Existence and Uniqueness for a Class of Nonlinear Higher-Order Partial Differential Equations in the Complex Plane

O. COSTIN
Rutgers University
AND
S. TANVEER
Ohio State University

Abstract

We prove existence and uniqueness results for nonlinear third-order partial differential equations of the form

$$f_t - f_{yyy} = \sum_{j=0}^{3} b_j(y,t;\, f)\; f^{(j)} + r(y,t)$$

where superscript j denotes the j^{th} partial derivative with respect to y. The inhomogeneous term r, the coefficients b_j, and the initial condition $f(y,0)$ are required to vanish algebraically for large $|y|$ in a wide enough sector in the complex y-plane. By using methods related to Borel summation, a unique solution is shown to exist that is analytic in y for all large $|y|$ in a sector. Three partial differential equations arising in the context of Hele-Shaw fingering and dendritic crystal growth are shown to be of this form after appropriate transformation, and then precise results are obtained for them. The implications of the rigorous analysis on some similarity solutions, formerly hypothesized in two of these examples, are examined. © 2000 John Wiley & Sons, Inc.

1 Introduction

The theory of partial differential equations (PDEs) when one or more of the independent variables are in the complex plane appears to be largely undeveloped. The classic Cauchy-Kowalewski (C-K) theorem holds for a system of first-order equations (or those equivalent to it) when the quasi-linear equations have analytic coefficients and analytic initial data is specified on an analytic but noncharacteristic curve. Then the C-K theorem guarantees the local existence and uniqueness of analytic solutions. As is well known, its proof relies on the convergence of local power series expansions, and, without the given hypotheses, the power series may have zero radius of convergence and the C-K method does not yield solutions.

Communications on Pure and Applied Mathematics, Vol. LIII, 0001–0026 (2000)
© 2000 John Wiley & Sons, Inc.

Relatedly, not much is known in general for higher-order nonlinear partial differential equations in the complex plane, even for analytic initial conditions and analytic dependence of the coefficients. The only work we are aware of on nonlinear partial differential equations in the complex plane involving higher spatial derivatives is that of Sammartino and Caflisch [14, 15], who proved among other results the existence of a solution to nonlinear Prandtl boundary layer equations for analytic initial data. This work involved inversion of the heat operator $\partial_t - \partial_{YY}$ and using the abstract Cauchy-Kowalewski theorem for the resulting integral equation. Unfortunately, this methodology cannot be adapted to our problem. The coefficients of the highest (third-order) spatial derivatives in our equation depend on the unknown function as well. These terms cannot be controlled by inversion of a linear operator and estimates of the kernel, as used by Sammartino and Caflisch. Instead, the essence of the methodology introduced here is the use of large y asymptotics, conveniently expressed in terms of the behavior of the unknown function in the Borel transform variable p for small p. The choice of appropriate Banach spaces proves to be crucial, and after this choice the contraction mapping argument itself is not difficult.

One aim of the present paper is to obtain actual solutions with good smoothness and asymptotic properties for a class of PDEs when power series solutions may have a zero radius of convergence. Our approach, based on Borel summation techniques, provides at the same time appropriate existence and uniqueness results for a class of nonlinear PDEs in the complex domain.

Keeping in mind applications, we develop the framework for certain higher-order partial differential equations in a domain where one of the independent variables (y in this case) is complex, while the other (t) is real. While more sweeping generalizations are under way, the current paper is restricted in scope by the applications we have in mind and simplicity of exposition.

There is a class of nonlinear PDEs that have recently arisen in applications. The basic feature of the application problems is that in the absence of a regularization (like surface tension), the initial value problem in the real domain is relatively simple, yet ill-posed in the sense of Hadamard for any Sobolev norm on the real domain. However, the analytically continued equations into the complex spatial domain are well-posed, even without a regularization term. Earlier, Garabedian [7] recognized the conversion of an ill-posed elliptic initial value problem into a well-posed one by excursion into the complex plane in the spatial variable. Moore [12, 13], Caflisch and Orellana [1, 2], Caflisch and Semmes [4], and Caflisch et al. [3] have studied solutions to the complex plane equations that arise from simplifications of vortex sheet evolution (in fluid mechanical contexts). The initial value problem in these cases is ill-posed in the real domain, though well-posed in an appropriate class of analytic functions in a domain in the complex plane. Study of the complex equations proved useful since evidence [10] of finite time singularities in the real domain can be traced to earlier singularity formation in the complex domain.

In the physical context of Hele-Shaw dynamics, it was suggested [17] that it is fruitful to study the complex plane equations even when the initial value problem is well-posed in the physical domain through the addition of a small regularization term. The advantage of this procedure is that one can study small regularization effects by perturbing about the relatively simpler but well-posed zeroth-order problem. The ill-posedness of the unregularized problem in the real domain, shown earlier by Howison [8], is transferred into ill-posedness of the analytic continuation of initial data to the complex plane. However, when analytic initial data is specified in a domain in the complex plane such as to allow for isolated singularities, there is no ill-posedness of the zeroth-order approximation of the dynamics. This provides the basis for a perturbative study that includes small but nonzero regularization effects in the real domain. Consideration of an ensemble of complex initial conditions, subject to appropriate constraints on its behavior on the real axis, provide a way to understand the robust features of the dynamics when regularization effects are small.

Indeed this procedure has yielded information about how small surface tension can singularly perturb a smooth solution of the unregularized dynamics [16]. It has given scaling results on nonlinear dendritic processes as well [11]. However, much of the results derived so far are purely formal and rely fundamentally on the existence and uniqueness of analytic solutions to certain higher-order, nonlinear partial differential equations in a sector in the complex plane with imposed far-field matching conditions. Indeed, in a more general context, one can expect that whenever regularization appears in the form of a small coefficient multiplying the highest spatial derivative, the resulting asymptotic equation in the neighborhood of initial complex singularities will satisfy a higher-order nonlinear partial differential equation with sectorial far-field matching condition in the complex plane of the type shown in examples 1 through 3. Hence, there is a need to develop a general theory in this direction.

2 Problem Statement and Main Result

We seek to prove the existence and uniqueness of solutions $f(y,t)$ to the initial value problem for a general class of quasi-linear partial differential equations of the form:

$$f_t - f_{yyy} = \sum_{j=0}^{3} b_j(y,t;\ f)\, f^{(j)} + r(y,t) \quad \text{with } f(y,0) = f_I(y) \tag{2.1}$$

where the superscript $^{(j)}$ refers to the j^{th} derivative with respect to y. The inhomogeneous term $r(y,t)$ is a specified analytic function in the domain

$$\mathcal{D}_{\rho_0} = \{(y,t) : \arg y \in (-\tfrac{2\pi}{3}, \tfrac{2\pi}{3}),\ |y| > \rho_0 > 0,\ 0 \le t \le T\},$$

and it is assumed that in $\mathcal{D}_{\rho_0}$ there exist constants $\alpha_r \geq 1$ and A_r, with only A_r allowed to depend on T, such that

$$|y^{\alpha_r} r(y,t)| < A_r(T)\,. \tag{2.2}$$

Further in (2.1), the coefficients b_j may depend on the solution f—this is how nonlinearity in the problem arises. It is possible to extend the current theory to include dependence of b_j on $f^{(j)}$ as well, though for simplicity we will restrict ourselves only to dependence on f. Further, we restrict ourselves to the case where each b_j is given by a convergent series

$$b_j(y,t;\, f) = \sum_{k=0}^{\infty} b_{j,k}(y,t)\, f^k \tag{2.3}$$

for known $b_{j,k}$, analytic for y in $\mathcal{D}_{\rho_0}$. It will be assumed that in this domain, there exists some choice of positive constants β, α_j, and A_b, independent of j and k (with β and α_j independent of T as well), such that

$$|y^{\alpha_j + k\beta} b_{j,k}| < A_b(T)\,. \tag{2.4}$$

Further, the series (2.3) converges in the domain $\mathcal{D}_{\phi,\rho}$, defined as

$$\mathcal{D}_{\phi,\rho} = \tag{2.5}$$
$$\left\{(y,t) : \arg y \in \left(-\tfrac{\pi}{2} - \phi, \tfrac{\pi}{2} + \phi\right),\ |y| > \rho > \rho_0 \text{ where } 0 < \phi < \tfrac{\pi}{6},\ 0 \leq t \leq T\right\}$$

if

$$|f| < \rho^{\beta}\,. \tag{2.6}$$

CONDITION 2.1 The solution $f(y,t)$ we seek for (2.1) is required to be analytic for complex y in $\mathcal{D}_{\phi,\rho}$ for some $\rho > 0$ (to be determined later). In the same domain, the solution and the initial condition $f_I(y)$ must satisfy the condition

$$|y^{\alpha_r} f(y,t)| < A_f(T) \tag{2.7}$$

for some A_f that can only depend on T for $(y,t) \in \mathcal{D}_{\phi,\rho}$.

