TOPOLOGICAL CONSTRUCTION OF TRANSSERIES AND INTRODUCTION TO GENERALIZED BOREL SUMMABILITY

O COSTIN

Abstract. Transseries in the sense of Écalle are constructed using a topological approach. A general contractive mapping principle is formulated and proved, showing the closure of transseries under a wide class of operations.

In the second part we give an overview of results and methods reconstruction of actual functions and solutions of equations from transseries by generalized Borel summation with in ordinary and partial differential and difference equations.

MSC:

Contents

1. Introduction 1
1.1. Abstract multiseries 2
1.2. Topology on multiseries 5
1.3. Contractive operators 6
1.4. Inductive construction of logarithm-free transseries 9
1.5. The space $\mathcal{T}$ of general transseries 16
2. Equations in $\mathcal{T}$: examples 18
2.1. Multidimensional systems: transseries solutions at irregular singularities of rank one 20
3. Borel summation techniques 21
3.1. Borel summation of transseries: a first order example 22
3.2. Generalized Borel summation for rank one ODEs 26
3.3. Difference equations and PDEs 34
3.4. More general irregular singularities and multisummability 36
References 38

1. Introduction

In the first part we give a rigorous and concise construction of a space of transseries adequate for the study of a relatively large class of ordinary differential and difference equations. The main stages of the construction and the notations are similar to those in [15], but the technical details rely on a generalized contractivity principle which we prove for abstract multiseries (Theorem 15). We then discuss how various classes of problems are solved within transseries. The last part of the paper is devoted to a brief overview of generalized Borel summation, an isomorphism used to associate actual functions to transseries, in the context of ODEs, difference equations and the extension of some of these techniques to PDEs.

Various spaces of transseries were introduced roughly at the same time in logic (see [22]), analysis [14], [15], and the theory of surreal numbers [9]; similar structures were introduced independently by Berry and used as a powerful tool in applied mathematics, see [2, 3].

Informally, transseries are *asymptotic*[1]*, finitely generated* combinations powers, logarithms and exponentials. It is a remarkable fact that a wide class of functions can be asymptotically described in terms of them. As Hardy noted [19] "No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmic-exponential terms". This universality of representation can be seen as a byproduct of the fact that transseries are closed under many common operations.

It is convenient to take x^{-1}, $x \to +\infty$, as the small parameter in the transseries. When corresponding functions are considered we change variables so that the limit is again $x \to +\infty$.

A simple example of a transseries generated by x^{-1} and e^{-x} is

$$\sum_{k,m=0}^{\infty} c_{km} e^{-kx} x^{-m}$$

where $c_{km} \in \mathbb{C}$. This transseries is without logarithms or log-free, and has *level* one since it contains no iterated exponential; the simplest nontrivial transseries of level two is e^{e^x}. A more complicated example of a transseries of exponential level two, with level zero *generators* x^{-1} and $x^{-\sqrt{2}}$, level one generator $\exp(x)$, and level two generators $\exp(\sum_{k=0}^{\infty} c_k e^x x^{-k})$ and $\exp(-e^x)$ is

$$e^{\sum_{k=0}^{\infty} c_k e^x x^{-k}} + \sum_{k=0}^{\infty} d_k x^{-k\sqrt{2}} + e^{-e^x}$$

Some examples of transseries-like expressions for $x \to +\infty$, which are in fact *not* transseries, are $\sum_{k=0}^{\infty} x^k$ (it fails the asymptoticity condition) and $\sum_{k=0}^{\infty} e^{-e^{nx}}$ (it does not have finitely many generators, this property is described precisely in the sequel).

1.1. **Abstract multiseries.** The underlying structure behind the condition of asymptoticity is that of *well ordering*. In order to formalize transseries and study their properties, it is useful to first introduce and study more general abstract expansions, over a well ordered set.

1.1.1. *Totally ordered sets; well ordered sets.* Let A be an ordered set, with respect to $\leq$. If $x \not\leq y$ we write $x > y$ or $y < x$. A is **totally ordered** if any two elements are comparable, i.e., if for any $x, y \in A$ we have $x \geq y$ or $y \geq x$. If A is not totally ordered, it is called **partially ordered**.

The set A is **well ordered** with respect to $>$ if every nonempty totally ordered subset (*chain*) of A has a minimal element, i.e.

$$A' \subset A \implies \exists M \in A' \text{ such that } \forall x \in A',\ M \leq x$$

If any nonempty totally ordered subset of A has a **maximal** element, we say that A is well ordered with respect to $<$.

[1]An asymptotic expansion is one in which the terms are ordered decreasingly with respect to the order relation $\ll$, where $f_1(x) \ll f_2(x)$ if $f_1(x) = o(f_2(x))$

1.1.2. *Finite chain property.* A has the **finite descending chain** property if there is no *infinite strictly decreasing* sequence in A, in other words if $f:\mathbb{N}\mapsto A$ is decreasing, then f is constant for large n.

Proposition 1. *A is well ordered with respect to "$>$"* iff *it has the finite descending chain property.*

Proof. A strictly decreasing infinite sequence is obviously totally ordered and has no minimal element. For the converse, if there exists $A'\subset A$ such that $\forall\, x\in A'\,\exists\, y =: f(x)\in A'$, $f(x)<x$ then for $x_0\in A'$, the sequence $\{f^{(n)}(x_0)\}_{n\in\mathbb{N}}$ is an infinite descending chain in A. □

Example: multi-indices. $\mathbb{N}$ is well ordered with respect to $>$, and so is

$$\mathbb{N}^M-\mathbf{k}_0 := \{\mathbf{k}\in\mathbb{Z}^M : \mathbf{k}\ge -\mathbf{k}_0\}$$

with respect to the order relation $\mathbf{m}\ge\mathbf{n}$ iff $m_i\ge n_i\,\forall\, i\ge k$. Indeed an infinite descending sequence $\mathbf{n}_i$ would be infinitely descending on at least one component.

Proposition 2. *Let $\mathbf{k}_0\in\mathbb{Z}^M$ be fixed. Any infinite set A in $\mathbb{N}^M-\mathbf{k}_0$ contains a* strictly increasing *(infinite) sequence.*

Proof. The set A is unbounded, thus there must exist at least one component $i\le M$ so that the set $\{m_i : \mathbf{m}\in A\}$ is also unbounded; say $i=1$. We can then choose a sequence $S=\{\mathbf{m}_n\}_{n\in\mathbb{N}}$ so that $(\mathbf{m}_n)_1$ is strictly increasing. If the set $\{(\mathbf{m}_n)_j : \mathbf{m}\in A;\ j>1\}$ is bounded, then there is a subsequence S' of S so that $(m_2,...,m_n)_{n'}$ is a constant vector. Then S' is a strictly increasing sequence. Otherwise, one component, say $(\mathbf{m}_{n'})_2$ is unbounded, and we can choose a subsequence S'' so that $(m_1,m_2)_{n''}$ is increasing. The argument continues in this fashion until in at most M steps an increasing sequence is constructed. □

Corollary 3. *Any infinite set of multi-indices in $\mathbb{N}^M$ contains at least two comparable elements.*

Corollary 4. *Let A be a nonempty set of multi-indices in $\mathbb{N}^M-\mathbf{k}_0$. There exists a unique and* **finite** *minimizer set $\mathcal{M}_A$ such that none of its elements are comparable and for any $a'\in A$ there is an $a\in\mathcal{M}_A$ such that $a\le a'$.*

Proof. Consider the set C of all *maximal* totally ordered subsets of A (every chain is contained in a maximal chain; also, in view of countability, Zorn's lemma is not needed). Let $\mathcal{M}_A$ be the set of the least elements of these chains, i.e. $\mathcal{M}_A=\{\min c : c\in C\}$. Then $\mathcal{M}_A$ is finite. Indeed, otherwise, by Corollary 4 at least two elements in $\mathcal{M}_A$ such that $a_1'<a_2'$. But this contradicts the maximality of the chain whose least element was a_2'. It is clear that if $\mathcal{M}_A'$ is a minimizer then $\mathcal{M}_A'\supset\mathcal{M}_A$. Conversely if $m\in\mathcal{M}_A'\setminus\mathcal{M}_A$ then $m\not<a$, $\forall a\in\mathcal{M}_A$ contradicting the definition of $\mathcal{M}_A$.

□

1.1.3. *Definition and properties of abstract multiseries.* If $\mathcal{G}$ is a commutative group with an order relation, we call it an **abelian ordered group** if the order relation is compatible with the group operation, i.e., $a\le A$ and $b\le B\implies ab\le AB$ (e.g. $\mathbb{R}$ or $\mathbb{Z}^M$ with addition). Let $\mathcal{G}$ be an abelian ordered group and let $\mu:\mathbb{Z}^M\mapsto\mathcal{G}$ be a *decreasing group morphism*, i.e.,

(1) $\mu_{\mathbf{0}}=1$.
(2) $\mu_{\mathbf{k}_1+\mathbf{k}_2}=\mu_{\mathbf{k}_1}\mu_{\mathbf{k}_2}$.

(3) $\mathbf{k}_1 > \mathbf{k}_2 \implies \mu_{\mathbf{k}_1} < \mu_{\mathbf{k}_2}$. Then $\mu(\mathbb{Z})$ is the subgroup finitely generated by $\mu_{\mathbf{e}_j}; j = 1, ..., M$ where $\mathbf{e}_j$ is the unit vector in the direction j in $\mathbb{Z}^M$, and since $\mathbf{e}_j > 0$ it follows that

$$\mu_{\mathbf{e}_j} < 1, \ j = 1, ..., M$$

In view of our final goal, a simple example to keep in mind is the multiplicative group of *monomials* $\mathcal{G}_1$, generated by the functions $x^{-1/2}$, $x^{-1/3}$ and e^{-x}, for large positive x. The order relation on $\mathcal{G}_1$ is $\mu_1 < \mu_2$ if $|\mu_1(x)| < |\mu_2(x)|$ for all large x. When $M = 3$ we choose $\mu(\mathbf{k}) = x^{-k_1/2-k_2/3}e^{-k_3 x}$.

Remark 1. *The relation $\mu_{\mathbf{k}_1} = \mu_{\mathbf{k}_2}$ induces an equivalence relation on $\mathbb{Z}^r$; we denote it by $\equiv$.*

For instance in $\mathcal{G}_1$, since $1/2$ and $1/3$ are not rationally independent, there exist distinct $\mathbf{k}', \mathbf{k}$ so that $\mathbf{k}' \in \mathbb{N}^3 : \mu_{\mathbf{k}'} = \mu_{\mathbf{k}}$.

Remark 2. *Clearly any choice of μ_i with $\mu_i < 1$ for $i = 1, ..., M$ defines an order preserving morphism via $\mu(\mathbf{k}) = \prod_{i=1}^M \mu_i^{k_i}$.*

Ordered morphisms preserve well-ordering:

Proposition 5. *Let $P \subset \mathbb{Z}^M$ be* well ordered *(an important example for us is $P = \mathbb{N}^M - \mathbf{k}_0$) and μ an order preserving morphism. Then $\mu(P)$ is well ordered.*

Proof. Assuming the contrary, let $J = \{\mathbf{k}_n\}_{n\in\mathbb{N}}$ be such that $\boldsymbol{\mu}_J = \{\boldsymbol{\mu}_{\mathbf{k}_n}\}_{n\in\mathbb{N}}$ is an infinite strictly *ascending* chain in $\mu(P)$. Then the index set J is clearly infinite, and then, by Proposition 2 it has a strictly increasing subsequence J'. Then $\boldsymbol{\mu}_{J'}$ is a *descending* subsequence of $\boldsymbol{\mu}_J$, which is a contradiction. ☐

Corollary 6. *For any $\mathbf{k} \in \mathbb{Z}^M$, the set $\{\mathbf{k}' \in \mathbb{N}^M - \mathbf{k}_0 : \mu_{\mathbf{k}'} = \mu_{\mathbf{k}}\}$ is finite. In particular, given $\mathbf{k}''$, the set $\{\mathbf{k}, \mathbf{k}' \in \mathbb{N}^M - \mathbf{k}_0 : \mathbf{k} + \mathbf{k}' = \mathbf{k}''$ is finite.*

Proof. By Proposition 2, the contrary would imply the existence of an strictly increasing subsequence of $\mathbf{k}'$, for which then $\{\mu_{\mathbf{k}'}\}$ would be strictly decreasing, a contradiction. The last part follows if we take $\mu_{\mathbf{k}} = \mathbf{k}$. ☐

Proposition 7. *The space of formal series*[2]

$$\tilde{\mathcal{A}}(\mu_1, ..., \mu_M) = \tilde{\mathcal{A}} = \{\tilde{S} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\mu_{\mathbf{k}} : \mathbf{k}_0 \in \mathbb{Z}^M, \, c_{\mathbf{k}} \in \mathbb{C}\}$$

is an algebra with respect to componentwise multiplication by scalars, componentwise addition, and the inner multiplication

$$\tilde{S}\tilde{S}' = \sum_{\mathbf{k}\ge\mathbf{k}_0} \sum_{\mathbf{k}'\ge\mathbf{k}'_0} c_{\mathbf{k}}c_{\mathbf{k}'}\mu_{\mathbf{k}+\mathbf{k}'} = \sum_{\mathbf{k}''\ge\mathbf{k}_0+\mathbf{k}'_0} \mu_{\mathbf{k}''}c_{\mathbf{k}''} \tag{1}$$

where

$$c_{\mathbf{k}''} = \sum_{\substack{\mathbf{k}\ge\mathbf{k}_0, \mathbf{k}'\ge\mathbf{k}_0 \\ \mathbf{k}+\mathbf{k}'=\mathbf{k}''}} c_{\mathbf{k}}c_{\mathbf{k}'} \tag{2}$$

The same is true for $\tilde{\mathcal{A}}(\mu_1, ..., \mu_M)$ factored by the equivalence relation

$$\tilde{S} \equiv \tilde{S}' \iff \sum_{\mathbf{k}'\equiv\mathbf{k}} (c_{\mathbf{k}'} - c'_{\mathbf{k}'}) = 0 \ \forall \, \mathbf{k} \ge \mathbf{k}_0 \tag{3}$$

[2] i.e. the space of real or complex *functions* on $\mu(\mathbb{N}^M - \mathbf{k}_0)$ with usual addition and convolution (2).

replacing $\mathbf{k}+\mathbf{k}' = \mathbf{k}''$ *with* $\mathbf{k}+\mathbf{k}' \equiv \mathbf{k}''$ *in (1) (note that by Corollary 6 the equivalence classes have finitely many elements).*

Proof. Straightforward. □

We define $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ the linear subspace of $\tilde{\mathcal{A}}$ for which $\mathbf{k}_0$ is fixed.

Definition 8. *The sum*

$$\tilde{S}_c = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\mu_{\mathbf{k}}$$

is in collected *form if, by definition,* $c_{\mathbf{k}} \neq 0 \implies \mathbf{k} = \max\{\mathbf{k}' : \mathbf{k}' \equiv \mathbf{k}\}$, *where the maximum is with respect to the lexicographic order. (In other words the coefficients are collected and assigned to the earliest* μ *in its equivalence class.)*

It is then natural to represent the equivalence class (v. (3)) $\{\tilde{S}\}$ *of* $\tilde{S}$, *in* $\tilde{\mathcal{A}}_{\mathbf{k}_0}/\equiv$, *by* $\tilde{S}_c$.

Corollary 9. *By Proposition 7 every nonzero sum can be written in collected form.*

1.2. **Topology on multiseries.** A topology is introduced in the following way:

Definition 10. The sequence $\tilde{S}^{(n)}$ in $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ converges in the asymptotic topology if for any $\mathbf{k}$, $c_{\mathbf{k}}^{(n)}$ becomes constant (with respect to n) eventually. *This induces a natural topology on* $\tilde{\mathcal{A}}_{\mathbf{k}_0}/\equiv$.

This topology is metrizable. Indeed, any bounded $w : \mathbb{N}^M \mapsto (0,\infty)$ such that $w(\mathbf{k}) \to 0$ iff all $k_i \to \infty$ (e.g. $w(\mathbf{k}) = \sum_{i\le M} e^{-k_i}$) provides a translation-invariant distance

$$d(\tilde{S}^{(1)},\tilde{S}^{(2)}) = \sup_{\mathbf{k}\ge\mathbf{k}_0} \theta(c_{\mathbf{k}}^{(1)} - c_{\mathbf{k}}^{(2)})w(\mathbf{k}-\mathbf{k}_0)$$

where $\theta(x) = 0$ if $x = 0$ and is one otherwise. A Cauchy sequence in $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ is clearly convergent, and in this sense $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ is a complete topological algebra. From this point on, we assume $\mathcal{G}$ is a **totally ordered abelian group**. Let $\tilde{S} \in \tilde{\mathcal{A}}$.

