \infty$ we have

(33.4)

$$\psi' = z^B \prod_{i=1}^{n-1} (1 - a_i/z)^{\beta_i} = \zeta^{-B} g(\zeta); \ g(\zeta) = \prod_{i=1}^{n-1} (1 - a_i \zeta)^{\beta_i}, \ \zeta = 1/z$$

If $|\zeta| < \varepsilon < \max\{1/|a_i|, i = 1, ..., n-1\}$, g is analytic and $\operatorname{Re} g(\zeta) > 0$, implying that $\ln g$ is well defined. If z traverses $\partial \mathbb{D}_R$ from $+R$ to $-R$ where $R = 1/\varepsilon$ in a positive direction the change in $\arg \psi'$ comes solely from z^B and it equals πB, the same change in argument that $(z - A)^{1-\alpha_n}$ would produce. This "closes the polygon" with the last vertex $A_n = \lim_{z\to\infty} f(z)$. See figure. In the case of a closed polygon we see that ψ is bounded on $\mathbb{R}$. In the case of an open polygonal line, the arguments are similar.

Theorem 33.3. *(i) Let P be any polygon (closed or open) with n vertices $A_1, ..., A_n$ and interior angles $\pi\alpha_1, ..., \pi\alpha_n$. There is a choice of $a_1, ..., a_n \in \mathbb{R}$ and $C, C' \in \mathbb{C}$ such that the function*

$$\psi = z \mapsto C \int_0^z \prod_{k=1}^{n} (s - a_k)^{\alpha_k - 1} ds + C' \tag{33.5}$$

maps $\mathbb{H}^u$ into P (and $\psi(a_i) = A_i$).

(ii) The function mapping $\mathbb{D}$ conformally into P is given by

$$\psi = z \mapsto C \int_0^z \prod_{k=1}^{n} (b_k - s)^{\alpha_i - 1} ds + C' \tag{33.6}$$

for a choice of $b_1, ...b_n, C, C'$. (iii) Moreover, all transformations between $\mathbb{H}^u$ or $\mathbb{D}$ and polygons are of this form.

(iv) One of the points a_k can be chosen to be ∞. In this case the transformation is

$$\psi = z \mapsto C \int_0^z \prod_{k=1}^{n-1} (s - a'_k)^{\alpha_k - 1} ds + C' \tag{33.7}$$

(simply the point a_n is omitted). (v) The map from $\mathbb{D}$ to ext(P) taking 0 to ∞ is

$$\psi = z \mapsto C \int_0^z s^{-2} \prod_{k=1}^{n-1} (s - a'_k)^{\alpha_k - 1} ds \tag{33.8}$$

Note 33.4. *(i) The formulas* (33.5) *and* (33.6) *are the same, up to constants:* (33.5) *is invariant under a Cayley transform since the sum of exterior angles of a closed polygon is 2π. Do the calculation.*

(ii) One may ask what kind a conformal map is that in (v)? The answer is best seen if we think of the Riemann sphere as the target.

Remark 33.5. (i) It is important to note what freedom we have in such transformations. Suppose we want to map a triangle Δ. All triangles with same angles are similar, and a mapping between two similar triangles reduces to scaling, rotation and translation. Thus we need to understand one triangle with given angles α_i. We take say $a_1 = 0$ and $a_2 = 1$, use the α's, and see what triangle Δ_1 is obtained. Then, we can choose C and C' so that we remap Δ_1 to Δ. Thus we are able to map any triangle to $\mathbb{H}^u$, prescribing the position of the images of the vertices at will. See §33.2.

(ii) For $n > 3$, we can still place three points at will but the position of the fourth one etc cannot be chosen arbitrarily. We see that we have the freedom of $n - 1$ *real* constants, the $a'_i s$, and of two complex ones C and C', a total of $n + 3$ real constants, whereas an arbitrary closed polygon has $2n$ real constants as degrees of freedom (the position of its vertices); $2n > n + 3$ if $n > 3$. The a_k for $k \geqslant 4$ are called accessory parameters, and they are determined by the polygon and the values of $a_1, ..., a_3$; except for very symmetric cases (such as regular polygons), these parameters cannot be determined in closed form.

Given the disparity in the number of available parameters versus the degrees of freedom of the problem, it is not a priori clear that the Schwarz-Chrsistoffel transformation should always work (but it does).

First proof of the Schwarz-Christoffel formula. We analyze closed polygons, open ones being similar. Once we prove the formula for the UHP, the formula for $\mathbb{D}$ follows through a simple calculation applying the Cayley transform.

We can first arrange that one vertex is at zero and a second one at 1. Indeed, we can transform P by translation and multiplications by a constant (changing C, C' in the Theorem) into one geometrically similar to it, $\tilde{P}$ that has these properties. By composition to the right with $az+b$ we can arrange that 0 and 1 in $\mathbb{H}^u$ are mapped into 0 and 1 in $\tilde{P}$. Note that by Proposition 33.1 the function $F(z) = \psi'(z)\prod_{i=1}^{n-1}(z-a_i)^{1-\alpha_i}$ is analytic in $\overline{\mathbb{H}^u}$, does not change its phase except perhaps at a_i. But at every a_i, F is real and positive. Thus F is real on the real line and extends to an entire function. For the behavior at ∞ note that the transformation $\zeta = -1/z$, $\psi(z) = G(\zeta)$. $z \mapsto -1/z$ is an automorphism of $\mathbb{H}^u$ which maps ∞ to 0. Traversing $\mathbb{R}^+$ in z from 0 to ∞, returning to $-\infty$ on a "large" circle and moving in $(-\infty, 0)$ to the right corresponds to ζ traversing $(-\infty,\infty)$ from negative to positive values. To straighten the angle at a_n ($= 0$ in ζ) we need, as in Proposition 33.1 to multiply $G'(\zeta)$ by $\zeta^{1-\alpha_n}$: $\zeta^{1-\alpha_n}\frac{dG}{d\zeta} = H(1/z)$ where H is analytic in particular bounded. Since $\frac{d}{d\zeta}G = \zeta^{-2}\psi'(-1/\zeta)$ it follows that $z^{1+\alpha_n}\psi'$ is analytic at infinity.

Now, $1-\alpha_1+...+1-\alpha_n = 2$ implying $1-\alpha_1+...+1-\alpha_{n-1} = 1+\alpha_n$ and thus

$$(z-a_1)^{1-\alpha_1}\cdots(z-\alpha_{n-1})^{1-\alpha_{n-1}}\psi' \sim const.z^{1+\alpha_n}\psi' \text{ as } z\to\infty \tag{33.9}$$

is entire and bounded, thus constant and hence

$$(z-a_1)^{1-\alpha_1}\cdots(z-\alpha_{n-1})^{1-\alpha_{n-1}}\psi' = C \tag{33.10}$$

Corollary 33.6 (Analytic structure of ψ)**.** *The function ψ is ramified-analytic (cf. Definition 33.2). In a neighborhood of a_j we have*

$$\psi = C_{1j}x^{\alpha_j}H_j + C_{2j} \tag{33.11}$$

where H_j is holomorphic.

Proof. This follows by straightforward integration of (33.10). ∎

∎

"Geometric" proof of the Schwarz-Christoffel formula. This proof is largely based on [11]. A slightly different argument is used in [3]. We prove the statement for a closed polygon, the one for open ones being very similar. We use the notations in Fig. 28 and will assume first that

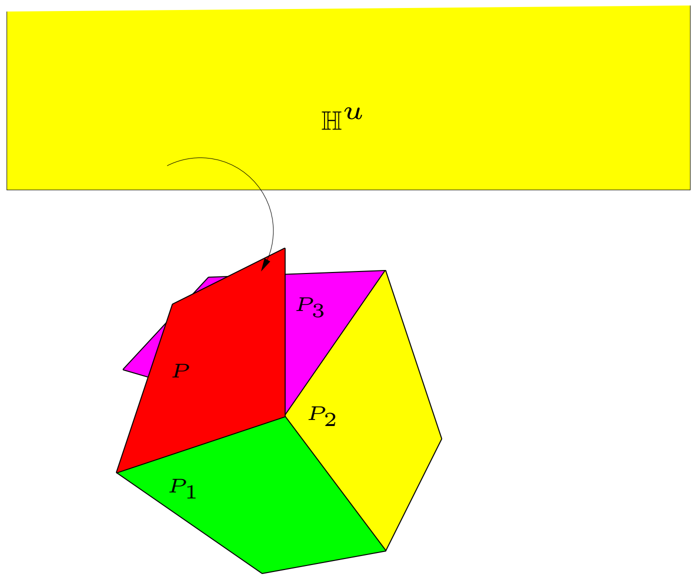


FIGURE 28. Successive reflections.

none of the points a_i is infinity (infinity must however go somewhere on ∂P).

Let ψ be the conformal map between the UHP and P. The figure shows various successive reflections of a polygon P across its sides. Since ψ transforms an interval I of $\mathbb{R}^+$ into the line segment ℓ bounding P and P_1, it has analytic continuation across I, and in fact the image of the UHP is P and that of the LHP is P_1. This continuation can be reflected back, and we obtain P_2. By simple geometry, the polygons P and P_2 are Euclidian transformations of each-other, of the form $P_2 = aP + c$ for some constants a and c with $|a| = 1$, whereas P and P_1 are related by a flip. We note also that ψ is analytic at all points except for $a_1, ..., a_n$, where it is singular. Indeed, at a_i ψ is not conformal: it maps two collinear successive intervals of $\mathbb{R}$ into two segments forming an angle $\pi\alpha_i$. However, consider the auxiliary function

$$h(z) = [\psi(z) - \psi(a_i)]^{1/\alpha_i}$$

Then, h maps the two successive intervals into a straight line (clearly an analytic arc), and by Schwarz reflection, it is analytic at a_i. Hence,

$$\psi(z) = \psi(a_i) + [h(z)]^{\alpha_i} \tag{33.12}$$

Since $h(a_i) = 0$, a_i is a branch point of ψ. Consider the function

$$g(z) = \psi''(z)/\psi'(z)$$

Using (33.12) we see that g has a simple pole of residue $1-\alpha_i$ at a_i. Since a_i are the only singularities of g, g is single valued and extends to a meromorphic function in $\mathbb{C}$. Thus,

$$G(z) = g(z) + \sum_{j=0}^{n-1} \frac{1-\alpha_j}{z-a_j} \tag{33.13}$$

is entire. Since $\psi(\infty) \in \partial P$ but is different from $a_1, ..., a_n$, by the Schwarz reflection principle, ψ is analytic at infinity (what that means is best understood if we map the UHP conformally to $\mathbb{D}$: ∞ corresponds to some point on $\partial\mathbb{D}$, other than the images of the a_i). Hence, in a neighborhood of $z=\infty$ we have $\psi(z) = \sum_{k=0}^{\infty} c_k z^{-k}$, where $c_0 = \psi(\infty)$ whence $g(z) \sim const./z$ for large z and in particular it vanishes at ∞. Then G is an entire function that vanishes at infinity implying $G=0$ and we have

$$g(z) = -\sum_{j=0}^{n-1} \frac{1-\alpha_i}{z-a_k} \tag{33.14}$$

meaning

$$\frac{\psi''(z)}{\psi'(z)} = \sum_{j=0}^{n-1} \frac{\alpha_k - 1}{z-a_k} \tag{33.15}$$

Integrating 33.15, we get

$$\psi(z) = c_1 + c\int_0^z \prod_{k=1}^{n} (t-a_k)^{\alpha_k - 1} dt$$

and Theorem 33.3 (i) and (iii) follow, and, as discussed, a composition with the Cayley transform proves (ii). To make one of the $a_k = \infty$ we use an LFT substitution: $z = a_k - 1/z_1$, and the rest of the proof of (iv) is a simple calculation.

Finally, for the mapping of the exterior of the polygon, which contains the point at infinity, making g unbounded there, the formula has to be reworked. Assume that $\infty = \psi(0)$. Thus ψ is unbounded, but must be conformal at zero, and the only possibility is a first order pole: $\psi(z) = \psi_1(z)/z$ where ψ_1 is regular at 0 and $\psi_1(0) \neq 0$. Now $\psi''(z)/\psi'(z) + 2/z$ is regular and a calculation shows that

$$\frac{\psi''(z)}{\psi'(z)} - \sum_{i=1}^{n} \left(\frac{1-\alpha_i}{z-a_i} + \frac{1-\alpha_i}{a_i} \right) + \frac{2}{z}$$

is entire, bounded in $\mathbb{C}$ and zero at zero, hence it vanishes identically, and (v) follows. ∎

Note 33.7. The second order linear ODE obtained after multiplying (33.15) by f' is a *Fuchsian equation*: it has only *regular singularities* in $\hat{\mathbb{C}}$ [4].

33.2. **Conformal map of the UHP onto the interior of a triangle of angles $\pi\alpha, \pi\beta, \pi\gamma$.** We choose for convenience $a_1 = 0, a_2 = 1, a_3 = \infty$ resp. Theorem 33.3 (iv) gives

$$\psi(z) = A \int_0^z z^{\alpha-1}(1-z)^{\beta-1} dz + B \tag{33.16}$$

We choose $A = 1$. From the choice above, we know that the length of the first side is

$$\int_0^1 |\psi'(s)| ds = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1} ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

The lengths of the other sides can be most easily determined from the relation

$$\frac{a}{\sin \pi\alpha} = \frac{b}{\sin \pi\beta} = \frac{c}{\sin \pi\gamma}$$

We note that $\psi(z) = B_z(\alpha, \beta)$, the incomplete Beta function, which, for $\alpha, \beta \notin -\mathbb{N}$ satisfies

$$B_z(\alpha, \beta) = \Gamma(\alpha) z^{\alpha} \ {}_2F_1(\alpha, 1-\beta; \alpha+1; z)$$

where ${}_2F_1$ is the hypergeometric function.

33.3. **Schwarz triangle functions and hypergeometric functions.** We can attempt the reflection-continuation procedure of §36.1. We now imagine the reflections having a common vertex. To insure a single valued function upon successive reflections about the sides, we must return to the starting triangle with no overlap or gap. If the rotations preserve one vertex, for this to happen we need $\alpha_i = 1/n_i, \ n_i \in \mathbb{N}$. The constraints are thus:

$$\alpha + \beta + \gamma = 1; \quad 1/\alpha, 1/\beta, 1/\gamma \text{ positive integers} \tag{33.17}$$

Check that the only solutions are: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (equilateral triangle) $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ (half of an equilateral triangle) and $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ (isosceles right triangle). Then the reflected images cover the whole plane and the mapping functions are restrictions of meromorphic functions. These are special cases of the *Schwarz triangle functions*. The group of reflections is a special case of an infinite *Coxeter group*.

Each triangle function corresponds to an elliptic function. We will return to this topic later.

Differential equation. We can derive an equation with polynomial coefficients for ψ as follows. In $\psi' = z^{\alpha-1}(z-1)^{\beta-1}$ we write $\psi = z^{\alpha}\psi_1$ and get, after dividing by $z^{\alpha-1}$,

(33.18)
$$z\psi_1' + \alpha\psi_1 - (1-z)^{\beta-1} = 0 \Rightarrow \frac{z}{(1-z)^{\beta-1}}\psi_1' + \frac{\alpha}{(1-z)^{\beta-1}}\psi_1 - 1 = 0$$

which we differentiate one more time to eliminate the constant 1 and we get

$$z(1-z)\psi_1'' + [\alpha + 1 - (2-\beta-\alpha)z]\psi_1' + \alpha(\beta-1)\psi_1 = 0 \tag{33.19}$$

The differential equation for the hypergeometric function ${}_2F_1(a,b;c;z)$ is the *Riemann equation*

$$z(1-z)h'' + [c-(a+b+1)z]\,h' - abh = 0 \tag{33.20}$$

From the way we went from (33.18) to (33.19) (or by direct verification) we see that one solution is $z^{-\alpha}$. Comparing with (33.20) the second one is

$${}_2F_1(\alpha, 1-\beta; \alpha+1; z) \Rightarrow \psi = z^{\alpha} \cdot \ {}_2F_1(\alpha, 1-\beta, \alpha+1, z) \tag{33.21}$$

That is, in this case, the Schwarz-Christoffel transformation is a ratio of two independent solutions of the special hypergeometric equation (33.19).

34. Optional material: curvilinear triangles

In general, the map from $\mathbb{H}^u$ into a *curvilinear triangle*, one whose sides are arccircles is given by the ratio of two independent solutions of (33.20), where the angles α, β, γ are related to a, b, c by (cf. [11])

$$a = \tfrac{1}{2}(1+\beta-\alpha-\gamma),\ b = \tfrac{1}{2}(1-\alpha-\beta-\gamma),\ c = 1-\alpha \tag{34.22}$$

Note 34.8. To see qualitatively why that is, before we work out the mathematical details, since the sought-for function f transforms segments into arccircles, Möbius transforms of f map real segments into real segments. Möbius transforms are conformal wherever defined, so they preserve angles, and thus f should be ramified analytic at three points, say $\{0, 1, \infty\}$.

Secondly, note that any function which is real-analytic (with real values) on some interval I and such that $f' \neq 0$ on I conformally maps a neighborhood of I into a neighborhood of $f(I)$ (where, of course, $f(I)$ is an interval in $\mathbb{R}$).

Finally, this approach would apply to any curvilinear polygon, and the final ODE that we construct will still be second order linear, but the solutions are generally quite complicated.

To find an f as in Note 34.8, a natural candidate would be a ratio of two solutions of a real-valued linear second order ODE with analytic coefficients, whose solutions have only ramified singularities, at in $\{0, 1, \infty\}$. Such an ODE will have two linearly independent solutions, f_1 and f_2 [7]. Define $S = \{a_1 f_1 + a_2 f_2 | A_{1,2} \in \mathbb{C}\}$ (this of course is the space of all solutions). Then,

(1) $f_{1,2}$ are analytic in $\mathbb{C}^* \setminus \{0, 1, \infty\}$ while at the points $0, 1, \infty$ they are *ramified-analytic*;
(2) on each interval $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$ there are two positive linearly independent functions in S;
(3) S is invariant under Schwarz reflections [8]. This is because any analytic continuation of a solution is a solution, by permanence of relations.
(4) $(f_1/f_2)' \neq 0 \, \forall x \in \mathbb{R}$. [9]

Then some ratio F_1/F_2 of two linearly independent $F_{1,2} \in S$ maps the upper half plane into a curvilinear triangle, that is, one whose sides are line segments or arccircles (a line segment is a limiting case of an arccircle, so we will call arccircle too). Indeed, take F_1 and F_2 in S linearly independent, choose one of the intervals in (2) above, and let $\tilde{F}_1$ and $\tilde{F}_2$ the functions which are real valued on the interval I. We choose $\tilde{F}_1$ and $\tilde{F}_2$ such that $\tilde{F}_1/\tilde{F}_2$ is bounded at zero. By assumption, $F_{1,2} = A_{1,2}\tilde{F}_1 + B_{1,2}\tilde{F}_2$ and

$$\frac{F_1}{F_2} = \frac{1 + a\tilde{F}_1/\tilde{F}_2}{b + c\tilde{F}_1/\tilde{F}_2} \tag{34.23}$$

Now, $\tilde{F}_1/\tilde{F}_2$ is real-valued and one-to-one, and thus the right side of (34.23) is a Möbius transformation of a line segment: an arccircle. The image of $\mathbb{R}$ through F_1/F_2 consists of three arccircles, a general curvilinear triangle, provided that the singularities are such that F_1/F_2 are continuous on $\mathbb{R}$.

We now show that (33.20) is such an equation, for a, b, c satisfying (34.22). The following integral representation due to Euler can be

[7]Meaning that $A, B \in \mathbb{C}$ and $Af_1 + Bf_2 = 0 \Rightarrow A = B = 0$.

[8]Equivalently, the space generated by $f_{1,2}$, $S := \{C_1 f_1 + C_2 f_2 : C_1, C_2 \in \mathbb{C}\}$ is closed under continuations at the branch points (e.g., $f_1(ze^{2\pi i}) = C_1 f_1(z) + C_2(z)$).

[9]This in fact is implied by linear independence. Check this by first showing that two functions are linearly independent *iff* their Wronskian $W(f_1, f_2) := f_1' f_2 - f_2' f_1 \neq 0$.

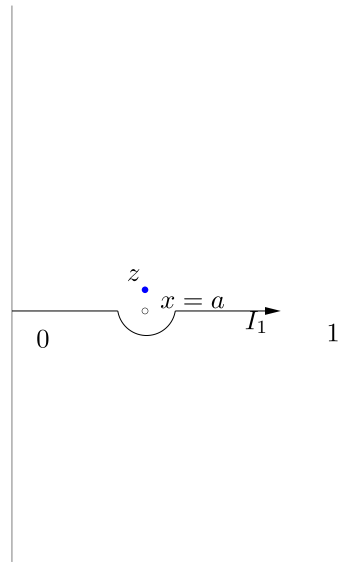


FIGURE 29. Hypergeometric contours

checked to solve (33.20):

$$ {}_2F_1(a,b;c;z) = K\int_0^1 H(z,x)dx, \tag{34.24}$$

where $H(z,x) = x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}$ assuming $c > b > 0$

The standard choice of K, immaterial here, is $K^{-1} = B(b, c-b)$ with B the Beta function [10]. For the integral to exist, we need

$$ b-1 > -1, c-b-1 > -1, -a > -1 \tag{34.25}$$

which are satisfied if $\alpha+\beta+\gamma < 1$, $\alpha,\beta,\gamma \in \mathbb{R}^+$, which is the case for a hyperbolic triangle (with concave sides) as can be verified from (34.22). Under this same condition, we have

$$ (b-1)+c-b-1-a < -1 \Rightarrow H(x,z) \sim s^{-p},\ p > 1 \text{ as } x \to \infty \tag{34.26}$$

Singularities of ${}_2F_1$ For general a, b, c the behavior of h in (33.20) at the singular points of the ODE, $0, 1, \infty$ [11] follows general results about regular singular points of ODEs (Frobenius theory, cf. [4]); we will not assume this here however. We will find the behavior of the solutions in three different way, to illustrate various approaches.

---

[10]The formula is valid under the more general condition $\mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0$, but here we only need real a, b, c.

[11]Writing the equation in the form $h'' + Q(x)h + R(x) = 0$, $\{0, 1\}$ are singular points of P, Q. Ditto after the change of variable $z = 1/t$, $y(z) = t^{-c}Y(t)$ at $t = 0$.

Directly from the integral representation, the behavior can be calculated in the following way.

Clearly, (34.24) and Corollary **??** imply that ${}_2F_1$ is analytic except possibly on $\mathbb{R}^+$. If $z \to a \in \overline{\mathbb{R}^+} \setminus \notin \{1\}$ then we can use the analyticity of the integrand at a to first homotopically deform the contour as shown in Fig. 29. The new integral (of course, equal to the original one) is manifestly analytic in z near a. Thus the only possible singularity is at 1 (since the integrand is manifestly analytic for small z, cf. Corollary **??**).

At $z = 0$ we get

$$\frac{{}_2F_1(a,b;c;z)}{K} = \int_0^1 x^{b-1}(1-x)^{c-b-1}dx > 0 \tag{34.27}$$

To find the type of behavior at $z = 1$ it is convenient to take $z = 1-\varepsilon$ and change the variable to $x = 1 - s$ in the integral:

$$K\int_0^1 h(z,(1-s))ds = K\int_0^1 (1-s)^{b-1}s^{c-b-1}(\varepsilon+s-\varepsilon s)^{-a}ds \tag{34.28}$$

By (34.26) We can then push the contour up toward $+i\infty$,

(34.29)

$$\int_0^1 h(z,(1-s))ds = \left(\int_0^{+i\infty} - \int_1^{+i\infty}\right)(1-s)^{b-1}s^{c-b-1}(\varepsilon+s-\varepsilon s)^{-a}ds$$

The second integral is analytic in ε by the same Corollary **??**.