It is clear that for large y such a solution f will indeed satisfy (2.6), the condition for the convergence of the infinite series in (2.1). The general theorem proved in this paper is the following:

THEOREM 2.2 *For any $T > 0$ and $0 < \phi < \frac{\pi}{6}$, there exists $\tilde{\rho}$ such that the partial differential equation* (2.1) *has a unique solution f that is analytic in y and $O(y^{-1})$ as $y \to \infty$ for $(y,t) \in \mathcal{D}_{\tilde{\rho},\phi}$. In fact, we have for this solution $f = O(y^{-\alpha_r})$ as $y \to \infty$.*

We note that uniqueness requires analyticity and decay properties of y in a large enough sector. The proof of Theorem 2.2 will have to await some definitions and lemmas. It is to be noted that from a formal argument if f is small, the dominant balance for large y is between f_t on the left of (2.1) and $r(y,t)$ on the right, indicating that $f(y,t) \sim f_I(y) + \int_0^t r(y,t)dt$. Since each of $f_I(y)$ and $r(y,t)$ decays

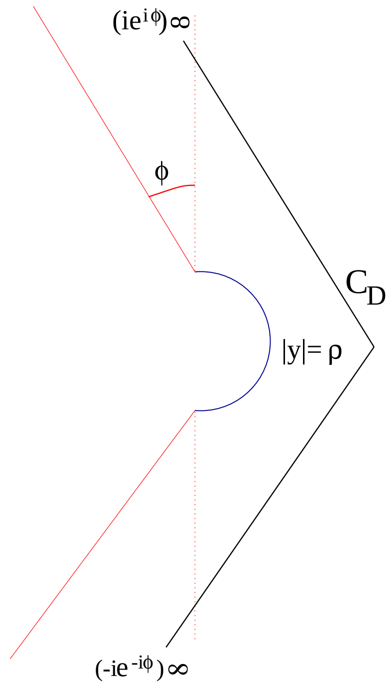

FIGURE 3.1. Contour C_D in the p-plane.

algebraically as $y \to \infty$ within $\mathcal{D}_{\rho_0}$ at a rate $y^{-\alpha_r}$, which for $\alpha_r > 1$ is much less than y^{-1}, this suggests that other terms in the differential equation (2.1) should not contribute. This is in fact shown rigorously for $|y|$ large with $\arg y \in (-\frac{2\pi}{3}, \frac{2\pi}{3})$. As shall be seen in the examples, this behavior for solution f is not valid outside this sector, where in general one can expect infinitely many singularities with an accumulation point at ∞.

3 Inverse Laplace Transform and Equivalent Integral Equation

The inverse Laplace transform $G(p,t)$ of a function $g(y,t)$ analytic in $\mathcal{D}_{\phi,\rho}$ and vanishing algebraically as $|y| \to \infty$ is given by

$$G(p,t) = \left[\mathcal{L}^{-1}\{g\}\right](p,t) \equiv \frac{1}{2\pi i} \int_{\mathcal{C}_D} e^{py} g(y,t) dy \,, \tag{3.1}$$

where $\mathcal{C}_D$ is a contour as shown in Figure 3.1 (or deformations thereof), entirely within the domain $\mathcal{D}_{\phi,\rho}$. We restrict p to the domain

$$\mathcal{S}_\phi \equiv \{p : \arg p \in (-\phi, \phi), \ 0 < |p| < \infty\}$$

It is easily seen that if $g(y,t) = y^{-\alpha}$ for $\alpha > 0$, then $G(p,t) = p^{\alpha-1}/\Gamma(\alpha)$. From the following lemma, it is clear that the same kind of behavior for the inverse

Laplace transform $G(p,t)$ can be expected for small p in $\mathcal{S}_\phi$, with a $y^{-\alpha}$ behavior of g at ∞.

LEMMA 3.1 *If $g(y,t)$ is analytic in y in $\mathcal{D}_{\phi,\rho}$ and satisfies*

$$|y^\alpha g(y,t)| < A(T) \tag{3.2}$$

for $\alpha \ge \alpha_0 > 0$, then for any $\delta \in (0,\phi)$ the inverse Laplace transform $G = \mathcal{L}^{-1}g$ exists in $\mathcal{S}_{\phi-\delta}$ and satisfies

$$|G(p,t)| < C\,\frac{A(T)}{\Gamma(\alpha)}|p|^{\alpha-1}e^{2|p|\rho} \tag{3.3}$$

for some $C = C(\delta,\alpha_0)$.

PROOF: We first consider the case when $2 \ge \alpha \ge \alpha_0$. Let C_{ρ_1} be the contour C_D in Figure 3.1 that passes through the point $\rho_1 + |p|^{-1}$ and given by $s = \rho_1 + |p|^{-1} + ir\exp(i\phi\,\mathrm{signum(r)})$ with $r \in (-\infty,\infty)$. Choosing $2\rho > \rho_1 > (2/\sqrt{3})\rho$, we have $|s| > \rho$ along the contour and therefore, with $\arg(p) = \theta \in (-\phi+\delta, \phi-\delta)$,

$$|g(s,t)| < A(T)|s|^{-\alpha} \quad \text{and} \quad |e^{sp}| \le e^{\rho_1|p|+1}e^{-|r||p|\sin|\phi-\theta|}\,.$$

Thus

$$\left|\int_{C_{\rho_1}} e^{sp}g(s,t)ds\right| \le 2A(T)e^{\rho_1|p|+1}\int_0^\infty \left|\rho_1+|p|^{-1}+ire^{i\phi}\right|^{-\alpha}e^{-|p|r\sin\delta}dr$$
$$\le \tilde{K}A(T)e^{\rho_1|p|}|\rho_1+|p|^{-1}|^{-\alpha}\int_0^\infty e^{-|p|r\sin\delta}\,dr$$
$$\le K\delta^{-1}|p|^{\alpha-1}e^{2\rho|p|} \tag{3.4}$$

where $\tilde{K}$ and K are constants independent of any parameter. Thus, the lemma follows for $2 \ge \alpha \ge \alpha_0$ if we note that $\Gamma(\alpha)$ is bounded in this range of α with the bound only depending on α_0.

For $\alpha > 2$, there exists an integer $k > 0$ so that $\alpha - k \in (1,2]$. Taking

$$(k-1)!\,h(y,t) = \int_\infty^y g(z,t)(y-z)^{k-1}\,dz$$

(clearly h is analytic in $\mathcal{D}_{\phi,\rho}$ and $h^{(k)}(y,t) = g(y,t)$), we get

$$h(y,t) = \frac{(-y)^k}{(k-1)!}\int_1^\infty g(yp,t)(p-1)^{k-1}\,dp$$
$$= \frac{(-1)^k y^{k-\alpha}}{(k-1)!}\int_1^\infty A(yp,t)p^{-\alpha}(p-1)^{k-1}\,dp$$

with $|A(yp,t)| < A(T)$, whence

$$|h(y,t)| < \frac{A(T)\Gamma(\alpha-k)}{|y|^{\alpha-k}\Gamma(\alpha)}\,.$$

From what has been already proved, with $\alpha - k$ playing the role of α,

$$|\mathcal{L}^{-1}\{h\}(p,t)| < C(\delta)\frac{A(T)}{\Gamma(\alpha)}|p|^{\alpha-k-1}e^{2|p|\rho}$$

Since $G(p,t) = (-1)^k p^k \mathcal{L}^{-1}\{h\}(p,t)$, by multiplying the above equation by $|p|^k$, the lemma follows for $\alpha > 2$ as well. □

Remarks. 1. As mentioned before, when $g(y,t) = A(t)y^{-\alpha}$ we have $G(p,t) = (A(t)/\Gamma(\alpha))p^{\alpha-1}$. In this case, the exponential factor in (3.3) can be omitted because of the algebraic behavior of $g(y,t)$ for all y. This result is relevant for Examples 1 through 3.

2. The constant 2ρ in the exponential bound can be lowered to anything exceeding ρ, but (3.3) suffices for our purposes.
3. Corollary 3.2 implies that for any $p \in \mathcal{S}_\phi$, the inverse Laplace transform exists for the specified functions $r(y,t)$ and $b_{j,k}(y,t)$, as well as the solution $f(y,t)$ to (2.1), whose existence is shown in the sequel.
4. Conversely, if $G(p,t)$ is any integrable function satisfying the exponential bound in (3.3), it is clear that the Laplace transform along a ray

$$\mathcal{L}_\theta G \equiv \int_0^{\infty e^{i\theta}} dp\, e^{-py} G(p,t) \tag{3.5}$$

exists and defines an analytic function of y in the half-plane $\Re[e^{i\theta}y] > 2\rho$ for $\theta \in (-\phi,\phi)$.
5. The next corollary shows that there exist bounds for $\mathcal{L}^{-1}\{b_{j,k}\}$ and $\mathcal{L}^{-1}\{r\}$ independent of $\arg p$ in $\mathcal{S}_\phi$ because of the assumed analyticity and decay properties in the region $\mathcal{D}_{\rho_0}$, which contains $\mathcal{D}_{\phi,\rho}$.

COROLLARY 3.2 *The inverse Laplace transform of the coefficient functions $b_{j,k}$ and the inhomogeneous function $r(y,t)$ satisfy the following upper bounds for any $p \in \mathcal{S}_\phi$:*

$$|B_{j,k}(p,t)| < \frac{C_1(\phi,\alpha_j)}{\Gamma(\alpha_j + k\beta)} A_b(T)|p|^{k\beta+\alpha_j-1}e^{2\rho_0|p|}\,, \tag{3.6}$$

$$|R(p,t)| < \frac{C_2(\phi)}{\Gamma(\alpha_r)} A_r(T)|p|^{\alpha_r-1}e^{2\rho_0|p|}\,. \tag{3.7}$$

PROOF: From the assumed conditions we see that $b_{j,k}$ is analytic in y over $\mathcal{D}_{\phi_1,\rho_0}$ for any ϕ_1 satisfying $\frac{\pi}{6} > \phi_1 > \phi$. So Lemma 3.1 can be applied for $g(y,t) = b_{j,k}$, with $\phi_1 = \phi + (\pi/6 - \phi)/2$ replacing ϕ, and with δ replaced by $\phi_1 - \phi = (\pi/6 - \phi)/2$; the same applies to $R(p,t)$, leading to (3.6) and (3.7). In the latter case, since $\alpha_r \geq 1$, α_0 in Lemma 3.1 can be chosen to be 1. Thus, one can choose C_2 to be independent of α_r, as indicated in (3.7). □

The formal inverse Laplace transform of (2.1) with respect to y is

$$F_t + p^3 F = \sum_{j=0}^{3} (-1)^j \sum_{k=0}^{\infty} \left[B_{j,k} * (p^j F) * F^{*k} \right](p,t) + R(p,t) \tag{3.8}$$

where the symbol $*$ stands for convolution (see also [5]). On formally integrating (3.8) with respect to t, we obtain the integral equation

(3.9)

$$\begin{aligned} F(p,t) &= \sum_{j=0}^{3} \sum_{k=0}^{\infty} \int_0^t (-1)^j e^{-p^3(t-\tau)} \left[(p^j F) * B_{j,k} * F^{*k} \right](p,\tau) d\tau + F_0(p,t) \\ &\equiv \mathcal{N} F(p,t) \end{aligned}$$

where

$$F_0(p,t) = e^{-p^3 t} F_I(p) + \int_0^t e^{-p^3(t-\tau)} R(p,\tau) d\tau \, . \tag{3.10}$$

Here $F_I = \mathcal{L}^{-1}\{f_I\}$.