Remark 3. *A subgroup of* $\mathcal{G}$ *generated by* n *elements* $\mu_1 < 1, ..., \mu_n < 1$ *is totally ordered and well ordered, and thus can be indexed by a set of ordinals* Ω, *in such a way that* $\omega_1 < \omega_2$ *implies* $\mu_{\omega_1} > \mu_{\omega_2}$. *A sum*

$$\tilde{S} = \sum_{\omega\in\Omega_{\tilde{S}}} c_\omega\mu_\omega \tag{4}$$

where we agree to omit from $\Omega_{\tilde{S}}$ *all ordinals for which* $c_\omega = 0$ *is called the* asymptotic *form of* $\tilde{S}$.

Definition 11. Dominant term. *Assume* $\tilde{S} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\mu_{\mathbf{k}} \in \mathcal{A}_{\mathbf{k}_0}$ *is presented in collected form. The set* $\mu_{\mathbf{k}} : c_{\mathbf{k}} \neq 0$ *is then totally ordered and must have a maximal element* $\mu_{\mathbf{k}_1}$, *by Proposition 5. We say that* $c_{\mathbf{k}_1}\mu_{\mathbf{k}_1} =: \mathrm{dom}(\tilde{S})$ *is the dominant term of* $\tilde{S}$ *and* $\mu_{\mathbf{k}_1} =: \mathrm{mag}(\tilde{S})$ *is the* (dominant) magnitude *of* $\tilde{S}$ *(equivalently,* $\mathrm{mag}(\tilde{S}) = \mu_{\min\Omega_{\tilde{S}}}$*). We allow for* $\mathrm{mag}(\tilde{S})$ *to be zero,* **iff** $\tilde{S} = 0$.

The following property is an immediate consequence of Corollar9:

Remark 4. *For any nonzero* $\tilde{S}$ *we can write*

$$\tilde{S} = c_{\mathbf{k}_1}\,\mathrm{mag}(\tilde{S})(1+\sum c'_{\mathbf{k}'}\mu_{\mathbf{k}'}) = \mathrm{dom}(\tilde{S})(1+\tilde{S}_1)$$

where all the terms in $\tilde{S}_1$ *are less than one.*

Remark 5. *The magnitude is continuous: if $\tilde{S}_{\mathbf{k}} \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$ and $\tilde{S}_{\mathbf{k}} \to \tilde{S}$ in the asymptotic topology, then* $\operatorname{mag}(\tilde{S}_{\mathbf{k}}) \to \operatorname{mag}(\tilde{S})$ *(i.e. $\exists \mathbf{k}_1$ so that* $\operatorname{mag}(\tilde{S}) = \operatorname{mag}(\tilde{S}_{\mathbf{k}})$, $\forall \mathbf{k} \geq \mathbf{k}_1$*).*

Proof. This follows immediately from the definition of the topology and of $\operatorname{mag}(\cdot)$. □

The proposition below discusses the closure of $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ under restricted *infinite sums.*

Proposition 12. *Let $\mathbf{j}_0, \mathbf{k}_0, \mathbf{l}_0 \in \mathbb{Z}^M$ with $\mathbf{k}_0 + \mathbf{l}_0 = \mathbf{j}_0$ and consider the sequence in $\tilde{\mathcal{A}}_{\mathbf{k}_0}$*

$$\tilde{S}^{(\mathbf{m})} = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}}^{(\mathbf{m})} \mu_{\mathbf{k}}$$

and a fixed $T \in \tilde{\mathcal{A}}_{\mathbf{l}_0}$,

$$T = \sum_{\mathbf{k} \geq \mathbf{l}_0} c'_{\mathbf{k}} \mu_{\mathbf{k}}$$

Then the sum ("blending")

$$T(\tilde{S}) := \sum_{\mathbf{k} \geq \mathbf{l}_0} c'_{\mathbf{k}} \mu_{\mathbf{k}} \cdot \tilde{S}^{(\mathbf{k})}$$

obtained by replacing each $\mu_{\mathbf{k}}$ in T by the product $\mu_{\mathbf{k}} \cdot \tilde{S}^{(\mathbf{k})}$ is well defined in $\tilde{\mathcal{A}}_{\mathbf{j}_0}$ as the limit as $\mathbf{k}' \uparrow \infty$ of the convergent sequence of truncates

$$T^{[\mathbf{k}']}(\tilde{S}) = \sum_{\mathbf{l}_0 \leq \mathbf{k} \leq \mathbf{k}'} c'_{\mathbf{k}} \mu_{\mathbf{k}} \cdot \tilde{S}^{(\mathbf{k})} = \sum_{\mathbf{j} \geq \mathbf{j}_0} \mu_{\mathbf{j}} \sum_{B_{\mathbf{j}}^{(\mathbf{k}')}} c'_{\mathbf{m}_1} c_{\mathbf{m}_2}^{(\mathbf{m}_1)} = \sum_{\mathbf{j} \geq \mathbf{j}_0} d_{\mathbf{j}}^{(\mathbf{k}')} \mu_{\mathbf{j}} \tag{5}$$

where

$$B_{\mathbf{j}}^{(\mathbf{k}')} = \{\mathbf{m}_1, \mathbf{m}_2 : \mathbf{m}_1 + \mathbf{m}_2 = \mathbf{j}, \mathbf{l}_0 \leq \mathbf{m}_1 \leq \mathbf{k}',\ \mathbf{m}_2 \geq \mathbf{k}_0\}$$

Proof. Given $\mathbf{j}$, the coefficient $d_{\mathbf{j}}^{(\mathbf{k}')}$ is constant for large $\mathbf{k}'$. Indeed, in the expression of $B_{\mathbf{j}}^{(\mathbf{k}')}$ we have $\mathbf{l}_0 \leq \mathbf{m}_2 = \mathbf{j} - \mathbf{m}_1 \leq \mathbf{j} - \mathbf{k}_0$ and similarly $\mathbf{k}_0 \leq \mathbf{m}_2 \leq \mathbf{j} - \mathbf{l}_0$ and therefore there is a bound independent of $\mathbf{k}'$ on the number of elements in the set $B_{\mathbf{j}}^{(\mathbf{k}')}$. On the other hand, we obviously have $B_{\mathbf{j}}^{(\mathbf{k}')} \subset B_{\mathbf{j}}^{(\mathbf{k}'')}$ if $\mathbf{k}'' > \mathbf{k}'$. Thus the set $B_{\mathbf{j}}^{(\mathbf{k}')}$ is constant if $\mathbf{k}'$ is large enough, and thus $d_{\mathbf{j}}^{(\mathbf{k}')}$ is constant for all large $\mathbf{k}'$, which means the sums $T^{[\mathbf{k}']}(\tilde{S})$ are convergent in the asymptotic topology. □

Note. The condition that $\operatorname{mag}(\tilde{S})^{(m)}$ decreases strictly in m does **not** suffice for $\sum_{m \geq 0} c_m \tilde{S}^{(m)}$ to be well defined. Indeed, the terms $\tilde{S}^{(m)} = x^{-m} + e^{-x} \in \mathcal{G}_1$ (cf. § 1.1.3) have strictly decreasing magnitudes and yet the formal expression $\sum_{m \geq 0}(x^{-m} + e^{-x})$ is meaningless.

1.3. **Contractive operators.**

Definition 13. *Let J be a linear operator from $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ or from one of its subspaces, to $\tilde{\mathcal{A}}_{\mathbf{k}_0}$,*

$$J\tilde{S} = J \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} \mu_{\mathbf{k}} = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} J \mu_{\mathbf{k}} \tag{6}$$

Then J is called asymptotically contractive on $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ if

$$J\mu_{\mathbf{j}} = \sum_{\mathbf{p}>0} c_{\mathbf{j};\mathbf{p}} \mu_{\mathbf{j}+\mathbf{p}} \tag{7}$$

thus

$$J\tilde{S} = \sum_{\mathbf{k}\geq\mathbf{k}_0} \mu_{\mathbf{k}} \sum_{\substack{\mathbf{j}+\mathbf{p}\equiv\mathbf{k} \\ \mathbf{p}>0;\mathbf{j}\geq\mathbf{k}_0}} c_{\mathbf{j};\mathbf{p}} c_{\mathbf{j}} \tag{8}$$

We note that by (7) and Proposition 12, J is well defined.

Definition 14. *The linear or nonlinear operator J is (asymptotically) contractive in the set $A \subset \mathcal{A}_{\mathbf{k}_0}$ if $J : A \mapsto A$ and the following condition holds. Let f_1 and f_2 in A be arbitrary and let*

$$f_1 - f_2 = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}} \mu_{\mathbf{k}} \tag{9}$$

Then

$$J(f_1) - J(f_2) = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}} \mu_{\mathbf{k}} \tilde{S}_{\mathbf{k}} \tag{10}$$

where $\operatorname{mag}(\tilde{S}_{\mathbf{k}}) = \mu_{\mathbf{p}_{\mathbf{k}}}$ *for some* $\mathbf{p}_{\mathbf{k}} > 0$.

Remark 6. *The sum of asymptotically contractive operators is contractive; the composition of contractive operators, whenever defined, is contractive.*

Theorem 15. *(i) If J is linear and contractive on $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ then for any $\tilde{S}_0 \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$ the fixed point equation $\tilde{S} = J\tilde{S} + \tilde{S}_0$ has a unique solution $\tilde{S} \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$.*

(ii) In general, if $A \in \mathcal{A}_{\mathbf{k}_0}$ is closed and $J : A \mapsto A$ is a (linear or nonlinear) contractive operator on A, then $\tilde{S} = J(\tilde{S})$ has a unique solution is A.

Proof. (i) Uniqueness: if we had two solutions $\tilde{S}_1$ and $\tilde{S}_2$ we would get $\operatorname{mag}(\tilde{S}_1 - \tilde{S}_2) = \operatorname{mag}(J(\tilde{S}_1 - \tilde{S}_2)) < \operatorname{mag}(\tilde{S}_1 - \tilde{S}_2)$ by (7). We show that $J^n\tilde{S}_0 \to 0$ in the asymptotic topology implying that $\sum_{n=0}^{\infty} J^n\tilde{S}_0$ is convergent. We have

$$J^n\tilde{S} = \sum_{\mathbf{k}\geq\mathbf{k}_0} \mu_{\mathbf{k}} \sum_{\substack{\mathbf{p_1}>0,\dots,\mathbf{p}_n>0;\mathbf{j}\geq\mathbf{k}_0 \\ \mathbf{j}+\mathbf{p}_1+\cdots+\mathbf{p}_n\equiv\mathbf{k}}} const_{\mathbf{p_1};\dots,\mathbf{p_n};\mathbf{j}} \tag{11}$$

and since $|\mathbf{p}_1 + \cdots + \mathbf{p}_n| > n$ and since the set $\{\mathbf{k}' : \mathbf{k}' \equiv \mathbf{k}\}$ is finite by Corollary 6, for each $\mathbf{k}$ the condition $\mathbf{j}+\mathbf{p}_1+\cdots+\mathbf{p}_n \equiv \mathbf{k}$ becomes impossible if n exceeds some n_0, and then the coefficient of $\mu_{\mathbf{k}}$ in $J^n\tilde{S}$ is zero for $n > n_0$.

(ii) Uniqueness follows in the same way as in the linear case. For existence, prove the convergence of the recurrence $f_{n+1} = J(f_n)$.

With $f_n - f_{n-1} = \delta_{n-1} = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}}^{(n-1)} \mu_{\mathbf{k}}$ we have, for some coefficients $C_{\mathbf{m};\mathbf{m}_1}^{(n-1)}$

$$\begin{aligned} \delta_n = J(f_{n-1} + \delta_{n-1}) - J(f_{n-1}) &= \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}}^{(n-1)} \mu_{\mathbf{k}} \tilde{S}_{\mathbf{k}}^{(n-1)} \\ &= \sum_{\mathbf{k}\geq\mathbf{k}_0} \mu_{\mathbf{k}} \sum_{\substack{\mathbf{m}+\mathbf{p}=\mathbf{k} \\ \mathbf{m}\geq\mathbf{k}_0;\mathbf{p}>0}} c_{\mathbf{m}}^{(n-1)} C_{\mathbf{m};\mathbf{p}}^{(n-1)} := \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}}^{(n)} \mu_{\mathbf{k}} \end{aligned} \tag{12}$$

and therefore δ_n has an expression similar to (11),

$$\delta_n = \sum_{\mathbf{k}\ge\mathbf{k}_0} \mu_{\mathbf{k}} \sum_{\substack{\mathbf{p_1}>0,\dots,\mathbf{p}_n>0;\mathbf{j}\ge\mathbf{k}_0\\ \mathbf{j}+\mathbf{p}_1+\dots+\mathbf{p}_n\equiv\mathbf{k}}} const_{\mathbf{p_1};\dots,\mathbf{p_n};\mathbf{j}}$$

Consequently $\delta_n \to 0$ and, as before, it follows that $\sum_n \delta_n$ converges. □

Corollary 16. *Let* $\tilde{S} \in \mathcal{A}_0$ *be arbitrary and* $\tilde{S}_n = \sum_{\mathbf{k}>0} c_{\mathbf{k};n}\mu_{\mathbf{k}} \in \mathcal{A}_0$ *for* $n \in \mathbb{N}$. *Then the operator defined by*

$$J(y) = \tilde{S} + \sum_{n\ge 2} \tilde{S}_{n-2} y^n$$

is contractive in the set $\{y : \operatorname{mag}(y) < 1\}$.

Proof. We have

$$J(y+\delta) - J(y) = \delta \sum_{n\ge 2} \tilde{S}_{n-2} \left(\sum_{j=1}^{n-1} y^j \delta^{n-j} \right)$$

□

1.3.1. *The field of finitely generated formal series.* Let $\mathcal{G}$ be a totally ordered abelian group. We now define the algebra:

(13) $$\tilde{\tilde{S}} = \bigcup_{\substack{\mathbf{k}_0\in\mathbb{Z}^M\\ \mu_1<1,\dots,\mu_M<1\\ M\in\mathbb{N}}} \tilde{\mathcal{A}}_{\mathbf{k}_0}(\mu_1,\dots,\mu_M)$$

modulo the obvious inclusions, and with the induced topology (convergence in $\tilde{\tilde{S}}$ means convergence in one of the $\tilde{\mathcal{A}}_{\mathbf{k}_0}(\mu_1,\dots,\mu_M)$).

Product form.

Proposition 17. *Any* $\tilde{S} \in \tilde{\tilde{S}}$ *can be written in the form*

(14) $$c \operatorname{mag}(\tilde{S}) \left(1 + \sum_{\mathbf{k}>0} c_{\mathbf{k}} \mu_{\mathbf{k}} \right)$$

i.e.,

(15) $$c \operatorname{mag}(\tilde{S})(1 + \tilde{S}_1)$$

where $\tilde{S}_1 \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$ *for some* $\mathbf{k}_0 > 0$ *(cf. also Remark 4) and* $\operatorname{mag}(\tilde{S}_1) < 1$.

Proof. We have, by Remark 4,

(16) $$\tilde{S} = c \operatorname{mag}(\tilde{S}) \left(1 + \sum_{\mathbf{k}\ge\mathbf{k}_0} c'_{\mathbf{k}} \mu_1^{k_1} \cdots \mu_M^{k_M} \operatorname{mag}(\tilde{S})^{-1} \right)$$

where all the elements in the last sum are less than one.

Let A be the set of multi-indices in the sum in (16) for which *some* $k_i < 0$ and let A' be its minimizer in the sense of Corollary 4, a finite set. We now consider the extended set of generators

$$\{\overline{\mu}_i : i \le M'\} := \{\nu \operatorname{mag}(\tilde{S})^{-1} : \nu = \mu_i \text{ with } i \le M \text{ or } \nu = \mu_{\mathbf{k}} \text{ with } \mathbf{k} \in A'\}$$

We clearly have $\overline{\mu}_i < 1$. By the definition of A', for any term in the sum in (16) either $\mathbf{k} > 0$ or else $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$ with $\mathbf{k}' \in A'$ and $\mathbf{k}'' \ge 0$. In both cases we have $c_{\mathbf{k}}\mu_{\mathbf{k}} = c_{\mathbf{k}'}\overline{\mu}_{\mathbf{k}'}$ with $\mathbf{k}' \in \mathbb{Z}^{M'}$ and $\mathbf{k}' > 0$. Thus $\tilde{S}$ can be rewritten in the form

$$c \operatorname{mag}(\tilde{S}) \left(1 + \sum_{\mathbf{k}>0} c_{\mathbf{k}}\mu_{\mathbf{k}}\right) \quad (\mathbf{k} \in \mathbb{N}^{M'})$$

where the assumptions of the Proposition are satisfied.

□

Proposition 18. *$\tilde{\tilde{S}}$ is a field.*

Proof. The only property that needs verification is the existence of a reciprocal for any nonzero $\tilde{S}$. Using Proposition 17 we only need to consider the case when

$$\tilde{S} = 1 + \sum_{\mathbf{k}>0} c_{\mathbf{k}}\mu^{\mathbf{k}}$$

Since multiplication by t is manifestly contractive (see § 1.3), $\tilde{S}^{-1}$ is the solution (unique by Proposition 15) of

$$\tilde{S}^{-1} = 1 - t\tilde{S}^{-1}$$

□

Closure under infinite sums.