In the first one, we change variable to $s = \varepsilon u$, to get

$$\begin{aligned} &-\varepsilon^{c-b-1-a+1}\int_0^{+i\infty}(1-\varepsilon u)^{b-1}u^{c-b-1}(1+u-\varepsilon u)^{-a}ds \\ &\qquad = -\varepsilon^{c-b-a}\int_0^{-\infty}(1-\varepsilon u)^{b-1}u^{c-b-1}(1+u-\varepsilon u)^{-a}ds \end{aligned} \tag{34.30}$$

where the contour change is justified by (34.26).

Exercise 34.9. *What is the phase of the last integral in* (34.30)*?*

The integral in (34.30) is analytic in ε and thus

$${}_2F_1(a,b;c;z) = A_1 + A_2(1-z)^{\gamma} \text{ where } A_{1,2} \text{ are analytic at } 1 \tag{34.31}$$

and $A_2(1) = \int_1^{+\infty} u^{c-b-1}(1+u)^{-a}ds > 0$. A second solution of (33.20) is obtained, cf. [11], by noticing that the substitution $h(z) = g(1-z)$ leads to the equation (33.20) with c replaced by $C = a+b+1-c$. The integral representation (34.24) still holds with c replaced by $C = 1-\gamma$, and converges under the same conditions on α, β, γ as (34.24). The behavior of the integrand at infinity is still of the form in (34.26).

Thus the same analysis applies, to show that there is a second solution which is analytic at $z = 1$ and has a singularity at zero, with $c - b - a$ replaced by $1 - c = \alpha$. This also suggests making the substitution $h_1 = z^{1-c}h = z^{\alpha}h$ directly into (33.20); we get an equation of the form (33.20), with A, B, C replaced by

(34.32) $$A = a + 1 - c, B = b + 1 - c, C = c - 2$$

which has an integral representation of the form (34.24) valid under the same conditions on α, β, γ. Thus, near $z = 0$ we have two linearly independent solutions, A_1 analytic and A_2 of the form $z^{\alpha}A(z)$ with A analytic and $A(0) \neq 0$. These are clearly linearly independent since the second solution is not analytic at zero. In the same way, there are two solutions, one analytic and positive at $z = 1$ and another one of the form $z^{\gamma}A(z)$, A analytic and positive for $z < 1$.

From the ODE. Here we are assuming knowledge of basic properties of linear ODEs: a linear combination (with constant coefficients) of solutions is a solution (this can be checked directly) and the fact that the space of solutions of a second order linear equation is two-dimensional (i.e., there are exactly two free constants, or, in other words, there are two linearly independent solutions which form a basis in the space of all solutions).

With F_{01} given by (34.24), we notice as before that F is analytic near zero and $F(0) > 0$. We look for a second solution in the form $F_{02} = F_{01}g$. The equation for g is

(34.33)
$$\frac{g''}{g'} = -2\frac{F'}{F} + q(x);\ \left(q(x) := \frac{(a+b+c)x - c}{x(1-x)}\right) = -2\frac{F'}{F} - \frac{c}{x} + A(x)$$
$$\Rightarrow g' = -\frac{c}{x}A_1(x) \Rightarrow F_{01}(x) = x^{-c+1}A_2(x) = x^{\alpha}A_2(x);$$

with A, A_1, A_2 analytic (check the conclusions above). Since F is real-valued for real $z \in (0,1)$, g is also real-valued, and it is an independent solution (it has a manifestly different behavior at zero). At $x = 1$ we make the substitution $z = 1 - y$ and we get

(34.34)
$$y(1-y)h'' + [C - (a+b+1)y]\,h' - abh = 0,\ C = a + b + c - 1 = 1 - \gamma$$

In the same way as above we get two independent real valued solutions for y real, $F_{10}(y)$ analytic and a second one F_{11} of the form $y^{\gamma}A_1$ with A_1 analytic. They are in general different from the solutions $F_{00}(y)$ and $F_{00}(y)$.

Finally the substitution $h(z) = z^{a}H(1/z)$ $z = 1/Z$ in (33.20) results in an equation of the same type as (33.20). The ratio of two solutions behaves like $z^{-\beta}$ for large z. Now, any transformation of the type F_1/F_2 where F_1 and F_2 are real maps the interval $(0,1)$ into an interval. A different choice f_1/f_2 is a Möbius transformation of F_1/F_2, and thus it maps the interval $(0,1)$ into an arccircle. Keeping one combination F_1/F_2 chosen so that $F_1/F_2 \to 0$ as $z \to \infty$ will then map $\mathbb{R}$ into a curvilinear triangle, of angles α, β, γ.

Note 34.10. In the simplest nondegenerate case, Frobenius theory allows to determine the behavior of solutions of linear meromorphic ODEs, Lf, with regular singularities, say at z_0 by a very simple method: take $L(z-z_0)^r$ and keep only the lowest power of $(z-z_0)$. This gives a quadratic equation for r. If the solutions $r_{1,2}$ do not differ by an integer, then $Lf = 0$ has two linearly independent solutions in the form $(z-z_0)^r_{1,2}A_{1,2}(z)$, A analytic at z_0.

Note 34.11. The equation satisfied by f_1/f_2, a ratio of solutions of the hypergeometric equation is

$$\{w,z\} = \frac{1-\alpha^2}{2z^2} + \frac{1-\gamma^2}{2(z-1)^2} + \frac{\alpha^2+\gamma^2-\beta^2-1}{2z(z-1)} = 0 \tag{34.35}$$

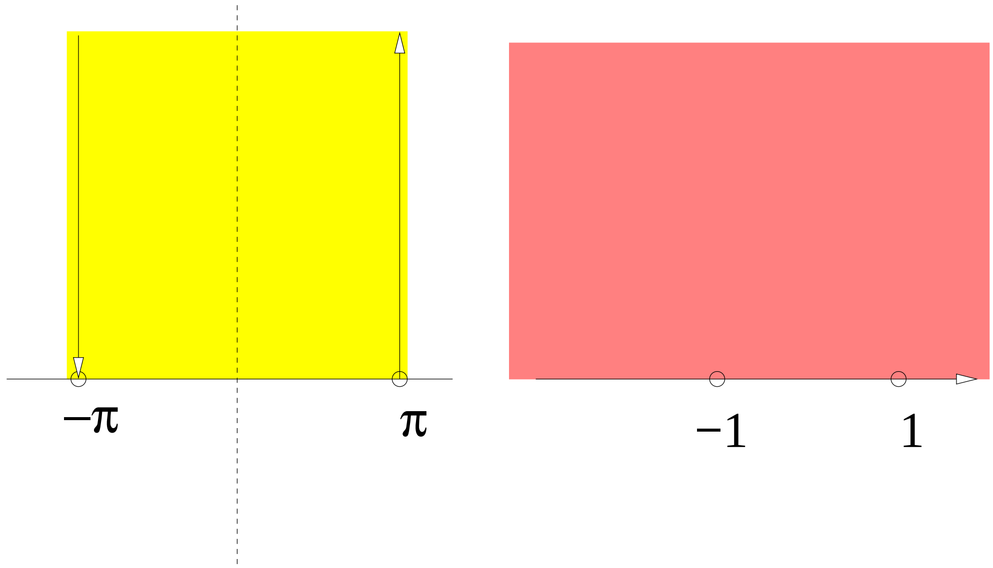
where $\{w,z\}$ is the important *Schwarzian derivative* [11]:

$$\{w,z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 \quad \left(w' = \frac{dw}{dz}\right) \tag{34.36}$$

The Schwarzian derivative is *invariant under any Möbius transformations of w* as expected from our discussion and can be checked by straightforward calculation.

35. Two other important examples of Schwarz-Christoffel transformations

35.1. **Another look at the sine function.** *Problem.* Map the strip indicated into $\mathbb{H}^u$ preserving the points marked with circles and the positive orientation.



Solution The α's at $-\pi$ and π are both 1/2. We note that Theorem 33.3 still applies, by passing the angles to this limit. We apply formula (33.5) with $a_1 = -1$, $a_2 = 1$ and the integrand is then $(s^2-1)^{-1/2}$. Eq. (33.5) therefore gives, for two arbitrary constants,

$$\psi = C \arcsin z + C' \tag{35.37}$$

and therefore our map $f = \psi^{-1}$ has the general form

$$\psi^{-1}(w) = \sin(cw + c') \tag{35.38}$$

We have now to choose c and c' to match the prescribed points. We must have $\sin(-\pi c + c') = -1$ and $\sin(c\pi + c') = -1$; the choice $c' = 0$ and $c = 1/2$ matches these conditions. We get

$$f(w) = \sin(w/2) \tag{35.39}$$

36. Mapping of a rectangle: Elliptic functions

Placing one of the a_i at infinity, and choosing the remaining three to be $0, 1, \rho$ where $\rho > 0$, we get the Schwarz-Christoffel formula for a rectangle in the form

$$F(z) = \int_0^z \frac{ds}{\sqrt{s(s-1)(s-\rho)}} \tag{36.1}$$

The double symmetry of the rectangle suggests a symmetric choice of a_i:

$$\psi(z,k) = \int_0^z \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \tag{36.2}$$

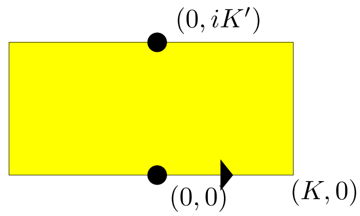


FIGURE 30. The fundamental rectangle.

Here $k \in (0,1)$. We have $\psi(z,k) = F(\arcsin z, k)$ where F is the incomplete elliptic integral of the first kind.

The branch of the square root is taken to be positive for $z \in (0,1)$ and otherwise it is obtained by analytic continuation in the UHP, passing to the limit $\operatorname{Im} z \to 0^+$ for $z \in \mathbb{R}$. Thus, as z grazes $\mathbb{R}^+$, the square root is first in $\mathbb{R}^+$, then changes to $i\mathbb{R}^+$ at 1 and then to $\mathbb{R}^-$ after $1/k$. Also, since the square root is even for small z, by analyticity this property is preserved along $\mathbb{R}$, and , for $k \in \mathbb{R}$, $\psi(z,k) = -\psi(-z,k)$. $\psi(\mathbb{H}^u)$ it is a rectangle with vertices

$$(-K,0),(K,0),(K,iK') \text{ and } (-K,iK) \tag{36.3}$$

where

$$K = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \tag{36.4}$$

$$iK' = \int_1^{1/k} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \tag{36.5}$$

Traversing $\mathbb{R}$ gives

$$\int_{-\infty}^{\infty} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} = 0 \tag{36.6}$$

(by symmetry, or since the contour can be pushed up to $i\infty$). Observe that for $s > 1/k$ we note that the square root is real and negative and we have

$$\sqrt{(1-s^2)(1-k^2s^2)} = -ks^2\sqrt{(1-s^{-2})(1-(ks)^{-2})} \tag{36.7}$$

(where the square roots on the right side are positive) and that the function $\sqrt{1-\zeta}$ $\zeta = 1/s^2$ is *analytic* for $|\zeta| < 1$. Thus $\psi(z,k)$ is analytic at infinity and for $z > 1/k$, writing $\int_0^z = \int_0^\infty - \int_z^\infty$ we get the expansion at infinity

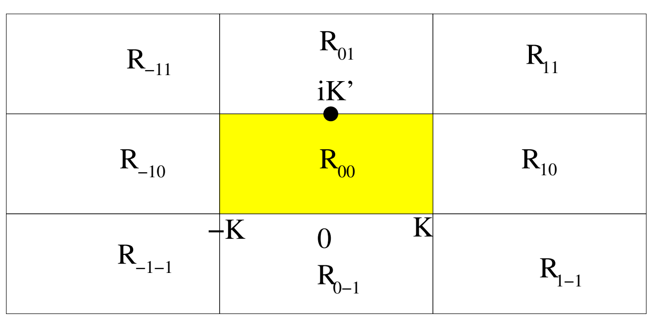
$$\psi(k,z) = iK' - \int_z^\infty -\frac{1}{ks^2} - \left(\frac{k^{-1}+k^{-3}}{2s^4} + \cdots\right) ds \\ = iK' + \frac{1}{kz} + \frac{k^{-1}+k^{-3}}{6z^3} \cdots \tag{36.8}$$

The value iK' is gotten by the symmetry of the function: when $z \to \infty$, exactly half of the rectangle has been traversed.

Exercise 36.1. *** Find changes of variables that connect* (36.6) *to* (36.1)

36.1. **Continuation to the whole of $\mathbb{C}$. Double periodicity.** As mentioned, after an even number of Schwarz reflections of ψ the function returns to its original value. Recall that an even number of successive Schwarz reflections results in ψ returning to its original value. This is due to the symmetry of the rectangle and would not hold for a general polygon; see the discussion in §32.4.1 The *inverse function* $\text{sn} = \psi^{-1}$ is, by this a doubly periodic function:

$$\text{sn}(z) = \text{sn}(z + 2K) = \text{sn}(z + 2iK') \tag{36.9}$$



37. Entire and Meromorphic functions

Analytic and meromorphic functions share with polynomials and rational functions a number of very useful properties, such as decomposition by partial fractions and root-factorization. These notions have to be carefully analyzed though, since questions of convergence arise.

37.1. **A historical context.** Finding the *exact* value of the sum

$$S := \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{37.1}$$

known as the "Basel problem" had been open for almost a century, in spite of efforts by great mathematicians of the time (the Bernoulli brothers, Leibniz, Goldbach, Stirling, Moivre and others) before Euler solved it in 1735 whe he was 28; because the problem stumped so many brilliant minds, this attracted a lot of attention. Euler's solution, well before there was any systematic theory of complex functions proceeds (roughly) as follows. Take the function

$$f(x) = \frac{\sin x}{x} \tag{37.2}$$

This has a Taylor series which converges for all $x \in \mathbb{C}$. *If* f were a polynomial with roots at $z = a_i$, then we would be able to write

$$f(x) = A \prod (x - a_i) \tag{37.3}$$

Assuming the same were true for an "infinite order polynomial" and noting that the roots of $\sin(x)/x$ are at $n\pi$, $n \in \mathbb{Z} \setminus \{0\}$ we get

$$\frac{\sin x}{x} = \prod_{n \geqslant 1} \left(1 - \frac{x}{n\pi}\right)\left(1 + \frac{x}{n\pi}\right) = \prod_{n \geqslant 1} \left(1 - \frac{x^2}{(n\pi)^2}\right) \tag{37.4}$$

(here the free constant must be 1, comparing the values at 0) Expanding out and collecting the coefficient of x^2 we get $\frac{\sin x}{x} = 1 - cx^2 + O(x^4)$ where

$$c = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \cdots = \frac{1}{\pi^2} S \tag{37.5}$$

On the other hand from the Maclaurin expansion of $f(x)$ we have $c = \frac{1}{3!}$. Thus

$$S = \frac{\pi^2}{6} \tag{37.6}$$

In fact, Euler went further and calculated $\sum \frac{1}{n^{2k}}$ for $k \geqslant 1$, in for all even k. This was mostly fine by the standards of the day, though it led to some criticisms that prompted more rigorous proofs later by Euler. Weierstrass was apparently inspired by this solution when he developed the theory of decomposition of entire functions as products. For a rigorous proof of (37.5) see §38.2

37.2. **Partial fraction decompositions.** First let $R = P_0/Q = P_1 + P/Q$ be a rational function. where P_i and Q are polynomials and $\deg(P) < \deg(Q)$. We aim at a partial fraction decomposition of R; if $\deg(Q) = 0$ there is nothing further to do. Otherwise let $z_1, ..., z_n$, $n \geqslant 1$, be the zeros of Q, where we don't count the multiplicities, and let m_j be the multiplicities of these roots. Let's look at the singular part of the Laurent expansion of P/Q at z_j:

$$\frac{P}{Q} = \sum_{k=1}^{m_j} \frac{c_{jk}}{(z - z_j)^k} + \text{analytic at } z_j \tag{37.7}$$

We claim that

$$\frac{P}{Q} = \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{c_{jk}}{(z - z_j)^k} \tag{37.8}$$

Indeed,

$$E(z) := \frac{P}{Q} - \sum_{j=1}^{n} \sum_{k=1}^{m_j} \frac{c_{jk}}{(z - z_j)^k} \tag{37.9}$$

is an entire function. By assumption, $P/Q \to 0$ as $z \to \infty$ and the rhs of (37.8) also, clearly, goes to zero as $z \to \infty$. Thus $E(z) \to 0$ as $z \to \infty$ and therefore $E \equiv 0$.

Second example. We examine in detail an example that has some of the features of the general case. The function

$$\frac{\pi^2}{\sin^2 \pi z} \tag{37.10}$$

has poles for $z_i = k$, $k \in \mathbb{Z}$, and the singular part of the Laurent series at $z = k$ is, as it can be quickly checked

$$\frac{1}{(z - k)^2} \tag{37.11}$$

We first note that

$$g(z) = \sum_{k \in \mathbb{Z}} \frac{1}{(z - k)^2}; \quad z \in \mathbb{C} \setminus \mathbb{Z}$$

converges uniformly and absolutely on compact sets in $\mathbb{C} \setminus \mathbb{Z}$ to an analytic function: take z_0 s.t. dist $(z, \mathbb{Z}) > \varepsilon$ and the disk $\overline{\mathbb{D}_{\varepsilon/2}(z_0)}$. By translation symmetry we may assume that $\lfloor x_0 \rfloor = \alpha \in [0, 1)$. We can write

$$g(z) = \sum_{|k| \leqslant 1} \frac{1}{(z - k)^2} + \sum_{|k| \geqslant 2} \frac{1}{(z - k)^2}$$

The second sum is bounded in absolute value by $\sum_{|k|\geqslant 1}\frac{1}{k^2}$, while the finite sum is a manifestly analytic function. Next we note that

$$\frac{\pi^2}{\sin^2 \pi z} - g(z) \tag{37.12}$$

extended by continuity at the integers is entire. This can be immediately checked by noting that the only singularities of $h(z) = \frac{\pi^2}{\sin^2 \pi z}$ are $z = k \in \mathbb{Z}$, and calculating the Laurent series of h at $z = k$.

Proposition 37.1. *We have*

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{k\in\mathbb{Z}} \frac{1}{(z-k)^2}; \quad z \notin \mathbb{Z} \tag{37.13}$$

Proof. In the notations introduced above the proposition, we saw that, after extending it by continuity at the removable singularities, the function $h - g$ is entire. This function is clearly periodic of period 1, and it is clearly bounded in any compact subset of the strip $\operatorname{Re} z \in [0, 1]$. By dominated convergence $g(z) \to 0$ as $\operatorname{Im} z \to \infty$. Now, with $z = x + iy$ we have

$$|\sin(x+iy)^2| = \frac{1}{2}\Big(\cosh(2y) - \cos(2x)\Big) \geqslant \frac{1}{2}\left(\cosh(2y) - 1\right)$$

It follows that $h - z$ is entire and vanishes at infinity, thus it is identically zero.

∎

Exercise 37.2. *By integrating* (37.13) *show that*

$$\cot(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{n^2\pi^2 - x^2}$$

By one more integration, show the product formula Euler used. (See also Lemma 38.5 below.

37.3. **The Mittag-Leffler theorem.** As such, the procedure in Proposition 37.1 does not extend in general. For instance, for the function Γ'/Γ the poles are located at the negative integers and have residue 1, and the partial fraction decomposition would not converge without some modifications. Furthermore, even when it does converge, a function can be determined by a sum over its poles only up to an entire function. For a general theorem, we need to address these issues.

The theorem below shows that for any sequence of one-sided Laurent series centered at the points $\{b_n\}_{n\in\mathbb{N}} \in \mathbb{C}$ with no accumulation point, there is a meromorphic function with exactly that singular behavior and analytic elsewhere. Conversely, any meromorphic function can be decomposed as a sum of its negative powers-part of its Laurent series at the poles and an entire function. More precisely,

Theorem 37.3 (Mittag-Leffler)**.** (i) Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers with no accumulation point (cf. Note 37.4) in $\mathbb{C}$ and let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials without constant term. Then there are meromorphic functions f in $\mathbb{C}$ whose only poles are at $z = b_n$ and having singular part $P_n((z-b_n)^{-1})$ at $z = b_n$.

(ii) Conversely let f be meromorphic in $\mathbb{C}$ whose poles are $\{b_n\}_{n\in\mathbb{N}}$ and singular part of the Laurent expansion $P_n((z-b_n)^{-1})$ at b_n, $n \in \mathbb{N}$. Then there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ and an entire function g such that

$$f = \sum_{n\in\mathbb{N}} \left[P_n\left(\frac{1}{z-b_n}\right) - p_n(z)\right] + g(z) := S(z) + g(z) \tag{37.14}$$

where the series converges uniformly on compact sets in $\mathbb{C} \setminus \{b_n\}_{n\in\mathbb{N}}$.

Note 37.4. *The condition that* $\{b_n\}_{n\in\mathbb{N}}$ *has no accumulation point in* $\mathbb{C}$ *automatically implies that* $|b_n| \to \infty$ *as* $n \to \infty$*.*

Note 37.5 (Idea of the construction)**.** The possible divergence of the infinite series $\sum_{n\in\mathbb{N}} P_n((z-b_n)^{-1})$ is due to the behavior for large n of the terms of the series. We fix z and take n large enough so that $|z/b_n| < 1$ For large n, $\frac{1}{z-b_n}$ roughly behaves like $-b_n^{-1}$. If we subtract out $-1/b_n$, the behavior is roughly $-zb_n^{-2}$. Continuing in this way, and keeping in mind that $|b_n| \to \infty$ as $n \to \infty$, we obtain for each n a bound $|z|^{m_n-1}|b_n|^{-m_n}$ ultimately achieving convergence.

Proof. We start by proving (i). We can assume without loss of generality that $b_n \neq 0$ for if say $b_1 = 0$ then we can prove the theorem for $\tilde{f} = f - P_1(1/z)$. Note first that the series at infinity in $1/b_n$ of $P_n(z-b_n)^{-1}$ and its series in z at zero coincide.

Let p_n be a Maclaurin polynomial of P_n such that

$$\left|P_n((z-b_n)^{-1}) - p_n(z)\right| \leqslant 2^{-n}; \quad \forall z \text{ s.t. } |z| < |b_n/4| \tag{37.15}$$

Now we look at the series

$$f_1 = \sum_{n=1}^{\infty} \left[P_n((z-b_n)^{-1}) - p_n(z)\right] \tag{37.16}$$

and fix an R and analyze the series for $z \in \mathbb{D}_R$. We split the sum (37.15) into two parts:

(37.17)

$$f_1 = \sum_{n:|b_n|\leqslant 4R} \left[P_n((z-b_n)^{-1}) - p_n(z)\right] + \sum_{n:|b_n|> 4R} \left[P_n((z-b_n)^{-1}) - p_n(z)\right]$$

The first sum is finite, while for the second one we have $|z| < |b_n/4|$ and the estimate (37.15) applies. Thus (37.16) is convergent away from the poles uniformly on compact sets.