Our strategy is to reduce the problem of existence and uniqueness of a solution of (2.1) to the problem of existence and uniqueness of a solution of (3.9) under appropriate conditions.

4 Solution to the Integral Equation

To establish the existence and uniqueness of solutions to the integral equation, we need to introduce an appropriate function class for both the solution and the coefficient functions.

DEFINITION 4.1 Denoting by $\overline{\mathcal{S}_\phi}$ the closure of $\mathcal{S}_\phi$, $\partial\mathcal{S}_\phi = \overline{\mathcal{S}_\phi} \setminus \mathcal{S}_\phi$, and $\mathcal{K} = \overline{\mathcal{S}_\phi} \times [0,T]$, we define for $\nu > 0$ (later to be taken appropriately large) the norm $\| \cdot \|_\nu$ as

$$\|G\|_\nu = M_0 \sup_{(p,t)\in\mathcal{K}} (1+|p|^2) e^{-\nu|p|} |G(p,t)| \tag{4.1}$$

where

$$M_0 = \sup_{s\geq 0} \left\{ \frac{2(1+s^2)\left(\ln(1+s^2) + s \arctan s\right)}{s(s^2+4)} \right\} = 3.76\ldots \, . \tag{4.2}$$

DEFINITION 4.2 We now define the following class of functions:

$$\mathcal{A}_\phi = \Big\{ F : F(\cdot,t) \text{ analytic in } \mathcal{S}_\phi \text{ and continuous in } \overline{\mathcal{S}_\phi} \text{ for } t \in [0,T] \text{ such that } \|F\|_\nu < \infty \Big\} \, .$$

It is clear that $\mathcal{A}_\phi$ forms a Banach space.

Comment 4.3. Note that given $G \in \mathcal{A}_\phi$, $g(y,t) = \mathcal{L}_\theta\{G\}$ exists for appropriately chosen θ when ρ is large enough so that $\rho\cos(\theta + \arg y) > \nu$, and that $|yg(y,t)|$ is bounded for y on any fixed ray in $\mathcal{D}_{\phi,\rho}$.

LEMMA 4.4 *For $\nu > 4\rho_0 + \alpha_r$, $F_I = \mathcal{L}^{-1}\{f_I\}$ satisfies*

$$\|F_I\|_\nu < C(\phi)A_{f_I}\left(\frac{\nu}{2}\right)^{-\alpha_r+1}$$

while $R = \mathcal{L}^{-1}r$ satisfies the relation

$$\|R\|_\nu < C(\phi)A_r(T)\left(\frac{\nu}{2}\right)^{-\alpha_r+1}$$

and therefore

$$\|F_0\|_\nu < C(\phi)(TA_r + A_{f_I})\left(\frac{\nu}{2}\right)^{-\alpha_r+1}. \tag{4.3}$$

PROOF: First note the bounds on R in Corollary 3.2. We also note that $\alpha_r \ge 1$ and that for $\nu > 4\rho_0 + \alpha_r$ we have

$$\begin{aligned}\sup_p \frac{|p|^{\alpha_r-1}}{\Gamma(\alpha_r)}e^{-(\nu-2\rho_0)|p|} &\le \frac{(\alpha_r-1)^{\alpha_r-1}}{\Gamma(\alpha_r)}e^{-\alpha_r+1}(\nu-2\rho_0)^{-\alpha_r+1}\\ &\le K\alpha_r^{-1/2}\left(\frac{\nu}{2}\right)^{-\alpha_r+1}\end{aligned}$$

where K is independent of any parameter. The latter inequality follows by accounting for Stirling's formula for $\Gamma(\alpha_r)$ for large α_r. Similarly,

$$\begin{aligned}\sup_p \frac{|p|^{\alpha_r+1}}{\Gamma(\alpha_r)}e^{-(\nu-2\rho_0)|p|} &\le \frac{(\alpha_r+1)^{\alpha_r+1}}{\Gamma(\alpha_r)}e^{-\alpha_r-1}(\nu-2\rho_0)^{-\alpha_r-1}\\ &\le K\alpha_r^{3/2}\left(\frac{\nu}{2}\right)^{-\alpha_r-1}.\end{aligned}$$

Using the definition of the ν-norm and the two equations above, the inequality for $\|R\|_\nu$ follows. Since $f_I(y)$ is required to satisfy the same bounds as $r(y,t)$, a similar inequality holds for $\|F_I\|_\nu$. Now, from the relation (3.10),

$$|F_0(p,t)| < |F_I(p)| + T\sup_{0\le t\le T}|R(p,t)|.$$

Therefore, (4.3) follows. ☐

Remark. Not all Laplace-transformable analytic functions in $\mathcal{D}_{\phi,\rho}$ belong to $\mathcal{A}_\phi$. For the applications we have in mind, the coefficients are not bounded near $p = 0$ and hence do not belong in $\mathcal{A}_\phi$. It is then useful to introduce the following function class:

DEFINITION 4.5

$$\mathcal{H} \equiv \left\{ H : H(p,t) \text{ analytic in } \mathcal{S}_\phi, \ |H(p,t)| < C|p|^{\alpha-1} e^{\rho|p|} \right\}$$

for some positive constants C, α, and ρ, which may depend on H.

LEMMA 4.6 *If $H \in \mathcal{H}$ and $F \in \mathcal{A}_\phi$, then for $\nu > \rho + 1$, $H * F$ belongs to $\mathcal{A}_\phi$ and satisfies the following inequality:*[1]

$$\|H * F\| \le \||H| * |F|\|_\nu \le C\Gamma(\alpha)(\nu-\rho)^{-\alpha}\|F\|_\nu \,. \tag{4.4}$$

PROOF: From elementary properties of convolution, it is clear that $H * F$ is analytic in $\mathcal{S}_\phi$ and is continuous on $\overline{\mathcal{S}_\phi}$. Let $\theta = \arg p$. It is to be noted that

$$|H * F(p)| \le ||H| * |F|(p)| \le \int_0^{|p|} |H(se^{i\theta})||F(p - se^{i\theta})|ds \,.$$

But

$$|H(se^{i\theta})| \le Cs^{\alpha-1}e^{|s|\rho}$$

and

$$\int_0^{|p|} s^{\alpha-1}e^{|s|\rho}|F(p-se^{i\theta})|ds \le \|F\|_\nu e^{\nu|p|}|p|^\alpha \int_0^1 \frac{s^{\alpha-1}e^{-(\nu-\rho)|p|s}}{M_0(1+|p|^2(1-s)^2)}\,ds \,. \tag{4.5}$$

If $|p|$ is large, noting that $\nu - \rho \ge 1$, we obtain from Watson's lemma,

$$\int_0^{|p|} s^{\alpha-1}e^{|s|\rho}|F(p-se^{i\theta})|ds \le K\Gamma(\alpha)\|F\|_\nu \frac{e^{\nu|p|}}{M_0(1+|p|^2)}|\nu-\rho|^{-\alpha} \,. \tag{4.6}$$

Now, for any other $|p|$, we obtain from (4.5),

$$\int_0^{|p|} s^{\alpha-1}e^{|s|\rho}|F(p-se^{i\theta})|ds \le K|\nu-\rho|^{-\alpha}\|F\|_\nu \frac{e^{\nu|p|}\Gamma(\alpha)}{M_0} \,.$$

Thus the relation (4.6) must hold in general because it subsumes the above relation when $|p|$ is not large. From this relation, (4.4) follows from applying the definition of $\|\cdot\|_\nu$. □

COROLLARY 4.7 *For $F \in \mathcal{A}_\phi$, and $\nu > 4\rho_0 + 1$,*

$$\|B_{j,k} * F\|_\nu \le \||B_{j,k}| * |F|\|_\nu \le KC_1(\phi,\alpha_j)\left(\frac{\nu}{2}\right)^{-k\beta-\alpha_j} A_b(T)\|F\|_\nu \,.$$

PROOF: The proof follows simply by using Lemma 4.6, with H replaced by $B_{j,k}$ and using the relations in Corollary 3.2. □

LEMMA 4.8 *For $F, G \in \mathcal{A}_\phi$ and $j \ge 0$,*

$$|(p^j F) * G(p,t)| \le \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)}\|F\|_\nu\|G\|_\nu \,. \tag{4.7}$$

[1] In the following equation, $\|\cdot\|_\nu$ is extended naturally to continuous functions in $\mathcal{K}$.