Corollary 19. *(i) Let $\mathbf{k}_0 > 0$ and $\tilde{S} \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$ and $\{c_n\}_{n\in\mathbb{N}} \in \mathbb{C}$ be any sequence. Then*

$$\sum_{n=0}^{\infty} c_n\tilde{S}^n \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$$

(ii) More generally, if $\tilde{S}_{01}, ..., \tilde{S}_{0M}$ are of the form $\tilde{S}_0$ and $\{c_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^M}$ is a multi-sequence of constants, then $\sum_{\mathbf{k}\ge 0} c_{\mathbf{k}}\tilde{S}_0^{\mathbf{k}} = \sum_{\mathbf{k}\ge 0} c_{\mathbf{k}}\tilde{S}_{01}^{k_1}\cdots\tilde{S}_{0M}^{k_M}$ is well defined.

Proof. For some $C \in \mathbb{C}$ we have $\tilde{S} = C \operatorname{mag}(\tilde{S})(1+t) = C(1+t)\mu$. Since $\mu < 1$ we have $\sum_{n=0}^{\infty} c_n C^n \mu^n \in \tilde{\mathcal{A}}_0(\mu)$ and by Proposition 12

$$\sum_{n=0}^{\infty} c_n\tilde{S}^n = \sum_{n=0}^{\infty} c_n C^n \mu_{n\mathbf{k}_1}(1+t)^n \in \tilde{\tilde{S}}$$

□

Formal series with real coefficients. Order relation. Let $\tilde{\tilde{S}}_{\mathbb{R}}$ the subfield consisting in $\tilde{S} \in \tilde{\tilde{S}}_{\mathbb{R}}$ which have real coefficients. We say that $\tilde{\tilde{S}}_{\mathbb{R}} \ni \tilde{S} > 0$ if $\operatorname{dom}(\tilde{S})/\operatorname{mag}(\tilde{S}) > 0$. Then every nonzero $\tilde{S} \in \tilde{\tilde{S}}_{\mathbb{R}}$ is either positive or else $-\tilde{S}$ is positive. This induces a *total* order relation on $\tilde{\tilde{S}}_{\mathbb{R}}$, by writing $\tilde{S}_1 > \tilde{S}_2$ if $\tilde{S}_1 - \tilde{S}_2 > 0$.

1.4. **Inductive construction of logarithm-free transseries.**

1.4.1. *Transseries.* Transseries are constructed as a special instance of abstract series in which the ablelian ordered group is constructed inductively.

In constructing spaces of transseries, one aims at constructing differential fields containing x^{-1}, closed under all operations of importance for a certain class of problems operations. Smaller closed spaces can be endowed with better overall properties.

The construction presented below differs in a number of technical respects from the one of Écalle, and the transseries space constructed here is smaller than his. Still some of the construction steps and the structure of the final object are similar enough to Écalle's, to justify using his terminology and notations.

1.4.2. *Écalle's notation.*

- $\sqcup$ —small transmonomial.
- $\sqcap$ —large transmonomial.
- $\Box$ —any transmonomial, large or small.
- $\sqcup\!\sqcup$ —small transseries.
- $\sqcap s \urcorner$ —large transseries.
- $\Box\!\Box$ —any transseries, small or large.

1.4.3. *Level 0: power series.* Let x be large and positive and let $\mathcal{G}$ be the totally ordered multiplicative group $(x^{\sigma}, \cdot, \ll), \sigma \in \mathbb{R}$, with $x^{\sigma_1} \ll x^{\sigma_2}$ if $x^{\sigma_1} = o(x^{\sigma_2})$ as $x \to \infty$, i.e., if $\sigma_1 < \sigma_2$. The space of level zero log-free transseries is by definition $\tilde{\mathcal{T}}^{[0]} = \tilde{\tilde{\tilde{S}}}(\mathcal{G})$. By Proposition 18, $\tilde{\mathcal{T}}^{[0]}$ is a field.

If $\tilde{T} \in \tilde{\mathcal{T}}^{[0]}$, then $\tilde{T} = \Box$ iff $\tilde{T} = x^{\sigma}$ for some $\sigma \neq 0$, $\tilde{T} = \sqcap$ if $\sigma > 0$ and $\tilde{T} = \sqcup$ if $\sigma < 0$.

The general element of $\tilde{\mathcal{T}}^{[0]}$ is a level zero transseries , $\Box\!\Box^{[0]}$ or $\Box\!\Box$ in short. We have

$$\Box\!\Box = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} \sqcup^{\mathbf{k}} \tag{17}$$

There are two order relations: $<$ and $\ll$ on $\tilde{T} \in \tilde{\mathcal{T}}^{[0]}$. We have $\Box\!\Box_1 \ll \Box\!\Box_2$ iff $\operatorname{mag}(\Box\!\Box_1) \ll \operatorname{mag}(\Box\!\Box_2)$ (the sign of the leading coefficient is immaterial) and $\Box\!\Box > 0$ if ($\Box\!\Box \neq 0$ and) the real number $\operatorname{dom}(\Box\!\Box)/\operatorname{mag}(\Box\!\Box)$ is positive.

Definition 20. A transseries is **small**, i.e. $\Box\!\Box = \sqcup\!\sqcup$ iff in (17) we have $c_{\mathbf{k}} = 0$ whenever $\sqcup^{\mathbf{k}} \not\ll 1$. Correspondingly, transseries is **large**, i.e. $\Box\!\Box = \sqcap s \urcorner$ iff in (17) we have $c_{\mathbf{k}} = 0$ whenever $\sqcup^{\mathbf{k}} \not\gg 1$. We note that $\Box\!\Box = \sqcup\!\sqcup$ iff $\operatorname{mag}(\Box\!\Box) \ll 1$ (there is an asymmetry: the condition $\operatorname{mag}(\Box\!\Box) \gg 1$ does *not* imply $\Box\!\Box = \sqcap s \urcorner$, since it does not prevent the presence of small terms in $\Box\!\Box$). *Any transseries can then be written uniquely as*

$$\begin{aligned} \Box\!\Box = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} \sqcup^{\mathbf{k}} &= \sum_{\mathbf{k} \geq \mathbf{k}_0;\ \sqcup^{\mathbf{k}} > 1} c_{\mathbf{k}} \sqcup^{\mathbf{k}} + const + \sum_{\mathbf{k} \geq \mathbf{k}_0;\ \sqcup^{\mathbf{k}} < 1} c_{\mathbf{k}} \sqcup^{\mathbf{k}} \\ &= \sqcap s \urcorner + const + \sqcup\!\sqcup := L(\Box\!\Box) + C(\Box\!\Box) + s(\Box\!\Box) \end{aligned} \tag{18}$$

1.4.4. *Level 1: Exponential power series.* The set $\mathcal{G}^{[1]}$ of transmonomials of exponentiality one consists by definition in the formal expressions

$$\Box^{[1]} = \Box^{[0]} \exp(\sqcap s \urcorner^{[0]}), \quad \Box^{[0]}, \sqcap s \urcorner^{[0]} \in \tilde{\mathcal{T}}^{[0]}$$

where we allow for $\sqcap s \urcorner^{[0]} = 0$ and set $\exp(0) = 1$. With respect to the operation

$$\Box_1^{[0]} \exp(\sqcap s \urcorner_1^{[0]}) \Box_2^{[0]} \exp(\sqcap s \urcorner_2^{[0]}) = (\Box_1^{[0]} \Box_2^{[0]}) \exp(\sqcap s \urcorner_1^{[0]} + \sqcap s \urcorner_2^{[0]})$$

we see that $\mathcal{G}^{[1]}$ is a commutative group.

The order relations are introduced in the following way.

$$\Box_1 \exp(\sqcap s \urcorner_1) \gg \Box_2 \exp(\sqcap s \urcorner_2) \quad \text{iff} \; (\sqcap s \urcorner_1 > \sqcap s \urcorner_2) \; \text{or} \; (\sqcap s \urcorner_1 = \sqcap s \urcorner_2 \text{ and } \Box_1 \gg \Box_2) \tag{19}$$

In particular, if $\sqcap s \urcorner_1^{[0]}$ is positive, then $\Box_1^{[0]} \exp(\sqcap s \urcorner_1^{[0]}) \gg 1$.

The second order relation, $>$, is defined by

$$\Box_1^{[0]} \exp(\sqcap s \urcorner_1^{[0]}) > 0 \iff \Box_1^{[0]} > 0$$

It is straightforward to check that $(\mathcal{G}^{[1]}, \cdot, \gg)$ is an abelian ordered group. The abelian ordered group of zero level monomials, $(\mathcal{G}^{[0]}, \cdot, \gg)$, is naturally identified with the set of transmonomials for which $\sqcap s \urcorner^{[0]} = 0$.

The space $\tilde{\mathcal{T}}^{[1]}$ of level one transseries is by definition $\tilde{\tilde{S}}\,(\mathcal{G}^{[1]})$. By Proposition 18, $\tilde{\mathcal{T}}^{[1]}$ is a field. By construction, the space $\tilde{\mathcal{T}}^{[0]}$ is embedded in $\tilde{\mathcal{T}}^{[1]}$. Formula (17) is the general expression of a level one transseries , where now $\sqcup$ is a transmonomial of level one. The two order relations on transseries are the ones induced by transmonomials, namely

$$\Box\!\Box \gg 1 \iff \operatorname{mag}(\Box\!\Box) \gg 1 \; \text{ and } \; \Box\!\Box > 0 \iff \operatorname{dom}(\Box\!\Box)/\operatorname{mag}(\Box\!\Box) > 0 \tag{20}$$

1.4.5. *Induction step: level n transseries.* Assuming the transseries of level $\le n-1$ are constructed, transseries of level n together with the order relation, are constructed exactly as in § 1.4.4, replacing [0] by $[n-1]$ and [1] by $[n]$. The group $\mathcal{G}^{[1]}$ of transmonomials of order at most n consists in expressions of the form

$$\Box^{[n]} = x^{\sigma} \exp(\sqcap s \urcorner^{[n-1]}) \tag{21}$$

where $\sqcap s \urcorner^{[n-1]}$ is either zero or a large transseries of level $n-1$ with the multiplication:

$$x^{\sigma_1} \exp(\sqcap s \urcorner_1^{[n-1]}) x^{\sigma_2} \exp(\sqcap s \urcorner_2^{[n-1]}) = x^{\sigma_1+\sigma_2} \exp(\sqcap s \urcorner_1^{[n-1]} + \sqcap s \urcorner_2^{[n-1]}) \tag{22}$$

The order relation is given by

$$x^{\sigma_1} \exp(\sqcap s \urcorner_1^{[n-1]}) \gg x^{\sigma_2} \exp(\sqcap s \urcorner_2^{[n-1]}) \iff \tag{23}$$

$$\left(\sqcap s \urcorner_1^{[n-1]} > \sqcap s \urcorner_2^{[n-1]}\right) \text{ or } \left(\sqcap s \urcorner_1^{[n-1]} = \sqcap s \urcorner_2^{[n-1]} \text{ and } \sigma_1 > \sigma_2\right) \tag{24}$$

$$\Box\!\Box^{[n]} = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}} (\sqcup^{[n]})^{\mathbf{k}} \tag{25}$$

As in § 1.4.4, $\tilde{\mathcal{T}}^{[n-1]}$ is naturally embedded in $\tilde{\mathcal{T}}^{[n]}$.

1.4.6. *General log-free transseries* , $\tilde{\tilde{\tilde{\mathcal{T}}}}$. This is the space of arbitrary level transseries, the inductive limit of the finite level spaces of transseries :

$$\tilde{\tilde{\tilde{\mathcal{T}}}} = \bigcup_{n=0}^{\infty} \tilde{\mathcal{T}}^{[n]}$$

Clearly $\tilde{\tilde{\tilde{\mathcal{T}}}}$ is a field. The order relation is the one inherited from $\tilde{\mathcal{T}}^{[n]}$. The topology is also that of an inductive limit, namely a sequence converges iff it converges in $\tilde{\mathcal{T}}^{[n]}$ for *some* n.

1.4.7. *Further properties of transseries. Definition.* The level $l(\Box\!\Box)$ of $\Box\!\Box$ is n if $\Box\!\Box \in \tilde{\mathcal{T}}^{[n]}$ and $\Box\!\Box \notin \tilde{\mathcal{T}}^{[n-1]}$.

Proposition 21. *If* $n = l(\sqcap s\urcorner_1) > l(\sqcap s\urcorner_2)$ *then* $\sqcap s\urcorner_1 \gg \sqcap s\urcorner_2$.

Proof. We may clearly take $n \geq 1$. Since (by definition) $\sqcap s\urcorner \gg 1$ we must have, in particular, $\mathrm{dom}(\sqcap s\urcorner) = cx^{\sigma}\exp(\sqcap s\urcorner')$ with $\sqcap s\urcorner' \geq 0$. By induction, and the assumption $l(\sqcap s\urcorner_1) = n$ we must have $\sqcap s\urcorner_1' > 0$ and $l(\sqcap s\urcorner_1') = n-1$. The proposition follows since, by again by the induction step, $\sqcap s\urcorner_1' \gg \sqcap s\urcorner_2'$. $\square$

Remark 7. *If* $\Box$ *is of level no less than* 1*, then either* $\Box$ *is large, and then* $\Box \gg x^{\alpha}$, $\forall\alpha\in\mathbb{R}$ *or else* $\Box$ *is small, and then* $\Box \ll x^{-\alpha}$, $\forall\alpha\in\mathbb{R}$.

Remark 8. *We can define* generating monomials *of* $0 \neq \Box\!\Box \in \tilde{\mathcal{T}}^{[n]}$ *a minimal subgroup* $\mathcal{G} = \mathcal{G}(\Box\!\Box)$ *of* $\mathcal{G}^{[n]}$ *with the following properties:*

- $\Box\!\Box \in \tilde{\tilde{\tilde{S}}}(\mathcal{G})$;
- $x_1^{\sigma}\exp(\sqcap s\urcorner_1) \in \mathcal{G}$ *implies* $x_1^{\sigma} \in \mathcal{G}$ *and, if* $\sqcap s\urcorner_1 \neq 0$*, then* $\mathcal{G} \supset \mathcal{G}(\sqcap s\urcorner_1)$.

By induction we see that $\mathcal{G}(\Box\!\Box)$ *is finitely generated for any* $\Box\!\Box \in \mathcal{T}^{[n]}$.

1.4.8. *Closure of* $\tilde{\tilde{\tilde{\mathcal{T}}}}$ *under composition and differentiation.*

Proposition 22. $\tilde{\tilde{\tilde{\mathcal{T}}}}$ *and* $\mathcal{T}^{[n]}$; $n \in \mathbb{N}$ *are differential fields.*

Proof. Differentiation $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}x}$ is introduced inductively on $\tilde{\tilde{\tilde{\mathcal{T}}}}$, as term by term differentiation, in the following way. Differentiation in $\tilde{\mathcal{T}}^{[0]}$ is defined as:

$$\mathcal{D}\Box\!\Box = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}} \mathcal{D}\sqcup^{\mathbf{k}} \tag{26}$$

where, as mentioned in §1.4.3 we have $\sqcup = x^{-\sigma}$ for some $\sigma \in \mathbb{R}^+$ and, in the natural way, we set $\mathcal{D}x^{-\sigma} = -\sigma x^{-\sigma-1}$. This makes $\mathcal{D}\Box\!\Box \in \tilde{\mathcal{T}}^{[0]}$, and the generating transmonomials of $\mathcal{D}\Box\!\Box$ are those of $\Box\!\Box$ together with x^{-1}.