(ii) Clearly, $f - f_1$ is entire. ▮

37.4. **Further examples.** We write the result in Exercise 37.2 in the form

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{2z}{z^2-n^2} \tag{37.18}$$

We can now use this identity to calculate easily some familiar sums. Note that the lhs of (37.18) has the Laurent expansion at $z = 0$

$$\pi \cot \pi z = \frac{1}{z} - \frac{\pi^2 z}{3} - \frac{\pi^4 z^3}{45} - \frac{2\pi^6 z^5}{945} - \cdots \tag{37.19}$$

Since the series on the rhs of (37.18) converges uniformly near $z = 0$, by Weierstrass's theorem it converges together with all derivatives. On the other hand we have

$$\frac{2z}{z^2-n^2} = -2\left(\frac{z}{n^2} + \frac{z^3}{n^4} + \frac{z^5}{n^6} + \cdots\right) \tag{37.20}$$

and we get immediately,

$$\sum_{n\geqslant 1} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n\geqslant 1} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n\geqslant 1} \frac{1}{n^6} = \frac{\pi^6}{945} \cdots \tag{37.21}$$

Exercise 37.6. ** The definition of the Bernoulli numbers B_k is*

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1} \tag{37.22}$$

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2^{2k-1} \frac{B_k}{(2k)!} \pi^{2k} \tag{37.23}$$

38. Infinite products

An infinite product is the limit

$$\prod_{n=1}^{\infty} p_n := \lim_{k\to\infty} \prod_{n=1}^{k} p_n = \lim_{k\to\infty} \Pi_k \tag{38.1}$$

We wish to express entire functions as infinite products. Then the following conventions are natural:

1. *Only finitely many p_j are nonzero.*

Hence there exists an n_0 such that $p_k \neq 0$ for any $k \geqslant n_0$ and we require:

2.

$$\prod_{k=n_0}^{n} p_k \to \ell \neq 0 \quad \text{as } n \to \infty$$

In issues of convergence we ignore the zero terms, so in the following we assume they simply do not exist. Write now $p_n = P_n/P_{n-1}$, with $P_0 = 1$ we see that the limit of Π_k is the same as the limit of the P_k, and since $P_k \to \ell \neq 0$ we have $P_{k+1}/P_k \to 1$ as $k \to \infty$. Thus $p_n \to 1$ is a necessary condition of convergence of the infinite product. We should then better write the products as

$$\prod_{n=1}^{\infty}(1 + a_n) \tag{38.2}$$

and then a necessary condition of convergence is $a_n \to 0$.

Theorem 38.1. *The infinite product (38.2) converges iff*

$$\sum_{n=1}^{\infty} \ln(1 + a_n) \tag{38.3}$$

converges. We omit, as before, the terms with $a_n = -1$. The log is defined with a cut along $(-\infty, 0]$, extended by continuity on the upper side of the cut.

Proof. If the sum (38.3) converges, then Π_n converges, since the exponential of a finite sum is a finite product.

In the opposite direction, a word of caution. We know that in the complex domain, $\ln ab$ is not always $\ln a + \ln b$. The limit of the sum will not, in general, be the log of the infinite product. So the reasoning is not that obvious. We write $(1 + a_n) = \rho_n e^{i\phi_n}$ and note that $\prod \rho_n$ is convergent, hence the real part of (3.19) converges. Since $(1 + a_{n+1})/(1 + a_n) \to 1$ we can inductively choose a multiple of $2\pi i$ in $\arg(1 + a_n)$ so that

$$\phi_{n+1} - \phi_n \to 0 \ \text{ as } \ n \to \infty \tag{38.4}$$

With this choice, we see that $\sum_k \phi_k$ must converge. Otherwise, there are two possibilities. If $S_n = \sum_{k=1}^{n} \phi_k$ is unbounded, then, by (38.4), $\text{dist}(\{S_n\}_{n\in\mathbb{N}}, 2\mathbb{Z} + 1) = \text{dist}(\{S_n\}_{n\in\mathbb{N}}, 2\mathbb{Z}) = 0$ entailing infinite oscillation and divergence of the product. If instead $\{S_n\}_{n\in\mathbb{N}}$ is confined to a compact set and has $a \neq b$ as accumulation points, then the conclusion is the same. Hence, the imaginary part of (3.19) converges as well. ∎

Absolute convergence is easier to control in terms of series. An infinite product is absolutely convergent, by definition, iff

$$\sum_{n=1}^{\infty} |\ln(1+a_n)| \tag{38.5}$$

is convergent.

Theorem 38.2. *The sum (38.5) is absolutely convergent iff* $\sum a_k$ *is absolutely convergent.*

Proof. Assume $\sum a_k$ converges absolutely. Then in particular $a_n \to 0$. Also, if $\sum_{n=1}^{\infty} \ln(1+a_n)$ converges absolutely then $\ln(1+a_n) \to 0$ and $a_n \to 0$. But (eliminating all the irrelevant zero terms which are zero) we have, as $n \to \infty$ $\lim_{n\to\infty} |a_n|^{-1} \ln(1+|a_n|) = 1$, and the rest follows from the limit ratio theorem. ∎

Note 38.3. *Conditional (not absolute) convergence of* $\sum a_n$ *and of* $\prod(1+a_n)$ *are unrelated notions. (Consider, e.g., the product* $\prod(1 - (-1)^n n^{-1/2})$*. Is the associated series* $\sum(-1)^n n^{-1/2}$ *convergent? Is the product convergent?)*

38.1. **Uniform convergence of products.**

Exercise 38.4. *Assume that* $p_n(z)$ *are analytic in the domain* $\mathcal{D}$*. Show that* $f(z) = \prod_{n\geqslant 1} p_n(z)$ *converges absolutely and uniformly on every compact set in the domain* $\mathcal{D}$ *iff*

$$f'(z) = \sum_{k=1}^{\infty} \prod_{n=1}^{\infty} \frac{p_k'}{p_k} p_n \tag{38.6}$$

converges absolutely and uniformly on every compact set in the domain $\mathcal{D}$ *(if some* p_k *are zero, the terms are understood in the sense of removable singularities).*

38.2. **Example: the** sin **function.**

Let us prove directly Euler's product formula for the sin function. A good product formula candidate is indeed

$$C\pi \sin \pi z \stackrel{?}{=} z \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \tag{38.7}$$

The constant C can only be $1/\pi^2$ if we look at the behavior near $z = 0$. Thus,

$$\frac{\sin \pi z}{\pi} \stackrel{?}{=} z \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \tag{38.8}$$

This equality of course needs to be proved, but this is not difficult.

First we note that the product on the rhs of (38.8) is absolutely and uniformly convergent on any compact z set; this can be easily checked. It thus defines an entire function $g(z)$. Motivated by the way we obtained this possible identity, let us look at the expression $f'/f - g'/g$ where $\pi f(z) = \sin \pi z$. We get, using Exercise 38.4,

$$f'/f - g'/g = \pi \cot \pi z - \frac{1}{z} + \sum_{n \in \mathbb{N} \setminus \{0\}} \frac{2z}{z^2 - n^2} = 0 \tag{38.9}$$

This means that

$$\frac{f'g - fg'}{fg} = 0 \tag{38.10}$$

in $\mathbb{C} \setminus \mathbb{Z}$ or, equivalently,

$$\frac{f'g - fg'}{g^2} = 0 = \left(\frac{f}{g}\right)' \tag{38.11}$$

or $f/g = const$; we already calculated the constant based on the behavior at zero, it is one. Thus indeed,

$$\frac{\sin \pi z}{\pi z} = \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \tag{38.12}$$

proving (37.5). More generally, we have the following:

Lemma 38.5. *Assume that $\{f_n\}$ are analytic and nonzero in a domain $\mathcal{D}$ containing zero, that $f_n(0) = 1$ and that $\sum_{k=1}^n f_k'/f_k$ converges uniformly on compact sets in $\mathcal{D}$. Then $\{\prod_{k=1}^n f_k\}_{n \in \mathbb{N}}$ converges uniformly on compact sets in $\mathcal{D}$, to a function analytic in $\mathcal{D}$.*

Proof. Under these assumptions, $h_n(z) = \int_0^z \sum_{k=1}^n f_k'(s)/f_k(s) ds$ also converges uniformly on compact sets in $\mathcal{D}$. Defining $g(z) = e^{-h_n(z)} \prod_{k=1}^n f_k$ and using Exercise 38.4 we have $g_n'(z) = 0$ and $g(0) = 1$. The rest is immediate. ∎

38.3. **Canonical products.** The simplest possible case is that in which we have a function with no zeros.

Theorem 38.6. *Assume f is entire and $f \neq 0$ in $\mathbb{C}$ Then f is of the form*

$$f = e^g \tag{38.13}$$

where g is also entire.

Proof. Since f'/f is entire and $\mathbb{C}$ is simply connected, $h(z) = \int_0^z f'(s)/f(s)ds$ is well defined and also entire. Now we note that $(fe^{-h})' = 0$ in $\mathbb{C}$ and thus $f = \exp(h + C)$ proving the result. Another proof is by using the monodromy theorem and the fact that $\log f$ has no singularities in $\mathbb{C}$. ∎

Assume now that f has finitely many zeros, a zero of order $m \geqslant 0$ at the origin, and the nonzero ones, possibly repeated are $a_1, ...a_n$.

Then

$$f = z^m \prod_{k=1}^{n} \left(1 - \frac{z}{a_n}\right) e^{g(z)}$$

where g is entire.

This is clear, since if we divide f by the prefactor of e^g we get an entire function with no zeros.

We cannot expect, in general, such a simple formula to hold if there are infinitely many zeros. Again we have to take care of convergence problems. This is done in a manner similar to that used in the Mittag-Leffler construction.

Theorem 38.7 (Weierstrass)**.** *(i) If* $\{a_n\}_{n\in\mathbb{N}}$ *is a sequence with no accumulation points, and* $\{m_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ *is a set of multiplicities then there exists an entire function with zeros at* a_n *of multiplicities* m_n *and no other zeros.*

(ii) Assume f *is an entire function with zeros at* a_n *of multiplicities* m_n*. Then there exist polynomials* q_n*, and an entire function* $g(z)$ *such that*

$$f(z) = e^{g(z)} z^m \prod_{n\in\mathbb{N}} \left[\left(1 - \frac{z}{a_n}\right)^{m_n} e^{q_n}\right] \tag{38.14}$$

Proof. This is a consequence of Mittag-Leffler. For the proof, we may assume that $f(0) \neq 0$, for otherwise, if f has a zero of order m at 0 we prove the theorem first for $\tilde{f}(z) = f(z)z^{-m}$ understood as usual after removing the singularity at zero.

Take a meromorphic function h with simple poles at a_n of residues m_n as provided by Mittag-Leffler:

$$h(z) = \sum_{n\in\mathbb{N}} \left(\frac{m_n}{z - a_n} - p_n\right) \tag{38.15}$$

We claim that $e^{\int_0^z h(s)ds}$ understood by analytic continuation from a neighborhood of zero has the desired properties. Indeed, in a neighborhood of 0we write

$$\frac{m_n}{z-a_n} = \frac{f_n'(z)}{f_n(z)} \quad \text{where} \quad f_n(z) = \left(1 - \frac{z}{a_n}\right) \tag{38.16}$$

we choose any branch of $\log(1 - z/a_n)$ (a multiple of $2\pi i$ will not matter); the natural choice is that given by the power series. We choose a disk of radius r $\mathbb{D}_r$ and choose N_r so that for $n \geqslant N_r$ $a_n \notin \mathbb{D}_r$. In $\mathbb{D}_r$ we have

(38.17)

$$\int_0^z h(s)ds = \sum_{n \leqslant N(r)} m_n \log(1 - z/a_n) - \int_0^z p_n(s)ds + \int_{n \geqslant N(r)} \left(\frac{m_n}{z-a_n} - p_n\right)$$

Once more, the integral is understood as being well defined near zero, and analytically continued to $\mathbb{D}_r$. In $\mathbb{D}_r$ we have

$$e^{\int_0^z h(s)ds} = \prod_{n \leqslant N(r)} (1 - z/a_n)^{m_n} e^{\int_{n \geqslant N(r)} \left(\frac{m_n}{s-a_n} - p_n(s)\right)ds} \tag{38.18}$$

where the exponential on the right side of (38.18) is analytic. For (ii) we note that if f is entire with the prescribed zeros and multiplicities and H is as in (i), then f/H is entire with no zeros and the result follows from Theorem 38.6. ∎

Note 38.8. *Historically, Weierstrass' factorization theorem came first. Gösta Mittag-Leffler was inspired by Weierstrass and found his decomposition theorem.*

Corollary 38.9. *Any meromorphic function is a ratio of entire functions.*

Proof. Let F be meromorphic with poles at b_n of order m_n. Let G be any entire function with zeros at b_n of order m_n. Then FG has only removable singularities. ∎

38.4. **Counting zeros of analytic functions. Jensen's formula.** The rate of growth of an analytic function is closely related to the density of zeros. A quite effective counting theorem is due to Jensen.

Theorem 38.10 (Jensen)**.** *Assume $f \not\equiv 0$ is analytic in the closed disk $\overline{\mathbb{D}_r}$ and $f(z) = cz^m g(z)$ with $m \geqslant 0$ and $g(0) = 1$. Let a_i be the nonzero roots of f in $\mathbb{D}_r$, repeated according to their multiplicity. Then*

$$\ln|c| = -m\ln r - \sum_{i=1}^{n} \ln\left(\frac{r}{|a_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta \tag{38.19}$$

Proof. The proof will follow quite easily from the case when $f \neq 0$ in $\mathbb{D}_r$. In this case we can define a consistent branch of $\ln f$ in $\mathbb{D}_r$ (see also the second proof of Theorem 38.6), and $\mathrm{Re}\ln f = \ln|f|$ is harmonic in $\mathbb{D}_r$. For $r' < r$, by the mean value theorem for harmonic functions, we have

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r'e^{i\theta})|d\theta \tag{38.20}$$

Since f is analytic in the closed disk and $\ln|x|$ is in $L^1(\mathbb{R})$, it is easy to see by dominated convergence (check) that (38.20) holds in the limit $r = r'$ too, even if there are zeros on the circle of radius r:

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta \tag{38.21}$$

Assume now f has zeros, with the convention in the statement of the theorem. We then build a function which has no zeros inside $\mathbb{D}_r$ and has the same absolute value for $|z| = r$. Such a function is

$$h(z) = \frac{r^m}{z^m} f(z) \prod_{i=1}^{n} \frac{r^2 - \overline{a_i} z}{r(a_i - z)} \tag{38.22}$$

(understood as usual with the removable singularities removed). Then

$$\ln|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|h(re^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})|d\theta \tag{38.23}$$

We have $h(0) = r^m c \prod_{i=1}^{n} \frac{r}{a_i}$ and the formula follows from the case with no roots. ∎

Corollary 38.11. *Assume f is analytic in the closed disk of radius R and $f(0) \neq 0$. Let $\nu(r)$ denote the number of zeros of f in the disk of radius $r \leqslant R$. Then*

$$\int_0^R \frac{\nu(x)}{x} dx \leqslant \ln \max_{|z|=R} |f(z)| - \ln|f(0)| \tag{38.24}$$

(Note that for some $\varepsilon > 0$, $\nu(x) = 0$ for $x < \varepsilon$.) Clearly, $\nu(x)$ is an increasing (discontinuous) function of x.

Proof. Note that

$$\ln(R/|a_i|) = \int_{|a_i|}^R \frac{dx}{x} = \int_0^R \chi([|a_i|, x])\frac{dx}{x}$$

Thus

$$\sum_{i=1}^{n} \ln\left(\frac{R}{|a_i|}\right) = \int_0^R \sum_{i=1}^{n} \chi([|a_i|, x])\frac{dx}{x} = \int_0^R \frac{\nu(x)}{x} dx$$

The rest follows immediately from (38.19). ▮

38.5. **Entire functions of finite order.** Let f be an entire function. We denote

$$\|f\|_R = \sup_{|z|\leqslant R} |f(z)| = \sup_{|z|=R} |f(z)$$

A function is of **order** $\leqslant \rho$ if for all R large enough we have

$$\forall \varepsilon > 0\ \exists c_\varepsilon \text{ s.t. } \|f\|_R \leqslant \exp(c_\varepsilon R^{\rho+\varepsilon}) \tag{38.25}$$

or equivalently

$$\forall \varepsilon > 0\ \ \ln \|f\|_R = O(R^{\rho+\varepsilon}) \tag{38.26}$$

Note 38.12. *We can always check the condition for $R \in \mathbb{N}$ large enough since $(N+1)^\rho = O(N^\rho)$.*

The function f has **strict order** $\leqslant \rho$ if there is some $c > 0$ such that for all R large enough we have

$$\|f\|_R \leqslant \exp(cR^\rho) \tag{38.27}$$

A function has **order equal** ρ if

$$\rho = \inf\{\rho' : \forall \varepsilon > 0\ \exists c_\varepsilon \text{ s.t. } \|f\|_R \leqslant \exp(c_\varepsilon R^{\rho'+\varepsilon})\}$$

A function has **strict order equal** ρ if

$$\rho = \inf\{\rho' : \forall R > 0, \|f\|_R \leqslant \exp(cR^{\rho'})\}$$

Exercise 38.13. *Check that $\cosh z^{1/2}$ has strict order $1/2$ and $\cos z^{1/4} + \cosh z^{1/4}$ has strict order $1/4$. Variations of this construction lets you find entire functions of any rational order. But can you find an entire function of exact order π?

Proposition 38.14. *Assume $f(z)$ is entire, and for large $|z|$ there are positive constants C, c and ρ such that $|f(z)| \leqslant Ce^{c|z|^\rho}$. Then, for $R > 0$ there is a $c_2 > 0$ such that*

$$\nu(R) \leqslant c_2 R^\rho \tag{38.28}$$

Indeed, the zeros at zero do not change the shape of the inequality, and we can thus assume $f(0) \neq 0$. Then,

$$c|R|^\rho \geqslant \int_0^R \frac{\nu(x)}{x} dx \geqslant \int_{R/2}^R \frac{\nu(x)}{x} dx \geqslant \frac{\nu(R/2)}{R}\frac{R}{2} \tag{38.29}$$

and the rest is immediate. The constants in the inequality can be optimized by choosing R/τ, $\tau > 1$ instead of $R/2$ and finding the best τ.

Theorem 38.15. *Let f be entire of strict order $\leqslant \rho$ and let $\{z_n\}$ be its nonzero zeros, repeated according to their multiplicity and ordered increasingly by their absolute value. Then for any $\varepsilon > 0$, $\{|z_n|^{-1}\}_{n\in\mathbb{N}} \in \ell^{\rho+\varepsilon}(\mathbb{N})$, i.e., the series*

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\rho+\varepsilon}} \tag{38.30}$$

is convergent.

Proof. We can obviously discard the roots with $|z_i| \leqslant 1$ which are in finite number. Without loss of generality we assume there are none. We have, with $N \in \mathbb{N}$ and estimating the sum by annuli,

$$\sum_{|z_n|\leqslant N} \frac{1}{|z|^{\rho+\varepsilon}} \leqslant \sum_{k=1}^{N} \frac{\nu(k+1)-\nu(k)}{k^{\rho+\varepsilon}} \tag{38.31}$$

we can now use the method of Abel summation by parts. We write

(38.32)
$$\frac{\nu(k+1)-\nu(k)}{k^{\rho+\varepsilon}} = \nu(k+1)\left(\frac{1}{k^{\rho+\varepsilon}} - \frac{1}{(k+1)^{\rho+\varepsilon}}\right) + \left(\frac{\nu(k+1)}{(k+1)^{\rho+\varepsilon}} - \frac{\nu(k)}{k^{\rho+\varepsilon}}\right)$$

and note that by summation, the terms in the last parenthesis cancel out to

$$\frac{\nu(N+1)}{(N+1)^{\rho+\varepsilon}} - \nu(1)$$

Note that by the mean value theorem we have for some $\gamma = \gamma(k)$

$$\frac{\nu(k+1)}{k^{\rho+\varepsilon}} - \frac{\nu(k+1)}{(k+1)^{\rho+\varepsilon}} = \frac{(\rho+\varepsilon)\nu(k+1)}{(k+\gamma)^{\rho+\varepsilon+1}} \leqslant \frac{Ck^{\rho}(\rho+\varepsilon)}{k^{\rho+\varepsilon+1}} \tag{38.33}$$

and the sum converges. ∎

38.6. **Estimating analytic functions by their real part.**

Theorem 38.16 (Borel-Carathéodory)**.** *Let $R > 0$ and assume $f = u + iv$ is analytic $\overline{\mathbb{D}_R}$. Let $A_R = \max_{|z|=R} u(z)$. Then for $r < R$ we have*

$$\max_{|z|\leqslant r} |f(z)| \leqslant \frac{2rA_R}{R-r} + \frac{R+r}{R-r}|f(0)| \tag{38.34}$$

Note that if f is entire and nonconstant, then $\|f\|_R \to \infty$ as $R \to \infty$. Then, since $|u| \leqslant |f|$, the theorem above shows that

$$\lim_{R\to\infty} \frac{\max_{|z|\leqslant R} u(z)}{\max_{|z|\leqslant R} |f(z)|} \geqslant \frac{1}{2} \tag{38.35}$$

Proof. Assume first that $f(0) = 0$. Then $u(0) = 0$ and by the mean value theorem $A_R \geqslant 0$. If $A_R = 0$ then by the same argument $u \equiv 0$ on $\partial\mathbb{D}_R$ and by Poisson's formula $u \equiv 0$ in $\mathbb{D}_R$. Then $v \equiv const = 0$ since $f(0) = 0$, thus $f \equiv 0$ and the formula holds trivially.

We now assume $A_R > 0$. Since the maximum of a harmonic function is reached on the boundary, we have $2A_R - u \geqslant u$ in $\mathbb{D}_R$ and the inequality is strict in the interior. Also note that if at some point $u < 0$, then again $2A_R - u = 2A_R + |u| \geqslant |u|$. In $\mathbb{D}_R$ we have

$$|2A_R - f| = \sqrt{(2A_R - u)^2 + v^2} > \sqrt{u^2 + v^2} = |f| \tag{38.36}$$

Hence the function

$$g(z) = \frac{1}{2A_R - f(z)} \frac{f(z)}{z} \tag{38.37}$$

is holomorphic in $\mathbb{D}_R$ and we have, by passing to the limit $|z| \to R$,

$$|g(z)| = \left| \frac{1}{2A_R - f(z)} \frac{f(z)}{z} \right| \leqslant \frac{1}{R} \tag{38.38}$$

hence

$$\left| \frac{f(z)}{z} \right| \leqslant \frac{1}{R} |2A_R - f(z)| \leqslant \frac{1}{R}(2A_R + |f(z)|) \tag{38.39}$$

Solving for $|f(z)|$ we get

$$|f(z)| \leqslant \frac{2|z|A_R}{R - |z|} \tag{38.40}$$

as claimed. The general case is obtained by applying this inequality to $f(z) - f(0)$ (exercise). ▮

Corollary 38.17. *Assume $\rho \geqslant 0$, $f = u + iv$ is entire and as $|z| \to \infty$ we have*

$$|u(z)| \leqslant C|z|^\rho \tag{38.41}$$

Then f is a polynomial of degree at most ρ.

Proof. From (38.35), for any $\varepsilon > 0$ we have, for large R $|f(z)| \leqslant (2C + \varepsilon)|z|^\rho$ and the rest is a simple exercise. ▮

39. Hadamard's theorem

Let $\rho > 0$ and let k_ρ be the smallest integer *strictly* greater than ρ, $k_\rho = \lfloor \rho \rfloor + 1$. We consider again the truncates of the series of $-\ln(1-z)$, namely, with $k = k_\rho$,

$$P_k(z) = z + \frac{z^2}{2} + \cdots + \frac{z^{k-1}}{k-1} \tag{39.1}$$

Theorem 39.1 (Hadamard)**.** *Let f be entire of order ρ, let z_n be its nonzero zeros and let $k = k_\rho$. Then, with $m \geqslant 0$ the order of the zero of f at zero, there is a polynomial h of degree $\leqslant \rho$ such that*

$$f(z) = e^{h(z)} z^m \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{z_n}\right) e^{P_k(z/z_n)} \right] = e^{h(z)} E(z) \tag{39.2}$$

The proof of this important theorem requires a number of intermediate results, notably the *minimum modulus theorem* proved in the following section, a very useful result in its own right.