PROOF: Let $p = |p|e^{i\theta}$. Then

$$\begin{aligned}\left|\left(p^j F\right) * G(p,t)\right| &= \left|\int_0^p \tilde{s}^j F(\tilde{s}) G(p-\tilde{s}) d\tilde{s}\right| \\ &\le \int_0^{|p|} ds\, s^j \left|F\left(se^{i\theta}\right)\right| \left|G\left(p - se^{i\theta}\right)\right| .\end{aligned} \tag{4.8}$$

Using the definition of $\|\cdot\|_\nu$, the above is bounded by

$$\frac{|p|^j}{M_0^2} e^{\nu|p|} \|F\|_\nu \|G\|_\nu \int_0^{|p|} \frac{ds}{(1+s^2)[1+(|p|-s)^2]} \le \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)} \|F\|_\nu \|G\|_\nu .$$

The last inequality follows from the fact that

$$\int_0^{|p|} \frac{1}{(1+s^2)[1+(|p|-s)^2]} = 2\frac{\ln(|p|^2+1) + |p|\tan^{-1}|p|}{|p|(|p|^2+4)}$$

and the definition of M_0 in (4.2). ☐

COROLLARY 4.9 *With the convolution $*$, $\mathcal{A}_\phi$ is a Banach algebra and furthermore,*

$$\|F * G\|_\nu \le \|F\|_\nu \|G\|_\nu . \tag{4.9}$$

PROOF: This follows by applying Lemma 4.8 for $j = 0$ and using the definition of $\|\cdot\|_\nu$. ☐

LEMMA 4.10 *For $\nu > 2\rho_0 + 1$,*

$$\left|\int_0^t (p^j F) * B_{j,k} * F^{*k} e^{-p^3(t-\tau)}\, d\tau\right| \le \frac{C(\phi)}{M_0(1+|p|^2)} \| |B_{j,k}| * |F| \|_\nu \|F\|_\nu^k e^{\nu|p|} T^{(3-j)/3} \tag{4.10}$$

where the constant C is independent of T but depends on ϕ.

PROOF: For $k \ge 1$, from Lemma 4.8, with $G = (B_{j,k} * F) * F^{*(k-1)}$, we obtain

$$\begin{aligned}|(p^j F) * B_{j,k} * F^{*k}| &\le \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)} \|B_{j,k} * F\|_\nu \|F\|_\nu^k \\ &\le \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)} \| |B_{j,k}| * |F| \|_\nu \|F\|_\nu^k .\end{aligned} \tag{4.11}$$

For $k = 0$, we note that

$$|(p^j F) * B_{j,0}| \le |p|^j ||F| * |B_{j,0}|| \le \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)} \| |B_{j,0}| * |F| \|_\nu .$$

Therefore, (4.11) holds for $k = 0$ as well. Thus, for any $k \geq 0$,

$$\left| \int_0^t (p^j F) * B_{j,k} * F^{*k} e^{-p^3(t-\tau)} d\tau \right| \leq \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)} \| |B_{j,k}| * |F| \|_\nu \|F\|_\nu^k \int_0^t e^{-|p|^3 \cos(3\theta)(t-\tau)} \, d\tau$$

where $p = |p| e^{i\theta}$. On integrating with respect to τ, we obtain

$$|p|^j \int_0^t e^{-|p|^3 \cos(3\theta)(t-\tau)} d\tau \leq \frac{T^{(3-j)/3}}{\cos^{j/3} 3\phi} \sup_\gamma \frac{1 - e^{-\gamma^3}}{\gamma^{3-j}} .$$

□

DEFINITION 4.11 For $H \in \mathcal{H}$, F and h in $\mathcal{A}_\phi$, define $h_0 = 0$ and for $k \geq 1$,

$$h_k \equiv H * \left[(F+h)^{*k} - F^{*k} \right] . \tag{4.12}$$

LEMMA 4.12 *For* $\nu > \rho + 1$,

$$\|h_k\|_\nu \leq k (\|F\|_\nu + \|h\|_\nu)^{k-1} \| |H| * |h| \|_\nu . \tag{4.13}$$

PROOF: We prove this by induction. $k = 0$ follows trivially since $h_0 = 0$. The case of $k = 1$ is obvious from (4.12). Assume the formula (4.13) holds for all $k \leq l$. Then

$$\begin{aligned} \left\| h_{l+1} \right\|_\nu &= \left\| H * (F+h) * (F+h)^{*l} - H * F * F^{*l} \right\|_\nu \\ &= \left\| H * h * (F+h)^{*l} + F * h_l \right\|_\nu . \end{aligned}$$

On using (4.13) for $k = l$, we get

$$\begin{aligned} &\leq \| |H| * |h| \|_\nu (\|F\|_\nu + \|h\|_\nu)^l + l \|F\|_\nu (\|F\|_\nu + \|h\|_\nu)^{l-1} \| |H| * |h| \|_\nu \\ &\leq (l+1)(\|F\|_\nu + \|h\|_\nu)^l \| |H| * |h| \|_\nu . \end{aligned}$$

Thus the formula (4.13) holds for $k = l + 1$. □

LEMMA 4.13 *For* F *and* h *in* $\mathcal{A}_\phi$, $\nu > 2\rho_0 + 1$, *and* $k \geq 1$,

$$\left| (p^j [F+h]) * B_{j,k} * (F+h)^{*k} - (p^j F) * B_{j,k} * F^{*k} \right| \leq \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)} (\|F\|_\nu + \|h\|_\nu)^{k-1} \{ k \|F\|_\nu \| |B_{j,k}| * |h| \|_\nu + \| B_{j,k} * (F+h) \|_\nu \|h\|_\nu \} . \tag{4.14}$$

PROOF: It is clear that

$$\left| (p^j [F+h]) * B_{j,k} * (F+h)^{*k} - (p^j F) * B_{j,k} * F^{*k} \right| \leq \left| (p^j h) * B_{j,k} * (F+h)^{*k} \right| + |(p^j F) * h_k| \tag{4.15}$$

where H is now replaced by $B_{j,k}$ in the definition of h_k in (4.12). Applying Lemma 4.8 and Corollary 4.9 to the first term, we obtain for $k \geq 1$

$$\left|(p^j h) * B_{j,k} * (F+h)^{*k}\right| \leq \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)} \|h\|_\nu \|B_{j,k} * (F+h)\|_\nu (\|F\|_\nu + \|h\|_\nu)^{k-1} .$$

On the other hand, applying Lemma 4.8 and Lemma 4.12, with $H = B_{j,k}$ and ρ replaced by $2\rho_0$, we obtain

$$|(p^j F) * h_k| \leq \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)} k \|F\|_\nu \|B_{j,k} * h\|_\nu [\|F\|_\nu + \|h\|_\nu]^{k-1} .$$

Combining the previous two equations, and using it in (4.15), we obtain the proof of Lemma 4.13 by noting that $\|B_{j,k} * h\|_\nu \leq \| |B_{j,k}| * |h| \|_\nu$. □

LEMMA 4.14 *For* $\nu > 2\rho_0 + 1$,

$$\left| \int_0^t \left\{ (p^j [F+h]) * B_{j,0} - (p^j F) * B_{j,0} \right\} e^{-p^3(t-\tau)} \, d\tau \right| \leq \frac{C(\phi) T^{(3-j)/3} e^{\nu|p|}}{M_0(1+|p|^2)} \| |B_{j,0}| * |h| \|_\nu . \tag{4.16}$$

PROOF: We note that

$$\begin{aligned} \left|(p^j [F+h]) * B_{j,0} - (p^j F) * B_{j,0}\right| &\leq |(p^j h) * B_{j,0}| \\ &\leq \frac{|p|^j e^{\nu|p|}}{M_0(1+|p|^2)} \| |B_{j,0}| * |h| \|_\nu . \end{aligned}$$

Further, as before in the proof of Lemma 4.10

$$\int_0^t |p|^j \left| e^{-p^3(t-\tau)} \right| d\tau \leq C(\phi) T^{(3-j)/3} .$$

Combining the two equations above, the lemma follows. □

LEMMA 4.15 *For* $\nu > 2\rho_0 + 1$ *and* $k \geq 1$,

$$\begin{aligned} &\left| \int_0^t \left\{ (p^j [F+h]) * B_{j,k} * (F+h)^{*k} - (p^j F) * B_{j,k} * F^{*k} \right\} e^{-p^3(t-\tau)} \, d\tau \right| \leq \\ &\qquad \frac{C(\phi) T^{(3-j)/3} e^{\nu|p|}}{M_0(1+|p|^2)} (\|F\|_\nu + \|h\|_\nu)^{k-1} \\ &\qquad \cdot \left\{ k \|F\|_\nu \| |B_{j,k}| * |h| \|_\nu + \|B_{j,k} * (F+h)\|_\nu \|h\|_\nu \right\} . \end{aligned} \tag{4.17}$$

PROOF: The proof is similar to that of Lemma 4.10, except Lemma 4.13 is used instead of Lemma 4.8. □

LEMMA 4.16 *For $F \in \mathcal{A}_\phi$ and $\nu > 4\rho_0 + 1$ large enough so that $(\nu/2)^{-\beta}\|F\|_\nu < 1$, $\mathcal{N}F$ satisfies the following bounds*:

$$\|\mathcal{N}F\|_\nu \le A_b(T)\sum_{j=0}^{3} C_j(\phi)T^{(3-j)/3}\left(\frac{\nu}{2}\right)^{-\alpha_j}\frac{\|F\|_\nu}{1-(\nu/2)^{-\beta}\|F\|_\nu} + \|F_0\|_\nu\,. \tag{4.18}$$

Further, for $h \in \mathcal{A}_\phi$ such that $(\nu/2)^{-\beta}(\|F\|_\nu + \|h\|_\nu) < 1$

$$\|\mathcal{N}(F+h) - \mathcal{N}F\|_\nu \le A_b(T)\sum_{j=0}^{3} C_j(\phi)\left(\frac{\nu}{2}\right)^{-\alpha_j} T^{(3-j)/3}\frac{\|h\|_\nu}{[1-(\nu/2)^{-\beta}(\|F\|_\nu+\|h\|_\nu)]^2}\,. \tag{4.19}$$