We assume by induction that differentiation $\mathcal{D} : \tilde{\mathcal{T}}^{[n-1]} \mapsto \tilde{\mathcal{T}}^{[n-1]}$ has been defined for all transseries of level at most $n-1$. (In particular, $\mathcal{D}\Box\!\Box$ is finitely generated.) We define

$$\mathcal{D}\left(\Box^{[n]}\right)=\mathcal{D}\left(x^{\sigma}\exp(\ulcorner s\urcorner^{[n-1]})\right)=\sigma x^{\sigma-1}\exp(\ulcorner s\urcorner^{[n-1]}) + x^{\sigma}\mathcal{D}\ulcorner s\urcorner^{[n-1]}\exp(\ulcorner s\urcorner^{[n-1]}) \tag{27}$$

A level n transseries is

$$\Box\!\Box=\sum_{\mathbf{k}\geq\mathbf{k}_0}c_{\mathbf{k}}\sqcup^{\mathbf{k}}=\sum_{\mathbf{k}\geq\mathbf{k}_0}c_{\mathbf{k}}\prod_{j=1}^{M}\sqcup_j^{k_j} \tag{28}$$

and we write in a natural way

$$\begin{aligned}\mathcal{D}\Box\!\Box^{[n]}&=\sum_{\mathbf{k}\geq\mathbf{k}_0}c_{\mathbf{k}}\sum_{m=1}^{M}k_m\sqcup_m^{k_m-1}\mathcal{D}\sqcup_m\prod_{m\neq j=1}^{M}\sqcup_j^{k_j}\\&=\sum_{m=1}^{M}\sqcup_m^{-1}\sum_{\mathbf{k}\geq\mathbf{k}_0}c_{\mathbf{k}}k_m\sqcup_m^{k_m-1}\mathcal{D}\sqcup_m\prod_{m\neq j=1}^{M}\sqcup_j^{k_j}\end{aligned} \tag{29}$$

and the result follows from the induction hypothesis, since

$$\sum_{\mathbf{k}\geq\mathbf{k}_0}c_{\mathbf{k}}k_m\sqcup^{\mathbf{k}}\frac{\mathcal{D}\sqcup_m}{\sqcup_m}=\frac{\mathcal{D}\sqcup_m}{\sqcup_m}\sum_{\mathbf{k}\geq\mathbf{k}_0}c_{\mathbf{k}}k_m\sqcup^{\mathbf{k}}\in\tilde{\mathcal{T}}^{[n]} \tag{30}$$

□

Corollary 23. *If $\mathcal{G}_{\Box\!\Box}$ is the group (finitely) generated by all generators in any of the levels of $\Box\!\Box$, then $\mathcal{D}\Box\!\Box$ is generated by the transmonomials of $\mathcal{G}_{\Box\!\Box}$ together possibly with x^{-1}. If $\Box\!\Box\neq\mathrm{Const.}$ then $l(\Box\!\Box)=l(\Box\!\Box')$.*

Proof. Immediate induction; cf. also the beginning of the proof of Proposition 22. □

The properties of differentiation are the usual ones:

Proposition 24. *$\mathcal{D}(fg)=g\mathcal{D}f+f\mathcal{D}g$, $\mathcal{D}const=0$ and $\mathcal{D}(f\circ g)=(\mathcal{D}f)\circ g\mathcal{D}g$ (for composition, see § 1.4.11).*

Proof. The proof is straightforward induction. □

In the space of transseries, differentiation is also compatible with the order relation, a property which is not true in general, in function spaces.

Proposition 25. *For any $\ulcorner s\urcorner_i, i=1,2$ and $\sqcup\!\sqcup_i, i=1,2$ we have*

$$\ulcorner s\urcorner_1\gg\ulcorner s\urcorner_2\Leftrightarrow\ulcorner s\urcorner_1'\gg\ulcorner s\urcorner_2'$$
$$\sqcup\!\sqcup_1\gg\sqcup\!\sqcup_2\Leftrightarrow\sqcup\!\sqcup_1'\gg\sqcup\!\sqcup_2'$$
$$\ulcorner s\urcorner_1'\gg\sqcup\!\sqcup_1'$$

Proof. The proof is by induction. It is true for power series, which are the level zero transseries . Assume the property holds for transseries of level $\leq n-1$, and first prove the result for *transmonomials* of order n, i.e. for the case when for $i=1,2$ $\Box\!\Box_i=\Box_i^{[n]}=x^{\sigma_i}\exp(\ulcorner s\urcorner_i)$, where at least one of $\ulcorner s\urcorner_i$ has level $n-1$.

We have to evaluate

$$\exp(\sqcap s\rceil_1 - \sqcap s\rceil_2)x^{\sigma_1-\sigma_2}\frac{\sqcap s\rceil_1' + \sigma_1 x^{-1}}{\sqcap s\rceil_2' + \sigma_2 x^{-1}}$$

and it is plain that we can assume without loss of generality that $\sqcap s\rceil_1 > 0$.

(1) If $l(\sqcap s\rceil_1 - \sqcap s\rceil_2) = n-1$ then $M \gg 1$ by Proposition 21. The remaining case is that for $i = 1, 2$ we have $l(\sqcap s\rceil_i) = n-1$ but $\sqcap s\rceil_i$ are equal or else $l(\sqcap s\rceil_1 - \sqcap s\rceil_2) \le n-2$ (which obviously requires $n \ge 2$). Let $\Delta = \sqcap s\rceil_1 - \sqcap s\rceil_2$. We have $l(\Delta) < l(\sqcap s\rceil_1)$ and also $\sqcap s\rceil_1 \gg x^{\alpha}$ for some $\alpha > 0$, thus by Proposition 21 and the induction hypothesis we have $\sqcap s\rceil_1' \gg \Delta' + \sigma x^{-1}$

$$\sqcap s\rceil_1' + \Delta' + \frac{\sigma_2}{x} = \sqcap s\rceil_1'(1 + \sqcup\!\sqcup)$$

thus

$$\exp(\sqcap s\rceil_1 - \sqcap s\rceil_2)x^{\sigma_1-\sigma_2}\frac{\sqcap s\rceil_1' + \sigma_1 x^{-1}}{\sqcap s\rceil_2' + \sigma_2 x^{-1}} = \exp(\sqcap s\rceil_1 - \sqcap s\rceil_2)x^{\sigma_1-\sigma_2}(1 + \sqcup\!\sqcup) \gg 1$$

(2) We now let $\square\!\square$ be arbitrary with the property $\operatorname{mag}(\square\!\square) \ne 1$ and use Proposition 17 to write

$$\square\!\square = c\square_0 + c\sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\square_0\sqcup^{\mathbf{k}} = c\square + c\sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\square_{\mathbf{k}}$$

where $\square_{\mathbf{k}} \ll \square_0$ and thus, by step (1) we have

$$\square\!\square' = c\square' + c\sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\square_{\mathbf{k}}' = c\square'(1 + \sqcup\!\sqcup)$$

The rest of the proof is immediate. □

Corollary 26. *We have* $\mathcal{D}\square\!\square = 0 \iff \square\!\square = \text{Const}$.

Proof. We have to show that if $\square\!\square = \sqcap s\rceil + \sqcup\!\sqcup \ne 0$ then $\square\!\square' \ne 0$. If $\sqcap s\rceil \ne 0$ then (for instance) $\sqcap s\rceil + \sqcup\!\sqcup \gg x^{-1} = \sqcup\!\sqcup$ and then $\sqcap s\rceil' + \sqcup\!\sqcup' \gg x^{-2} \ne 0$. If instead $\sqcap s\rceil = 0$ then $(1/\square\!\square) = \sqcap s\rceil_1 + \sqcup\!\sqcup_1 + c$ and we see that $(\sqcap s\rceil_1 + \sqcup\!\sqcup_1)' = 0$ which, by the above, implies $\sqcap s\rceil_1 = 0$ which gives $1/\sqcup\!\sqcup = \sqcup\!\sqcup_1$, a contradiction. □

Proposition 27. *Assume* $\square\!\square = \sqcap s\rceil$ *or* $\square\!\square = \sqcup\!\sqcup$. *Then:*
(i) If $l(\operatorname{mag}(\square\!\square)) \ge 1$ *then* $l(\operatorname{mag}(\square\!\square^{-1}\square\!\square')) < l(\operatorname{mag}(\square\!\square))$.
(ii) $\operatorname{dom}(\square\!\square') = \operatorname{dom}(\square\!\square)'(1 + \sqcup\!\sqcup)$.

Proof. Straightforward induction. □

1.4.9. *Transseries with complex coefficients.* Complex transseries $\mathcal{T}_{\mathbb{C}}$ are constructed in a similar way as real transseries, replacing everywhere $\sqcap s\rceil_1 > \sqcap s\rceil_2$ by $\Re\sqcap s\rceil_1 > \Re\sqcap s\rceil_2$. Thus there is only one order relation in $\mathcal{T}_{\mathbb{C}}$, $\gg$. Difficulties arise when exponentiating transseries whose dominant term is imaginary. Operations with complex transseries are then limited. We will only use complex transseries in contexts that will prevent these difficulties.

1.4.10. *Differential systems in $\mathcal{T}$.* The theory of differential equations in $\mathcal{T}$ is similar to the corresponding theory for functions.

Example. The general solution of the differential equation

$$f' + f = 1/x \tag{31}$$

in $\mathcal{T}$ (for $x \to +\infty$) is $\square\square(x;C) = \sum_{k=0}^{\infty} k! x^{-k} + Ce^{-x} = \square\square(x;0) + Ce^{-x}$.

Indeed, the fact that $\square\square(x;C)$ is a solution follows immediately from the definition of the operations in $\mathcal{T}$. To show uniqueness, assume $\square\square_1$ satisfies (31). Then $\square\square_2 = \square\square_1 - \square\square(x;0)$ is a solution of $\mathcal{D}\square\square + \square\square = 0$. Then $\square\square_2 = e^x \square\square$ satisfies $\mathcal{D}\square\square_2 = 0$ i.e., $\square\square_2 = \text{Const}$.

The particular solution $\square\square(x;0)$ is the unique solution of the equation $f = 1/x - \mathcal{D}f$ which is manifestly contractive in the space of level zero transseries (cf. § 1.3). However this same equation is not contractive for transseries of positive level, (because e.g. $\mathcal{D}e^x = e^x$); this could also have been anticipated noting that the solution is not unique.

1.4.11. *Restricted composition.* The right composition $\square\square_1 \circ \square\square_2$ is defined on $\tilde{\tilde{\mathcal{T}}}$, if $\text{mag}(\square\square_2) \gg 1$ and $\text{dom}(\square\square_2) > 0$. The definition is inductive.

We first define the power and the exponential of a transseries . Assume powers and exponentials have been defined for all transseries of level $\le n-1$. Let $\square\square = c\,\text{mag}(\square\square)(1 + \sqcup\sqcup) \in \tilde{\mathcal{T}}^{[n]}$ be any transseries such that $c > 0$, cf. Proposition 17. By the definition of $\text{mag}(\cdot)$ and (25), $\text{mag}(\square\square)$ is a transmonomial, $\text{mag}(\square\square) = \sqcap^{[n-1]} \exp(\sqcap s \urcorner^{[n-1]})$. We let

$$\begin{aligned} \square\square^{\sigma} &= c^{\sigma} \left(\sqcap^{[n-1]}\right)^{\sigma} \exp(\sigma \sqcap s \urcorner^{[n-1]})(1 + \sqcup\sqcup)^{\sigma} \\ &= c^{\sigma} \left(\sqcap^{[n-1]}\right)^{\sigma} \exp(\sigma \sqcap s \urcorner^{[n-1]})(1 + \sqcup\sqcup)^{\sigma} \\ &= c^{\sigma} \left(\sqcap^{[n-1]}\right)^{\sigma} \exp(\sigma \sqcap s \urcorner^{[n-1]}) \sum_{k=0}^{\infty} \binom{n}{\sigma} \sqcup\sqcup^{n} \end{aligned} \tag{32}$$

where $\binom{n}{\sigma}$ are the generalized binomial coefficients, the infinite sum is well defined, by Proposition 19, and thus $\square\square^{\sigma}$ is well defined as well. Then, if $\boldsymbol{\sigma} \in (\mathbb{R}^+)^M$ and if $\square\square^{[0]} = \sum_{\mathbf{k} \ge \mathbf{k}_0} c_{\mathbf{k}} x^{-\boldsymbol{\sigma}\cdot\mathbf{k}}$ is a level zero transseries , we write

$$\square\square^{[0]} \circ \square\square = \sum_{\mathbf{k} \ge \mathbf{k}_0} c_{\mathbf{k}} (\square\square^{-1})^{\boldsymbol{\sigma}\cdot\mathbf{k}}$$

which is well defined by Proposition 19 (ii) and Proposition 17. We note that, under our assumptions for $\square\square$, $\square\square^{[0]} \circ \square\square > 0$ is positive iff $\square\square^{[0]} > 0$

Similarly, we write cf. Definition 20

$$\begin{aligned} \exp(\square\square) &= e^{L(\square\square) + C(\square\square) + s(\square\square)} \\ &= e^{C(\square\square)} e^{L(\square\square)} \sum_{k=0}^{\infty} \frac{s(\square\square)^n}{n!} = C' \square^{[n+1]} \square\square^{[n]} \end{aligned} \tag{33}$$

well defined by the definition of a transmonomial, Proposition 19 (ii) and Proposition 17. Now the definition of general composition is straightforward induction.

We assume that composition is defined at all $\le n-1$ levels, and that in addition $\square\square^{[n-1]} \circ \square\square > 0$ if $\square\square^{[n-1]} > 0$. Then

$$\square\square^{[n]} \circ \square\square = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} (\sqcup^{[n]} \circ \square\square)^{\mathbf{k}} \tag{34}$$

$$= \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} (\square^{[n-1]} \circ \square\square)^{\mathbf{k}} \exp(-\sqcap s \urcorner^{[n-1]} \circ \square\square)$$

$$= \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} (\square^{[n-1]} \circ \square\square)^{\mathbf{k}} \Big[\exp(-L(\sqcap s \urcorner^{[n-1]} \circ \square\square) C \exp(-s(\sqcap s \urcorner^{[n-1]} \circ \square\square)\Big]$$

$$= \sum_{\mathbf{k}\ge\mathbf{k}_0} c'_{\mathbf{k}} (\square^{[n-1]} \circ \square\square)^{\mathbf{k}} \sqcup^{[n]} \exp(-s(\sqcap s \urcorner^{[n-1]} \circ \square\square)$$

and the last sum exists by the induction hypothesis and Proposition 12.

1.5. **The space $\mathcal{T}$ of general transseries.** We define

$$L_n(x) = \underbrace{\log\log \dots \log(x)}_{n\ times} \tag{35}$$

$$E_n(x) = \underbrace{\exp\exp \dots \exp(x)}_{n\ times} \tag{36}$$

$$\tag{37}$$

with the convention $E_0(x) = L_0(x) = x$.
We write $\exp(\ln x) = x$ and then any log-free transseries can be written as $\square\square(x) = \square\square \circ E_n(L_n(x))$. This defines right composition with L_n in this trivial case, as $\square\square_1 \circ L_n(x)) = (\square\square \circ E_n) \circ L_n(x) := \square\square(x)$.

More generally, we define $\mathcal{T}$, the space of general transseries , as a set of formal compositions

$$\mathcal{T} = \{\square\square \circ L_n : \square\square \in \tilde{\tilde{\mathcal{T}}}\}$$

with the algebraic operations (symbolized below by $*$) inherited from $\tilde{\tilde{\mathcal{T}}}$ by

$$(\square\square_1 \circ L_n) * (\square\square_2 \circ L_{n+k}) = [(\square\square_1 \circ E_k) * \square\square_2] \circ L_{n+k} \tag{38}$$

and using (38), differentiation is defined by

$$\mathcal{D}(\square\square \circ L_n) = \left[(\prod_{k=0}^{n-1} L_k)^{-1} \right] (\mathcal{D}\square\square) \circ L_n$$

Proposition 28. *$\mathcal{T}$ is an ordered differential field, closed under restricted composition.*

Proof. The proof is straightforward, by substitution from the results in § 1.4. □

We will denote generically the elements of $\mathcal{T}$ with the same symbols that we used for $\tilde{\tilde{\mathcal{T}}}$.

Proposition 29. *$\mathcal{T}$ is closed under integration.*

Proof. The idea behind the construction of $\mathcal{D}^{-1}$ is the following: we first find an invertible operator J which is *to leading order* $\mathcal{D}^{-1}$; then the equation for the correction will be contractive. Let $\Box\!\Box = \sum_{\mathbf{k}\geq\mathbf{k}_0} \sqcup^{\mathbf{k}} \circ L_n$. To unify the treatment, it is convenient to use the identity

$$\int_x \Box\!\Box(s)ds = \int_{L_{n+2}(x)} \left(\Box\!\Box \circ E_{n+2}\right)(t) \prod_{j\leq n+1} E_j(t)\mathrm{d}t = \int_{L_{n+2}(x)} \Box\!\Box_1(t)\mathrm{d}t$$

where the last integrand, $\Box\!\Box_1(t)$ is a log-free transseries and moreover

$$\Box\!\Box_1(t) = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}} \sqcup_1^{k_1} \cdots \sqcup_M^{k_M} = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}} e^{-k_1 \sqcap s\rceil_1 - \ldots - k_M \sqcap s\rceil_M}$$

The case $\mathbf{k} = 0$ is trivial and it thus suffices to find $\mathcal{D}^{-1}e^{-\sqcap s\rceil}$, where $n = l(\sqcap s\rceil) \geq 1$. Then $\sqcap s\rceil \gg x^m$ for any m and thus also $\mathcal{D}\sqcap s\rceil \gg x^m$ for all m. Therefore, since $\mathcal{D}e^{-\sqcap s\rceil} = -(\mathcal{D}\sqcap s\rceil)e^{-\sqcap s\rceil}$ we expect that $\mathrm{dom}(\mathcal{D}^{-1}e^{-\sqcap s\rceil}) = -(\mathcal{D}\sqcap s\rceil)^{-1}e^{-\sqcap s\rceil}$ and we look for a Δ such that

$$\mathcal{D}^{-1}e^{-\sqcap s\rceil} = -\frac{e^{-\sqcap s\rceil}}{\mathcal{D}\sqcap s\rceil}(1+\Delta) \tag{39}$$

Then Δ satisfies the equation

$$\Delta = -\frac{\mathcal{D}^2\sqcap s\rceil}{(\mathcal{D}\sqcap s\rceil)^2} - \frac{\mathcal{D}^2\sqcap s\rceil}{(\mathcal{D}\sqcap s\rceil)^2}\Delta + (\mathcal{D}\sqcap s\rceil)^{-1}\mathcal{D}\Delta \tag{40}$$

By Propositions 21, Corollary 23 and Proposition 27, (40) is contractive in $\mathcal{T}^{[n]}$. The proof now follows from Proposition 12.