Lemma 39.2. *Let ε be such that $\lambda := \rho + \varepsilon < k_\rho := k$. There is a $c > 0$ such that*

$$|(1-\zeta)\exp P_k(\zeta)| \leqslant \exp(c|\zeta|^\lambda) \tag{39.3}$$

Proof. For $|\zeta| \leqslant 1/2$ we have

$$\ln(1-\zeta) + P_k(\zeta) = \sum_{n=k}^{\infty} \frac{\zeta^n}{n} = \zeta^k C_k; \ |C_k| \leqslant \sum_{n=k}^{\infty} 2^{-n} \leqslant 2 \tag{39.4}$$

$$\Rightarrow (1-\zeta)e^{P_k(\zeta)} \leqslant e^{2|\zeta|^k} \leqslant e^{2|\zeta|^\lambda} \tag{39.5}$$

For $|\zeta| \in [1/2, 1]$ we have

$$\begin{aligned} |(1-\zeta)\exp P_k(\zeta)| &\leqslant \frac{1}{2} \exp\left[|\zeta|^k \left(\frac{1}{|\zeta|^{k-1}} + \cdots + \frac{1}{|\zeta|(k-1)} \right) \right] \\ &\leqslant \frac{1}{2}\exp(2^k|\zeta|^k) \leqslant \frac{1}{2}\exp(2^k|\zeta|^\lambda) \end{aligned} \tag{39.6}$$

For $|\zeta| > 1$ we have

(39.7)

$$|(1-\zeta)\exp P_k(\zeta)| \leqslant |(1-\zeta)| \exp\left[|\zeta|^{k-1} \left(\frac{1}{k-1} + \cdots + \frac{1}{|\zeta|^{k-2}} \right) \right]$$

$$\leqslant \exp\left(k|\zeta|^{k-1} + \ln|1+|\zeta||\right) \leqslant \exp\left(k|\zeta|^\lambda + \ln|1+|\zeta||\right) \leqslant \exp\left(C_2|\zeta|^\lambda\right)$$

for some C_2 independent of $\zeta, |\zeta| > 1$. This is because $t^{-\lambda}\ln(1+t)$ is continuous on $[1, \infty)$ and goes to zero at infinity (fill in the details). ∎

39.1. **Canonical products.** Take any sequence $\{z_n\}_n$ where the terms are ordered by absolute value, with the property that for some $\rho > 0$ and any $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\rho+\varepsilon}} < \infty \tag{39.8}$$

Definition 39.3. *The canonical product determined by the sequence $\{z_n\}$, denoted by $E^{(k)}(z, \{z_n\})$ or simply $E(z)$ is defined by*

$$E(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp[P_k(z/z_n)] \tag{39.9}$$

Theorem 39.4. *$E(z)$ is an entire function of order $\leqslant \rho$.*

Proof. Take again any ε be such that $\lambda := \rho + \varepsilon < k_\rho$. Then, by Lemma 39.2 we have
(39.10)

$$|E(z)| \leqslant \prod_{n=1}^{\infty} \exp(c|z/z_n|^{\lambda}) = \exp\left(c|z|^{\lambda} \sum_{k=1}^{\infty} |z_n|^{-\lambda}\right) \leqslant \exp\left(c_1|z|^{\lambda}\right)$$

proving, in the process, uniform convergence of the product. ∎

40. The minimum modulus theorem; end of proof of Theorem 39.1

This important theorem tells us, roughly, that if a function does not grow too fast it cannot decrease too quickly either, aside from zeros. More precisely we have

Theorem 40.1 (Minimum modulus theorem)**.** *Let f be an entire function of order $\leqslant \rho$. As before, let $\{z_n\}$ be its zeros with $|z_i| > 1$, repeated according to their multiplicity and let $\varepsilon > 0$. At every root, remove a disk $\mathbb{D}_{r_n}(z_n)$ with $r_n = |z_n|^{-\rho-\varepsilon}$ $U = \mathbb{C} \setminus \cup_n \mathbb{D}_{r_n}(z_n)$. Then, there is a constant c such that for all $z \in U$ we have*

$$|f(z)| \geqslant \exp(-c|z|^{\rho+\varepsilon}) \quad or \quad \frac{1}{|f(z)|} = O(\exp(|z|^{\rho+\varepsilon})) \tag{40.1}$$

Proof. We start with the case when the entire function is a canonical product. We take $|z| = r$ and write

$$E(z) = \prod_{|z_n|<2r} E_k(z, z_n) \prod_{|z_n|\geqslant 2r} E_k(z, z_n) \tag{40.2}$$

and estimate the two terms separately. We note that in the second product, all ratios $\zeta =: \zeta_n = z/z_n$ have the property $|\zeta| \leqslant 1/2$. Taking one term of the product, we have to estimate *below*

$$E(\zeta) = (1-\zeta)e^{P(\zeta)} \tag{40.3}$$

Since $|\zeta| \leqslant 1/2$, $\ln(1-\zeta)$ exists; we take the principal branch and write

$$\begin{aligned}\left|(1-\zeta)e^{P(\zeta)}\right| &= \left|e^{\ln(1-\zeta)+P(\zeta)}\right| = \left|\exp\left(-\sum_{n=k}^{\infty}\frac{\zeta^n}{n}\right)\right| \\ &= \exp\left(-\mathrm{Re}\sum_{n=k}^{\infty}\frac{\zeta^n}{n}\right) \geqslant \exp\left(-\left|\sum_{n=k}^{\infty}\frac{\zeta^n}{n}\right|\right) \geqslant \exp\left(-\sum_{n=k}^{\infty}\frac{|\zeta|^n}{n}\right) \\ &\geqslant \exp\left(-|\zeta|^k\sum_{n=0}^{\infty}\frac{(1/2)^n}{n+k}\right) \geqslant e^{-2|\zeta|^k} \geqslant e^{-2|\zeta|^{\rho+\varepsilon}}\end{aligned} \tag{40.4}$$

Thus for the second product in (40.2) we have

$$\prod_{|z_n|\geqslant 2r} E_k(z,z_n) \geqslant \exp\left(-2|z|^{\rho+\varepsilon}\sum_{|z_n|\geqslant 2r}\frac{1}{|z_n|^{\rho+\varepsilon}}\right) \geqslant e^{-c|z|^{\rho+\varepsilon}} \tag{40.5}$$

since the infinite sum converges by Theorem 38.15.

We split the remaining region into $z_k \in [1,r]$ and $z_k \in (r,2r)$. Here the factors $(1-z/z_k)$ have to be bounded from below, and it is here that we use the conditions on the removed disks. We have, on $[1,r]$,

(40.6)

$$\prod_{|z_n|\leqslant r}|1-z/z_n| = \frac{|z-z_n|}{|z_n|} \geqslant \prod_{|z_n|\leqslant r}|z_n|^{-\rho-\varepsilon-1} \geqslant \prod_{|z_n|\leqslant r} r^{-\rho-\varepsilon-1} = \left(r^{-\rho-\varepsilon-1}\right)^{\nu(r)} \gtrsim e^{-r^{\rho+\varepsilon}\ln r(\rho+\varepsilon+1)} \geqslant e^{-c_1 r^{\rho+\varepsilon'}}$$

for some C_1 and $\varepsilon' > \varepsilon$, since $(r^{\varepsilon-\varepsilon'}\ln r \to 0$ as $r \to \infty$.

On $(r,2r)$ we have

$$|1-z/z_n| = \frac{|z-z_n|}{|z_n|} \geqslant |z_n|^{-\rho-\varepsilon-1} \geqslant (2r)^{-\rho-\varepsilon-1} \tag{40.7}$$

and thus

(40.8)

$$\prod_{|z_n|<2r}|1-z/z_n| \geqslant [(2r)^{-\rho-\varepsilon-1}]^{\nu(2r)} = e^{-\nu(2r)(\rho+\varepsilon+1)\ln(2r)} \geqslant e^{-c_6 r^{\rho+\varepsilon'}}$$

for any $\varepsilon' > \varepsilon$ if r is large enough.

We now examine the convergence improving factors, for $|z_n| < 2r$.

$$\left|\sum_{|z_n|<2r} P_k(z/z_n)\right| \leqslant \left|\sum_{r<|z_n|<2r} P_k(z/z_n)\right| + \left|\sum_{|z_n|\leqslant r} P_k(z/z_n)\right| \tag{40.9}$$

For the first term on the right we note that when $|z/z_n| = |\zeta_n| =: |\zeta| < 1$ and we have

$$\left|\sum_{n=1}^{k-1} \frac{\zeta^n}{n}\right| \leqslant \sum_{n=1}^{k-1} \frac{1}{n} =: c_1 \tag{40.10}$$

and thus

$$\sum_{r<|z_n|<2r} |P_k(z/z_n)| \leqslant \nu(2r)c_1 \leqslant c_2 r^{\rho+\varepsilon} \tag{40.11}$$

For the second term on the right of (40.9) we note that $|z/z_n| \geqslant 1$ and thus, with $\zeta = z/z_n$ we have

$$\left|\sum_{n=1}^{k-1} \frac{\zeta^n}{n}\right| \leqslant |\zeta|^{k-1} \sum_{n=1}^{k-1} \frac{1}{n} =: c_1|\zeta|^{k-1} = c_1 r^{k-1}|z_n|^{-k+1} \tag{40.12}$$

We use Abel summation by parts (we are careful that r is not necessarily an integer)

$$\begin{aligned}
\sum_{|z_n|\leqslant r} |z_n|^{-k+1} &\leqslant \sum_{m\leqslant r} \frac{\nu(m+1)-\nu(m)}{m^{k-1}} \\
&= \sum_{m\leqslant r} \nu(m+1)\left(\frac{1}{m^{k-1}} - \frac{1}{(m+1)^{k-1}}\right) + \frac{\nu(r+1)}{r^{k-1}} - \nu(1) \\
&\leqslant \sum_{m\leqslant r} \nu(m+1)\left(\frac{1}{m^{k-1}} - \frac{1}{(m+1)^{k-1}}\right) + \frac{\nu(r+1)}{r^{k-1}} \\
&\leqslant \sum_{m\leqslant r} \frac{kCm^{\rho+\varepsilon}}{m^k} + c_3 r^{\rho+\varepsilon-k+1} \\
&\leqslant C_1 \sum_{m\leqslant r} m^{\rho+\varepsilon-k} + c_3 r^{\rho+\varepsilon-k+1} \leqslant C_3 r^{\rho+\varepsilon-k+1}
\end{aligned} \tag{40.13}$$

where we majorized the sum by an integral in the usual way. Multiplying by $c_1 r^{k-1}$ we get that the second term on the right of (40.9) is bounded by

$$C_4 r^{\rho+\varepsilon} \tag{40.14}$$

We now finish the proof of Theorem 39.1.

Proof. We take $\varepsilon > 0$ and $s = \rho+\varepsilon$. We order the roots nondecreasingly by $|z_n|$. For each root z_n we consider the annulus $A_n = \{z : |z| \in [|z_n| - 2|z_n|^{-s}, |z_n| + 2|z_n|^{-s}]$ (for the purpose of this argument, we can as well assume that all roots are on $\mathbb{R}^+$, since the angular position is irrelevant). Consider $J^c := \mathbb{R}^+ \setminus J$ where J is the union of all intersections of the A_n with $\mathbb{R}^+$. Since the Lebesgue measure of J does not exceed $4\sum_n |z_n|^{-s} < \infty$, there exist arbitrarily large numbers in the complement J^c. Take r be any number in J^c and consider the circle $\partial\mathbb{D}_r$. Consider the function $g = f/E$. g is clearly an entire function with no zeros. Then, by Theorem 38.6, $g = e^h$ with h entire. Since $\operatorname{Re} h \leqslant (C_f + C_E)r^{\rho+\varepsilon}$ for some $C_f + C_E$ independent of r in $\mathbb{D}_r$ for arbitrarily large r (check), we have by Corollary 38.17 that h is a polynomial of degree at most $\rho + \varepsilon$. Since ε is arbitrary, h is a polynomial of degree at most ρ. ∎

To finish the proof of the minimum modulus principle, we use Hadamard's theorem and the fact that e^{-h} satisfies the required bounds. (Exercise: fill in the details.) ∎

Example 40.2. Let us show that $f(z) = e^z - z$ has infinitely many roots in $\mathbb{C}$. Indeed, first note that $f(z)$ has order 1 since $|z| \leqslant e^{|z|}$ for all z. Suppose f had finitely many zeros. Then

$$e^z - z = P(z)e^{h(z)} \tag{40.15}$$

where $P(z)$ is a polynomial and $h(z)$ is a polynomial of degree one, and without loss of generality we can take $h(z) = cz$, $c = \alpha + i\beta$. As $z = t \to +\infty$ we have

$$P(t)e^{(c-1)t} = 1 - te^{-t} \to 1 \tag{40.16}$$

In particular $|P(t)|e^{(\alpha-1)t} \to 1$ which is only possible if $\alpha = 1$. But then $|P(t)| \to 1$ which is only possible if $P(t) = const = e^{i\phi}$. We are then left with

$$e^{i(\phi+t\beta)} = \cos(t\beta + \phi) + i\sin(t\beta + \phi) \to 1 \quad \text{as} \quad t \to +\infty \tag{40.17}$$

which clearly implies $\beta = 0$. Then $e^{i\phi} = 1$. We are left with the identity

$$e^z - z = e^z \ \forall z \tag{40.18}$$

which is obviously false.

Exercise 40.3. ** Let $P \not\equiv 0$ be a polynomial. Show that the equation $e^z = P(z)$ has infinitely many roots in $\mathbb{C}$.*

Exercise 40.4. *** (i) Rederive formula (38.12) using Hadamard's theorem.*

(ii) Write down a product formula of the function

$$f(z) = \sin z + 3\sin(3z) + 5\sin(5z) + 7\sin(7z)$$

The final formula should be explicit except for arcsins of roots of a cubic polynomial.

40.1. **Some applications.**

Corollary 40.5 (Borel)**.** *Assume that ρ is not an integer and f has strict order ρ. Then f takes every value in $\mathbb{C}$ infinitely many times.*

Proof. It suffices to show that such a function has infinitely many zeros, since f and $f - z_0$ have the same strict order. Assume to get a contradiction f had finitely many zeros. Then $g(z) = f(z)\prod_{i=1}^{n}(z - z_i)^{-1}$ would be entire, with no zeros, and as it is easy to check, of strict order ρ. Then $g = e^h$ with h a polynomial whose degree can only be an integer. ∎

Definition 40.6. *Let $\exp_n$ be the exponential composed with itself n times.*

Corollary 40.7 (A weak form of Picard's theorem)**.** *A nonconstant entire function which is bounded by $\exp_n(C|z|)$ for some n and large z takes every value with at most one exception.*

Proof. We prove this by induction on n. We first show that a nonconstant entire function of finite order takes every value with at most one exception. Assume a is an exceptional (*lacunary*) value. Then $f(z) - a$ is entire with no zeros, thus of the form e^h with h a polynomial, $f = e^h - a$. If the degree of h is zero, then f is a constant. Otherwise, we must show that $e^h - a$ takes all values with at most one exception ($-a$ of course), or, which is the same, e^h takes all values with at most one exception. The equation $e^h = b$, $b \neq 0$ is solved if $h - \ln b$ has roots, which is true by the fundamental theorem of algebra.

Assume now the property holds for $n \leqslant k-1$ and we wish to prove it for $n = k$. Let f be an entire function bounded by $\exp^{(n)}(C|z|)$ which avoids the value a. Then $f - a$ is entire with no zeros, $f - a = e^h$ with h entire. It is easy to show that h is bounded by $\exp^{(n-1)}(C|z|)$. Thus it avoids at most one value, by the induction hypothesis. The equation $e^h - a = b$, for $b \neq -a$ always has a solution. Indeed, if $\ln(b+a)$ is not an avoided value of h this is obvious. On the other hand, if $\ln(b+a)$ is avoided by h, then again by the induction hypothesis $\ln(b+a) + 2\pi i$ is not avoided. ∎

Exercise 40.8. ** Show that the equation

$$\cos(z) = z^4 + 5z^2 + 13 \tag{40.19}$$

has infinitely many roots in $\mathbb{C}$.

Exercise 40.9. ** *(Bonus)* Show that the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{40.20}$$

takes every complex value infinitely many times. (Hint: using L'Hospital show that $\operatorname{erf}(is)/(e^{s^2}/s) \to const.$ as $s \to +\infty$.)

We will return to the error function later and use asymptotic methods to locate, for large x, these special points.

41. The Phragmén-Lindelöf Theorem

Theorem 41.1 (Phragmén-Lindelöf)**.** *Let U be the open sector between two rays from the origin, forming an angle π/β, $\beta > \frac{1}{2}$. Assume f is analytic in U, and continuous in $\overline{U}$, and for some $C_1, C_2, M > 0$ and $\alpha \in (0, \beta)$ it satisfies the estimates*

$$|f(z)| \leqslant C_1 e^{C_2 |z|^\alpha}; \quad z \in U; \quad |f(z)| \leqslant M; \quad z \in \partial U \tag{41.1}$$

Then

$$|f(z)| \leqslant M; \quad z \in U \tag{41.2}$$

Proof. By a rotation we can make $U = \{z : 2|\arg z| < \pi/\beta\}$. Making a cut in the complement of U we can define an analytic branch of the log in U and, with it, an analytic branch of z^β. By taking $g = f(z^{1/\beta})$, we can assume without loss of generality that $\beta = 1$ and $\alpha \in (0, 1)$ and then $U = \{z : |\arg z| < \pi/2\}$. Let $\alpha' \in (\alpha, 1)$, $\varepsilon > 0$ and consider the analytic function

$$h(z) = e^{-\varepsilon z^{\alpha'}} f(z) \tag{41.3}$$

Since $|e^{-\varepsilon z^{\alpha'}}| < 1$ in U (check) and $|e^{-\varepsilon z^{\alpha'} + C_2 z^\alpha}| \to 0$ as $|z| \to \infty$ on the half circle $|z| = R, \operatorname{Re} z \geqslant 0$ (check), the usual maximum modulus principle shows that $|h| < M$ in $\overline{U}$. The proof is completed by taking $\varepsilon \to 0$. ∎

41.1. **An application to Laplace transforms.** We will study Laplace and inverse Laplace transforms in more detail later. For now let $F \in L^1(\mathbb{R})$. Then by Fubini and dominated convergence, the Laplace transform

$$\mathcal{L}F := \int_0^\infty e^{-px} F(p)dp \tag{41.4}$$

is well defined and continuous in x in the closed $\mathbb{H}^+$ and analytic in the open RHP. (Obviously, we could allow $Fe^{-|\alpha|p} \in L^1$ and then $\mathcal{L}F$ would be defined for Re $x > |\alpha|$.) F is uniquely defined by its Laplace transform, as seen below.

Lemma 41.2 (Uniqueness)**.** *Assume $F \in L^1(\mathbb{R}^+)$ and $\mathcal{L}F = 0$ for a set of x with an accumulation point in $\mathbb{H}^+$. Then $F = 0$ a.e. on $[0, \infty)$.*

Proof. By analyticity, $\mathcal{L}F = 0$ in the open RHP and by continuity, for $s \in \mathbb{R}$, $\mathcal{L}F(is) = 0 = \hat{\mathcal{F}}F$ where $\hat{\mathcal{F}}F$ is the Fourier transform of F (extended by zero for negative values of p). Since $F \in L^1$ and $0 = \hat{\mathcal{F}}F \in L^1$, by the known Fourier inversion formula [7], $F = 0$ a.e. ∎

More however can be said. We can draw interesting conclusions about F just from the rate of decay of $\mathcal{L}F$.

Proposition 41.3 (Lower bound on decay rates of Laplace transforms)**.** *Assume $F \in L^1(\mathbb{R}^+)$ and for some $\varepsilon > 0$ we have*

$$\mathcal{L}F(x) = O(e^{-\varepsilon x}) \quad as \quad x \to +\infty \tag{41.5}$$

Then $F = 0$ a.e. on $[0, \varepsilon]$.

Proof. We write

$$\int_0^\infty e^{-px} F(p)dp = \int_0^\varepsilon e^{-px} F(p)dp + \int_\varepsilon^\infty e^{-px} F(p)dp \tag{41.6}$$

we note that

$$\left| \int_\varepsilon^\infty e^{-px} F(p)dp \right| \leqslant e^{-\varepsilon x} \int_\varepsilon^\infty |F(p)|dp \leqslant e^{-p\varepsilon} \|F\|_1 = O(e^{-\varepsilon x}) \tag{41.7}$$

Therefore

$$g(x) = \int_0^\varepsilon e^{-px} F(p)dp = O(e^{-\varepsilon x}) \quad \text{as} \quad x \to +\infty \tag{41.8}$$

The function g is entire. Let $h(x) = e^{\varepsilon x} g(x)$. Then by assumption h is entire and uniformly bounded for $x \in \mathbb{R}$ (since by assumption, for some x_0 and all $x > x_0$ we have $h \leqslant C$ and by continuity $\max |h| < \infty$ on $[0, x_0]$). The function is also manifestly bounded for $x \in i\mathbb{R}$ (by

$\|F\|_1$). By Phragmén-Lindelöf (first applied in the first quadrant and then in the fourth quadrant, with $\beta = 2, \alpha = 1$) h is bounded in the closed RHP. Now, for $x = -s < 0$ we have

$$e^{-s\varepsilon} \int_0^\varepsilon e^{sp} F(p) dp \leqslant \int_0^\varepsilon |F(p)| \leqslant \|F\|_1 \tag{41.9}$$

Again by Phragmén-Lindelöf (again applied twice) h is bounded in the closed $\mathbb{H}_l$ thus bounded in $\mathbb{C}$, and it is therefore a constant. But, by the Riemann-Lebesgue lemma, $h \to 0$ for $x = is$ when $s \to +\infty$. Thus $h \equiv 0$. But then, with χ_A the characteristic function of A,

$$\int_0^\varepsilon F(p) e^{-isp} dp = \hat{\mathcal{F}}(\chi_{[0,\varepsilon]} F) = 0 \tag{41.10}$$

for all $s \in \mathbb{R}$ entailing the conclusion. ∎

Corollary 41.4. *Assume $F \in L^1$ and $\mathcal{L}F = O(e^{-Ax})$ as $x \to +\infty$ for all $A > 0$. Then $F = 0$.*

Proof. This is straightforward. ∎

As we see, uniqueness of the Laplace transform can be reduced to estimates. Also, no two different $L^1(\mathbb{R}^+)$ functions, real–analytic on $(0, \infty)$, can have Laplace transforms within exponentially small corrections of each–other. This will play an important role later on.

41.2. **A Laplace inversion formula.**

Theorem 41.5. *Assume $c \geqslant 0$, $f(z)$ is analytic in the closed half plane $H_c := \{z : \text{Re } z \geqslant c\}$. Assume further that $\sup_{c' \geqslant c} |f(c' + it)| \leqslant g(t)$ with $g(t) \in L^1(\mathbb{R})$. Let*

$$F(p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} f(x) dx =: (\mathcal{L}^{-1} F)(p) \tag{41.11}$$

Then for any $x \in \{z : \text{Re } z > c\}$ we have

$$\mathcal{L}F = \int_0^\infty e^{-px} F(p) dp = f(x) \tag{41.12}$$

Proof. Note that for any $x' = x_1' + iy_1' \in \{z : \text{Re } z > c\}$

(41.13)
$$\int_0^\infty dp \int_{c-i\infty}^{c+i\infty} \left| e^{p(s-x')} f(s) \right| d|s| \;\leqslant\; \int_0^\infty dp e^{p(c-x_1')} \|g\|_1 \;\leqslant\; \frac{\|g\|_1}{x_1' - c}$$

and thus, by Fubini we can interchange the orders of integration:

$$U(x') = \int_0^\infty e^{-px'} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} f(x) dx \tag{41.14}$$
$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dx f(x) \int_0^\infty dp e^{-px'+px} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(x)}{x'-x} dx$$

Since $g \in L^1$ there must exist subsequences $\tau_n, -\tau_n'$ tending to ∞ such that $|g(\tau_n)| \to 0$. Let $x' > \operatorname{Re} x = x_1$ and consider the box $B_n = \{z : \operatorname{Re} z \in [x_1, x'], \operatorname{Im} z \in [-\tau_n', \tau_n]\}$ with positive orientation. We have

$$\int_{B_n} \frac{f(s)}{x'-s} ds = -f(x') \tag{41.15}$$

while, by construction,

$$\lim_{n\to\infty} \int_{B_n} \frac{f(s)}{x'-s} ds = \int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x'-s} ds - \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{x'-s} dx \tag{41.16}$$

On the other hand, by dominated convergence, we have

$$\int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x'-s} ds \to 0 \quad \text{as} \quad x' \to \infty \tag{41.17}$$

∎

41.3. **Abstract Stokes phenomena.** This theorem shows that if an analytic function decays rapidly along some direction, then it increases "correspondingly" rapidly along a complementary direction. The following is reminiscent of a theorem by Carlson [12].