PROOF: On inspection of (3.9) and using Lemma 4.10, it follows that

$$\|\mathcal{N}F\|_\nu \le \sum_{j=0}^{3} C_j(\phi)T^{(3-j)/3}\sum_{k=0}^{\infty}\||B_{j,k}| * |F|\|_\nu \|F\|_\nu^k + \|F_0\|_\nu\,. \tag{4.20}$$

Now using Corollary 4.7 and noting the dependence of C_1 on j through α_j, (4.18) follows. As far as (4.19), from inspection of (3.9) and using Lemmas 4.14 and 4.15, we get

$$\begin{aligned}
&\|\mathcal{N}(F+h) - \mathcal{N}F\|_\nu \\
&\quad\le \sum_{j=0}^{3} C_j(\phi)T^{(3-j)/3} \\
&\qquad\cdot\Bigg(\||B_{j,0}| * |h|\|_\nu + \sum_{k=1}^{\infty}(\|F\|_\nu + \|h\|_\nu)^{k-1} \\
&\qquad\quad\cdot\left\{k\|F\|_\nu \||B_{j,k}| * |h|\|_\nu + \|B_{j,k} * (F+h)\|_\nu \|h\|_\nu\right\}\Bigg)\,.
\end{aligned} \tag{4.21}$$

On using Corollary 4.7 and noting the dependence of C_1 on j through α_j, (4.19) follows. ☐

Remark. Lemma 4.16 is the key to showing the existence and uniqueness of solutions in $\mathcal{A}_\phi$ to (3.9), since it provides the conditions for the nonlinear operator $\mathcal{N}$ to map a ball into itself as well the necessary contractivity condition.

LEMMA 4.17 *If there exists some $b > 1$ so that*

$$\left(\frac{\nu}{2}\right)^{-\beta} b\|F_0\|_\nu < 1 \quad \textit{and} \quad A_b(T)\sum_{j=0}^{3}\frac{C_j(\phi)(\nu/2)^{-\alpha_j}T^{(3-j)/3}}{1-(\nu/2)^{-\beta}b\|F_0\|_\nu} < 1 - \frac{1}{b} \tag{4.22}$$

then the nonlinear mapping $\mathcal{N}$ maps a ball of radius $b\|F_0\|_\nu$ back into itself. Further, if

$$A_b(T)\sum_{j=0}^{3}\frac{C_j(\phi)(\nu/2)^{-\alpha_j}T^{(3-j)/3}}{[1-(\nu/2)^{-\beta}b\|F_0\|_\nu]^2}<1\,, \tag{4.23}$$

then $\mathcal{N}$ is a contraction there.

PROOF: This is a simple application of Lemma 4.16 if we simply note that $\|F\|_\nu^k < b^k\|F_0\|_\nu^k$. ◻

LEMMA 4.18 *For any given $T>0$ and ϕ in the interval $(0,\pi/6)$, for all sufficiently large ν, there exists a unique $F\in\mathcal{A}_\phi$ that satisfies the integral equation* (3.9).

PROOF: We choose $b=2$. It is clear from the bounds on $\|F_0\|_\nu$ in Lemma 4.4 that for given T, since $\alpha_r\geq 1$, $b(\nu/2)^{-\beta}\|F_0\|_\nu<1$ for all sufficiently large ν. Further, it is clear on inspection that both conditions (4.22) and (4.23) are satisfied for all sufficiently large ν. The lemma now follows from the fixed-point theorem. ◻

4.1 Behavior of Solution F_s near $p=0$

PROPOSITION 4.19 *For some $K_1>0$ and small p we have $|F_s|<K_1|p|^{\alpha_r-1}$ and thus $|f_s|<K_2|y|^{-\alpha_r}$ for some $K_2>0$ as $|y|\to\infty$ in $\mathcal{D}_{\rho,\phi}$.*

PROOF: Convergence in $\|\cdot\|_\nu$ implies uniform convergence on compact subsets of $\mathcal{K}$, and we can interchange summation and integration in (3.9). With F_s the unique solution of (3.9), we let

$$G_j=\sum_{k=0}^{\infty}(-1)^j B_{j,k}*F_s^{*k}$$

and define the linear operator $\mathcal{G}$ by

$$\mathcal{G}Q=\int_0^t e^{-p^3(t-\tau)}\sum_{j=0}^{3}(p^jQ)*G_j\,d\tau\,.$$

Clearly F_s also satisfies the linear equation

$$F_s=\mathcal{G}F_s+F_0 \quad\text{or}\quad F_s=(1-\mathcal{G})^{-1}F_0\,.$$

For $a>0$ small enough, define $\overline{\mathcal{S}}_a=\overline{\mathcal{S}}\cap\{p:|p|\leq a\}$. Since F_s is continuous in $\overline{\mathcal{S}}$, we have $\lim_{a\downarrow 0}\|\mathcal{G}\|=0$, where the norm is taken over $C(\overline{\mathcal{S}}_a)$.

By (2.2), (2.7), (3.10), and Lemma 3.1, we see that $\|F_0\|_\infty\leq K_3|a|^{\alpha_r-1}$ in $\overline{\mathcal{S}}_a$ for some $K_3>0$ independent of a. Then, as $a\downarrow 0$, we have

$$\max_{\overline{\mathcal{S}}_a}|F(p,t)|=\|F\|\leq(1-\|\mathcal{G}\|)^{-1}\max_{\overline{\mathcal{S}}_a}\|F_0\|\leq 2K_3|a|^{\alpha_r-1}\,,$$

and thus for small p we have $|F(p,t)| \le 2K_3|p|^{\alpha_r - 1}$ and the proposition follows. $\square$

PROOF OF THEOREM 2.2: Lemma 3.1 shows that if f is any solution of (2.1) satisfying Condition 2.1, then $\mathcal{L}^{-1}\{f\} \in \mathcal{A}_{\phi-\delta}$ for $0 < \delta < \phi$ for ν sufficiently large. For large y, the series (2.3) converges uniformly and thus $F = \mathcal{L}^{-1}\{f\}$ satisfies (3.9), which by Lemma 4.18 has a unique solution in $\mathcal{A}_\phi$ for any $\phi \in (0, \pi/6)$. Conversely, if $F_s \in \mathcal{A}_{\tilde{\phi}}$ is the solution of (3.9) for $\nu > \nu_1$, then from Comment 4.3, $f_s = \mathcal{L}F_s$ is analytic in $\mathcal{D}_{\phi,\rho}$ for $0 < \phi < \tilde{\phi} < \pi/6$, for sufficiently large ρ, where, in addition, from Proposition 4.19, $f_s = O(y^{-\alpha_r})$. This implies that the series in (2.1) converges uniformly, and by properties of the Laplace transform, f_s solves (2.1) and satisfies Condition 2.1. $\square$

Remark. Theorem 2.2 can be applied directly to each example in the following three sections to give existence and uniqueness of a solution in $\mathcal{D}_{\phi,\rho}$ for any given time T, provided ρ is large enough. However, this general theorem does not provide the specific dependence of ρ on T. In the following sections, we not only show that Theorem 2.2 can be applied to the examples given, but use the specific information on $b_{j,k}(y,t)$ and $r(y,t)$ to obtain dependence of ρ on T. This requires additional case-specific lemmas and theorems.

5 Example 1: Isotropic Inner Hele-Shaw Equation

This example comes in the context of solving the leading-order inner equation for a complex singularity of the conformal mapping function corresponding to an evolving Hele-Shaw flow [17, equations 5.5–5.9] as well as a two-dimensional dendrite in the small Peclet number limit, when surface energy anisotropy is small [11, equation A44] after a transformation). Consider the PDE

$$H_t = H^3 H_{xxx} \tag{5.1}$$

with the initial condition

$$H(x,0) = x^\gamma \tag{5.2}$$

where $0 < \gamma < 1$ (note that this γ is related to the β defined in [17] through $\gamma = \beta/2$), and the far-field matching condition

(5.3)
$$H(x,t) = x^\gamma + O\big(x^{4\gamma-3}\big) \quad \text{as } |x| \to \infty \text{ for } \arg x \in \left(-\frac{2\pi}{3(1-\gamma)}, \frac{2\pi}{3(1-\gamma)}\right).$$

Here γ is real and in the interval $(0,1)$.