□

In the following we also use the notation $\mathcal{D}\Box\!\Box = \Box\!\Box'$ and we write $\mathcal{P}$ for the antiderivative $\mathcal{D}^{-1}$ constructed above.

Proposition 30. *$\mathcal{P}$ is an antiderivative without constant terms, i.e,*

$$\mathcal{P}\Box\!\Box = \sqcap s\rceil + \sqcup\!\sqcup$$

Proof. This follows from Proposition 21, together with the fact that Δ, $\sqcap s\rceil$ and $\mathcal{D}$ in (39) belong to $\mathcal{T}^{[n]}$. □

Proposition 31. *We have*

$$\begin{aligned}\mathcal{P}(\Box\!\Box_1 + \Box\!\Box_2) &= \mathcal{P}\Box\!\Box_1 + \mathcal{P}\Box\!\Box_2\\ (\mathcal{P}\Box\!\Box)' &= \Box\!\Box;\ \mathcal{P}\Box\!\Box' = \Box\!\Box(0)\\ \mathcal{P}(\Box\!\Box_1\Box\!\Box_2') &= \Box\!\Box_1\Box\!\Box_2 - \mathcal{P}(\Box\!\Box_1'\Box\!\Box_2)\\ \Box\!\Box_1 \gg \Box\!\Box_2 &\implies \mathcal{P}\Box\!\Box_1 \gg \mathcal{P}\Box\!\Box_2\\ \Box\!\Box > 0 &\implies \mathcal{P}\Box\!\Box > 0 \end{aligned} \tag{41}$$

where

$$\Box\!\Box = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}} \sqcup^{\mathbf{k}} \implies \Box\!\Box(0) = \sum_{\mathbf{k}\geq\mathbf{k}_0;\mathbf{k}\neq 0} c_{\mathbf{k}} \sqcup^{\mathbf{k}}$$

Proof. All the properties are straightforward; preservation of inequalities uses Proposition 25. □

Remark 9. *Let* $\sqcup\!\sqcup_0 \in \mathcal{T}$*. The operators defined by*

$$J_1(\square\!\square) = \mathcal{P}(e^{-x}(\text{Const.} + \sqcup\!\sqcup_0)\square\!\square(x)) \tag{42}$$

$$J_2(\square\!\square) = e^{\pm x}x^{\sigma}\mathcal{P}(x^{-2}x^{-\sigma}e^{\mp x}(\text{Const.} + \sqcup\!\sqcup_0)\square\!\square(x)) \tag{43}$$

are contractive on $\mathcal{T}$*.*

Proof. For (42) it is enough to show contractivity of $\mathcal{P}(e^{-x}\cdot)$. This is a straightforward calculation similar to the proof of Proposition 29. We have for some n $\square\!\square(x) = \sum_{\mathbf{k}\ge\mathbf{k}_0} \sqcup^{\mathbf{k}}(L_n(x))$ where $\sqcup_j \in \tilde{\tilde{\tilde{\mathcal{T}}}}$.

$$\mathcal{P}e^{-x}(\square \circ L_n) = \mathcal{P}\left(e^{-E_{n+2}} \prod_{1\le j\le n+2} E_j \exp(\sqcap s \urcorner \circ E_2)\right) \circ L_{n+2} \tag{44}$$
$$= \left[\frac{e^{-E_{n+2}} \prod_{1\le j\le n+2} E_j \exp(\sqcap s \urcorner \circ E_2)}{-E'_{n+2} + \sum_{0\le j\le n+1} E'_j + \sqcap s \urcorner' \circ E'_2}(1 + \sqcup\!\sqcup)\right] \circ L_{n+2}$$
$$\ll \prod_{1\le j\le n+2} E_j \exp(\sqcap s \urcorner \circ E_2)$$

The proof of (ii) is similar.

□

2. Equations in $\mathcal{T}$: examples

Remark 10. *The general contractivity principle stated in Theorem 15, which we have used in proving closure of transseries with respect to a number of operations can be used to show closure under more general equations. Our main focus is on differential systems.*

2.0.1. *Nonlinear ODEs in* $\mathcal{T}$*.* We start with an example, a first order equation:

$$f' = J_1(f) = F_0(x^{-1}) - f - \frac{\beta}{x}f - g(x^{-1}, f) \tag{45}$$

where

$$F_0(x^{-1}) = \sum_{k\ge 2} \frac{F_{0k}}{x^k}$$
$$g(x^{-1}, f) = \sum_{k\ge 0;\ l\ge 1} g_{kl}x^{-k}f^l \tag{46}$$

where the sums are assumed to converge and $g_{01} = g_{11} = 0$.
We see that J_1 is well defined if $f = \sqcup\!\sqcup \in \mathcal{T}$ (cf. Proposition 12), and it is under this assumption that we study J_1.[3]

(1). Solutions of (46) in $\tilde{\mathcal{T}}^{[0]}$. The equation

[3]If there are infinitely many nonzero terms in the sum in (46), J_1 is not in $\mathcal{T}$ if $f \gg 1$ (since, in this case, $\text{mag}(f^n)$ is unbounded).

$$f = J_2(f) = -\mathcal{D}f + F_0(x^{-1}) - \frac{\beta}{x}f - g(x^{-1}, f) \tag{47}$$

is contractive in $\tilde{\mathcal{T}}^{[0]}$ (this follows immediately from §1.3). Thus there exists in $\tilde{\mathcal{T}}^{[0]}$ a unique solution $\tilde{f}_0$. Since (47) is also contractive in the subspace of $\tilde{\mathcal{T}}^{[0]}$ of series of the form $\sum_{k=2}^{\infty} \frac{c_k}{x^k}$ we have

$$\tilde{f}_0 = \sum_{k=2}^{\infty} \frac{c_k}{x^k} \tag{48}$$

Note. The iteration $f_{n+1} = J_1 f_n$, $f_1 = x^{-1}$ is convergent in $\mathcal{T}$ and, if $f_i = \sum_{k=2}^{i} c_k^{[i]} x^{-k}$ then $c_k^{[i]} = c_k$ for $k \le i$, and this is a very convenient way to calculate the coefficients c_i.

(2) Let now $\delta = f - \tilde{f}_0$. Then

$$\begin{aligned} \delta' &= -\delta - \frac{\beta}{x}\delta - g(x^{-1}, \tilde{f}_0 + \delta) + g(x^{-1}, \tilde{f}_0 + \delta) \\ &= -\delta - \frac{\beta}{x}\delta + \sum_{k\ge 0;\ l\ge 1} c_{kl} x^{-k} \delta^l \end{aligned} \tag{49}$$

with

$$c_{01} = c_{11} = 0 \tag{50}$$

or

$$\frac{\delta'}{\delta} = -1 - \frac{\beta}{x} - \sum_{k\ge 2} \frac{c_{k1}}{x^k} + \sum_{k\ge 0;\ l\ge 1} c_{k;l+1} x^{-k} \delta^l \tag{51}$$

Since by assumption $\delta \ll 1$ we have

$$\ln \delta = C_0 - x + \beta \ln x + \sum_{k\ge 1} \frac{c_{k+1;1}}{kx^k} + x \sqcup\!\sqcup(x)$$

and thus $\delta \ll \exp(-cx)$ for any $c < 1$ so that

$$\ln \delta = C_0 - x + \beta \ln x + \sum_{k\ge 1} \frac{c_{k+1;1}}{kx^k} + \exp(-cx) \sqcup\!\sqcup(x)$$

whence, by composition with exp we get

$$\delta = C_1 x^{\beta} e^{-x} \sum_{k\ge 1} \frac{d_{k+1;1}}{kx^k} + \exp(-cx) \sqcup\!\sqcup(x)$$

Equation (51) implies

$$\delta = C x^{\beta} e^{-x} \tilde{y}_0 \exp\left(\int \sum_{k\ge 0;\ l\ge 1} c_{k;l+1} x^{-k} \delta^l \right); \quad \left(\tilde{y}_0 = \sum_{k\ge 0} \frac{d_{k+1;1}}{kx^k} \right) \tag{52}$$

and (52) is contractive by Remark 6 and Remark 9. In particular, for every C there is a unique $\delta(x; C)$ satisfying (52).

Proposition 32. *The general transseries solution of (45) is* $\tilde{f}_0 + \delta$ *where*

$$\delta = \sum_{k=1}^{\infty} C^k x^{\beta k} e^{-kx} \tilde{f}_k(x) \tag{53}$$

with $\tilde{f}_k \in \tilde{\mathcal{T}}^{[0]}$ *and*

$$\tilde{f}_k(x) = \sum_{j=0}^{\infty} \frac{f_{k;j}}{x^j}$$

Proof. This is a straightforward consequence of discussion of this section and of (52) . ☐

2.0.2. *Formal linearization.* Let $z = Cx^\beta e^{-x}$. We have $C(x,\delta) = x^{-\beta} e^x \sum_{k\geq 1} \delta^k \tilde{g}_k(x)$. A direct calculation shows that $C' = C_x + C_\delta \delta' = 0$. The transformation $(x \mapsto x; y \mapsto C(x, y - f_0))$ formally linearizes (45).

2.1. **Multidimensional systems: transseries solutions at irregular singularities of rank one.** Consider the differential system

$$\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n \tag{54}$$

We look at solutions $\mathbf{y}$ such that $\mathbf{y}(x) \to 0$ as $x \to \infty$ along some direction $d = \{x \in \mathbb{C} : \arg(x) = \varphi\}$. The following conditions are assumed
(a1) The function $\mathbf{f}$ is analytic at $(0,0)$.
(a2) Nonresonance: the eigenvalues λ_i of the linearization

$$\hat{\Lambda} := -\left(\frac{\partial f_i}{\partial y_j}(0,0)\right)_{i,j=1,2,\ldots n} \tag{55}$$

are linearly independent over $\mathbb{Z}$ (in particular nonzero) and such that $\arg \lambda_i$ are different from each other (i.e., the Stokes lines are distinct; we will require somewhat less restrictive conditions, see § 3.1).

By relatively straightforward algebra, ([23] and also § 3.2 where all this is exemplified in a two-dimensional case), the system (54) can then be brought to the form

$$\mathbf{y}' = -\hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x^{-1}, \mathbf{y}) \tag{56}$$

where $\hat{\Lambda} = \mathrm{diag}\{\lambda_i\}$, $\hat{A} = \mathrm{diag}\{\alpha_i\}$ are constant matrices, $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}, \mathbf{y}^2)$, $(x \to \infty, \mathbf{y} \to 0)$.

Remark 11. *(i) If* $\mathbf{g}(x^{-1}, \mathbf{y}) \equiv 0$*. In this case the system (56) is linear and has the general transseries solution*

$$\mathbf{y} = \mathrm{e}^{-x\hat{\Lambda}} \mathbf{C} x^{\hat{A}}$$

(ii) More generally, if $\mathbf{g}(x^{-1}, \mathbf{y}) = \mathbf{G}(x^{-1})$ *is a transseries, then the general solution of (56) is*

$$\mathbf{y} = \mathrm{e}^{-x\hat{\Lambda}} x^{\hat{A}} \mathbf{C} + \mathrm{e}^{-x\hat{\Lambda}} x^{\hat{A}} \mathcal{P}\left(\mathrm{e}^{x\hat{\Lambda}} x^{-\hat{A}} \mathbf{g}\right) \tag{57}$$

Proof. In both cases the system is diagonal and the result follows immediately from the case when $n = 1$, i.e. from Proposition 32. □

The general solution of (56) in $\mathcal{T}_{\mathbb{C}}$ (cf. §1.4.9) is an $n_1 \le n$ parameter transseries, as shown in the sequel.

Proposition 33. *Let d be a ray in $\mathbb{C}$. The general solution of (56) in $\mathcal{T}_{\mathbb{C}}$ with the restriction $\mathbf{y} \ll 1$ is of the form*

$$\tilde{\mathbf{y}}(x) = \sum_{\mathbf{k}\ge 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \tilde{\mathbf{s}}_{\mathbf{k}}(x) = \sum_{\mathbf{k}\ge 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{m}_0\cdot\mathbf{k}} \tilde{\mathbf{y}}_{\mathbf{k}}(x) \tag{58}$$

where $C_i = 0$ for all i so that $\mathrm{e}^{-\lambda_i x} \not\to 0$ as $x \to \infty$ in d.

Proof. If $\mathbf{y}$ is a solution of (56) then we have, by Remark 11

$$\mathbf{y} = \mathrm{e}^{-x\hat{\Lambda}} x^{\hat{A}} \mathbf{C} + \mathrm{e}^{-x\hat{\Lambda}} x^{\hat{A}} \mathcal{P}\Big(\mathrm{e}^{x\hat{\Lambda}} x^{-\hat{A}} \mathbf{g}(x^{-1}, \mathbf{y})\Big) \tag{59}$$

for some $\mathbf{C}$. Since $\mathbf{y} \ll 1$ we have $\mathbf{g}(x^{-1}, \mathbf{y}) \ll 1$ and thus

$$\mathcal{P}\Big(\mathrm{e}^{x\hat{\Lambda}} x^{-\hat{A}} \mathbf{g}(x^{-1}, \mathbf{y})\Big) \ll \mathrm{e}^{x\hat{\Lambda}} x^{-\hat{A}}$$

Again since $\mathbf{y} \ll 1$, we then have $C_i = 0$ for all i for which $e^{-\lambda_i x} \not\ll 1$.
Note. With the condition $\mathbf{y} \ll 1$, eq. (59) has a unique solution.

Indeed, the difference of two solutions $\mathbf{y}_1 - \mathbf{y}_2$ satisfies the equation

$$\mathbf{y}_1 - \mathbf{y}_2 = \mathrm{e}^{-x\hat{\Lambda}} x^{\hat{A}} \mathcal{P}\Big(\mathrm{e}^{x\hat{\Lambda}} x^{-\hat{A}} \big[\mathbf{g}(x^{-1}, \mathbf{y}_1) - \mathbf{g}(x^{-1}, \mathbf{y}_2)\big]\Big) \tag{60}$$

Since $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}, \mathbf{y}^2)$ we have

$$\mathbf{g}(x^{-1}, \mathbf{y}_1) - \mathbf{g}(x^{-1}, \mathbf{y}_2) = O(x^{-2}\boldsymbol{\delta}, |\mathbf{y}||\boldsymbol{\delta}|)$$

which by Proposition 31 implies $\boldsymbol{\delta} = o(\boldsymbol{\delta})$, i.e., $\delta = 0$.

Using Remark 9 it is easy to check that (60) is an asymptotically contractive equation, in the space of $\mathbf{y}$ which are $\ll x^{-2}$ thus it has a solution $\mathbf{y}^{[0]}$ with this property. Since the previous note shows the solution of (56) with $\mathbf{y} \ll 1$ is unique, we have $\mathbf{y} = \mathbf{y}^{[0]}$. Formula (58) is obtained by straightforward iteration of (60). □

3. Borel summation techniques

3.0.1. *Introduction.* In this section we discuss a Borel summation–induced isomorphism between transseries and functions in the setting §2.1. The goal is to show, on simple examples, how Borel summation is proved and used. For more general results we refer to [12].

Definition 34. *A Borel-summable series $\tilde{y} := \sum_{k=K}^{\infty} y_k x^{-k}$, $K \in \mathbb{Z}$ is a formal power series with the following properties*

- (i) *the truncated Borel transform $Y = \mathcal{B}\tilde{y} := \sum_{k>0} \frac{y_k}{(k-1)!} t^{k-1}$ of $\tilde{y}$ has a nonzero radius of convergence,*
- (ii) *Y can be analytically continued along $[0, +\infty)$ and*
- (iii) *the analytic continuation Y grows at most exponentially along $[0, +\infty)$ and is therefore Laplace transformable along $[0, +\infty)$.*

The Borel sum y of $\tilde{y}$ is then given by

$$y = \mathcal{L}\mathcal{B}\tilde{y} := \sum_{k=K}^{0} y_k x^{-k} + \mathcal{L}Y, \tag{61}$$

where the sum is understood to be zero if $K > 0$ and $\mathcal{L}$ denotes the usual Laplace transform.

Example The formal solution for large x of the equation $f' - f = x^{-1}$ is Borel summable. Indeed

$$\mathcal{L}\mathcal{B}\sum_{k=0}^{\infty} \frac{-k!}{(-x)^{k+1}} = \mathcal{L}\frac{1}{1+p} = \int_0^{\infty} \frac{e^{-px}}{1+p} dp$$

and it can be checked that the Borel sum is a solution of the given equation; see also (i) in the note below.