Theorem 41.6. *Assume $f \not\equiv 0$ is analytic in the closed $\mathbb{H}^+$ and that for all $a > 0$ we have $f(t) = O(e^{-at})$ for $t \in \mathbb{R}^+, t \to \infty$. Then, for all $b > 0$ the function*

$$e^{-bz} f(z) \tag{41.18}$$

is unbounded in the closed $\mathbb{H}^+$.

Proof. Assume that for some $b > 0$ we had $|e^{-bz} f(z)| < M$ in the closed RHP. Then, the function

$$\psi(z) = \frac{e^{-bz} f(z)}{(z+1)^2} \tag{41.19}$$

satisfies the assumptions of Theorem 41.5. But then $\psi(z) = \mathcal{L}\mathcal{L}^{-1}\psi(z)$ satisfies the assumptions of Corollary (41.4) and $\psi \equiv 0$. ∎

Let $\alpha > 2$.

Corollary 41.7. *Assume $f \not\equiv 0$ is analytic in the closed sector $S = \{z : 2|\arg z| \leqslant \pi/\alpha\}$, $\alpha > \frac{1}{2}$ and that $f(t) \leqslant Ce^{-t^\beta}$ with $\beta > \alpha$ for $t \in \mathbb{R}^+$. Then for any $\beta' < \beta$ there exists a subsequence $z_n \in S$ such that*

$$\left| f(z_n)e^{-z_n^{\beta'}} \right| \to \infty \quad \text{as} \quad n \to \infty \tag{41.20}$$

Proof. This follows from Theorem 41.6 by simple changes of variables. ∎

Exercise 41.8. ** Carry out the details of the preceding proof.*

42. Elliptic functions

This material is based on [3] and [11], with some simplifications and additions.

Elliptic functions are doubly periodic function, and play an important role in analysis, algebra, and number theory. We have already encountered an example of an elliptic function, sn, in §36.1.

42.1. The period module.

We have already established (see §29.1) that, for a function f to have two distinct periods ω_1 and ω_2, we must have $\omega_1/\omega_2 \notin \mathbb{R}$. If ω_1 and ω_2 are periods of f, so is, as you can check, any element of $\Lambda = \Lambda_{\omega_1,\omega_2}$, and this lattice is called *the period module* of f.

We will only analyze doubly periodic functions which are *meromorphic.*

42.2. General properties of elliptic functions.

In general, we will not necessarily assume that Λ contains all the periods of f. The equivalence relation $z_1 = z_2 \mod \Lambda$ means, naturally, that $z_1 - z_2 \in \Lambda$. Since the values of f only depend on congruence classes, we can regard it as a function defined on these congruence classes. One way to do that is to restrict f to a parallelogram P_a with vertices $a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2$ for some a. To represent all the values of f, we need to include part of the boundary of the parallelogram in P_a, for instance, two adjacent sides. The exact choice of a is immaterial, but it is convenient to choose an a such that f has no poles on the boundary.

Theorem 42.9. *An elliptic function without poles is a constant.*

Proof. Such a function is entire and, by double periodicity, bounded. ∎

By definition, poles of an analytic function are isolated, and thus there are only finitely many in a fundamental parallelogram. When we count poles or zeros of elliptic functions, we only count those in a fundamental parallelogram.

Theorem 42.10. *The sum of the residues of an elliptic function is zero.*

Proof. We choose as discussed a fundamental parallelogram P_a such that no poles lie on ∂P_a. The sum of the residues is given by

$$\frac{1}{2\pi i}\int_{\partial P_a} f(s)ds \tag{42.21}$$

This integral is zero since the integrals on opposite sides cancel each-other. ▮

Theorem 42.11. *If f is a non-constant elliptic function, then the number of poles equals the number of zeros of f.*

Proof. The function $g = f'/f$ is also meromorphic and doubly periodic, with poles of residue -1 for each pole of f and $+1$ for each zero. Now Theorem 42.11 follows from Theorem 42.10. ▮

Note 42.12. *The three theorems above are due to Liouville.*

Corollary 42.13. *Any value of f is assumed equally many times.*

Proof. If $c \in \mathbb{C}$, then $f - c$ is elliptic, and has the same poles as f. ▮

Theorem 42.14. *Let $a_1, ..., a_n$ be the zeros of the elliptic function f and $b_1, ..., b_n$ its poles in the period parallelogram. Then $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$* mod Λ.

Proof. Take a parallelogram as in Theorem 42.10 and note that

$$\sum_{i=1}^n a_i - \sum_{i=1}^n b_i = \frac{1}{2\pi i}\int_{\partial P_a} \frac{sf'(s)}{f(s)}ds \tag{42.22}$$

Note also that the values of s, and thus of the integrand, on opposite sides of the parallelogram differ by the corresponding period, ω_i of f, yielding

(42.23)
$$\frac{1}{2\pi i}\left(\int_a^{a+\omega_1} + \int_{a+\omega_1+\omega_2}^{a+\omega_2}\right)\frac{sf'(s)}{f(s)}ds = -\frac{\omega_2}{2\pi i}\int_a^{a+\omega_1}\frac{f'(s)}{f(s)}ds = -\frac{\omega_2}{2\pi i}N_1$$

where N_1, an integer, is the winding number of 0 w.r.t. the *closed* curve $f([a, a+\omega_1])$. Repeating with the other sides, we see that

$$\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = n_1\omega_1 + n_2\omega_2, \quad \text{for some } n_1, n_2 \in \mathbb{Z} \tag{42.24}$$

∎

42.3. **The Weierstrass ellipic functions.** The starting point of the Weierstrass theory are elliptic functions with one double pole per parallelogram, that we convene to place at the origin. Since multiplication by a constant does not change the theory of such functions, we choose the singular part to be just z^{-2}. The function f must be even. Indeed, $f(z) - f(-z)$ is elliptic without poles, thus a constant, and since $f(\omega_1/2) = f(-\omega_1/2)$ the constant is zero. Since adding a constant to f is also irrelevant, we choose it so that the constant term in the Laurent expansion is zero. With this choice, we have $f = \wp$, the Weierstrass elliptic function:

$$\wp(z) = z^{-2} + \sum_{k\geqslant 1} a_k z^{2k} \tag{42.25}$$

However, all of the above is contingent on the existence of such a function which we must prove:

Theorem 42.15. *The function*

$$\wp(z) = \frac{1}{z^2} + \sum_{0\neq\omega\in\Lambda_{\omega_1,\omega_2}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \tag{42.26}$$

is doubly periodic and has a Laurent expansion of the form (42.25) *at the origin.*

Proof. Until we finish the proof, we write $f(z)$ for the sum on the right side of (42.26). Let us first check convergence of the sum on compact sets away from Λ. We note that $|n_1\omega_1 + n_2\omega_2| = |\omega_1||n_1 + n_2\tau|$. Since $\tau \notin \mathbb{R}$, for fixed z and large $|n_1| + |n_2|$ we have, for some constants C', C

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| \leqslant \frac{C'|z|}{|\omega|^3} \leqslant \frac{C|z|}{(n_1^2 + n_2^2)^{3/2}} \tag{42.27}$$

Uniform and absolute convergence now follows from the integral test since

$$\int_{r\geqslant R} \int_0^{2\pi} \frac{1}{r^3} r dr d\phi$$

converges. Next we check periodicity. Due to the correction terms $-\omega^{-2}$ needed to ensure convergence, it is tedious to check this directly. Instead, we can differentiate term by term in (42.26) and get

$$f'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3} \tag{42.28}$$

Clearly f' is doubly periodic; it follows that $f(z+\omega_1)-f(z)=c_1$ and $f(z+\omega_2)-f(z)=c_2$ for some constants c_1 and c_2. Recall that f is even. Because of that, choosing $z_1=-\omega_1/2$ and $z_2=-\omega_2/2$ we see that $c_1=c_2=0$. It is easy to check that the Laurent expansion at 0 of f is of the form (42.25). ∎

42.3.1. *The Weierstrass zeta function.* Since $\wp$ has zero residues, its antiderivative is also a meromorphic function. We choose the antiderivative $-\zeta(z)$ which is *odd*. Explicitly,

$$\zeta(z) = \frac{1}{z}+\sum_{\omega\neq 0}\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right) \tag{42.29}$$

The Maclaurin series in z of the term in parenthesis, equal to the expansion in ω^{-1} as $\omega\to\infty$, is

$$\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2} = -\omega^{-1}\sum_{k\geqslant 2}(z/\omega)^k \tag{42.30}$$

from which convergence and parity follow. Summing over periods we get

$$\zeta(z) = \frac{1}{z}-\sum_{k\geqslant 2}G_k z^{2k-1} \tag{42.31}$$

where

$$G_k = \sum_{\omega\neq 0}\frac{1}{\omega^{2k}} \tag{42.32}$$

Since $\wp$ is doubly periodic, we must have $\zeta(z+\omega_i)-\zeta(z)=\eta_i$, $i=1,2$, where η_i are constants. Since ζ has one simple pole of residue one per parallelogram, we have

$$\frac{1}{2\pi i}\int_{\partial P_a}\zeta(s)ds = 2\pi i \tag{42.33}$$

By adding the contributions of the opposite sides of the parallelogram, using the relation $\zeta(z+\omega_i)-\zeta(z)=\eta_i$ and comparing with (42.33) we get *Legendre's relation*

$$\eta_1\omega_2-\eta_2\omega_1 = 2\pi i \tag{42.34}$$

42.3.2. *The differential equation of* $\wp$. Using the definition $-\zeta' = \wp$ and (42.31) (or simply the definition of $\wp$) we get

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty}(2k-1)G_k z^{2k-2} \tag{42.35}$$

From here, a calculation shows that the singular part of the combination $f = (\wp')^2 - 4\wp^3 + 60G_2\wp$ vanishes. Since f is a doubly periodic function, it must be a constant, equal to $f(0)$ which the same calculation shows it equals $-140G_3$. Writing $+60G_2 = g_2$ and $140G_3 = g_3$ we get the separable ODE

$$\wp'(z)^2 = 4\wp^3 - g_2\wp - g_3 \tag{42.36}$$

with solution

$$z - z_0 = \int_{\wp(z_0)}^{\wp(z)} \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}} \tag{42.37}$$

along a path which avoids the zeros of the denominator.

We note that the roots of the cubic are distinct. We can see this in two ways. One is to note that the integral is elementary if two or three roots coincide. If three coincide, we get $\wp(s) = 4(s+C)^{-2} + 1$ whose inverse is not periodic, and if only two coincide, we get an elementary simply periodic function. But there is a more consequential way to show and use that.

42.4. **The modular function λ.** Let e_1, e_2, e_3 be the roots of $\wp$ and write its differential equation in the form

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \tag{42.38}$$

We see that the roots of $\wp$ are zeros of $\wp'$. Since $\wp$ is periodic and even, for any period ω of $\wp$ we have $\wp(\omega - z) = \wp(z)$ which implies $\wp'(\omega - z) = -\wp'(z)$ implying that $\wp'(\omega/2) = 0$. If we take for ω the values $\omega_1, \omega_2, \omega_1 + \omega_2$ we get the three roots of $\wp'$ which lie in the fundamental parallelogram and are distinct,

$$e_1 = \wp(\omega_1/2),\ e_2 = \wp(\omega_2/2),\ e_3 = \wp(\tfrac{1}{2}(\omega_1 + \omega_2)) \tag{42.39}$$

If, say, $e_1 = e_2$ then we would have $\wp(\omega_1/2) = \wp(\omega_2/2)$; since $\omega_1/2$ is distinct from $\omega_2/2$ and $\wp$ takes each value exactly twice, this is a contradiction (otherwise $(\wp')^2$ cannot be of order 6).

Calculating e_1 from (42.26), we see that the e_i are homogeneous of degree -2 in ω_1, ω_2, that is, replacing ω_1, ω_2 by $\lambda\omega_1, \lambda\omega_2$ we have $e_i \mapsto \lambda^{-2} e_i$. Based on this, we conclude that the λ function,

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2} \tag{42.40}$$

depends only on $\tau = \omega_2/\omega_1$ as indicated.

We next note, again from (42.26) that

$$\omega_1^2 e_1 = 4 + \sum_{\omega\neq 0}\left(\frac{1}{(\frac{1}{2} - n_1 - n_2\tau)^2} - \frac{1}{(n_1 + n_2\tau)^2}\right) \tag{42.41}$$

which is analytic for $\operatorname{Im}\tau > 0$ (and also for $\operatorname{Im}\tau < 0$). The same conclusion is reached for e_2, e_3. Since $e_1 \neq e_2$ and $e_2 \neq e_3$, λ *is analytic in the UHP and does not take the values* 0 *or* 1.

42.5. **The action of the PSL$(2,\mathbb{Z})$ on λ.** Recall that PSL$(2,\mathbb{Z})$ is an automorphism of Λ, and since $\wp$ is periodic with any $\omega \in \Lambda$, if we take

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$$

and change basis to

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = M \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \tag{42.42}$$

the roots of $\wp$ cannot change; they can only be permuted. Recall also that PSL$(2,\mathbb{Z})$ is generated by $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponding to $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$ resp. Choosing $M = M_1$ in (42.42) has the effect $\frac{1}{2}\omega_2' = \frac{1}{2}(\omega_1 + \omega_2)$ and $\omega_1' = \omega_1$. This means e_2 is fixed and and e_3 and e_1 are interchanged. Choosing $M = M_2$ we get $\omega_2'/2 = -\omega_1/2$ and $\omega_1'/2 = -\omega_2/2$, in which case e_1 and e_2 are interchanged and e_3 is fixed. It follows that λ satisfies the functional equations

$$\lambda(\tau+1) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}; \quad \lambda(-1/\tau) = 1 - \lambda(\tau) \tag{42.43}$$

Both transformations on the right side are involutions. It follows that $\lambda(\tau + 2) = \lambda(\tau)$. In turn, this implies that λ is an analytic function of $e^{i\pi\tau}$.

Now, if the matrix $M \in PL(2,\mathbb{Z})$ is of the form

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \tag{42.44}$$

then

$$\lambda\left(\frac{a\tau + b}{c\tau + d}\right) = \lambda(\tau)$$

This is simply because $\omega_1'/2 = \omega_1/2 + \tilde{\omega}_1$ and $\omega_2'/2 = \omega_2/2 + \tilde{\omega}_2$ where $\tilde{\omega}_{1,2} \in \Lambda$. Hence, if $M_2 \equiv M_2 \mod 2$, then the modular transformations induced on λ by M_1 and M_2 are identical.

The subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ defined by the equivalence mod 2 is called the congruence subgroup mod 2.

Exercise 42.16. *Using* (42.43)*, check that*

(42.45)
$$\lambda\left(\frac{a\tau+b}{c\tau+d}\right)=\begin{cases}\lambda(\tau), & (a,b,c,d)\equiv(1,0,0,1) \pmod 2\\ 1-\lambda(\tau), & (a,b,c,d)\equiv(0,1,1,0) \pmod 2\\ \frac{1}{\lambda(\tau)}, & (a,b,c,d)\equiv(1,0,1,1) \pmod 2\\ \frac{1}{1-\lambda(\tau)}, & (a,b,c,d)\equiv(0,1,1,1) \pmod 2\\ \frac{\lambda(\tau)-1}{\lambda(\tau)}, & (a,b,c,d)\equiv(1,1,1,0) \pmod 2\\ \frac{\lambda(\tau)}{\lambda(\tau)-1}, & (a,b,c,d)\equiv(1,1,0,1) \pmod 2\end{cases};$$
$$\tau\in\mathbb{H} \text{ and } \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z})$$

We note that the transformations above generate the so-called **anharmonic group***:*

$$\left\{\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}\frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}\right\}$$

Recall that by the definition of e_1, e_2, e_3 *and the double periodicity of* $\wp$ *any transformation as in* (42.45) *can only permute the* e_i*. The table above can also be obtained based on this observation.*

42.6. **The conformal mapping of λ.** We can without loss of generality normalize $\omega_1=1, \omega_2=\tau$. In this case we have

$$e_3-e_2 = \sum_{m,n\in\mathbb{Z}}\left[\frac{1}{(m-\frac12+(n+\frac12)\tau)^2}-\frac{1}{(m+(n-\frac12)\tau)^2}\right]$$
(42.46)
$$e_1-e_2 = \sum_{m,n\in\mathbb{Z}}\left[\frac{1}{(m-\frac12+n\tau)^2}-\frac{1}{(m+(n-\frac12)\tau)^2}\right]$$

As discussed, these functions are analytic in the UHP. Note also that the sums <u>are even</u>, implying that *both of them are real on* $i\mathbb{R}^+$. Proposition 37.1 gives

$$\frac{\pi^2}{\sin^2\pi z}=\sum_{m\in\mathbb{Z}}\frac{1}{(z-m)^2}$$

and we see that

$$e_3 - e_2 = \pi^2 \sum_{n\in\mathbb{Z}} \left(\frac{1}{\cos^2 \pi(n-\frac12)\tau} - \frac{1}{\sin^2 \pi(n-\frac12)\tau} \right)$$

(42.47)

$$e_1 - e_2 = \pi^2 \sum_{n\in\mathbb{Z}} \left(\frac{1}{\cos^2 \pi n\tau} - \frac{1}{\sin^2 \pi(n-\frac12)\tau} \right)$$

which converge uniformly in $\{z : \operatorname{Im} z \geqslant \delta > 0\}$. This implies that λ is real on $i\mathbb{R}^+$.

Theorem 42.17. *The function λ is a conformal map between the domain Ω bounded by the lines $i[0,\infty)$, the circle of radius $\frac12$ centered at $\frac12$ and the line $1+i[0,\infty)$ and the UHP.*

Proof. We first follow the mapping of the boundary of Ω and then apply the argument principle. Dominated convergence shows that, as $\operatorname{Im}\tau \to \infty$, $e_3 - e_2 \to 0$ and $e_1 - e_2 \to \pi^2$ (the latter due to the term with $n = 0$). We have $\lambda(\tau) \to 0$ as $\operatorname{Im}\tau \to \infty$ uniformly in $\operatorname{Re}\tau$. More precisely, from (42.47) we see that

$$e_3 - e_2 = 2\pi^2 \left[\frac{4e^{\pi i\tau}}{(1+e^{\pi i\tau})^2} + \frac{4e^{\pi i\tau}}{(1-e^{\pi i\tau})^2} \right] (1 + h(\tau)) \tag{42.48}$$

where $h \to 0$ as $\operatorname{Im}\tau \to \infty$. Hence

$$\lambda(\tau)e^{-i\pi\tau} \to 16 \quad \text{as} \quad \operatorname{Im}\tau \to +\infty \tag{42.49}$$

From (42.43), $\lambda(-1/\tau) = 1 - \lambda(\tau)$, we see that $\lambda \to 1$ as $\tau \to 0$ along $i\mathbb{R}^+$; hence 0 is mapped to 1.

The map $\frac{\tau}{\tau+1} \in Aut(\mathbb{H}^u)$ and takes $i[0,\infty)$ to the half circle in Fig. 31.

We have $\lambda(\frac{\tau}{\tau+1}) = \frac{1}{\lambda(\tau)}$. This implies that 0 is mapped to 1, as before and, as $\tau \to i\infty$ we have $\frac{\tau}{\tau+1} \to 1$, hence 1 is mapped to $+\infty$. Finally, we have

$$\lambda(1+it) = \frac{\lambda(it)}{\lambda(it)-1} = -\frac{1}{\lambda(\frac{i}{t})} \tag{42.50}$$

This means that $1 + i0^+$ is mapped to $-\infty$ and $1 + i\infty$ is mapped to 0. Hence the boundary of Ω is mapped to $\mathbb{R}$.

Now we use the argument principle and Note 18.74. By (42.50) λ takes any small value only once. Indeed, we cannot have $\lambda \to 0$ inside Ω since $\lambda \neq 0$. Near infinity λ is clearly injective, and the only point on $\partial\Omega$ where 0 is taken is at ∞. Hence, by Note 18.74 every value in the UHP is taken exactly once. The same reasoning shows that λ

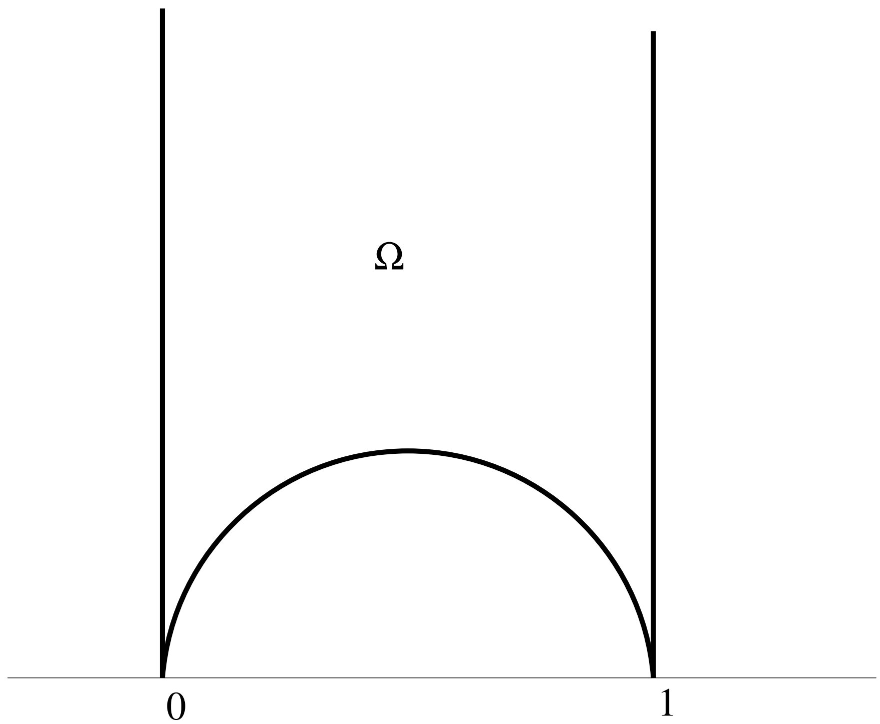


FIGURE 31. The domain Ω

takes no value in the LHP: Indeed, small values can only be taken if $\operatorname{Im} \tau \to +\infty$ but there λ is in the UHP. ∎

43. The uniformization theorem

Theorem 43.18. *Every simply connected Riemann surface M is conformally equivalent to one of three Riemann surfaces: $\mathbb{D}$, $\mathbb{C}$, or $\hat{\mathbb{C}}$.*

(see [8]). Recall the hyperbolic plane. Since the UHP is conformally equivalent through a Cayley transform to $\mathbb{D}$, $\mathbb{D}$ also has a metric of constant negative curvature. The conformal map in Theorem 43.18 induces a constant curvature metric on any simply connected Riemann surface M.