The transformations

$$y = \frac{x^{1-\gamma}}{1-\gamma}, \quad H = x^\gamma(1 + y^{-1} f(y,t)), \tag{5.4}$$

bring the equation to the form (2.1) with

$$r(y,t) = -\gamma(\gamma-1)^{-2}(\gamma-2)y^{-2}\,, \tag{5.5}$$

and the only nonzero coefficients are $\{b_{i,j} : i, j = 0,1,2,3\}$, which are presented below as a matrix:

$$\begin{bmatrix} \dfrac{22\gamma-11\gamma^2-6}{y^3(\gamma-1)^2} & 9\dfrac{6\gamma-3\gamma^2-2}{y^4(\gamma-1)^2} & \dfrac{50\gamma-25\gamma^2-18}{y^5(\gamma-1)^2} & 2\dfrac{(1-2\gamma)(2\gamma-3)}{y^6(\gamma-1)^2} \\ \dfrac{7\gamma^2-14\gamma+6}{(\gamma-1)^2y^2} & 3\dfrac{7\gamma^2-14\gamma+6}{y^3(\gamma-1)^2} & 3\dfrac{7\gamma^2-14\gamma+6}{y^4(\gamma-1)^2} & \dfrac{7\gamma^2-14\gamma+6}{y^5(\gamma-1)^2} \\ -3y^{-1} & -9y^{-2} & -9y^{-3} & -3y^{-4} \\ 0 & 3y^{-1} & 3y^{-2} & y^{-3} \end{bmatrix}.$$

The initial condition (5.2) translates as

$$f_I(y) = 0\,. \tag{5.6}$$

Also, condition (5.3) implies that as $y \to \infty$ for $(y,t) \in \mathcal{D}_{\phi,\rho}$ (as defined earlier),

$$f(y,t) = O(y^{-2})\,. \tag{5.7}$$

From the expressions for $b_{j,k}$ above, it is possible to calculate $B_{j,k}$ explicitly. For our purpose, it is enough to note that aside from $B_{3,0}$, which is identically zero, we can write

$$|B_{j,k}(p,t)| < C|p|^{2-j+k}$$

for a constant C independent of j and k, as well as T, and therefore from Lemma 4.4 we conclude that for $F \in \mathcal{A}_\phi$, for $(j,k) \neq (3,0)$,

$$\| |B_{j,k}| * |F| \|_\nu < C\nu^{-3+j-k}\|F\|_\nu\,. \tag{5.8}$$

We also note from Lemma 4.4 that

$$\|F_0(p,t)\|_\nu < A_r\nu^{-1}T \tag{5.9}$$

where A_r is independent of T. Since only a finite number of $B_{j,k}$ are nonzero, it is better to use properties (4.20) and (4.21) directly to obtain conditions for the fixed-point theorem to apply

$$\frac{1}{b} + C\sum_{j=0}^{3}{}' \sum_{k=0}^{3} b^k A_r^k T^k \nu^{-2k+j-3} T^{(3-j)/3} < 1\,, \tag{5.10}$$

$$\sum_{j=0}^{3}{}' \sum_{k=0}^{3} b^k (k+1) A_r^k T^k \nu^{-2k+j-3} T^{(3-j)/3} < 1\,. \tag{5.11}$$

Here the primes in the summation symbol in (5.10) and (5.11) mean that the term $j = 3$, $k = 0$ is missing. Each of these conditions (5.10) and (5.11) are satisfied for $\nu^{-1}T^{1/3}$ sufficiently small for any choice of $b > 1$. The condition that T/ν^3 is less than some number translates into T/ρ^3 being sufficiently small; i.e., we are restricted to a region of space in the x-plane where $T/x^{3(1-\gamma)}$ is sufficiently small. It was noted earlier (Kadanoff, private communication, 1991, and independently by Howison [9] for a special case) that there was a similarity solution to (5.1) of the form

$$H(x,t) = t^{\gamma/[3(1-\gamma)]} h\left(\frac{x}{t^{1/[3(1-\gamma)]}}\right). \tag{5.12}$$

The resulting ordinary differential equation for $h(\eta)$ was solved numerically [17], and a first few singularities (in order of distance from the origin) of H were thus determined. It was surmised that these solutions form an infinite set that straddle the boundary of the sector $|\arg x| < 2\pi/[3(1-\gamma)]$, over which one can specify the asymptotic condition $h(\eta) \sim \eta^\gamma$ that one needs to satisfy initial and far-field conditions (5.2) and (5.3). Later these conclusions were confirmed rigorously by Fokas and Tanveer [6], who transformed the equation into Painlevé P_{II} and used isomonodromic approaches for integrable systems to confirm the earlier behavior seen numerically.

These results are also amenable to exponential asymptotic methods and formal trans-series association with actual functions, which have recently been worked out [5]. The latter method is more general than the isomonodromic methods since they apply equally well for integrable and nonintegrable equations. The application of our existence and uniqueness results mean that the only solution to the initial value problem for the PDE are those with a similarity structure given by (5.12).

From what has been discussed so far and proved in this paper, an interesting aspect of the complex plane initial value problem is that that the initial condition (5.2) is not recovered as $t \to 0^+$ except in the sector $|\arg x| < 2\pi/[3(1-\gamma)]$. This follows from the equivalence of small t with large x in the similarity solution (5.12).

6 Example 2: Hele-Shaw Inner Equation near a Zero

The second example also comes from Hele-Shaw flow [17, equations (6.10)–(6.12)] as well as dendritic crystal growth for weak undercooling and for weak anisotropic surface energy [11]. In the asymptotic limit of surface tension tending to zero, it was determined that in the neighborhood of an initial zero, the local governing equation is

$$H_t + H_x = H^3 H_{xxx} - \frac{1}{2} H^3. \tag{6.1}$$

The initial condition is

$$H(x,0) = x^{-1/2}. \tag{6.2}$$

The far-field matching condition with the "outer" asymptotic solution is

$$H(x,t) = x^{-1/2} + O(x^{-5}) \tag{6.3}$$

as $|x| \to \infty$ with $\arg x \in (-\frac{4}{9}\pi, \frac{4}{9}\pi)$. It is to be noted that in this case, unlike case I, there are no similarity solutions that satisfy both the initial and asymptotic boundary conditions. We introduce the transformation

$$x = t + \left(\frac{3y}{2}\right)^{2/3}, \quad H(x,t) = x^{-1/2} + x^{-3/2} y^{-1} f(y,t)\,. \tag{6.4}$$

Note that if x is large enough, $y \sim \frac{2}{3}x^{3/2}$. The initial condition (6.2) implies that

$$f_I(y) = 0\,, \tag{6.5}$$

and the asymptotic far-field condition (6.3) implies

$$f(y,t) = O(y^{-4/3}) \tag{6.6}$$

as $y \to \infty$ in some $\mathcal{D}_{\phi,\rho}$. Under the change of variables, the PDE is of the form (2.1) with

$$r(y,t) = -\frac{15y}{8x^{7/2}}\,, \tag{6.7}$$

and each $b_{j,k}$ containing one or more terms of the form $x^{-\beta}y^{-\delta}$, with $\beta > 0$ and $\frac{2}{3}\beta + \delta > 0$. The exact expressions for $b_{j,k}$ are given in the appendix.

Remark. Since the conditions for Theorem 2.2 hold, we may simply apply it and obtain a unique analytic solution over a sector $\mathcal{D}_{\phi,\rho}$ for a fixed ϕ satisfying conditions $0 < \phi < \pi/6$. The theorem, when applied to (6.1), would imply that for any T, there exists a unique analytic solution $H(x,t)$ in the sector $\arg x \in (-\pi/3 - 3\phi/2, -\pi/3 + 3\phi/2)$ provided $|x|$ is large enough. Since this is true for any ϕ in the interval $(0, \pi/6)$, the theorem establishes the existence of a unique analytic solution assumed before [17]. However, the restriction on how large x has to be depends on T, and because of the generality, Theorem 2.2 does not give a precise dependence on T. Finding this constraint is the objective of the rest of this section.

LEMMA 6.1 *Let $g(y,t) = x^{-\beta}y^{-\delta}$, where x is given by* (6.4), $\beta > 0$, *and* $\frac{2}{3}\beta + \delta > 0$; *then for any $p \in \mathcal{S}_\phi$,*

$$|G(p,t)| \leq C p^{\frac{2}{3}\beta + \delta - 1} \tag{6.8}$$

where C is independent of ν and T but can depend on β and δ. Also, if $\frac{2}{3}\beta + \delta > 1$, then for $\nu > 1$,

$$\|G\|_\nu \leq C\nu^{-2\beta/3 - \delta + 1}\,. \tag{6.9}$$

PROOF: Given the relation between x and y, we note that we may write

$$g(y,t) = y^{-2\beta/3 - \delta} h\big(ty^{-2/3}\big)$$

where

$$h(s) = \left(\frac{3}{2}\right)^{-2\beta/3} \left(1 + \frac{2^{2/3}}{3^{2/3}}s\right)^{-\beta}\,.$$

It is to be noted that for $\arg s$ bounded away from $\pm\pi$, $h(s)$ is uniformly bounded. It is also clear that

$$G(p,t) = \mathcal{L}^{-1} g[p,t] = \frac{1}{2\pi i} p^{2\beta/3+\delta-1} \left(\int_C e^s h\big(t p^{2/3} s^{-2/3}\big) ds \right)$$

where the contour C is similar to that shown in Figure 3.1 except in the s-plane. The intersection point of C on the real s-axis will be chosen to be 1. It is clear that for $p \in \mathcal{S}_\phi$ and s on the contour C, $\arg(tp^{2/3}s^{-2/3}) \in (-\frac{5}{9}\pi, \frac{5}{9}\pi)$. Therefore h is uniformly bounded and

$$|G(p,t)| < C|p|^{\frac{2}{3}\beta+\delta-1} \int_0^\infty e^{-r/2}\, dr \, .$$

Hence the lemma follows. ☐

Remark. Note that the preceding lemma gives a much sharper result than applying the more general Lemma 3.1, because we made specific use of the form of the function $g(y,t)$.

COROLLARY 6.2 $\|R\|_\nu < C\nu^{-1/3}$.

PROOF: This follows simply from the expression for r and application of Lemma 6.1. ☐

COROLLARY 6.3 *For some C independent of T and p,*

(i) $|B_{0,0}| < C|p|^2$, $|B_{0,1}| < C|p|^{4/3}$, $|B_{0,2}| < C|p|^3$, *and* $|B_{0,3}| < C|p|^7$.
(ii) $|B_{1,0}| < C|p|$, $|B_{1,1}| < C|p|^{8/3}$, $|B_{1,2}| < C|p|^{13/3}$, *and* $|B_{1,3}| < C|p|^6$.
(iii) $|B_{2,0}| < C$, $|B_{2,1}| < C|p|^{5/3}$, $|B_{2,2}| < C|p|^{10/3}$, *and* $|B_{2,3}| < C|p|^5$.
(iv) $|B_{3,0}| < CT|p|^{-1/3}$, $|B_{3,1}| < C|p|^{2/3}$, $|B_{3,2}| < C|p|^{7/3}$, *and* $|B_{3,3}| < C|p|^4$.