Note 35. (i) It can be shown that Borel summation is an extended isomorphism: in particular it commutes with algebraic operations, including multiplication and complex conjugation, and with differentiation [1]. This can be expected from the fact that, formally, Borel summation is the composition of Laplace transform with its inverse, and the identity obviously commutes with the operations mentioned above.

(ii) Borel summation has to be generalized in a number of ways in order to be used for solving more general equations. Indeed, an equation as simple as $f' + f = x^{-1}$ has a formal solution which is not Borel summable: $\mathcal{B}\sum_{k=0}^{\infty} k! x^{-k-1} = (1-p)^{-1}$ is not Laplace transformable. If the Laplace transform integral is taken along a ray above or below $\mathbb{R}^+$, then $\mathcal{L}\mathcal{B}(1-p)^{-1}$ is a solution of the given equation. None of these path choices however yields a real valued function, whereas the formal series has real coefficients. The resulting summation operator would not be an isomorphism since commutation with complex conjugation fails. In this simple example the half sum of the upper and lower integrals is real valued and solves the equation, but this ad hoc procedure would not work for nonlinear equations. More complicated averages need to be introduced, see §3.2.3.

3.1. **Borel summation of transseries: a first order example.** Consider the (first order) differential equation (45) with, say, $F_0 = x^{-2}$ and $g = af^2 + bf^3$:

$$f' + (1 - \beta x^{-1})f = x^{-2} + af^2 + bf^3 \tag{62}$$

and a, b some constants. We first look at solutions, both formal and actual, which go to zero as $x \to \infty$ in some direction in $\mathbb{C}$.

We have seen in § 2.0.1 that the solution $\tilde{f}_0$ as a formal power series (i.e., in $\mathcal{T}^{[0]}$), is unique and $\tilde{f}_0 = \sum_{k=2}^{\infty} c_k x^{-k}$. By Remark 32 the general transseries solution is

$$\tilde{f} = \sum_{k=0}^{\infty} \xi^k \tilde{f}_k(x) \qquad (\xi := Ce^{-x}x^{\beta}) \tag{63}$$

where $\tilde{f}_k$ are integer power series. It is important to note that in (63) only the constant C depends on the solution. We will see that the $\tilde{f}_k$ are simultaneously Borel summable, to $f_k = \mathcal{L}\mathcal{B}\tilde{f}_k$, and that the sum

$$f = \sum_{k=0}^{\infty} \xi^k f_k(x) \tag{64}$$

is *convergent* and provides the general solution of (62) with the property $f \to 0$ as $x \to \infty$ in some direction in which (63) is a valid complex transseries.

Assuming for the moment we proved that (64) indeed provides a solution of (62) for any C it is not difficult to show that there are no further solutions:

Lemma 36. *If f_1 is any solution of (62) with the stated condition in some direction at infinity, then $f_1 - f_0 = Ce^{-x}x^{\beta}(1 + o(1))$ as $x \to \infty$. If $C = 0$ then $f_1 = f_2$.*

Proof. Writing the equation for $\delta = f_1 - f_0$, multiplying with the integrating factor of the "dominant" part of the equation, $Ce^t t^{-\beta}$, and integrating we get

$$\delta = Ce^{-x}x^{\beta} + e^{-x}x^{\beta}\int_a^x e^t t^{-\beta}\left[\left(2af_0 + 3bf_0^2\right)\delta + (a + 3bf_0)\delta^2 + \delta^3\right] dt \tag{65}$$

which is contractive in the sup norm on (x_0, ∞), for $|x_0|$ large enough, in a ball of radius $\epsilon > 0$ small enough. The solution of (65) is thus unique and it is easy to see that $\delta = C_1 e^{-x}x^{\beta}(1 + o(1))$ for large x (C_1 is not in general equal to C). ☐

Corollary 37. *Formula (64) provides the most general solution of (62) with the property $f \to 0$ in some direction in $\mathbb{C}$.*

A convenient way to generate $\tilde{f}_0$ is the iteration(47), which we start with $\tilde{f}_0^{[0]} = 0$. Denoting $\delta^{[k]} = \tilde{f}_0^{[k+1]} - \tilde{f}_0^{[k]}$ we have

$$\delta^{[k]} = (-\mathcal{D} - \beta x^{-1} + O(x^{-2}))\delta^{[k-1]}$$

whence $\delta^{[k]} \sim const.\Gamma(k-\beta)x^{-k}$ and thus Borel summation appears natural. Since $\tilde{f}_0$ is defined through a differential equation it is natural to Borel transform the equation itself. The result is

$$-pF + F = p - \beta F * 1 + aF^{*2} + bF^{*3} \tag{66}$$

where convolution is defined by

$$(f * g)(p) = \int_0^p f(s)g(p-s)ds \tag{67}$$

and we write

$$F^{*k} = \underbrace{F * F * \cdots * F}_{\text{k times}}$$

A solution F that is Laplace transformable along any ray *other than* $\mathbb{R}^+$ is obtained by noting that the equation

$$F = (1-p)^{-1}\left(p - \beta F * 1 + aF^{*2} + bF^{*3})\right) = \mathcal{N}(F) \tag{68}$$

is contractive in an appropriate norm.

Proposition 38. *The space L^1_ν of functions along a ray $d = \{p : \arg(p) = a$, such that $\|f\|_\nu < \infty$ with the norm $\|f\|_\nu = \int_{t \in d} |f(t)|e^{-\nu|t|}d|t|$ is a Banach algebra with respect to convolution.*

Proof. All the properties are verified in a straightforward manner. In particular, we have $\|f * g\|_\nu = \|f\|_\nu \|g\|_\nu$. ☐

Therefore, with $\rho = \|F\|_\nu$ and $d_1 = \text{dist}(1, d) \neq 0$ we have

$$\begin{aligned} (69) \quad \|\mathcal{N}(F)\|_\nu &= \left\|(1-p)^{-1}\left(p - \beta F * 1 + aF^{*2} + bF^{*3}\right)\right\|_\nu \\ &\leq \frac{1}{d_1}\left(\|p\|_\nu + |\beta|\rho\|1\|_\nu + |a|\rho^2 + |b|\rho^3\right) \\ &= \frac{1}{d_1}\left(\frac{1}{\nu^2} + \frac{|\beta|\rho}{\nu} + |a|\rho^2 + |b|\rho^3\right) \to 0 \text{ as } \nu \to \infty \end{aligned}$$

and clearly, if ν and $1/\epsilon$ are large enough, the image $\mathcal{N}B_\epsilon$ of the ball $B_\epsilon = \{F : \|F\|_\nu < \epsilon\}$ is contained in B_ϵ. Similarly, it can be seen that $\mathcal{N}$ is contractive in B_ϵ. Thus the following conclusion.

Proposition 39. *There is a unique solution of (68) in* $\cup_{\nu \geq \nu_0} L^1_\nu$.

This however does not yet imply Borel summability of the series $\tilde{f}_0$; to show this we must prove appropriate analyticity properties for F and this can be done in essentially the same manner.

Proposition 40. *The space of analytic functions in a region of the form*

$$S_M = \{p : \arg(p) \in (a_1, a_2)) \not\ni 0 \textit{ and } |p| < M\} \cup \{p : |p| < 1 - \epsilon\}$$

vanishing at $p = 0$*, continuous in* $\overline{S}$*, and such that* $\|f\|_\nu < \infty$ *with the norm* $\|f\|_{\nu;\infty} = M^{-1} \sup_{\overline{S}} |f(p)| e^{-\nu|p|}$ *is a Banach algebra with respect to convolution. In addition, if* $g(\cdot e^{i\varphi}) \in L^1_\nu(\mathbb{R}^+)$ *for any* $\varphi \in (a_1, a_2)$ *and* g *is analytic in* S *then we have*

$$(70) \qquad \|f * g\|_{\nu;\infty} \leq \|f\|_{\nu;\infty} \|g\|_{\nu;1}$$

(71)

The function F_0 *is analytic in* $\mathcal{S}_M$*, Laplace transformable in any direction other than* $\mathbb{R}^+$ *and* $y_0 = \mathcal{L}\{F_0\}$ *is a solution of (62).*

Proof. For large enough ν and for any M there is a unique analytic solution in S_M, F_0, and F_0 is thus independent of M. Since we have $|F_0(p)| \leq |p| e^{\nu|p|}$ for $p \in S = \cup_{M>0} S_M$ it follows that $F_0 = F$. Using Proposition 39 the proof is complete. ☐

3.1.1. *Summability of the transseries* $\tilde{f}$. A straightforward calculation shows that the series $\tilde{f}_k$ in (63) satisfy the system of equations

$$(72) \qquad \begin{aligned} \tilde{f}_0' + (1 - \beta x^{-1})\tilde{f}_0 &= x^{-2} + a\tilde{f}_0^2 + b\tilde{f}_0^3 \\ \tilde{f}_1' + (2a\tilde{f}_0 + 3b\tilde{f}_0^2)\tilde{f}_1 &= 0 \\ \tilde{f}_k' + \Big((1-k)(1 - \beta x^{-1}) + 2a\tilde{f}_0 + 3b\tilde{f}_0^2\Big)\tilde{f}_k &= a\sum{}^{\dagger} \tilde{f}_{k_1}\tilde{f}_{k_2} + b\sum{}^{\ddagger} \tilde{f}_{k_1}\tilde{f}_{k_2}\tilde{f}_{k_3} \end{aligned}$$

where in the sum $\sum^\dagger$ the indices satisfy $k_1 + k_2 = k$; $k_1 > 0; k_2 > 0$ while in $\sum^\ddagger$ the condition is $k_1 + k_2 + k_3 = k$; $k_i > 0$. We have already seen that $\tilde{f}_0$ is summable. The equation for $\tilde{f}_1$ is special, and we treat it separately. To ensure

Borel transformability, since $\tilde{f}_1 = O(1)$, it is convenient to take $\tilde{f}_1 = x^2\tilde{y}_1$, which gives

$$\tilde{y}_1' + 2x^{-1}\tilde{y}_1 + (2a\tilde{f}_0 + 3b\tilde{f}_0^2)\tilde{y}_1 = 0$$

which, in Borel transform, with $Y_1 = \mathcal{B}\tilde{y}_1$,

$$-pY_1 + 2\int_0^p Y_1(s)ds + (2a_0F_0 + 3bF_0^{*2}) * Y_1 = 0 \tag{73}$$

which implies, denoting $2a_0F_0 + 3bF_0^{*2} = G$

$$-pY_1' + Y_1 = -G' * Y_1 \tag{74}$$

in which the leading behavior of Y_1 is expected to be $Y_1 = p + \dots$ and thus the dominanat balance is between the terms on the l.h.s. of (75). In integral form we have, with $Y_1 = pQ$,

$$Q = 1 + \int_0^p dss^{-2}\int_0^s uG'(u)Q(s-u)du = 1 + \int_0^p\int_0^1 vG'(sv)Q(sv-v)dvds \tag{75}$$

It is easy to see that (75) is contractive in a space of analytic functions for $|p| < \epsilon$ if ϵ is small. To find exponential bounds for large p we first restrict to a ray $p = te^{i\varphi}$ with $\varphi \neq 0$.

For $x = pe^{-i\varphi} \in [0, \infty)$ it is useful to write $Q = Q_0 + Q_1$ where $Q_0 = 0$ for $|p| > \epsilon$ and $Q_1 = 0$ for $|p| < \epsilon$. Noting that $\int_0^s Q_1(t)G'(s-t)dt = 0$ if $|s| < \epsilon$, the equation for Q_1 takes the form, for $|p| > \epsilon$,

$$Q_1 = F(p) + \int_{\epsilon e^{i\varphi}}^p \int_{\epsilon e^{i\varphi}}^s s^{-2}G'(u)Q_1(s-u)\,\mathrm{d}u\,\mathrm{d}s$$

where

$$F(p) = 1 + \int_0^p \int_0^{\epsilon e^{i\varphi}} s^{-2}G'(s-u)Q_0(u)\,\mathrm{d}u\,\mathrm{d}s$$

Taking $Y(p) = Q_1(\epsilon e^{i\varphi} + p)$ we get a convolution equation of the form

$$Y = F_1 + Y * \{(s + \epsilon e^{i\varphi})^{-2}\} * F_2$$

which is manifestly contractive in the norm $\|\cdot\|_\nu$ for large ν. Since Q_0 is manifestly in $\mathcal{A}_\nu$ for any ν, it follows that $Q \in \mathcal{A}_\nu$ as well.

For $k > 1$ it is convenient to take $\tilde{f}_k = x^{2k}\tilde{y}_k$ and we have

$$\tilde{y}_k' + \Big(1 - k + (2k + (k-1)\beta)x^{-1} + 2a\tilde{y}_0 + 3b\tilde{y}_0^2\Big)\tilde{y}_k = a{\sum}^{\dagger}\tilde{y}_{k_1}\tilde{y}_{k_2} + b{\sum}^{\ddagger}\tilde{y}_{k_1}\tilde{y}_{k_2}\tilde{y}_{k_3} \tag{76}$$

(cf. (72)) which, after Borel transform becomes

(77)

$$Y_k = (p+k-1)^{-1}\left((2k+k\beta-\beta+G)*Y_k + a\sum\nolimits^{\dagger} Y_{k_1}*Y_{k_2} + b\sum\nolimits^{\ddagger} Y_{k_1}*Y_{k_2}*Y_{k_3}\right)$$
$$= \mathcal{J}Y_k + (p+k-1)^{-1}a\sum\nolimits^{\dagger} Y_{k_1}*Y_{k_2} + b\sum\nolimits^{\ddagger} Y_{k_1}*Y_{k_2}*Y_{k_3}$$

Note now that the operator on the r.h.s. of (78) is contractive in the norms introduced, for large ν. It is then clear that Y_k are analytic and Laplace transformable for large enough ν. It remains to show they are simultaneously Laplace transformable.

Note that for large ν, denoting $\|Y_k\| = x_k$ we have, with $\lambda_\nu = \max\{\|\mathcal{J}\|, \|Y_1\|_\nu\}$ arbitrarily small if ν is large,

$$x_k \le \lambda_\nu x_k + a\sum\nolimits^{\dagger} x_{k_1}x_{k_2} + b\sum\nolimits^{\ddagger} x_{k_1}x_{k_2}x_{k_3} \tag{78}$$

and the coefficients x_k are majorized by the Taylor coefficients of the analytic solution of the algebraic equation

$$\psi(z) = \lambda_\nu z + \lambda_\nu\psi(z) + a\psi(z)^2 + b\psi(z)^3$$

Thus, for large enough ν we have that $x_k \le \rho_\nu^k$ ($\rho_\nu = o(1)$ for large ν) and thus, for x large enough

$$\|\mathcal{L}\{Y_k\}\|_\infty \le \rho_\nu^k$$

(An inductive calculation shows that $Y_k = O(p^{2k-1})$ for small p.) The sum

$$\sum_{k=0}^{\infty}(\xi x^2)^k\mathcal{L}\{Y_k\}$$

is then uniformly convergent. It is easy to see that it gives therefore a solution of (62).

3.2. **Generalized Borel summation for rank one ODEs.** We look at the differential system (54) under the same assumptions and normalization as in § 2.1.

Further normalizing transformations. For convenience, we rescale x and reorder the components of $\mathbf{y}$ so that

(n3) $\lambda_1 = 1$, and, with $\varphi_i = \arg(\lambda_i)$, we have $\varphi_i < \varphi_j$ if $i < j$. To simplify notations, we formulate some of our results relative to λ_1; they can be easily adapted to any other eigenvalue.

To unify the treatment we make, by taking $\mathbf{y} = \mathbf{y}_1 x^{-N}$ for some $N > 0$,

(n4) $\Re(\beta_j) < 0,\ j = 1, 2, \ldots, n$.

Note: there is an asymmetry at this point: the opposite inequality cannot be achieved, in general, as simply and without violating analyticity at infinity. In some instances this transformation is not convenient since it makes more difficult the study of certain properties, see [13].

Finally, through a transformation of the form $\mathbf{y} \leftrightarrow \mathbf{y} - \sum_{k=1}^{M}\mathbf{a}_k x^{-k}$ we arrange that

(n5) $\mathbf{f}_0 = O(x^{-M-1})$ and $\mathbf{g}(x,\mathbf{y}) = O(\mathbf{y}^2, x^{-M-1}\mathbf{y})$. We choose $M > 1 + \max_i \Re(-\beta_i)$.

Formal solutions. The transseries solutions of (54) were studied in § 2.1. More generally, there is an n–paramter family of formal exponential series solutions of

(54):

$$\tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\ge 0;|\mathbf{k}|>0} C_1^{k_1}\cdots C_n^{k_n}\mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x}x^{\mathbf{k}\cdot\mathbf{m}}\tilde{\mathbf{y}}_{\mathbf{k}} \tag{79}$$

(see [23] below) where $m_i = 1 - \lfloor\beta_i\rfloor$, ($\lfloor\cdot\rfloor$ = integer part), $\mathbf{C}\in\mathbb{C}^n$ is an arbitrary vector of constants, and $\tilde{\mathbf{y}}_{\mathbf{k}} = x^{-\mathbf{k}(\boldsymbol{\beta}+\mathbf{m})}\sum_{l=0}^{\infty}\mathbf{a}_{\mathbf{k};l}x^{-l}$ are formal power series.