If M is compact, then its *universal cover* is $\mathbb{D}$ iff it is a hyperbolic surface of genus greater than 1 with non-abelian fundamental group; its universal cover is $\mathbb{C}$ iff it has genus 1: the complex tori and the elliptic curves with fundamental group $\mathbb{Z}_2$; the universal cover is the Riemann sphere iff M has genus zero, meaning M is the Riemann sphere itself, with trivial fundamental group. Recall also that the only functions analytic on $\hat{\mathbb{C}}$ are rational functions. For a rational function to be injective, it has to be a LFT, and these belong to $\mathrm{Aut}(\hat{\mathbb{C}})$.

If M is not simply connected , we have the following generalization (see [5], with a proof in [8])

Theorem 43.19. *Let M be a non-simply connected Riemann surface. Then, the following is a complete list of isomorphic classes:*

(A) M is conformally isomorphic to the punctured plane, or compact and of genus 1, and isomorphic to $\mathbb{C}/\Lambda_{1,\tau}$, with $\operatorname{Im}\tau > 0$.

(B) M is conformally isomorphic to $\mathbb{D}/G$ where $G \in Aut(\mathbb{D})$ acts freely and properly discontinuously on $\mathbb{D}$ and apart from the identity no LFT in G has fixed points in $\mathbb{D}$; furthermore, G is isomorphic to $\pi_1(M)$ as groups.

Exercise 43.20 (Ahlfors)**.** *Show that the Klein $j-$invariant*

$$j(\tau) = 256\frac{(1-x)^3}{x^2}$$

where $x = \lambda(1-\lambda)$ is invariant under the modular group.

43.1. **An example: M, the universal cover of $\hat{\mathbb{C}}\backslash\{0, 1, \infty\}$.** We use the function λ to illustrate the uniformization of this Riemann surface on the upper half plane; of course, composition with a Cayley transform achieves uniformization on $\mathbb{D}$. The fundamental group $\pi_1(M)$ is the free group with two generators, a_0, a_1. We think of M in terms of equivalence classes of curves over $\mathbb{C} \setminus \{0, 1\}$, winding around $0, 1$ in the way prescribed by the associated word, e.g. $a_0 a_1 a_0^{-1} a_1, \ldots$

Exercise 43.21. *Show that a Schwarz reflection of a line in a circle is a line or a circle. Show that reflecting Ω successively across boundary arcs as shown in Fig. 33 covers the UHP.*

First we note that each domain obtained by Schwarz reflections as shown in Fig. 33 is mapped by λ conformally, alternatively, to the UHP and LHP. Since the $1 + i\mathbb{R}^+$ is mapped to $(-\infty, 0)$, it means that a reflection of Ω itself across its right boundary is a reflection of the UHP across the segment $(-\infty, 0)$. Likewise, a reflection across the left boundary corresponds to reflecting the UHP across $(0, 1)$ and reflection across the circle is a reflection of the UHP across $(1, \infty)$. Clearly, all these are analytic arcs, and it means that $\psi = \lambda^{-1}$ has analytic continuation through these arcs. Continuing all these reflections, we see that ψ has analytic continuation along any curve in the universal cover, and has singularities only at $\{0, 1, \infty\}$. Hence:

Proposition 43.22. *ψ has analytic continuation on M.*

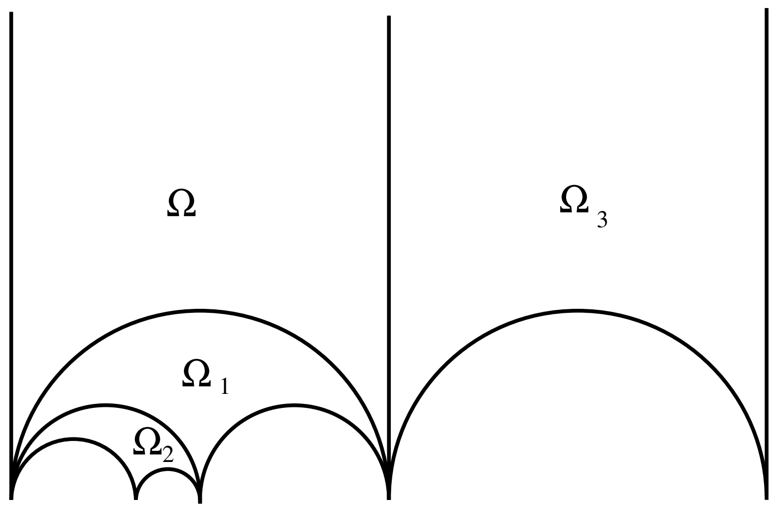


FIGURE 32. Successive reflections

Furthermore, the word *abc...* corresponds uniquely to a set of successive reflections in the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. Indeed, we see that two different words $abcd, ...$ and $abcD...$ correspond to reflections across distinct arcs in the figure (here the difference is in the fact that d and D correspond to reflections across different arcs). Hence, two different points on M are mapped by ψ to different points in the upper half plane. Using the result in Exercise 43.21 we see that

Proposition 43.23. *ψ a conformal map from M to the UHP.*

Exercise 43.24. *Clearly, λ is conformal from the UHP to M. Show that $\mathbb{R}$ is a natural boundary for λ. Use Exercises 42.16 and 29.21 to give a different proof of this fact.*

44. The little Picard theorem

Theorem 44.25 (Little Picard theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is entire and there are two points $a \neq b$ which are not in the range of f, then f is a constant.*

Points that are not in the range of f are called *lacunary values.*

Proof. Assume that f is entire and has two lacunary values. Without loss of generality, we can assume that these are 0 and 1. Let $h = C \circ \psi \circ f$ where C is the Cayley transform and ψ is as in Proposition 43.23. Since the only singularities of ψ are at 0 and 1, ψ takes M to the UHP and C is analytic in the UHP, h is entire with values in $\mathbb{D}$, meaning h, hence f are constants. ∎

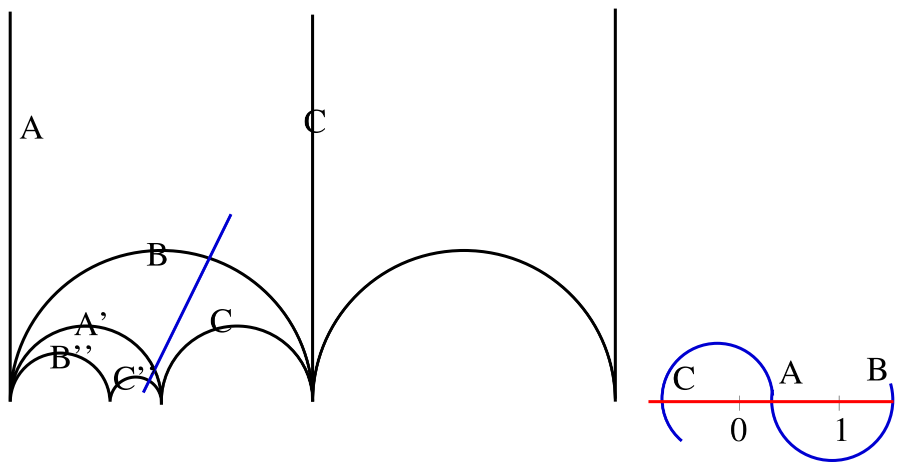


FIGURE 33. A winding curve in $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ becomes an open curve in the UHP.

Theorem 44.26 (Montel's fundamental normality test)**.** *Let $\mathcal{D}$ be a domain in $\mathbb{C}$, let α and β be two distinct complex numbers, and let $\mathcal{F}$ be the set of holomorphic functions in $\mathcal{D}$ whose range omits the two values α and β. Then $\mathcal{F}$ is a normal family in $\hat{\mathbb{C}}$.*

A stronger version, using the uniformization theorem is ([5])

Theorem 44.27. *Let M be a Riemann surface and let $\mathcal{F}$ be the set of holomorphic functions on M with values in $\hat{\mathbb{C}}$ whose range omits the three values α β and γ. Then $\mathcal{F}$ is a normal family in $\hat{\mathbb{C}}$.*

Proof of Theorem 44.26. We first arrange the following: $\alpha = 0, \beta = 1$, and, since normality is a local property, that $\mathcal{D} = \mathbb{D}$.

With $\psi = \lambda^{-1}$, let $\eta = C \circ \psi$ where C is the Cayley transform; the function η is a conformal map from M, the universal cover of $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ to $\mathbb{D}$.

Let $f \in \mathcal{F}$ and $a \in \mathbb{D}$ and choose any branch of $\eta(f(a))$. We note, as in the uniformization proof, that for that branch, $\tilde{f} = \eta \circ f$ defined by analytic continuation from a to $\mathbb{D}$ is analytic in $\mathbb{D}$.

To construct a convergent subsequence of a given sequence $\{f_n \in \mathcal{F}\}_{n\in\mathbb{N}}$ we first note that for any $a \in \mathbb{D}$, there is a subsequence of $\{f_n(a)\}$ which converges in $\hat{\mathbb{C}}$, which, by passing to this subsequence, we can assume is $\{f_n(a)\}_{n\in\mathbb{N}}$ itself. If for all a and any subsequence the limit is infinity, then f_n converges to the constant $\infty \in \mathbb{C}$ proving the result. Otherwise, there is a point, say $a = 0$ s.t. a subsequence of $\{f_n(0)\}_{n\in\mathbb{N}}$ converges to $\ell \in \mathbb{C}$. We assume first that $\ell \notin \{0, 1\}$.

Note now that $\tilde{f}_n, n \in \mathbb{N}$ have range in $\overline{\mathbb{D}}$, hence they are a normal family and by Theorem 31.7, they have a convergent subsequence, that by passing again to a subsequence, we can assume is $\{\tilde{f}_n\}_{n\in\mathbb{N}}$; let the limit be g. If g is a constant, then $g = \eta(\ell) \in \mathbb{D}$. Otherwise, since $\operatorname{ran}(g) \subset \overline{\mathbb{D}}$, by the open mapping theorem, $\operatorname{ran}(g) \subset \mathbb{D}$. Now we note that $\eta^{-1} \circ g$ is well-defined, and that $\lim_{n\to\infty} f_n = \lim_{n\to\infty} \eta^{-1}\tilde{f}_n = \eta^{-1} \circ g$, which finishes the proof it $\ell \notin \{0, 1\}$.

The remaining case is $\ell = 1$ since, if $\ell = 0$, then we can take the family $1 - f, f \in \mathcal{F}$. Since $f_n \neq 0$, there is a *holomorphic* branch of the square root such that $h_n = \sqrt{f_n} \to -1$. A holomorphic branch of the square root cannot take both values 1 and -1, so h_n also avoid 1 and -1. For h_n the proof above applies to extract a convergent subsequence, which by squaring gives the desired subsequence of f_n. ∎

Note 44.28. The fundamental normality test plays an important role in complex dynamics, see [5].

We are now going to prove the celebrated Great Picard's theorem:

Theorem 44.29. *If an analytic function f has an essential singularity at a point z_0, then on any punctured neighborhood of z_0, f takes on all possible complex values, with at most a single exception, infinitely often.*

Proof. Without loss of generality, we take $z_0 = 0$, assume that f is analytic in $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ and assume, to get a contradiction, that f omits the values 0 and 1. Let $\{z_n\}_{n\in\mathbb{N}}$ be any sequence converging to 0 and $f_n(z) = f(z_n z)$. By Theorem 44.26 the family $\{f_n\}_{n\in\mathbb{N}}$ is normal. Hence we can extract a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ convergent on compact sets in $\mathbb{D}_r \setminus \{0\}$ to, say g which is analytic in $\mathbb{D}_r \setminus \{0\}$ with values in $\hat{\mathbb{C}}$. Assume that g is not the constant $\infty \in \hat{\mathbb{C}}$. Note that, for s in the annulus between the circle of radius $12|z_{n_k}|$ and the circle of radius $\frac{1}{4}|z_n|$, we have $f(s) - g(s') \to 0$ as $n \to \infty$ and s' is in the annulus between the circle of radius $\frac{1}{2}$ and the circle of radius $\frac{1}{4}$. Hence f is bounded along any sequence converging to zero, contradiction. If $g = \infty$ then $1/f$ has a removable singularity at 0, thus f has a pole at zero, again a contradiction. ∎

45. Riemann-Hilbert problems: an introduction

An impressive number of problems coming from integrable world can be reduced to so-called Riemann Hilbert problems, and for many of them the only known way to get a closed form solution is via the associated Riemann-Hilbert problem.

Problems which can be solved with R-H techniques include

(1) integrable models such as the transcendental Painlevé equations e.g. $y'' = 6y^2 + x$ (P_I) and many others;

(2) integrable PDEs s.a. nonlinear initial value problem for the KdV (Korteweg–deVries) equation

(45.1)

$$u_t+u_{xxx}+uu_x = 0;\;\; u(x,0) = u_0(x),\; u \to 0 \text{ as } |x| \to \infty \;\; (x \in \mathbb{R}, t \in \mathbb{R}^+)$$

(3) inverse scattering problems: find, from the scattering data the potential $q(x)$ in the time–independent Schrödinger equation

$$-\psi_{xx} + (k^2 + q(x))\psi = 0 \tag{45.2}$$

(4) questions in orthogonal polynomials, random matrices, and combinatorial probability;

(5) integral equations of the type

$$f(t) + \int_0^\infty \alpha(t-t')f(t')dt' = \beta(t) \tag{45.3}$$

(under suitable integrability conditions);

(6) finding the inverse Radon transform, a transform which is measured in tomography.

45.1. **A simple Riemann-Hilbert problem.** Perhaps the simplest R-H problem is: given a simple smooth contour $\mathcal{C}$ and $f(t)$ a suitably regular function on $\mathcal{C}$, find analytic functions Φ^+, Φ^-, defined to the left and right of $\mathcal{C}$ such that the limits of $\Phi^\pm$ on $\mathcal{C}$ exist and satisfy

$$\Phi^+(z) - \Phi^-(z) = f(z) \tag{45.4}$$

Note 45.1. Φ *is clearly determined up to an entire function. To hope for a unique solution one needs to impose more conditions on* Φ*, such as behavior at infinity and other special points.*

46. Cauchy type integrals

We recall that a function is Hölder continuous of order β on a smooth curve $\mathcal{C}$ if

$$\exists\, \beta > 0 \text{ and } C > 0 \text{ s.t. } \forall x, y \in \mathcal{C},\; |f(x) - f(y)| \leqslant C|x-y|^\beta \tag{46.1}$$

The condition implies continuity if $\beta > 0$ and it is nontrivial if $\beta \leqslant 1$ (if $\beta > 1$ then $df/ds = 0$).

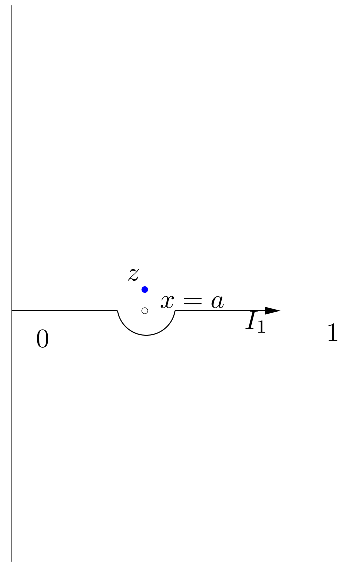


FIGURE 34. Analyticity of Φ when ϕ is analytic in a neighborhood of $\mathcal{C}$.

Let $\mathcal{C}$ for now be a compact smooth curve and ϕ be Hölder continuous on $\mathcal{C}$. Then the function

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\phi(s)}{s-z}ds \tag{46.2}$$

is manifestly analytic for $z \notin \mathcal{C}$ (you can check this by Morera's theorem using Fubini, and in a good number of other ways).

Revisit §10.2 and Definition 32.10.

46.1. Asymptotic behavior of $\Phi(z)$ for large z.

Exercise 46.1. *Assume $\mathcal{C}$ is compact. Show that $\Phi(z)$ is* analytic at infinity *in z and that*

$$\Phi(z) = -\left(\frac{1}{2\pi i}\int_{\mathcal{C}} \phi(s)ds\right)\ \frac{1}{z}\left(1 + g(1/z)\right) \quad \text{as} \quad z \to \infty \tag{46.3}$$

as where $g(1/z) \to 0$ as $z \to \infty$.

46.2. Regularity and singularities.

Let us first take a simple example, in which ϕ is analytic in a neighborhood $\mathcal{N}$ of $(0,1)$ (note that this does not exclude multi-valued functions singular at zero and one s.a. $\ln z(1-z)$).

Then Φ has analytic continuation from $\mathbb{H}^u$ and $\mathbb{H}_l$ through $\mathcal{C}\setminus\{c_i\}$ in $\mathcal{N}$, where c_i are endpoints of $\mathcal{C}$ (if any). (These analytic continuations are in general different.) Indeed, to perform analytic continuation in $z \in \mathbb{H}^u \cup (\mathcal{N}\setminus\{c_i\})$, we can first deform the contour as shown in deform as shown in Fig. 34. To calculate the values of Φ on $\mathcal{C}$, we can use Exercise 10.49 to see that $A(a) = \pm\frac{1}{2}\phi(a) + \frac{1}{2\pi i}PV\int_0^1 \phi(s)(s-t)^{-1}ds$.

The same, interestingly, holds more generally:

Theorem 46.2 (Plemelj's formulas)**.** *Assume ϕ is Hölder continuous of exponent β on the simple smooth curve $\mathcal{C}$. let t be an interior point of $\mathcal{C}$. Then,*

(i)

$$PV\int \frac{\phi(s)}{s-t}ds \tag{46.4}$$

(cf. Definition 10.50) exists.

(ii) Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence approaching t from the left (right). If $\mathcal{C}$ is not bounded, assume also that $\phi \in L^1(\mathcal{C})$. Then, with the $\pm$ sign being $+$ for left limit and $-$ for right limit,

$$\lim_{n\to\infty} \Phi(z_n) = \Phi^{\pm}(t) \tag{46.5}$$

where

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\phi(t) + \frac{1}{2\pi i}PV\int \frac{\phi(s)}{s-t}ds \tag{46.6}$$

and

Note 46.3. *(i) The properties in the theorem are local; it is enough to prove them for compact pieces of $\mathcal{C}$.*

(ii) A similar statement can be obviously made when t is approached along a curve, since all limits along subsequences coincide.

We leave it as an exercise to extend the proof from the case when $\mathcal{C}$ is a piece of $\mathbb{R}$, say $[a,b]$ with $t = 0 \in (a,b)$, approached from above, to a more general smooth bounded curve (open or not): parametrize the curve.

Proof. We can assume without loss of generality that $0 \in [a,b]$, $t = 0$ and $|a| > |b|$.

(i) We write $\phi(s) = (\phi(s) - \phi(0)) + \phi(0)$ and show that both PV integrals exist. By symmetry,

$$PV\int_{-b}^{b} s^{-1}\phi(0)ds = 0 \tag{46.7}$$

implying

$$PV\int_a^b \frac{\phi(0)}{s}ds = \int_a^{-b} \frac{\phi(0)}{s}ds \tag{46.8}$$

We then note that $|\phi(s) - \phi(0)|/|s| = (|\phi(s) - \phi(0)|s|^{-\beta})|s|^{\beta-1} \in L^1([a,b])$, hence

$$PV\int_a^b \frac{\phi(s)-\phi(0)}{s}ds = \int_a^b \frac{\phi(s)-\phi(0)}{s}ds \tag{46.9}$$

(ii) Take $\varepsilon > 0$ small enough, n large enough, and $0 < c \in (a,b)$, change variable to $u = s - x_n$ and break the interval of integration into $[a - x_n, -c], [-c, c]$ and $[c, b - x_n]$.

$$\lim_{n\to\infty}\left(\int_{a-x_n}^{-c} + \int_c^{b-x_n}\right)\frac{\phi(u+x_n)du}{u-iy_n} = \left(\int_a^{-c} + \int_c^b\right)\frac{\phi(u)du}{u} \tag{46.10}$$

We decompose the integral on $[-c, c]$ into

$$\int_{-c}^{c}\frac{\phi(u+x_n)-\phi(x_n)}{|u|^\beta}\frac{|u|^\beta du}{u-iy_n} + \phi(x_n)\int_C \frac{du}{u} \tag{46.11}$$

where in the last integral we homotopically deformed the contour into a half-circle of radius c in the LHP centered at zero. For all n, the integrand in the first integral is bounded in absolute value, up to a constant, by $|u|^{\beta-1} \in L^1([-c,c])$ while the second integral evaluates to πi. The result now follows by using dominated convergence and the symmetry argument in (i). ∎

The following result follows immediately.

Theorem 46.4 (Existence)**.** *Under the conditions of Theorem 46.2, the function in* (46.2) *solves the Riemann-Hilbert problem in §45.1.*

Note 46.5. *The Hölder condition can be replaced with the weaker Dini condition, with the same proof.*

Note 46.6. (a) The function defined by the Cauchy type integral (46.2) is called **sectionally analytic**. With the convention about the sides of the curve mentioned before, functions that are boundary values of Cauchy type integrals are sometimes denoted $\oplus$ and $\ominus$ functions. respectively.

(b) Branch jumps of analytic functions are an essential ingredient in Sato's theory of hyperfunctions.

46.3. **Examples.**

46.3.1. *A very simple example.* Find a function Φ analytic in $\mathbb{C} \setminus \partial\mathbb{D}$ such that along $\partial\mathbb{D}$ we have

$$\Phi^+(t) - \Phi^-(t) = 1 \tag{46.12}$$

Note that for this problem the set of analyticity of Φ is a union of two disjoint domains. Since Φ is discontinuous across $\partial\mathbb{D}$, we are dealing with two separate analytic functions. Plemelj's formula reads

$$\Phi(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{1}{s-z} ds \tag{46.13}$$

Clearly, if $z \in \mathbb{D}$ (which is to the left of $\partial\mathbb{D}$ oriented positively), we have

$$\Phi_{\text{in}}(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{1}{s-z} ds = 1 \tag{46.14}$$

Likewise, if z is outside $\mathbb{D}$ we have

$$\Phi_{\text{out}}(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{1}{s-z} ds = 0 \tag{46.15}$$

Both Φ_{in} and Φ_{out} are analytic, but not analytic continuations of each-other, so in this case our sectionally analytic function is really a pair of distinct analytic functions. We leave the question of uniqueness to the next subsection when the contour is open and which leads to a more interesting discussion.

46.3.2. *Another simple example.* Find a function Φ analytic in $\mathbb{C} \setminus [-1, 1]$ such the upper and lower limits across $[-1, 1]$ satisfy

$$\Phi^+(z) - \Phi^-(z) = 1 \tag{46.16}$$

46.3.3. *A solution.* According to Plemelj's formulas a function satisfying (46.16) is given by

$$\Phi(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{1}{s-z} ds \tag{46.17}$$

This formula shows Φ is analytic on the universal cover of $\hat{\mathbb{C}} \setminus \{-1, 1\}$ (Φ is analytic at infinity). We get explicitly

$$\Phi(z) = -\frac{1}{2\pi i} \left(\ln(z-1) - \ln(z+1) \right) \tag{46.18}$$

where the log is determined by the condition $\Phi(z) < 0$ for $z > 1$. It is useful to check directly on (46.18) that $\Phi^+ - \Phi^- = 1$ on $(-1, 1)$

FIGURE 35. Contour used for applying Morera's theorem.

46.3.4. *Calculating principal value integrals.* Plemelj's formulas help us calculate principal values integrals as well, sometimes in a simpler way. Let $\mathcal{C}$ be a simple smooth closed curve and assume that $f(z)$ is analytic in $\mathrm{Int}(\mathcal{C})$ and Hölder continuous in the closure of $\mathrm{Int}(\mathcal{C})$. Then,

$$\frac{1}{2\pi i} PV \int_{\mathcal{C}} \frac{f(s)}{s-t} ds = \frac{1}{2} f(t) \tag{46.19}$$

a "limiting case" of a Cauchy formula.

46.3.5. *Uniqueness issues.* What obvious freedoms do we have in solving the problem in §46.3.2? As mentioned, we can add to Φ any entire function. More generally, we can add any function analytic in $\mathbb{C} \setminus \{-1, 1\}$ (note that this means single-valued). The following theorem shows that these are all the freedoms.