PROOF: These inequalities follow immediately from the expressions of the coefficients $b_{j,k}$ in the appendix and Lemma 6.1. ☐

COROLLARY 6.4 *For C independent of T, p, j, and k and for $\nu > 1$,*

$$\| |B_{j,k}| * |F| \|_\nu < C\nu^{2j/9-2/3-k/3} \|F\|_\nu \quad \text{for } (j,k) \neq (3,0)$$
$$\| |B_{3,0}| * |F| \|_\nu < CT\nu^{-2/3} \|F\|_\nu \, .$$

PROOF: The estimates follow immediately on examination of the upper bounds on $B_{j,k}$ and using Lemma 4.6 with $\rho = 0$. ☐

Remark. Application of Lemma 4.6 to the results of Corollary 6.3 leads to stronger bounds; however, the bounds indicated in Corollary 6.4 suffice for our purposes.

LEMMA 6.5 *For any $0 < \phi < \pi/6$, the integral equation* (3.9), *with $B_{j,k}$ and R as determined in this section, has a unique solution $F_s \in \mathcal{A}_\phi$ provided $T\nu^{-2/3} < \varepsilon$, where ε (depending only on ϕ) is small. Further, $F_s = O(p^{1/3})$ as $p \to 0$ in $\mathcal{S}_\phi$.*

PROOF: Since $F_0(p,t)$ is given by (3.10) and $F_I(y) = 0$, it follows that $\|F_0\|_\nu < CT\nu^{-1/3}$. Further, from the estimates of Corollary 6.4, it follows from (4.20) that the condition for mapping a ball of radius $b\|F_0\|_\nu$ back into itself is now given by

$$\frac{1}{b} + C \sum_{j,k=0}^{3}{}' \left(T\nu^{-2/3}\right)^{(3-j)/3} \left(bT\nu^{-2/3}\right)^k + CT\nu^{-2/3} < 1$$

where $\sum'$ denotes the summation without the $(j,k) = (3,0)$ term. From (4.21), the contractivity requirement becomes

$$C \sum_{j,k=0}^{3}{}' (k+1) b^k \left(T\nu^{-2/3}\right)^{(3-j)/3} \left(bT\nu^{-2/3}\right)^k + CT\nu^{-2/3} < 1\,.$$

It is clear that both conditions are satisfied for some $b > 1$ (say $b = 2$) if $T\nu^{-2/3}$ is chosen sufficiently small. This ensures existence of a solution in the Banach space $\mathcal{A}_\phi$. Since C in the above equations depends on ϕ, the upper bound of $T\nu^{-2/3}$ for which the solution is guaranteed to exist depends on ϕ. Further, since $r(y,t) = O(y^{-4/3})$, by applying Proposition 4.19, it follows that $F_s(p,t) = O(p^{1/3})$ as $p \to 0$ in $\mathcal{S}_\phi$. ☐

THEOREM 6.6 *For any* $0 < \phi < \frac{\pi}{6}$*, there exists a unique solution* $H(x,t)$ *satisfying* (6.1)–(6.3) *that is analytic in* x *in the domain*

$$\left\{(x,t) : |x| > \tilde{\rho},\ \arg x \in \left(-\tfrac{4}{9}\pi + \tfrac{2}{3}\phi, \tfrac{4}{9}\pi - \tfrac{2}{3}\phi\right),\ 0 \le t \le T\right\}$$

provided $T\tilde{\rho}^{-1} < \varepsilon$*, with* ε *(depending only on* ϕ*) small enough. Thus, for any* t*, when* $\arg x \in (-\frac{4}{9}\pi, \frac{4}{9}\pi)$*, there exists a unique analytic solution satisfying* (6.1)–(6.3) *when* $t/|x|$ *is small enough.*

PROOF: By applying the equivalence between the solution to the integral equation (3.9), $F_s \in \mathcal{A}_\phi$ for $0 < \phi < \pi/6$, to the analytic solution f_s of the partial differential equation (2.1) satisfying condition 2.1 (as shown in the proof of Theorem 2.2), it follows that in this particular example a solution f_s exists for $(y,t) \in \mathcal{D}_{\phi,\rho}$, provided $T\rho^{-2/3} < \varepsilon$ and that $f_s = O(y^{-4/3})$ as $y \to \infty$ in $\mathcal{D}_{\phi,\rho}$. Theorem 6.6 follows merely from noting the change of variable (y,t,f) to (x,t,H) once we choose $\tilde{\rho} = \rho^{2/3}$ and use the relation $y \sim \frac{2}{3}x^{3/2}$ for large x. ☐

7 Example 3: Strongly Anisotropic Inner Equation

For strong anisotropic surface energy, the analytically continued conformal mapping function that maps the upper half-plane to the exterior of a one-sided, two-dimensional dendritic interface for small Peclet number satisfies, upon transformation, the following leading-order inner equation near a singularity of particular type (see [11], equation (A16), after some elementary transformations):

$$H_t = H^{1/3} H_{xxx} \tag{7.1}$$

with initial and boundary conditions

$$H(x,0) = x^{-9\delta}, \tag{7.2}$$

$$H(x,t) = x^{-9\delta} + O(x^{-12\delta-3}) \quad \text{as} |x| \to \infty, \tag{7.3}$$

for $\arg x \in (-2\pi/[3(1+\delta)], 2\pi/[3(1+\delta)])$ where $\delta > 0$. If we introduce the transformation

$$y = \frac{x^{\delta+1}}{1+\delta}, \quad H(x,t) = x^{-9\delta}\left(1 + \frac{f(y,t)}{y}\right), \tag{7.4}$$

then we obtain an equation of the form (2.1) with

$$r(y,t) = \frac{9\delta(9\delta+1)(9\delta+2)}{(\delta+1)^3 y^2} \tag{7.5}$$

$$\begin{aligned} b_0(f,y,t) = &\frac{9\delta(9\delta+1)(9\delta+2)\{(1+f/y)^{4/3}-1\}y}{(\delta+1)^3 y^3 f} \\ &+ (54\delta^2 + 277\delta + 32)\frac{(1+f/y)^{1/3}}{(\delta+1)^2 y^3} \end{aligned} \tag{7.6}$$

$$b_1(f,y,t) = \left(-\frac{217\delta+26}{(1+\delta)^2} - \frac{48\delta}{\delta+1} - 6\right)\left(1+\frac{f}{y}\right)^{1/3} y^{-2} \tag{7.7}$$

$$b_2(f,y,t) = \frac{3(9\delta+1)(1+f/y)^{1/3}}{y(\delta+1)} \tag{7.8}$$

$$b_3(f,y,t) = 1 - \left(1+\frac{f}{y}\right)^{1/3}. \tag{7.9}$$

Using the series expansions in f/y, it follows that

$$\begin{aligned} b_{0,k} &= \frac{9\delta(9\delta+1)(9\delta+2)\binom{4/3}{k+1}}{(\delta+1)^3 y^{4+k}} + \frac{(54\delta^2+277\delta+32)\binom{1/3}{k+1}}{(\delta+1)^2 y^{3+k}}, \\ b_{1,k} &= \left\{-\frac{217\delta+26}{(1+\delta)^2} - \frac{48\delta}{\delta+1} - 6\right\}\binom{1/3}{k} y^{-k-2}, \\ b_{2,k} &= \frac{3(9\delta+1)\binom{1/3}{k}}{y^{k+1}(\delta+1)}, \\ b_{3,0} &= 0, \\ b_{3,k} &= -\binom{1/3}{k} y^{-k} \quad \text{for } k \geq 1. \end{aligned}$$

It is clear that

$$B_{0,k} = \frac{9\delta(9\delta+1)(9\delta+2)\binom{4/3}{k+1}p^{k+3}}{(\delta+1)^3(k+3)!} + \frac{(54\delta^2+277\delta+32)\binom{1/3}{k}p^{k+2}}{(\delta+1)^2(k+2)!},$$
$$B_{1,k} = \left\{-\frac{217\delta+26}{(1+\delta)^2} - \frac{48\delta}{\delta+1} - 6\right\}\binom{1/3}{k}\frac{p^{k+1}}{(k+1)!},$$
$$B_{2,k} = \frac{3(9\delta+1)\binom{1/3}{k}p^k}{(\delta+1)k!},$$
$$B_{3,0} = 0,$$
$$B_{3,k} = -\binom{1/3}{k}\frac{p^{k-1}}{(k-1)!} \quad \text{for } k \geq 1.$$

Thus the following estimates hold:

$$\| |B_{j,k}| * |F| \|_\nu < C\nu^{-k+j-3}\|F\|_\nu$$

where C is a constant that can be made independent of j, k, and T. Therefore, using the above relation, from (4.20), the condition for mapping a ball of radius $b\|F_0\|_\nu$ back into itself becomes

$$C\sum_{j=0}^{3}{}' \sum_{k=0}^{\infty}(\nu^{-3}T)^{(3-j)/3}(\nu^{-1}b\|F_0\|_\nu)^k + \frac{1}{b} < 1$$

where the $\sum'$ indicates that the $j=3$, $k=0$ term is missing from the summation. Applying the estimates of this section to (4.21), the contraction condition is

$$C\sum_{j=0}^{3}{}' \sum_{k=0}^{\infty}(\nu^{-3}T)^{(3-j)/3}(k+1)(\nu^{-1}b\|F_0\|_\nu)^k < 1.$$