3.2.1. *Summability.* We give a brief overview of the results in [12]. The details of the proof follow the general strategy presented in §3.1. Let

$$\mathcal{W} = \{p\in\mathbb{C} : p\ne k\lambda_i\,,\forall k\in\mathbb{N}, i=1,2,\ldots,n\} \tag{80}$$

(see Fig. 2) The directions $d_j = \{p : \arg(p) = \varphi_j\}, j = 1,2,\ldots,n$ are the *Stokes lines.*

We construct over $\mathcal{W}$ a surface $\mathcal{R}$, consisting of homotopy classes of smooth curves in $\mathcal{W}$ starting at the origin, moving away from it, and crossing at most one Stokes line, at most once (Fig. 2):

$$\mathcal{R} := \Big\{\gamma : (0,1)\mapsto\mathcal{W} :\ \gamma(0_+) = 0;\ \frac{\mathrm{d}}{\mathrm{d}t}|\gamma(t)| > 0;\ \arg(\gamma(t))\text{ monotonic}\Big\} \tag{81}$$

Define $\mathcal{R}_1$ as the restriction of $\mathcal{R}$ to $\arg(\gamma)\in(\psi_n-2\pi,\psi_2)$ where $\psi_n = \max\{-\pi/2,\varphi_n-2\pi\}$ and $\psi_2 = \min\{\pi/2,\varphi_2\}$.

Fig 2. *The paths near λ_2 belong to $\mathcal{R}$.*
The paths near λ_1 relate to the balanced average

3.2.2. *Singularities of $\mathbf{Y}_{\mathbf{k}}$.* The Borel transforms of $\tilde{\mathbf{y}}_{\mathbf{k}}$ are analytic at zero and the only possible singularities are at multiples of the eigenvalues λ_j of the linearized system. These singularities are "regular" in the sense that there exist *convergent* local expansions at these singularities in terms of powers and possibly logarithms. The following theorem makes this statement precise.

Theorem 41 ([12])**.** *(i) $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$ is analytic in $\mathcal{R}\cup\{0\}$. The singularities of $\mathbf{Y}_0$ (which are contained in the set $\{l\lambda_j : l\in\mathbb{N}^+, j=1,2,\ldots,n\}$) are described as follows. For $l\in\mathbb{N}^+$ and small z*

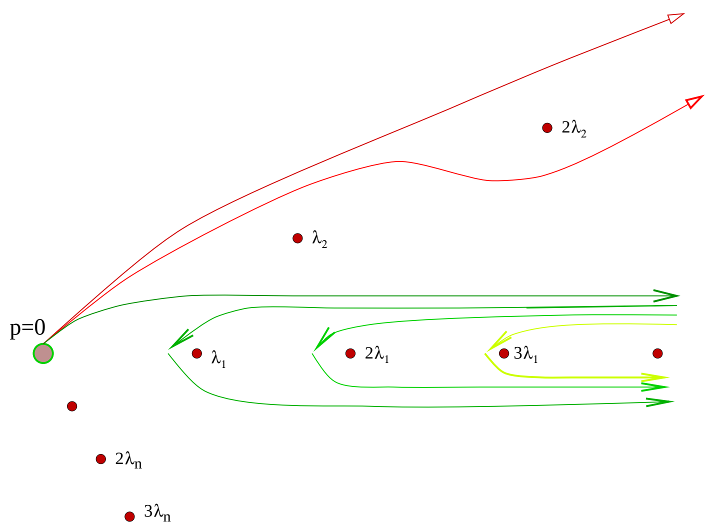
$$\mathbf{Y}_0^{\pm}(z+l\lambda_j) = \pm\Big[(\pm S_j)^l\ln(z)^{0,1}\mathbf{Y}_{l\mathbf{e}_j}(z)\Big]^{(lm_j)} + \mathbf{B}_{lj}(z) = \Big[z^{l\beta_j'-1}\ln z^{0,1}\,\mathbf{A}_{lj}(z)\Big]^{(lm_j)} + \mathbf{B}_{lj}(z)\ \ (l=1,2,\ldots) \tag{82}$$

where the power of $\ln(z)$ is one iff $l\beta_j\in\mathbb{Z}$, and $\mathbf{A}_{lj},\mathbf{B}_{lj}$ are analytic for small z. The functions $\mathbf{Y}_{\mathbf{k}}$ are, in addition, analytic at $p = l\lambda_j$, $l\in\mathbb{N}^+$, iff, exceptionally,

$$S_j = r_j\Gamma(\beta_j')\left(\mathbf{A}_{1,j}\right)_j(0) = 0 \tag{83}$$

where $r_j = 1-\mathrm{e}^{2\pi\mathrm{i}(\beta_j'-1)}$ if $l\beta_j\notin\mathbb{Z}$ and $r_j = -2\pi\mathrm{i}$ otherwise. The S_j are Stokes constants, see theorem 45.

(ii) $\mathbf{Y}_{\mathbf{k}} = \mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$, $|\mathbf{k}|>1$, are analytic in $\mathcal{R}\backslash\{-\mathbf{k}'\cdot\boldsymbol{\lambda}+\lambda_i : \mathbf{k}'\le\mathbf{k}, 1\le i\le n\}$. For $l\in\mathbb{N}$ and p near $l\lambda_j$, $j=1,2,\ldots,n$ there exist $\mathbf{A} = \mathbf{A}_{\mathbf{k}jl}$ and $\mathbf{B} = \mathbf{B}_{\mathbf{k}jl}$ analytic at zero so that (z is as above)



$$\mathbf{Y}_{\mathbf{k}}^{\pm}(z+l\lambda_j) = \pm\left[(\pm S_j)^l \binom{k_j+l}{l} \ln(z)^{0,1}\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(z)\right]^{(lm_j)} + l\mathbf{B}_{\mathbf{k}lj}(z) = \left[z^{\mathbf{k}\cdot\boldsymbol{\beta}'+l\beta_j'-1}(\ln z)^{0,1}\,\mathbf{A}_{\mathbf{k}lj}(z)\right]^{(lm_j)} + l\mathbf{B}_{\mathbf{k}lj}(z)\ (l=0,1,2,\ldots) \tag{84}$$

where the power of $\ln z$ *is* 0 *iff* $l=0$ *or* $\mathbf{k}\cdot\boldsymbol{\beta}+l\beta_j-1\notin\mathbb{Z}$ *and* $\mathbf{A}_{\mathbf{k}0j}=\mathbf{e}_j/\Gamma(\beta_j')$. *Near* $p\in\{-\mathbf{k}'\cdot\boldsymbol{\lambda}:0\prec\mathbf{k}'\le\mathbf{k}\}$, *(where* $\mathbf{Y}_0$ *is analytic)* $\mathbf{Y}_{\mathbf{k}}$, $\mathbf{k}\neq 0$ *have convergent Puiseux series.*

3.2.3. *Averaging.* It is possible to take Laplace transforms of $\mathbf{Y_0}$ along a ray avoiding $\mathbb{R}^+$ from above or below. However it can be checked that, assuming $\mathbf{Y_0}$ is singular at $p=1$ and that the series $\tilde{y}_0$ has real coefficients, neither of them would yield a real valued function.

Let $\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$ be extended along d_j by the "balanced average" of analytic continuations

$$\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}} = \mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}^{+} + \sum_{j=1}^{\infty} \frac{1}{2^j} \left(\mathbf{Y}_{\mathbf{k}}^{-} - \mathbf{Y}_{\mathbf{k}}^{-(j-1)+} \right) \tag{85}$$

The sum above coincides with the one in which $+$ is exchanged with $-$, accounting for the reality-preserving property. Clearly, if $\mathbf{Y}_{\mathbf{k}}$ is analytic along d_j, then the terms in the infinite sum vanish and $\mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}$; we also let $\mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}$ if $d \neq d_j$, where again $\mathbf{Y}_k$ is analytic. It follows from (85) and theorem 43 below that the Laplace integral of $\mathbf{Y}_{\mathbf{k}}^{ba}$ along $\mathbb{R}^+$ can deformed into contours as those depicted in Fig. 2, with weight 2^{-k} for a contour turning around $(k+1)\lambda_1$. More generally, we consider the averages

$$\mathcal{B}_{\alpha}\tilde{\mathbf{y}}_{\mathbf{k}} = \mathbf{Y}_{\mathbf{k}}^{\alpha} = \mathbf{Y}_{\mathbf{k}}^{+} + \sum_{j=1}^{\infty} \alpha^j \left(\mathbf{Y}_{\mathbf{k}}^{-} - \mathbf{Y}_{\mathbf{k}}^{-(j-1)+} \right) \tag{86}$$

and correspondingly

$$(\mathcal{L}\mathcal{B})_{\alpha}\tilde{\mathbf{y}}_{\mathbf{k}} := \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{\alpha} \tag{87}$$

With $\alpha \in \mathbb{R}$, this represents the most general family of averages of Borel summation formulas which commute with complex conjugation, with the algebraic and analytic operations and have good continuity properties [12]. The value $\alpha = 1/2$ is special in that it is the only one compatible with optimal truncation.

Note 42. *For rank one ODEs the balanced average mentioned above can be shown to coincide with the "median" summation of Ecalle, who has introduced and studied a wide variety of averages* [17]*, suitable in more complicated settings.*

Theorem 43. *(i) The branches of $(\mathbf{Y}_{\mathbf{k}})_{\gamma}$ in $\mathcal{R}_1$ have limits in a C^*-algebra of distributions, $\mathcal{D}'_{m,\nu}(\mathbb{R}^+) \subset \mathcal{D}'$. Their Laplace transforms in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ $\mathcal{L}(\mathbf{Y}_{\mathbf{k}})_{\gamma}$ exist simultaneously and with $x \in \mathcal{S}_x$ and for any $\delta > 0$ there is a constant K and an x_1 large enough, so that for $\Re(x) > x_1$ we have $|\mathcal{L}(\mathbf{Y}_{\mathbf{k}})_{\gamma}(x)| \leq K\delta^{|\mathbf{k}|}$.*

In addition, $\mathbf{Y}_{\mathbf{k}}(p\mathrm{e}^{\mathrm{i}\varphi})$ are continuous in φ with respect to the $\mathcal{D}'_{m,\nu}$ topology, (separately) on $[\psi_n - 2\pi, 0]$ and $[0, \psi_2]$.

If $m > \max_i(m_i)$ and $l < \min_i |\lambda_i|$ then $\mathbf{Y}_0(p\mathrm{e}^{\mathrm{i}\varphi})$ is continuous in $\varphi \in [0, 2\pi]\backslash\{\varphi_i : i \leq n\}$ in the $\mathcal{D}'_{m,\nu}(\mathbb{R}^+, l)$ topology and has (at most) jump discontinuities for $\varphi = \varphi_i$. For each $\mathbf{k}$, $|\mathbf{k}| \geq 1$ and any K there is an $l > 0$ and an m such that $\mathbf{Y}_k(p\mathrm{e}^{\mathrm{i}\varphi})$ are continuous in $\varphi \in [0, 2\pi]\backslash\{\varphi_i; -\mathbf{k}' \cdot \boldsymbol{\lambda} + \lambda_i : i \leq n, \mathbf{k}' \leq \mathbf{k}\}$ in the $\mathcal{D}'_{m,\nu}((0, K), l)$ topology and have (at most) jump discontinuities on the boundary.

(ii) The sum (85) converges in $\mathcal{D}'_{m,\nu}$ (and coincides with the analytic continuation of $\mathbf{Y}_{\mathbf{k}}$ when $\mathbf{Y}_{\mathbf{k}}$ is analytic along $\mathbb{R}^+$). For any δ there is a large enough x_1 independent of $\mathbf{k}$ *so that $\mathbf{Y}_{\mathbf{k}}^{ba}(p)$ with $p \in \mathcal{R}_1$ are Laplace transformable for $\Re(xp) > x_1$ and furthermore $|(\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba})(x)| \leq \delta^{|\mathbf{k}|}$. In addition, if $d \neq \mathbb{R}^+$, then for large ν, $\mathbf{Y}_{\mathbf{k}} \in L^1_{\nu}(d)$.*

The functions $\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba}$ are analytic for $\Re(xp) > x_1$. For any $\mathbf{C} \in \mathbb{C}^{n_1}$ there is an $x_1(\mathbf{C})$ large enough so that the sum

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^{ba} + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba} \tag{88}$$

converges uniformly for $\Re(xp) > x_1(\mathbf{C})$, *and* $\mathbf{y}$ *is a solution of (56). When the direction of* p *is not the real axis then, by definition,* $\mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}$, $\mathcal{L}$ *is the usual Laplace transform and (88) becomes*

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}} \tag{89}$$

In addition, $\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba} \sim \tilde{\mathbf{y}}_{\mathbf{k}}$ *for large* x *in the half plane* $\Re(xp) > x_1$, *for all* $\mathbf{k}$, *uniformly.*

iii) The general solution of (56) that is asymptotic to $\tilde{\mathbf{y}}_0$ *for large* x *along a ray in* S_x *can be equivalently written in the form (88) or as*

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^{\pm} + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{\pm} \tag{90}$$

for some $\mathbf{C}$ *(depending on the solution and chosen form). With the convention binding the directions of* x *and* p *and the representation form being fixed the representation of a solution is unique.*

Theorem 44. *i) For all* $\mathbf{k}$ *and* $\Re(p) > j, \Im(p) > 0$ *as well as in* $\mathcal{D}'_{m,\nu}$ *we have*

$$\mathbf{Y}_{\mathbf{k}}^{\pm j\mp}(p) - \mathbf{Y}_{\mathbf{k}}^{\pm(j-1)\mp}(p) = (\pm S_1)^j \binom{k_1+j}{j} \left(\mathbf{Y}^{\pm}_{\mathbf{k}+j\mathbf{e}_1}(p-j)\right)^{(mj)} \tag{91}$$

and also,

$$\mathbf{Y}_{\mathbf{k}}^{\pm} = \mathbf{Y}_{\mathbf{k}}^{\mp} + \sum_{j\ge 1} \binom{j+k}{k} (\pm S_1)^j (\mathbf{Y}^{\mp}_{\mathbf{k}+j\mathbf{e}_1}(p-j))^{(mj)} \tag{92}$$

ii) Local Stokes transition.
Consider the expression of a fixed solution $\mathbf{y}$ *of (56) as a Borel summed transseries (88). As* $\arg(x)$ *varies, (88) changes only through* $\mathbf{C}$, *and that change occurs when the Stokes lines are crossed. We have, in the neighborhood of* $\mathbb{R}^+$, *with* S_1 *defined in (83):*

$$\mathbf{C}(\xi) = \begin{cases} \mathbf{C}^- = \mathbf{C}(-0) & \textit{for } \xi < 0 \\ \mathbf{C}^0 = \mathbf{C}(-0) + \frac{1}{2} S_1 \mathbf{e}_1 & \textit{for } \xi = 0 \\ \mathbf{C}^+ = \mathbf{C}(-0) + S_1 \mathbf{e}_1 & \textit{for } \xi > 0 \end{cases} \tag{93}$$

Remark 12. *In view of (91) the different analytic continuations of* $\mathbf{Y}_0$ *along paths crossing* $\mathbb{R}^+$ *at most once can be expressed in terms of* $\mathbf{Y}_{j\mathbf{e}_1}$. *The most general formal solution of (56) that can be formed in terms of* $\mathbf{Y}_{j\mathbf{e}_j}$ *with* $j \ge 0$ *is (79) with* $C_1 = \alpha$ *arbitrary and* $C_j = 0$ *for* $j \ne 1$. *Any true solution of (56) based on such a transseries is given in (90) with* $\mathbf{C}$ *as above. Any average* $\mathcal{A}\mathbf{Y}_0$ *along paths going forward in* $\mathbb{R}^+$ *such that* $\mathcal{L}\mathcal{A}\mathbf{Y}_0$ *is thus of the form (87).*

Theorem 45. *Assume only λ_1 lies in the right half plane. Let $\gamma^{\pm}$ be two paths in the right half plane, near the positive/ negative imaginary axis such that $|x^{-\beta_1+1}\mathrm{e}^{-x\lambda_1}| \to 1$ as $x \to \infty$ along $\gamma^{\pm}$. Consider the solution $\mathbf{y}$ of (56) given in (88) with $\mathbf{C} = C\mathbf{e}_1$ and where the path of integration is $p \in \mathbb{R}^+$. Then*

$$\mathbf{y} = (C \pm \frac{1}{2}S_1)\mathbf{e}_1 x^{-\beta_1+1}\mathrm{e}^{-x\lambda_1}(1+o(1)) \tag{94}$$

for large x along $\gamma^{\pm}$, where S_1 is the same as in (83), (93).