Theorem 46.7 (Uniqueness)**.** *Consider the problem in §45.1 with the following further conditions:*

$$\Phi(z) \to 0 \text{ as } |z| \to \infty; \tag{1}$$

$$(1 \mp z)\Phi(z) \to 0 \text{ as } z \to \pm 1 \tag{2}$$

Then the solution is unique, namely (46.2)*.*

Proof. The Φ in (46.2) satisfies this condition, as it is easy to verify. Assume Φ_1 is another solution with the same properties. Then $f = \Phi - \Phi_1$ is analytic in $\mathbb{C} \setminus [-1, 1]$ and continuous on $(-1, 1)$, entailing continuity in $\mathbb{C} \setminus \{-1, 1\}$. Show that the contour integral of $\Phi - \Phi_1$ on a circle of radius ε around 1 and -1 is zero, hence it vanishes on any closed contour in $\mathbb{C}$ (see Fig. 35) implying, by Morera, that $\Phi - \Phi_1$ is entire. Since $\lim_{z\to\infty}(\Phi(z) - \Phi_1(z)) = 0$, $\Phi - \Phi_1 = 0$. ∎

Exercise 46.8. Check using Plemelj's formulas that any simple smooth curve $\mathcal{C}$ in $\mathbb{C}$ is the natural boundary of a large class analytic of functions, and furthermore the limit of such functions on $\mathcal{C}$ can be as smooth as we want, short of analytic. Use the Riemann mapping theorem to show that any simple Jordan curve is a natural boundary of a large class of analytic functions.

We see from

46.3.6. *Degree of a function at infinity.* By definition Φ has degree k at infinity if for some $C \neq 0$ we have

$$\Phi(z) = Cz^k + O(z^{k-1}) \quad \text{as} \quad z \to \infty \tag{46.20}$$

The function Φ has finite degree at infinity if $\Phi = o(z^m)$ for some m.

47. More general scalar R-H problems

47.1. **Scalar homogeneous R-H problems.**

47.1.1. *Scalar homogeneous R-H problems.* This is a problem of the type

$$\Phi^+ = g\Phi^- \quad \text{on} \quad \mathcal{C} \tag{47.1}$$

where $\mathcal{C}$ is a smooth simple closed contour, g nonzero on $\mathcal{C}$ and satisfying a Hölder condition on $\mathcal{C}$. We are looking for solutions of degree m at infinity.

Solution to (47.1) First we note that if Φ is a solution and H is entire, then by homogeneity ΦH is also a solution.

Assume that the *index of g w.r.t.* $\mathcal{C}$ is k. Without loss of generality we assume $0 \in$Int$(\mathcal{C})$. We can rewrite the problem as

$$\Phi^+(t) = (t^{-k}g(t))(t^k\Phi^-) \quad \text{on} \quad \mathcal{C} \tag{47.2}$$

Formally for now, taking the log of both sides, we get

$$\ln\Phi^+(t) = \ln(t^{-k}g(t)) + \ln(t^k\Phi^-) \quad \text{on} \quad \mathcal{C} \tag{47.3}$$

or finally, with obvious notation,

$$\Gamma^+(t) = f(t) + \Gamma^-(t) \quad \text{on} \quad \mathcal{C} \tag{47.4}$$

The reason we formed the combination $t^{-k}g(t)$, where we choose k to be the index of ϕ w.r.t $\mathcal{C}$, is to ensure Hölder continuity of f. Otherwise, since $\arg\phi$ changes by $2k\pi$ upon traversing $\mathcal{C}$, f would have a jump discontinuity somewhere on $\mathcal{C}$.

A solution to (47.4) is given by Plemelj's formulas:

$$\Gamma(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(s)}{s-z}ds \tag{47.5}$$

The exterior of $\mathcal{C}$ corresponds to the negative sign and we have $\Gamma^- = O(z^{-1})$ for large z, and thus $\exp(\Gamma^-) = z^k\Phi^- = 1 + o(1)$ for large z, hence $\Phi^- = z^{-k} + o(z^{-k})$. We finally get a solution of degree m at infinity by multiplying by P_{m+k}, a polynomial of degree $m+k$. The solution of (47.1) is then

$$\Phi = \begin{cases} P_{m+k}(z)\exp(\Gamma(z)), & z \text{ inside } \mathcal{C} \\ z^{-k}P_{m+k}(z)\exp(\Gamma(z)), & z \text{ outside } \mathcal{C} \end{cases} \tag{47.6}$$

We will not, for reasons of space, discuss uniqueness issues here.

47.1.2. *Ingomogeneous R-H problems.* These are equations of the form

$$\Phi^+ = g\Phi^- + f \tag{47.7}$$

Where f and g are Hölder continuous. These can be brought to Plemelj's formulas in the following way. We first solve the homogeneous problem

$$\Psi^+ = g\Psi^- \tag{47.8}$$

which we dealt with in §47.1.1, and look for a solution of (47.7) in the form $\Phi = X\Psi$. We get
(47.9)

$$X^+\Psi^+ = gX^-\Psi^- + f \Rightarrow X^-g\Psi^+ = X^-g\Psi^- + f \Rightarrow \Psi^+ - \Psi^- = \frac{f}{gX^-}$$

which is of the form we already solved.

47.2. **Applications.**

47.2.1. *Ingomogeneous singular integral equations.* These are equations of the form

$$a(t)\phi(t) + b(t)PV\int_{\mathcal{C}} \frac{\phi(s)}{s-z}ds = c(t) \tag{47.10}$$

with a, b, c Hölder continuous and the further condition $i\pi a(t) \pm b(t) \neq 0$. We attempt to write, guided by Plemelj's formulas

$$\phi(t) = \Phi^+(t) - \Phi^-(t) \tag{47.11}$$

and

$$\Phi(z) = \frac{1}{2\pi i}\int \frac{\phi(s)}{s-z}ds \tag{47.12}$$

and then

$$PV\int_{\mathcal{C}} \frac{\phi(s)}{s-t}ds = i\pi\Big[\Phi^+(t) + \Phi^-(t)\Big] \tag{47.13}$$

where, of course ϕ is still unknown. The equation becomes

$$a(t)\Big[\Phi^+(t) - \Phi^-(t)\Big] + b(t)i\pi\Big[\Phi^+(t) + \Phi^-(t)\Big] = c(t) \tag{47.14}$$

or

$$\Phi^+(t)(a(t)+b(t)i\pi)+\Phi^-(t)(b(t)i\pi-a(t))=c(t) \tag{47.15}$$

or, finally,

$$\Phi^+(t)=\frac{a(t)-b(t)i\pi}{a(t)+b(t)i\pi}\Phi^-(t)+\frac{c(t)}{a(t)+b(t)i\pi} \tag{47.16}$$

which is of the form (47.7) which we addressed already. Care must be taken that the chosen solution Φ is such that $\Phi^+ + \Phi^-$ has the behavior (46.3) at infinity. Then the substitution is a posteriori justified. We did not discuss whether there are other solutions of the integral equation. A complete discussion of this and related equations can be found in [10].

We choose one of the applications in [1], the solution of the Dirichlet problem for the Laplacian in the upper half plane, with condition f on the boundary, $\mathbb{R}$. The problem can be reformulated as a Riemann-Hilbert problem of the form (47.16), see [1], but in this case, the solution is obtained easily from Plemelj's formulas.

For this purpose we look for an analytic function in $\mathbb{H}^u$ generated by u. As we know, this is

$$\Phi^+ = u + iv \tag{47.17}$$

where v is the harmonic conjugate of u, unique up to a constant. Now note that the function

$$\Phi(z)=\frac{1}{\pi i}\int_{\mathbb{R}}\frac{f(s)}{s-z}ds \tag{47.18}$$

is analytic in the upper half plane and, if f is Hölder continuous, then by Plemelj's formulas we have

$$\lim_{z\downarrow t\in\mathbb{R}}\Phi(z)=f(t)+\frac{1}{\pi i}PV\int_{\mathbb{R}}\frac{f(s)}{s-t}ds \tag{47.19}$$

In particular $u=\operatorname{Re}\Phi$ is harmonic and has the limit $f(x,y)$ as $(x,y)\to(t,0)$. Now, simply writing $z=x+iy$ and taking the real part we get the solution in the Poisson kernel form,

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\tau)d\tau}{(\tau-x)^2+y^2} \tag{47.20}$$

The condition that f is Hölder can be relaxed to mere continuity as follows. To better adapt to the limit $y\to 0$ we change variable to $t=x+\beta y$ and obtain

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x+\beta y)}{\beta^2+1}d\beta \tag{47.21}$$

which, by dominated convergence, has $f(x)$ as the limit when $y \to 0$.

48. Asymptotic series

We have seen in the Schwarz-Christoffel section that the behavior of analytic functions near a point of nonanalyticity can be given by a series in noninteger powers of the distance to the singularity. The behavior can be more complicated, containing exponentially small corrections, logarithmic terms and so on. The series themselves may have zero radius of convergence. It is not the purpose of this part of the course to classify these behaviors, but it can be done for a fairly large class of functions. Here we look how simple behaviors can be determined for relatively simple functions.

Example 48.1. *Consider the following integral related to the so–called error function*

$$F(z) = e^{z^{-2}} \int_0^z s^{-2} e^{-s^{-2}} ds$$

It is clear that the integral converges at the origin, if the origin is approached through real values (see also the change of variable below). **Definition of** $F(z)$. *We define the integral* to $z \in \mathbb{C}$ as being taken on a curve γ with $\gamma'(0) > 0$, and define $F(0) = 0$.

Check that this is a consistent definition and the resulting function is analytic except at $z = 0$ (this is essentially the contents of Exercise 48.3 below.

What about the behavior at $z = 0$? It depends on the direction in which 0 is approached! Let's look more carefully. Replace z by $1/x$, make a corresponding change of variable in the integral and you are led to

$$E(x) = e^{x^2} \int_x^\infty e^{-s^2} ds =: \frac{\sqrt{\pi}}{2} e^{x^2} \operatorname{erfc}(x) \tag{48.1}$$

Let us take x (and thus z) real and integrate by parts m times

(48.2)

$$E(x) = \frac{1}{2x} - \frac{e^{x^2}}{2} \int_x^\infty \frac{e^{-s^2}}{s^2} ds = \frac{1}{2x} - \frac{1}{4x^3} + \frac{3e^{x^2}}{4} \int_x^\infty \frac{e^{-s^2}}{s^4} ds = \dots$$
$$= \sum_{k=0}^{m-1} \frac{(-1)^k}{2\sqrt{\pi}} \frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} + \frac{(-1)^m e^{x^2} \Gamma(m+\frac{1}{2})}{\sqrt{\pi}} \int_x^\infty \frac{e^{-s^2}}{s^{2m}} ds$$

On the other hand, we have, by L'Hospital

$$(48.3)\qquad \left(\int_x^\infty \frac{e^{-s^2}}{s^{2m}}ds\right)\left(\frac{e^{-x^2}}{x^{2m+1}}\right)^{-1} \to \frac{1}{2} \text{ as } x\to\infty$$

and the last term in (48.2) is $O(x^{-2m-1})$ as well. On the other hand it is also clear that the series in (48.2) is alternating and thus

$$(48.4)\qquad \sum_{k=0}^{m-1}\frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} \leqslant E(x) \leqslant \sum_{k=0}^{m}\frac{(-1)^k}{2\sqrt{\pi}}\frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}}$$

if m is even.

Remark 48.2. *Using (48.3) and Exercise 48.13 below we conclude that $F(z)$ has a Taylor series at zero,*

$$(48.5)\qquad \tilde{F}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{2\sqrt{\pi}}\Gamma(k+\frac{1}{2})z^{2m+1}$$

that $F(z)$ is C^∞ on $\mathbb{R}$ and analytic away from zero.

Exercise 48.3. *** Show that $z=0$ is an isolated singularity of $F(z)$. Using Remark 48.2, show that F is unbounded as 0 is approached along some directions in the complex plane.*

Notes (1) It is *not* the Laurent series of f at 0 that we calculated! Laurent series converge. Think carefully about this distinction and why the positive index coefficients do not coincide.

(2) The rate of convergence of the Laurent series is slower as 0 is approached, quickly becoming numerically useless. By contrast, the precision gotten from (48.4) near zero is such that for $z=0.1$ the error in calculating f is of order 10^{-45} ! However, of course (48.4) is divergent and it cannot be used to calculate *exactly* for any nontrivial value of z.

(3) We have illustrated here a simple method of evaluating the behavior of integrals, the method of integration by parts.

48.1. **More general asymptotic series.** Classical asymptotic analysis typically deals with the qualitative and quantitative description of the behavior of a function close to a point, usually a singular point of the function. This description is provided in the form of an *asymptotic expansion.* The expansion certainly depends on the point studied and, as we have noted, often on the direction along which the point is approached (in the case of several variables, it also depends on the

relation between the variables as the point is approached). If the direction matters, it is often convenient to change variables to place the special point at infinity.
Asymptotic expansions are formal series[12] of simpler functions f_k,

$$\tilde{f} = \sum_{k=0}^{\infty} f_k(t) \tag{48.6}$$

in which each successive term is much smaller than its predecessors (one variable is assumed for clarity). For instance if the limiting point is t_0 approached from above along the real line this requirement is written

$$f_{k+1}(t) = o(f_k(t)) \quad \text{or} \quad f_{k+1}(t) \ll f_k(t) \quad \text{as} \quad t \downarrow t_0 \tag{48.7}$$

denoting

$$\lim_{t \to t_0^+} f_{k+1}(t)/f_k(t) = 0 \tag{48.8}$$

We will often use the variable x when the limiting point is $+\infty$ and z when the limiting point is zero. Simple examples are the Taylor series, e.g.

$$\sin z + e^{-\frac{1}{z}} \sim z - \frac{z^3}{6} + \dots \quad (z \to 0^+)$$

and the expansion in the Stirling formula

$$\ln \Gamma(x) \sim x \ln x - x - \frac{1}{2} \ln x + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}}, \ x \to +\infty$$

where B_k are the Bernoulli numbers.

(The asymptotic expansions in the examples above are the formal sums following the "$\sim$" sign, the meaning of which will be explained shortly.)

Examples of expansions that are *not* asymptotic expansions are

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (x \to +\infty)$$

[12] That is, there are no convergence requirements. More precisely, they are defined as sequences $\{f_k\}_{k \in \mathbb{N} \cup \{0\}}$, the operations being defined in the same way as if they represented convergent series; see also §48.2.

which converges to $\exp(x)$, but it is not an asymptotic series for large x since it fails (48.7); another example is the series

$$\sum_{k=0}^{\infty} \frac{x^{-k}}{k!} + e^{-x} \quad (x \to +\infty) \tag{48.9}$$

(because of the exponential terms, this is not an ordered *simple series* satisfying (48.7)). Note however expansion (48.9), *does* satisfies all requirements in the *left* half plane, if we write e^{-x} in the first position.

We also note that in this particular case the first series is convergent, and if we replace (48.9) by

$$e^{1/x} + e^{-x} \tag{48.10}$$

then (48.10) *is* a valid asymptotic expansion, of a very simple kind, with two nonzero terms. Since convergence is relative to a topology, this elementary remark will play a crucial role when we will speak of Borel summation.

Functions asymptotic to a series, in the sense of Poincaré. The relation $f \sim \tilde{f}$ between an actual function and a formal expansion is defined as a sequence of limits:

Definition 48.4. *A function f is asymptotic to the formal series $\tilde{f}$ as $t \to t_0^+$ if*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) =: f(t) - \tilde{f}^{[N]}(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{48.11}$$

We note that condition (48.11) can then be also written as

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{48.12}$$

where $g(t) = O(h(t))$ means $\limsup_{t\to t_0^+} |g(t)/h(t)| < \infty$. Indeed, this follows from (48.11) and the fact that $f(t) - \sum_{k=0}^{N+1} \tilde{f}_k(t) = o(\tilde{f}_{N+1}(t))$.

48.2. **Asymptotic power series.** In many instances the functions f_k are exponentials, powers and logarithms. This is not simply a matter of choice or an accident, but reflects some important fact about the relation between asymptotic expansions and functions which will be clarified later.

A special role is played by power series which are series of the form

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k, \ \ z \to 0^+ \tag{48.13}$$

With the transformation $z = t - t_0$ (or $z = x^{-1}$) the series can be centered at t_0 (or $+\infty$, respectively).

Remark. If a c_k is zero then Definition 48.4 fails trivially in which case (48.13) is not an asymptotic series. This motivates the following definition.

Definition 48.5 (Asymptotic power series)**.** *A function possesses an asymptotic* power *series if*

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \tag{48.14}$$

We use the boldface notation $\boldsymbol{\sim}$ for the stronger asymptoticity condition in (48.11) when confusion is possible.

Example Check that the Taylor series of an analytic function at zero is its asymptotic series there.

In the sense of (48.14), the asymptotic power series at zero of e^{-1/x^2} is the zero series. It is however surely not the case that e^{-1/x^2} behaves like zero as $x \to 0$ on $\mathbb{R}$. Rather, in this case, the asymptotic *behavior* of e^{-1/x^2} is e^{-1/x^2} itself (only exponentials and powers involved).

Asymptotic power series form an algebra; addition of asymptotic power series is defined in the usual way:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} c'_k z^k = \sum_{k=0}^{\infty} (Ac_k + Bc'_k) z^k$$

while multiplication is defined as in the convergent case

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} c'_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j c'_{k-j}\right) z^k$$

Remark 48.6. *If the series $\tilde{f}$ is convergent and f is its sum (note the ambiguity of the "sum" notation) $f = \sum_{k=0}^{\infty} c_k z^k$ then $f \sim \tilde{f}$.*

The proof of this remark follows directly from the definition of convergence.

Lemma 48.7. *(Uniqueness of the asymptotic series to a function) If $f(z) \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k z^k$ as $z \to 0$ then the $\tilde{f}_k$ are unique.*

Proof. Assume that we also have $f(z) \sim \tilde{F} = \sum_{k=0}^{\infty} \tilde{F}_k z^k$. We then have (cf. (48.11))

$$\tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = o(z^N)$$

which is impossible unless $g_N(z) = \tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = 0$, since g_N is a polynomial of degree N in z. ∎

Corollary 48.8. *The asymptotic series at the origin of an analytic function is its Taylor series at zero. More generally, if F has a Taylor series at 0 then that series is its asymptotic series as well.*

The proof of the following lemma is immediate:

Lemma 48.9. *(Algebraic properties of asymptoticity to a power series) If $f \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ and $g \sim \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$ then*
(i) $Af + Bg \sim A\tilde{f} + B\tilde{g}$
(ii) $fg \sim \tilde{f}\tilde{g}$

Sometimes it is convenient to check a formally weaker condition of asymptoticity:

Lemma 48.10. *Let $\tilde{f} = \sum_{n=0}^{\infty} a_n z^n$. If f is such that there exists a sequence $p_n \to \infty$ such that*

$$\Big(\forall n \exists p_n\Big) \ \ s.t. \ \ f(z) - \tilde{f}^{[p_n]}(z) = o(z^n) \quad as \quad z \to 0$$

then $f \sim \tilde{f}$.

Proof. We let m be arbitrary and choose $n > m$ such that $p_n > m$. We have

$$f(z) - \tilde{f}^{[m]} = (f(z) - \tilde{f}^{[p_n]}) + (\tilde{f}^{[p_n]} - \tilde{f}^{[m]}) = o(z^m) \ (z \to 0)$$

by assumption and since $\tilde{f}^{[p_n]} - \tilde{f}^{[m]}$ is a polynomial for which the smallest power is z^{m+1} (we are dealing with truncates of the same series). ∎

48.3. **Integration and differentiation of asymptotic power series.** While asymptotic power series can be safely integrated term by term as the next proposition shows, differentiation is more delicate. In suitable spaces of functions and expansions, we will see the asymmetry largely disappears if we are dealing with analytic functions in suitable regions.

Anyway, for the moment note that the function $e^{-1/z}\sin(e^{1/z^2})$ is asymptotic to the zero power series as $z \to 0^+$ although the derivative is unbounded and thus not asymptotic to the zero series.

Proposition 48.11. *Assume f is integrable near $z = 0$ and that*

$$f(z) \sim \tilde{f}(z) = \sum_{k=0}^{\infty} \tilde{f}_k z^k$$

Then

$$\int_0^z f(s)ds \sim \int \tilde{f} := \sum_{k=0}^{\infty} \frac{\tilde{f}_k}{k+1} z^{k+1}$$

Proof. This follows from the fact that $\int_0^z o(s^n)ds = o(z^{n+1})$ as can be seen by immediate estimates. ∎

Asymptotic power series of analytic function, if they are valid in wide enough regions can be differentiated.

Asymptotics in a strip. Assume $f(x)$ is analytic in the strip $S_a = \{x : |x| > R, |\mathrm{Im}\,(x)| < a\}$. Let $\alpha < a$ and and $S_\alpha = \{x : |x| > R, |\mathrm{Im}\,(x)| < \alpha\}$ and assume that

$$f(x) \sim \tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k} \quad (|x| \to \infty, x \in S_\alpha) \tag{48.15}$$

It is assumed that that the limits implied in (48.15) hold uniformly in the given strip.

Proposition 48.12. *If (48.15) holds, then, for $\alpha' < \alpha$ we have*

$$f'(x) \sim \tilde{f}'(x) := \sum_{k=0}^{\infty} -\frac{kc_k}{x^{k+1}} \quad (|x| \to \infty, x \in S_{\alpha'})$$

Proof. We have $f(x) = \tilde{f}^{[N]}(x) + g_N(x)$ where clearly g is analytic in S_a and $|g_N(x)| \leqslant Const.|x|^{-N-1}$ in S_α. But then, for $x \in S_{\alpha'}$ and $\delta = \frac{1}{2}(\alpha - \alpha')$ we get

$$\begin{aligned} |g_N'(x)| = \frac{1}{2\pi}\left|\oint_{|x-s|=\delta} \frac{g_N(s)ds}{(s-x)^2}\right| &\leqslant \frac{1}{\delta}\frac{Const.}{(|x|-|\delta|)^{N+1}} \\ &= O(x^{-N-1}) \quad (|x| \to \infty,\ x \in S_{\alpha'}) \end{aligned}$$

By Lemma 48.10, the proof follows. ∎

Exercise 48.13. *** Show that if $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ and $f'(x) \to L$ as $x \downarrow 0$, then f is differentiable to the right at zero and this derivative equals L. Use this fact, Proposition 48.12 and induction to show that the Taylor series at the origin of $F(z)$ is indeed given by (48.5).*

48.4. **Watson's Lemma.** In many instances integral representations of functions are amenable to Laplace transforms

$$(\mathcal{L}F)(x) := \int_0^\infty e^{-xp} F(p)dp \tag{48.16}$$

The behavior of $\mathcal{L}F$ for large x relates to the behavior for small p of F.

It is shown in the later parts of this book that solutions of generic analytic differential equations, under mild assumptions can be conveniently expressed in terms of Laplace transforms.

For the error function note that

$$\int_N^\infty e^{-s^2} ds = N \int_1^\infty e^{-N^2u^2} du = \frac{\sqrt{x}e^{-x}}{2} \int_0^\infty \frac{e^{-xp}}{\sqrt{p+1}} dp; \quad x = N^2$$

For the Gamma function,

$$\Gamma(n+1) = \int_0^\infty e^{-t} t^n dt = n^{n+1} \int_0^\infty e^{n(-s+\ln s)} ds \tag{48.17}$$

writing $\int_0^\infty = \int_0^1 + \int_1^\infty$ we can make the substitution $s = t+1$, $t - \ln(1+t) = p$ in each of the two integrals and obtain

$$\Gamma(n+1) = n^{n+1} e^{-n} \int_0^\infty e^{-np} G(p) dp; \tag{48.18}$$
$$G(p) = \frac{1}{1 + W(0, -e^{-u-1})} - \frac{1}{1 + W(-1, -e^{-u-1})}$$

where W is the Lambert function, the inverse function of xe^x; $W(0,\cdot)$ is the principal branch of W and $W(-1,\cdot)$ is the -1 branch.