Since $\|F_0\|_\nu < KT\nu^{-1}$, a sufficient condition for use of the contraction mapping theorem is that

$$bKT\nu^{-2} < 1,$$

i.e., that $T\nu^{-2}$ is small enough. Note that in that case $T\nu^{-3}$ is automatically small when ν is sufficiently large. The restriction $T\nu^{-2}$ small means that the differential equation (7.1), with conditions (7.2)–(7.3), has a unique analytic solution for any x with $\arg(x) \in (-2\pi/[3(1+\delta)], 2\pi/[3(1+\delta)])$ in a region where $tx^{-2\delta-2}$ is small enough. However, (7.1) admits a similarity solution

$$H(x,t) = t^{-\frac{3\delta}{1+\delta}} q\left(\frac{x}{t^{1/[3(1+\delta)]}}\right),$$

and $q(\eta)$ solves an ordinary differential equation and the asymptotic boundary condition

$$q(\eta) \sim \eta^{-9\delta} \quad \text{for } \eta \to \infty,\ \arg\eta \in \left(-\frac{2\pi}{3(1+\delta)}, \frac{2\pi}{3(1+\delta)}\right).$$

Uniqueness means that the similarity solution is the only solution. However, the restriction $tx^{-2\delta-2} \ll 1$ is suboptimal; from the similarity structure of the solution, one expects analyticity in a sector for $tx^{-3\delta-3} \ll 1$; i.e., the integral equation (3.9) should have a unique solution for $T\nu^{-3}$ sufficiently small. The condition for convergence of the infinite series involving the norm of F is stronger than the original condition for the convergence of the infinite series in f that appear in the expansion of $(1 + f/y)^{1/3}$. This is the reason for the suboptimal estimates in this example involving infinite series. Nonetheless, the uniqueness results even for a restricted range prove that the similarity solution determined earlier is the only solution to the problem.

8 Conclusion

We have proved existence and uniqueness of solution to a class of third-order, nonlinear partial differential equations in a sector of the complex spatial variable y for sufficiently large $|y|$. Our technique is akin to Borel summation, which has demonstrated its effectiveness in the analysis of general classes of nonlinear ODEs [5]. The class of PDEs for which existence and uniqueness has been proved contains three examples that arise in the context of Hele-Shaw fluid flow and dendritic crystal growth. The uniqueness results show that the similarity solutions assumed earlier for Examples 1 and 3 are the only ones that satisfy the given initial and far-field matching conditions. Accordingly, the singularities of the PDE solutions are those that correspond to singularities of the similarity solutions.

Appendix: Expressions for $b_{j,k}$ for Example 2

$$b_{0,0} = -\frac{35}{6}\frac{1}{x^{3/2}y^2} - \frac{75}{4}x^{-9/2} - \frac{45}{8}\frac{(12)^{1/3}}{x^{7/2}y^{2/3}} - \frac{15}{4}\frac{(18)^{1/3}}{x^{5/2}y^{4/3}},$$

$$b_{0,1} = -\frac{35}{2}\frac{1}{x^{5/2}y^3} - \frac{45}{x^{11/2}y} - \frac{3}{2x^2y} - \frac{45}{4}\frac{(18)^{1/3}}{x^{7/2}y^{7/3}} - \frac{135}{8}\frac{(12)^{1/3}}{x^{9/2}y^{5/3}},$$

$$b_{0,2} = -\frac{165}{4}\frac{1}{x^{13/2}y^2} - \frac{135}{8}\frac{(12)^{1/3}}{x^{11/2}y^{8/3}} - \frac{1}{2x^3y^2} - \frac{35}{2}\frac{1}{x^{7/2}y^4} - \frac{45}{4}\frac{(18)^{1/3}}{x^{9/2}y^{10/3}},$$

$$b_{0,3} = -\frac{45}{8}\frac{(12)^{1/3}}{x^{13/2}y^{11/3}} - \frac{35}{6}\frac{1}{x^{9/2}y^5} - \frac{105}{8}\frac{1}{x^{15/2}y^3} - \frac{15}{4}\frac{(18)^{1/3}}{x^{11/2}y^{13/3}},$$

$$b_{1,0} = \frac{15}{4}\frac{(18)^{1/3}}{x^{5/2}\sqrt[3]{y}} + \frac{45}{8}\frac{\sqrt[3]{y}(12)^{1/3}}{x^{7/2}} + \frac{35}{6}\frac{1}{x^{3/2}y},$$

$$b_{1,1} = \frac{35}{2}\frac{1}{x^{5/2}y^2} + \frac{45}{4}\frac{(18)^{1/3}}{x^{7/2}y^{4/3}} + \frac{135}{8}\frac{(12)^{1/3}}{x^{9/2}y^{2/3}},$$

$$b_{1,2} = \frac{35}{2}\frac{1}{x^{7/2}y^3} + \frac{135}{8}\frac{(12)^{1/3}}{x^{11/2}y^{5/3}} + \frac{45}{4}\frac{(18)^{1/3}}{x^{9/2}y^{7/3}},$$

$$b_{1,3} = \frac{35}{6}\frac{1}{x^{9/2}y^4} + \frac{15}{4}\frac{(18)^{1/3}}{x^{11/2}y^{10/3}} + \frac{45}{8}\frac{(12)^{1/3}}{x^{13/2}y^{8/3}}\,,$$

$$b_{2,0} = -3x^{-3/2} - \frac{9(18)^{1/3}y^{2/3}}{4x^{5/2}}\,,$$

$$b_{2,1} = -\frac{9}{x^{5/2}y} - \frac{27}{4}\frac{(18)^{1/3}}{x^{7/2}\sqrt[3]{y}}\,,$$

$$b_{2,2} = -\frac{9}{x^{7/2}y^2} - \frac{27(18)^{1/3}}{4x^{9/2}y^{4/3}}\,,$$

$$b_{2,3} = -\frac{3}{x^{9/2}y^3} - \frac{9(18)^{1/3}}{4x^{11/2}y^{7/3}}\,,$$

$$b_{3,0} = -1 + \frac{3y}{2x^{3/2}}\,,$$

$$b_{3,1} = \frac{9}{2}x^{-5/2}\,,$$

$$b_{3,2} = \frac{9}{2x^{7/2}y}\,,$$

$$b_{3,3} = \frac{3}{2x^{9/2}y^2}\,.$$

Acknowledgments. O. C. was supported in part by the National Science Foundation (NSF-DMS-9704968) as well as by the Ohio State University Math Research Institute, while S. T. acknowledges partial support from the National Science Foundation (NSF-DMS-9803358) and NASA (NAG 3-1947).

Bibliography

[1] Caflisch, R. E.; Orellana, O. F. Long time existence for a slightly perturbed vortex sheet. *Comm. Pure Appl. Math.* **39** (1986), no. 6, 807–838.

[2] Caflisch, R. E.; Orellana, O. F. Singular solutions and ill-posedness for the evolution of vortex sheets. *SIAM J. Math. Anal.* **20** (1989), no. 2, 293–307.

[3] Caflisch, R. E.; Orellana, O. F.; Siegel, M. A localized approximation method for vortical flows. *SIAM J. Appl. Math.* **50** (1990), no. 6, 1517–1532.

[4] Caflisch, R. E.; Semmes, S. A nonlinear approximation for vortex sheet evolution and singularity formation. *Phys. D* **41** (1990), no. 2, 197–207.

[5] Costin, O. On Borel summation and Stokes phenomena for rank-1 nonlinear systems of ordinary differential equations. *Duke Math. J.* **93** (1998), no. 2, 289–344.

[6] Fokas, A. S.; Tanveer, S. A Hele-Shaw problem and the second Painlevé transcendent. *Math. Proc. Cambridge Philos. Soc.* **124** (1998), no. 1, 169–191.

[7] Garabedian, P. R. Stability of Cauchy's problem in space for analytic systems of arbitrary type. *J. Math. Mech.* **9** (1960), 905–914.

[8] Howison, S. D. Cusp development in Hele-Shaw flow with a free surface. *SIAM J. Appl. Math.* **46** (1986), no. 1, 20–26.

[9] Howison, S. D. Complex variable methods in Hele-Shaw moving boundary problems. *European J. Appl. Math.* **3** (1992), no. 3, 209–224.

[10] Krasny, R. A study of singularity formation in a vortex sheet by the point-vortex approximation. *J. Fluid Mech.* **167** (1986), 65–93.

[11] Kunka, M. D.; Foster, M. R.; Tanveer, S. Dendritic crystal growth for weak undercooling. II. Surface energy effects on nonlinear evolution. *Phys. Rev. E (3)* **59** (1999), no. 1, part B, 673–710.

[12] Moore, D. W. The spontaneous appearance of a singularity in the shape of an evolving vortex sheet. *Proc. Roy. Soc. London Ser. A* **365** (1979), no. 1720, 105–119.

[13] Moore, D. W. Numerical and analytical aspects of Helmholtz instability. *Theoretical and applied mechanics (Lyngby, 1984)*, 263–274. North-Holland, Amsterdam–New York, 1985.

[14] Sammartino, M.; Caflisch, R. E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.* **192** (1998), no. 2, 433–461.

[15] Sammartino, M.; Caflisch, R. E. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.* **192** (1998), no. 2, 463–491.

[16] Siegel, M.; Tanveer, S.; Dai, W.-S. Singular effects of surface tension in evolving Hele-Shaw flows. *J. Fluid Mech.* **323** (1996), 201–236.

[17] Tanveer, S. Evolution of Hele-Shaw interface for small surface tension. *Philos. Trans. Roy. Soc. London Ser. A* **343** (1993), no. 1668, 155–204.

OVIDIU COSTIN
Rutgers University
Department of Mathematics
Busch Campus-Hill Center
110 Frelinghuysen Road
Piscataway, NJ 08854
E-mail: costin@math.rutgers.edu

SALEH TANVEER
Ohio State University
Department of Mathematics
100 Math Tower
231 W 18th Avenue
Columbus, OH 43210
E-mail: tanveer@math.ohio-state.edu

Received September 1999.
Revised March 2000.