Watson's Lemma

This important tool states that the asymptotic series at infinity of $(\mathcal{L}F)(x)$ is obtained by formal term-by-term integration of the asymptotic series of $F(p)$ for small p, provided F has such a series.

Lemma 48.14. *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^\infty c_k p^{k\beta_1 + \beta_2 - 1}$ as $p \to 0^+$ for some constants β_i with* $\mathrm{Re}\,(\beta_i) > 0$, *$i = 1, 2$. Then*

$$\mathcal{L}F \sim \sum_{k=0}^{\infty} c_k \Gamma(k\beta_1 + \beta_2) x^{-k\beta_1-\beta_2}$$

along any ray ρ in the open right half plane H.

Proof. Induction, using the simpler version, Lemma 48.15, proved below. □

Lemma 48.15. *Let $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\phi}$, $\rho > 0$, $\phi \in (-\pi/2, \pi/2)$ and assume*

$$F(p) \sim p^{\beta} \quad \text{as } p \to 0^+$$

with $\mathrm{Re}\,(\beta) > -1$. *Then*

$$\int_0^{\infty} F(p) e^{-px} dp \sim \Gamma(\beta+1) x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. If $U(p) = p^{-\beta} F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let χ_A be the characteristic function of the set A and $\phi = \arg(x)$. We choose C and a positive so that $|F(p)| < C|p^{\beta}|$ on $[0, a]$. We write $\int_0^{\infty} = \int_0^a + \int_a^{\infty}$. The second integral will not contribute to the power series asymptotics since

$$\left| \int_a^{\infty} F(p) e^{-px} \mathrm{d}p \right| \leqslant e^{-|x| a \cos\phi} \|F\|_1 \; = o(x^{-n}) \text{ for any } n \in \mathbb{N} \tag{48.19}$$

For the first integral, after the change of variable $s = p|x|$ we get, by dominated convergence,

$$x^{\beta+1} \int_0^a F(p) e^{-px} \mathrm{d}p \tag{48.20}$$

$$= e^{i\phi(\beta+1)} \int_0^{\infty} s^{\beta} U(s/|x|) \chi_{[0,a]}(s/|x|) e^{-se^{i\phi}} \mathrm{d}s \to \Gamma(\beta+1) \text{ as } |x| \to \infty$$

48.5. **The Gamma function.** We can find the analytic properties of G without resorting to the Lambert function representation. The function $t - \ln(1+t)$ has a minimum at zero and it is monotonic on $(-1, 0)$ and $(0, \infty)$; the equation $t - \ln(1+t) = p$, $p > 0$ has two solutions t_- and t_+ on $(-1, 0)$ and $(0, \infty)$. We thus have

$$\int_{-1}^{\infty} e^{-n(t-\ln(1+t))} dt = \int_0^{\infty} G(p) e^{-np} dp; \; G(p) = \frac{dt_+}{dp} - \frac{dt_-}{dp} \tag{48.21}$$

Remark 48.16. *The function G is analytic in $\sqrt{p}$ and thus $G'(p)$ has a convergent Puiseux series*

$$\sum_{k=-1}^{\infty} c_k p^{k/2} = \sqrt{2} p^{-1/2} + \frac{\sqrt{2}}{6} p^{1/2} + \frac{\sqrt{2}}{216} p^{3/2} - \frac{139\sqrt{2}}{97200} p^{5/2} + \ldots$$

Thus, by Watson's Lemma, for large n we have

$$n! \sim \sqrt{2\pi n}n^n e^{-n}\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}+...\right) \tag{48.22}$$

Proof. Note that $t-\ln(1+t) = t^2U(t)/2$ where $U(0)=1$ and $U(t)$ is analytic for small t; with the natural branch of the square root, $\sqrt{U(t)} = H(t)$ is also analytic. We rewrite $t-\ln(1+t) = p$ as $tH(t) = \pm\sqrt{2}\sigma$ where $\sigma^2 = p$. Since $(tH(t))'_{|t=0} = 1$ the implicit function theorem ensures the existence of two functions $t_{\pm}(\sigma)$ (corresponding to the two choices of sign) which are analytic in σ. The concrete expansion may be gotten by implicit differentiation in $tH(t) = \pm\sqrt{2}\sigma$, for instance. ∎

49. The Painlevé property

In the theory of ODEs, one distinguishes between fixed and movable singularities. The location of a fixed singularity is common to most solutions of a given ODE and originates in a singularity of the equation itself. A simple example is $y' = 1/x, y(1) = y_1$ whose solutions are $\ln x + y_1$ with 0 a branch point of all solutions. In nonlinear equations, the position of singularities may depend on the initial condition. Consider for example the ODE $y' = y^2+1, y(0) = a$ whose general solution is $y = \tan(x+\arctan a)$ with singularities at $x = (k+1/2)\pi - \arctan a$ whose location is a function of a. A singularity of a solution of an ODE that is not a pole is called a *critical point*.

Broadly speaking, an equation is integrable if there exist enough *constants of motion* (functions that are constant along trajectories), whose knowledge completely determines the trajectories equation. In classical mechanics, conservative models are such systems: the energy, momentum, angular momentum etc are conserved and fully determine the trajectories; for example in a harmonic oscillator $mx'' = -kx$ the total energy $\frac{1}{2}mv^2 + \frac{1}{2}kx^2$ is conserved, and the trajectories are the ellipses $(x')^2 + \frac{k}{m}x^2 = C$.

In the history of nonlinear integrable equations, the work of Fuchs, Kowalewsky and Painlevé stand out. In 1884 L. Fuchs showed that amongst the first-order equations of the form $y' = F(y,x)$ with F rational in y and analytic in x the only equations without movable critical points are Riccati equations, and these are integrable: they reduce to linear second order ODEs.

Motivated by the work of Fuchs, S. Kovalevskaya looked for and found the choices of parameters for which the governing equations of the motion of a rigid body about a fixed point under the influence of gravity admit no movable critical points. She went further and solved explicitly these special equations, thereby finding new integrable cases of the motion of a gyroscope. For this outstanding work, she was awarded the Bordin Prize in 1888.

Around 1900, Paul Painlevé studied second order differential equations with no movable singularities. He found that up to certain transformations, every such equation of the form $y'' = R(y', y, x)$ with R rational can be put into one of fifty canonical forms. Painlevé (1900, 1902) found that forty-four of the fifty equations are reducible in the sense that they can be solved in terms of previously known functions, leaving just six equations requiring the introduction of new special functions to solve them.

An equation whose only movable singularities are poles is now known as having the Painlevé property. All first and second order ODEs with the Painlevé property are integrable: they can be linearized, either by changes of variables or through reformulation as a Riemann-Hilbert problem.

I will explain the intuition behind this integrability. Consider a general system of ODEs, $y' = F(y, x);\ y(x_0) = y_0, y \in \mathbb{C}^n$, where F is analytic and denote the solution $Y(x; y_0, x_0)$. Denote the solution of the equation with $x \mapsto -x$, $y' = -F(x, y)$ by $\tilde{Y}$. It is easy to see that $\tilde{Y}(x_0; Y(x), x) = y_0$, a constant. The function $\tilde{Y}(x_0; Y(x), x)$ is thus a constant along trajectories, thus *locally* a constant of motion. It is an $n - \textit{dimensional}$ function, and its knowledge of course completely determines the solution of the equation. But the word local is crucial here. If the equation is nonlinear, there are generally movable critical points, and the analytic continuation of this local constant of motion generates a Riemann surface that would also depend on the initial condition in a way that makes it unusable.

Formally checking for the Painlevé property is quite easy: one looks for the type of singularities compatible with an equation and checks whether the equation admits a convergent one-sided Laurent expansion at the singular point.

We take as an example the Painlevé P1 equation $y'' = y^2 + x$ and variations of it, say $y'' = y^2 + x^2$. Assuming that x_0 is a movable singular point, we first look at the leading power, $(x - x_0)^p$, at the singular point. We easily determine that the only power compatible with the equation is $p = -2$. Inserting $y = \sum_{k=-2}^{\infty} c_k x^k$ in $y'' - y^2 - x$, with $c_{-2} \neq 0$ and imposing the condition that the coefficients of the left side vanish we find $c_{-2} = 6, c_{-1} = c_0 = 0, c_2 = x_0/10, c_3 = -1/6$, c_4 is arbitrary and we can prove inductively that the equations for $c_k, k > 4$ have a unique solution, and the Laurent series converges. The same procedure in $y'' - y^2 - x^2$ however shows that the coefficient of $(x - x_0)^2$ is always one, and the solution cannot be meromorphic. A simple analysis shows that generically, singular solutions are not meromorphic.

Note 50.17. Composing with a Cayley transform, we can rephrase Montel's normality Theorem 31.7 we see that we can replace $\sup\{|f(z)| : z \in K, f \in \mathcal{F}\} = m(K) < \infty$ with $\forall f \in \mathcal{F},\ f : K \mapsto \mathbb{H}^+$.

50.1. **Appendix to Chapter 8.**

50.2. **Some facts about the topology of $\mathbb{C}$.** From a topological point of view, $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$. Namely, we identify the point $z = x + iy$ with the pair $(x, y) = (\operatorname{Re} z, \operatorname{Im} z)$ and the length $|z|$ with $|(x, y)| = \sqrt{x^2 + y^2}$, inducing a distance $d(z_1, z_2) := |z_1 - z_2|$. Since $\max(|x|, |y|) \leqslant \sqrt{x^2 + y^2} \leqslant |x| + |y|$ a sequence $\{z_n\}_{n\in\mathbb{N}}$ is Cauchy **iff** both $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy and $z_n \to 0$ as $n \to \infty$ **iff** $x_n \to 0$ **and** $y_n \to 0$ as $n \to \infty$; similarly, a sequence convergence in $\mathbb{C}$ **iff** it is Cauchy.

Consequently, if a series $\sum_{n=0}^{\infty} c_n z_0^n$ converges, then $c_n z_0^n \to 0$ as $n \to \infty$. Check that this implies that the real-valued series $\sum_{n=0}^{\infty} |c_n||z_1|^n$ converges for any $|z_1| < |z_0|$, that is $\sum_{n=0}^{\infty} c_n z_1^n$ converges uniformly and absolutely for $|z| < |z_0|$. This is Abel's theorem in the complex domain. The only difference from the real analysis counterpart is that $|z| < |z_0|$ is a disk instead of an open interval.

A region of $\mathbb{C}$ is called *open* if it contains together with any point z_0 all sufficiently close points, that is, it also contains a nonempty disk centered at z_0; intuitively, an open set is a region without its boundary. For example an *open disk*

$$\mathbb{D}(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\} \tag{50.23}$$

a *punctured disk*

$$\mathbb{D}_p(z_0, r) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$$

the upper half plane $\mathbb{H}^u := \{z : \operatorname{Im}(z) > 0\}$ and $\mathbb{C}$ are open, as is, trivially, the empty set $\emptyset$, but a *closed disk*

$$\overline{\mathbb{D}}(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| \leqslant r\}$$

is not open. The exterior of a closed disk, $\{z \in \mathbb{C} \mid |z - z_0| > r\}$, is open. A finite intersection of open sets is open. Clearly a set in $\mathbb{C}$ is open iff it is a (finite or infinite) union of open disks.

More generally, a **topology on a space** X consists of a family $\mathcal{O}$ of sets defined as open, which should have the following properties: (1) $X, \emptyset \in \mathcal{O}$, (2) $O_1, O_2 \in \mathcal{O} \Rightarrow O_1 \cap O_2 \in \mathcal{O}$ and (3) any union, finite or infinite of open sets $O_\alpha \in \mathcal{O}$ is open: $\cup_\alpha O_\alpha \in \mathcal{O}$. Complements of open sets are called *closed* sets. The whole X is both open and closed; so is its complement, $\emptyset$. The family $\mathcal{O}$ in the case of $\mathbb{C}$ can be taken to be the collection of all unions of open disks (50.23), for all $z_0 \in \mathbb{C}, r \in [0, \infty]$.

The *boundary* of the set S in $\mathbb{C}$, denoted ∂S, consists of all points z in $\mathbb{C}$ for which there are sequences contained in S which converge to z, as well as sequences in the exterior of S convergent to z. For example, the boundary of a disk is the circle surrounding it:

$$\partial\mathbb{D}(z_0, r) = C(z_0, r) := \{z \in \mathbb{C} | \, |z - z_0| = r\}$$

Also $\partial\overline{\mathbb{D}}(z_0, r) = C(z_0, r)$, but $\partial\mathbb{D}_p(z_0, r) = C(z_0, r) \cup \{0\}$.

A point z in $\mathbb{C}$ is an *accumulation point of the set S* if there is a sequence of points in S converging to z.

Note that a set O is open iff it contains no points in ∂O. At the opposite end, if a set contains *all* of its boundary points then it is closed.

If a set is defined by (finitely many) inequalities involving continuous functions, then the set is open only if all the inequalities are strict ($<$, $>$, or $\neq$), and it is closed if all are $\leqslant, \geqslant$ or $=$; the boundary is obtained by replacing all inequalities by equalities.

If X is a topological space and $X_1 \subset X$ the *induced topology* in X_1 is $\{X_1 \cap O | O \in \mathcal{O}\}$.

50.2.1. *Connected sets.* An open set O is *connected* if it is not the union of two disjoint nonempty open sets. More generally, a subset $X_1 \in X$ is connected if it is *not* the disjoint union of two nonempty sets that are open in the induced topology on X_1. Equivalently, there is no subset of X_1 which is both open and closed in the induced topology (other than X_1 and the empty set). For example any disk in $\mathbb{C}$ is connected, and so is a punctured disk. See also Proposition 50.19 below.

A *domain* in $\mathbb{C}$ is by definition an open connected set.

Exercise 50.18. *Is the annulus $\{z \in \mathbb{C} | \; r \leq |z| < R\}$ open? closed? connected? What is its boundary?*

A curve in $\mathbb{R}^2$ is often given using a parametrization, as the image of a pair of *continuous* real functions: $\{(x(t), y(t)) : t \in [a, b]\}$. The same curve can obviously be the image many different maps. If at least one of these is differentiable, then the curve is called differentiable; $\{x(t) + iy(t) : t \in [a, b]\}$ is the corresponding curve in $\mathbb{C}$.

A set S with the property that any two points is S can be connected by a curve in S is called *path connected*; it can be shown that a path connected set is necessarily connected. But the converse is not true, for example $S = \{(x, \sin\frac{1}{x}) \,|\, x > 0\} \cup \{(0, 0)\}$ is connected, but not path connected. But:

Proposition 50.19. *Domains $\mathcal{D} \subset \mathbb{C}$ are path connected. The path can be chosen to be a polygonal line.*

Proof. Indeed, let $z, w \in \mathcal{D}$ be two arbitrary points. Collect the points which are path connected to z:

$$\mathcal{D}_z = \left\{u \in \mathcal{D} \,\middle|\, \exists\gamma : [a, b] \to \mathcal{D} \text{ continuous, with } \gamma(a) = z,\, \gamma(b) = u\right\}$$

Then $\mathcal{D}_z$ is open since for any $u \in \mathcal{D}_z$, there is a disk $D_\varepsilon(z)$ included in $\mathcal{D}$ (since $\mathcal{D}$ is open), and then z can be path connected to any point in this disk (the path connecting z to u followed by the segment form u to any point in the disk), hence $\mathcal{D}_z$ contains a disk centered at u. By the same argument also $\mathcal{D}_w$ is open, and since $\mathcal{D}$ is connected then there must be a point $u \in \mathcal{D}_z \cap \mathcal{D}_w$. But then the path going from z to u followed by the path from u to w connects z to w.

The path connecting two points can be chosen to be a polygonal line: Indeed, let $\gamma : [a, b] \to \mathcal{D}$ continuous, so that $\gamma(a) = z$ and $\gamma(b) = w$. Since $\mathcal{D}$ is open, every point along the path is contained in a disk included in $\mathcal{D}$: for all $t \in [a, b]$ there is $\varepsilon_t > 0$ so that $\mathbb{D}(\gamma(t), \varepsilon_t) \subset \mathcal{D}$. Since the image of γ is compact, and is included in the union of all these disk, then it is included in a finite number of them: there are $t_1, \ldots, t_n$ so that $\gamma([a, b]) \subset \cup_{k=1}^{n} \mathbb{D}(\gamma(t_k), \varepsilon_{t_k}) \subset \mathcal{D}$ and now γ can be replaced by segments in each disk. To be more precise in this construction, let $t_0 = a, t_{n+1} = b$ and let $\varepsilon_0, \varepsilon_{n+1} > 0$ so that $\mathbb{D}(\gamma(t_k), \varepsilon_{t_k}) \subset \mathcal{D}$ for $k = 0$ and $k = n + 1$. Then $\gamma([a, b]) \subset \cup_{k=0}^{n+1} \mathbb{D}(\gamma(t_k), \varepsilon_{t_k}) \subset \mathcal{D}$. We can remove any disk of the covering that is completed include in another disk, and we number the t_k in increasing order. Then the segments $[\gamma(t_{k-1}), \gamma(t_k)]$ are included in $\mathbb{D}(\gamma(t_{k-1}), \varepsilon_{t_{k-1}}) \cup \mathbb{D}(\gamma(t_k), \varepsilon_{t_k}) \subset \mathcal{D}$ and form a polygonal line joining z and w. ☐

A *rectifiable curve* is a continuous curve $t \mapsto \gamma(t)$ (defined for $t \in [a, b]$) with finite length, meaning that the sup of the length of polygonal lines joining points of γ is finite. In $\mathbb{C}$ this means:

$$\sup\left\{\sum_{i=0}^{n} |\gamma(t_i) - \gamma(t_{i+1})| \,:\, 0 = t_0 < t_i < t_{i+1} < t_n = b \forall i \in 1..n-1, \forall n \in \mathbb{N}\right\} < \infty$$

A piecewise differentiable curve with integrable γ' is easily checked to be rectifiable, and the length, defined by the sup above, also equals

$$l(\gamma) = \int_a^b |\gamma'(t)| dt$$

DEFINE winding number

50.3. **Proof of the Ascoli-Arzelà theorem.** *Necessity* (i) Suppose $\mathcal{F}$ is not equicontinuous on some compact K. Then on K
(50.24)

$$\exists(\varepsilon > 0,\ \{z_n\},\{z'_n\},\{f_n\})\ s.t.(|z_n - z'_n| \to 0\ \&\ d(f_n(z_n), f_n(z'_n) > \varepsilon)$$

Since K is compact and $\mathcal{F}$ is normal from any sequence we can extract a convergent subsequence, which w.l.o.g. we can assume to be $\{z_n\},\{z'_n\},\{f_n\}$ themselves. Let $z_n \to z$, $f_n \to f$ ($z'_n \to z$ too). The limit f is continuous, thus uniformly continuous. We have

$$\lim_{n\to\infty} \sup_{x\in K} d'(f(x), f_n(x)) = 0$$

thus for n large enough,

$$(50.25) \quad d'(f_n(z'_n), f(z'_n)) < \frac{\varepsilon}{4},\ d'(f(z'_n), f(z)) < \frac{\varepsilon}{4}, \\ d'(f(z), f(z_n)) < \frac{\varepsilon}{4} \quad \text{and} \quad d'(f(z_n), f_n(z_n)) < \frac{\varepsilon}{4}$$

implying by the triangle inequality,

$$d'(f_n(z'_n), f_n(z_n)) < \varepsilon$$

a contradiction.

(ii) Fix z and take $K = \overline{\{f(z) : f \in \mathcal{F}\}}$. Take a sequence $\{w_n\} \subset K$ By the definition of K, if $w_n \in K$ $\exists f_n \in \mathcal{F}$ such that $d(f_n(z), w_n) < 1/n$. By the normality of $\mathcal{F}$, there exists a subsequence of functions, w.l.o.g. $\{f_n\}$ themselves, $f_n \to f$. But then $w_n \to f(z)$ $\Box$.

Sufficiency. The sufficiency of the two conditions is shown by Cantor's famous diagonal argument. Let $\{f_n\} \subset \mathcal{F}$. We take a countable everywhere dense set $\mathcal{Q} = \{z_k\}$ of points in Ω, e.g., those with rational coordinates and we let $\mathcal{K}$ be any compact in Ω. Take $z_1 \in \mathcal{Q}$. By (ii), there is a convergent subsequence $\{f_{n_{j1}}(z_1)\}_{j\in\mathbb{N}}$. Take now $z_2 \in \mathcal{Q}$. From $\{f_{n_{j1}}(z_2)\}$we can extract a subsequence $\{f_{n_{j2}}(z_2)\}_{j\in\mathbb{N}}$ which converges as well. So $\{f_{n_{j2}}(z)\}_{j\in\mathbb{N}}$ converges both at z_1 and z_2. Inductively we find a subsequence $\{f_{n_{jm}}(z)\}_{j\in\mathbb{N}}$ such that it converges at the points $z_1, ..., z_m$. But then, the subsequence $\{g_j\} := \{f_{n_{jj}}\}$ converges at all points in $\mathcal{Q}$. We aim to show that g_j *converges uniformly* in any compact set $K \in \Omega$. By equicontinuity,
(50.26)

$$\forall\varepsilon > 0\, \exists\delta\, s.t.\ \forall(a, b, f) \in K^2 \times \mathcal{F}(|a - b| < \delta \Rightarrow d(f(a), f(b)) < \frac{\varepsilon}{3})$$

Consider a finite covering of K by balls of radius $\delta/2$. Since $\mathcal{Q}$ is everywhere dense, there is a z_k in each of these balls. They are finitely many, so that for $l, m > n_0$,

$$d(g_l(z_k), g_m(z_k)) < \frac{\varepsilon}{3} \tag{50.27}$$

On the other hand, any $a \in \mathcal{K}$ is, by construction, at distance at most δ from some z_k and thus by (50.26) (for any $f \in \mathcal{F}$, in particular) for g_{ni}, g_{nj} we have

$$d(g_l(a), g_l(z_k)) < \frac{\varepsilon}{3} \tag{50.28}$$

$$d(g_m(a), g_m(z_k)) < \frac{\varepsilon}{3} \tag{50.29}$$

We thus see by the triangle inequality that

$$d(g_l(a), g_m(a)) < \varepsilon \tag{50.30}$$

Thus $g_n(a)$ converges. Convergence is uniform since the pair ε, δ is independent of a. □

51. Dominated convergence theorem

We state this theorem only for the real line

References

[1] Mark J. Ablowitz and Athanassios S. Fokas, Complex variables, 2nd edition, Cambridge University Press, 2003.

[2] M Abramowitz, I A Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables New York : Wiley-Interscience (1984).

[3] L. Ahlfors, Complex Analysis, McGraw-Hill (1979).

[4] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, (1955). (2009).

[5] J. Milnor, Complex Dynamics, Annals of Mathematics Studies, Third Edition, (2006).

[6] O. Costin and M. Huang, Geometric construction and analytic representation of Julia sets of polynomial maps, Nonlinearity 24, 1311–1327 (2011)

[7] Rudin, Real and Complex Analysis

[8] W. Schlag, A Course in Complex Analysis and Riemann Surfaces,Graduate Studies in Mathematics vol. 154, American Mathematical Society, 2014.

[9] S. Lang, Complex Analysis, fourth edition, (1999).

[10] N.I. Muskhelishvili, Singular integral equations, Dover (2008).

[11] Z. Nehari, Conformal Mapping, Dover (1975).

[12] E. C. Titchmarsh, The theory of functions, Oxford University Press, 1964.

[13]

[14] By !Original:Kilom691Vector: Alexander Hulpke - Own work based on: ModularGroup-FundamentalDomain-01.png, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=59963451