Proposition 46. *i) Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be solutions of (56) so that $\mathbf{y}_{1,2} \sim \tilde{\mathbf{y}}_0$ for large x in an open sector S (or in some direction d); then $\mathbf{y}_1 - \mathbf{y}_2 = \sum_j C_j \mathrm{e}^{-\lambda_{i_j}x}x^{-\beta_{i_j}}(\mathbf{e}_{i_j} + o(1))$ for some constants C_j, where the indices run over the eigenvalues λ_{i_j} with the property $\Re(\lambda_{i_j}x) > 0$ in S (or d). If $\mathbf{y}_1 - \mathbf{y}_2 = o(\mathrm{e}^{-\lambda_{i_j}x}x^{-\beta_{i_j}})$ for all j, then $\mathbf{y}_1 = \mathbf{y}_2$.*

ii) Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be solutions of (54) and assume that $\mathbf{y}_1 - \mathbf{y}_2$ has differentiable asymptotics of the form $\mathbf{K}a\exp(-ax)x^b(1+o(1))$ with $\Re(ax) > 0$ and $\mathbf{K} \neq 0$, for large x. Then $a = \lambda_i$ for some i.

iii) Let $\mathbf{U}_\mathbf{k} \in \mathcal{T}_{\{\cdot\}}$ for all $\mathbf{k}$, $|\mathbf{k}| > 1$. Assume in addition that for large ν there is a function $\delta(\nu)$ vanishing as $\nu \to \infty$ such that

$$\sup_{\mathbf{k}} \delta^{-|\mathbf{k}|} \int_d \left|\mathbf{U}_\mathbf{k}(p)\mathrm{e}^{-\nu p}\right| \mathrm{d}|p| < K < \infty \tag{95}$$

Then, if $\mathbf{y}_1, \mathbf{y}_2$ are solutions of (56) in S where in addition

$$\mathbf{y}_1 - \mathbf{y}_2 = \sum_{|\mathbf{k}|>1} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{m}\cdot\mathbf{k}} \int_d \mathbf{U}_\mathbf{k}(p)\exp(-xp)\mathrm{d}p \tag{96}$$

where $\boldsymbol{\lambda}, x$ are as in (n6), then $\mathbf{y}_1 = \mathbf{y}_2$, and $\mathbf{U}_\mathbf{k} = 0$ for all $\mathbf{k}$, $|\mathbf{k}| > 1$.

Given $\mathbf{y}$, the value of C_i can change only when $\xi + \arg(\lambda_i - \mathbf{k}\cdot\boldsymbol{\lambda}) = 0$, $k_i \in \mathbb{N}\cup\{0\}$, i.e. when crossing one of the (finitely many by (c1)) Stokes lines. The procedure is similar to the medianization proposed by Écalle, but (due to the structure of (56)) requires substantially fewer analytic continuation paths. Resurgence relations are found and in addition we provide a complete description, needed in applications, of the singularity structure of the Borel transforms of $\tilde{\mathbf{y}}_\mathbf{k}$.

3.2.4. *More general equations amenable to first rank. Examples.* Many classical equations which are not presented in the form studied in § 2.1 can be brought to that form by elementary transformations. We look at a few examples, to illustrate this and the normalization process.

(1). The equation

$$f' + 2xf = 1 \tag{97}$$

for large x is not of the form (54). It can nevertheless be brought to that form by simple changes of variables. The normalizing change of variables of a given is most conveniently obtained in the following way.

A transformation bringing the equation to its normal form also brings its transseries solutions to the form (58).

It is simpler to look for substitutions with this latter property, and then the first step is to find the transseries solutions of the equation.

At level zero, differentiation is contractive and thus, within power series there is a unique solution of (97), obtained as a fizxed point of

$$f = \frac{1}{x} - \frac{1}{x} f' \tag{98}$$

Denoting this series by f_1 we look for further transseries solutions in the form $f = f_1 + \delta$. We get

$$\delta' + 2x\delta = 0 \tag{99}$$

where we look for higher level transseries solutions (we choose to ignore the fact that (99) is solvable explicitly). We therefore take $\delta = e^w$ and look for level zero solutions. We have $w' + 2x = 0$ with the one parameter family of solutions $w = -x^2/2 + K$. Therefore

$$f = f_1 + Ce^{-x^2/2} \tag{100}$$

The exponent is supposed to be linear in x which suggests the change of variable $x^2 = z$. In terms of z we get

$$g' + g = \frac{1}{2\sqrt{z}} \tag{101}$$

which is not of the required form because of the noninteger power of z. This can be easily take care of by the substitution $g = \sqrt{z}h$ which leads to

$$h' + \left(1 + \frac{1}{2z}\right)h = \frac{1}{2z} \tag{102}$$

(2) A simple Schrodinger equation in one dimension (the harmonic oscillator).

$$y'' - x^2 y = \lambda y \tag{103}$$

A convenient way to find transseries solutions is to use a WKB-like procedure. Substituting $y(x) = e^{g(x)}$ in (103) one obtains

$$g' = \pm\sqrt{\lambda + x^2 - g''}$$

which is a contractive equation in the space $\tilde{\mathcal{T}}^{[0]}$, cf. § 2.0.1. It follows in particular that $\tilde{g}' = \pm x + O(1/x)$ and thus $\ln y(x) = \pm\frac{1}{2}x^2 + O(\ln x)$. Since for (54) the exponentials have linear exponents, the natural variable of (103), for our purpose, is $t = \frac{1}{2}x^2$. In this variable, (103) becomes

$$h'' + \frac{1}{2t}h' - \left(1 + \frac{\lambda}{2t}\right)h = 0 \tag{104}$$

which in vector form, with $y_1 = h + h', y_2 = h - h'$, reads

$$\mathbf{y}' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{4t} \begin{pmatrix} \lambda - 1 & \lambda + 1 \\ 1 - \lambda & -1 - \lambda \end{pmatrix} \right\} \mathbf{y} = \left(\hat{\Lambda} + \frac{1}{t}\hat{C}\right)\mathbf{y} \tag{105}$$

The matrix $\hat{C}$ can (always) be diagonalized, by taking $\mathbf{y} = (I + t^{-1}\hat{S})\mathbf{Y}$ for a suitable matrix $\hat{S}$, and the diagonalized matrix is $\hat{B} = \mathrm{diag}(C_{11}, C_{22})$. Indeed, the equation for $\hat{S}$ is $\hat{\Lambda}\hat{S} - \hat{S}\hat{\Lambda} = \hat{B} - \hat{C}$, which can be solved whenever the eigenvalues of $\hat{\Lambda}$ are distinct. The normal form of (105) is thus

$$\mathbf{y}' = \left(\hat{\Lambda} + \frac{1}{t}\hat{C}\right)\mathbf{y} + t^{-2}\hat{g}(t^{-1})\mathbf{y} \tag{106}$$

with $\hat{g}$ analytic.

(3) A nonintegrable case of Abel's equation

$$w' = w^3 - z \tag{107}$$

Power series solutions. As before, (107) written in the form

$$w = (w' + z)^{1/3}$$

is contractive in $\tilde{\mathcal{T}}^{[0]}$. By iteration, one obtains the power series formal solutions $\tilde{w}_0 = Az^{1/3}(1 + \sum_{k=1}^{\infty} \tilde{w}_{0,k} z^{-5k/3})$ $(A^3 = 1)$.

General transseries solutions of (107). Once we determined $\tilde{w}_0$ we look for possible small corrections to $\tilde{w}_0$; since these are usually exponentially small, formal WKB is again useful. We substitute $w = \tilde{w}_0 + e^g$. The equation for g

$$\tilde{g} = C + \mathcal{P}\left(3\tilde{w}_0^2 + 3\tilde{w}_0 e^{\tilde{g}} + e^{2\tilde{g}}\right)$$

is contractive in any sector where $\Re(\tilde{w}_0) < 0$ and in this case $e^{\tilde{g}} \propto z^{2/3} \exp\left(\frac{9}{5}A^2 z^{5/3}\right)$

Since the exponentials in a transseries solution of a normalized system have *linear* exponent, with negative real part, the independent variable should be $x = -(9/5)A^2 z^{5/3}$ and $\Re(x) > 0$. Then $\tilde{w}_0 = x^{1/5} \sum_{k=0}^{\infty} w_{0;k} x^{-k}$, which compared to (58) suggests the change of dependent variable $w(z) = Kx^{1/5}h(x)$. Choosing for convenience $K = A^{3/5}(-135)^{1/5}$ yields

$$h' + \frac{1}{5x}h + 3h^3 - \frac{1}{9} = 0 \tag{108}$$

The next step is to achieve leading behavior $O(x^{-2})$. This is easily done by subtracting out the leading behavior of h (which can be found by maximal balance, as above). With $h = y + 1/3 - x^{-1}/15$ we get the normal form

$$y' = -\left(1 - \frac{1}{5x}\right)y + g(y, x^{-1}) \tag{109}$$

where

$$g(y, x^{-1}) = -3(y^2 + y^3) + \frac{3y^2}{5x} - \frac{1}{15x^2} - \frac{y}{25x^2} + \frac{1}{3^2 5^3 x^3} \tag{110}$$

(3) The Painlevé equation P1.

$$\frac{d^2y}{dz^2} = 6y^2 + z \tag{111}$$

The transformations needed to normalize (111) are derived in the same way as in Example 1. After the change of variables

$$x = \frac{(-24z)^{5/4}}{30};\ y(z) = \sqrt{\frac{-z}{6}}\left(1 - \frac{4}{25x^2} + h(x)\right)$$

P1 becomes

$$h'' + \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625x^4} = 0 \tag{112}$$

Written as a system, with $\mathbf{y} = (h, h')$ this equation satisfies the assumptions in § 2.1 with $\lambda_{1,2} = \pm 1$, $\alpha_{1,2} = -1/2$, and then $\xi(x) = C\mathrm{e}^{-x}x^{-1/2}$.

(4) The Painlevé equation P2.

This equation reads:

$$y'' = 2y^3 + xy + \alpha \tag{113}$$

This example also shows that for a given equation distinct solution manifolds associated to distinct asymptotic behaviors may lead to different normalizations. After the change of variables

$$x = (3t/2)^{2/3}; \quad y(x) = x^{-1}(t\,h(t) - \alpha)$$

one obtains the normal form equation

$$h'' + \frac{h'}{t} - \left(1 + \frac{24\alpha^2 + 1}{9t^2}\right)h - \frac{8}{9}h^3 + \frac{8\alpha}{3t}h^2 + \frac{8(\alpha^3 - \alpha)}{9t^3} = 0 \tag{114}$$

Distinct normalizations (and sets of solutions) are provided by

$$x = (At)^{2/3};\ y(x) = (At)^{1/3}\left(w(t) - B + \frac{\alpha}{2At}\right)$$

if $A^2 = -9/8, B^2 = -1/2$. In this case,

$$w'' + \frac{w'}{t} + w\left(1 + \frac{3B\alpha}{tA} - \frac{1 - 6\alpha^2}{9t^2}\right)w \\ - \left(3B - \frac{3\alpha}{2tA}\right)w^2 + w^3 + \frac{1}{9t^2}\left(B(1 + 6\alpha^2) - t^{-1}\alpha(\alpha^2 - 4)\right) \tag{115}$$

so that

$$\lambda_1 = 1, \alpha_1 = -\frac{1}{2} - \frac{3}{2}\frac{B\alpha}{A}$$

The first normalization applies for the manifold of solutions such that $y \sim -\frac{\alpha}{x}$ (for $\alpha = 0$ y is exponentially small and behaves like an Airy function) while the second one corresponds to $y \sim -B - \frac{\alpha}{2}x^{-3/2}$.

3.3. **Difference equations and PDEs.**

3.3.1. *Difference equations.* Transseries and Borel summation techniques can be successfully used for difference equations. Consider difference systems of equations which can be brought to the form

$$\mathbf{x}(n+1) = \hat{\Lambda}\left(I + \frac{1}{n}\hat{A}\right)\mathbf{x}(n) + \mathbf{g}(n, \mathbf{x}(n)) \tag{116}$$

where $\hat{\Lambda}$ and $\hat{A}$ are constant coefficient matrices, $\mathbf{g}$ is convergently given for small $\mathbf{x}$ by

$$\mathbf{g}(n, \mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{g}_{\mathbf{k}}(n)\mathbf{x}^{\mathbf{k}} \tag{117}$$

with $\mathbf{g}_{\mathbf{k}}(n)$ analytic in n at infinity and

$$\mathbf{g}_{\mathbf{k}}(n) = O(n^{-2}) \text{ as } n \to \infty, \text{ if } \sum_{j=1}^{m} k_j \le 1 \tag{118}$$

under nonresonance conditions: Let $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ and $\mathbf{a} = (a_1, ..., a_n)$ where $e^{-\mu_k}$ are the eigenvalues of $\hat{\Lambda}$ and the a_k are the eigenvalues of $\hat{A}$. Then the nonresonance condition is

$$(\mathbf{k} \cdot \boldsymbol{\mu} = 0 \mod 2\pi i \text{ with } \mathbf{k} \in \mathbb{Z}^{m_1}) \Leftrightarrow \mathbf{k} = 0. \tag{119}$$

We consider the solutions of (116) which are small as n becomes large. Braaksma and Kuik [7] [20] showed that the recurrences (116) posess l-parameter transseries solutions of the form

$$\tilde{\mathbf{x}}(t) := \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu}t} t^{\mathbf{k}\cdot\mathbf{a}} \tilde{\mathbf{x}}_{\mathbf{k}}(t) \tag{120}$$

with $t = n$ where $\tilde{\mathbf{x}}_{\mathbf{k}}(n)$ are formal power series in powers of n^{-1} and $l \le m$ is chosen such that, after reordering the indices, we have $\Re(\mu_j) > 0$ for $1 \le j \le l$.

It is shown in [7], [20] that these transseries are generalized Borel summable in any direction and Borel summable in all except m of them and that

$$\mathbf{x}(n) = \sum_{\mathbf{k}\in\mathbb{N}^l} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu}n} n^{\mathbf{k}\cdot\mathbf{a}} \mathbf{x}_{\mathbf{k}}(n) \tag{121}$$

is a solution of (116), if $n > y_0$, t_0 large enough.

3.3.2. *PDEs.*

3.3.3. *Existence, uniqueness, regularity, asymptotic behavior.* For partial differential equations with analytic coefficients which can be transformed to equations in which the differentiation order in a distinguished variable, say time, is no less than the one with respect to the other variable(s), under some other natural assumptions, Cauchy-Kowalevski theory (C-K) applies and gives existence and uniqueness of the initial value problem. A number of evolution equations do not satisfy these assumptions and even if formal power series solutions exist their radius of convergence is zero. The paper [11] provides a C-K type theory in such cases, providing existence, uniqueness and regularity of the solutions. Roughly, convergence is replaced by Borel summability, although the theory is more general.

Unlike in C-K, solutions of nonlinear evolution equations develop singularities which can be more readily studied from the local behavior near $t = 0$, and this is useful in determining and proving spontaneous blow-up.

In the following, $\partial_{\mathbf{x}}^{\mathbf{j}} \equiv \partial_{x_1}^{j_1}\partial_{x_2}^{j_2}...\partial_{x_d}^{j_d}$, $|\mathbf{j}| = j_1 + j_2 + ... + j_d$, $\mathbf{x}$ is in a poly-sector $\mathcal{S} = \{\mathbf{x} : |\arg x_i| < \frac{\pi}{2} + \varphi; |\mathbf{x}| > a\}$ in $\mathbb{C}^d$ where $\varphi < \frac{\pi}{2n}$, $\mathbf{g}\left(\mathbf{x}, t, \{\mathbf{y_j}\}_{|\mathbf{j}|=0}^{n-1}\right)$ is a function analytic in $\{\mathbf{y_j}\}_{|\mathbf{j}|=0}^{n-1}$ near $\mathbf{0}$ vanishing as $|\mathbf{x}| \to \infty$. The results in [11] hold for n-th order nonlinear *quasilinear* partial differential equations of the form

$$\mathbf{u}_t + \mathcal{P}(\partial_{\mathbf{x}}^{\mathbf{j}})\mathbf{u} + \mathbf{g}\left(\mathbf{x}, t, \{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\}\right) = 0 \tag{122}$$

where $\mathbf{u} \in \mathbb{C}^m$, for large $|\mathbf{x}|$ in S. Generically, the constant coefficient operator $\mathcal{P}(\partial_{\mathbf{x}})$ in the linearization of $\mathbf{g}(\infty, t, \cdot)$ is diagonalizable. It is then taken to be diagonal, with eigenvalues $\mathcal{P}_j$. $\mathcal{P}$ is subject to the requirement that for all $j \le m$ and $\mathbf{p} \ne 0$ in $\mathbb{C}^d$ with $|\arg p_i| \le \varphi$ we have

$$\Re \mathcal{P}_j^{[n]}(-\mathbf{p}) > 0 \tag{123}$$

where $\mathcal{P}^{[n]}(\partial_{\mathbf{x}})$ is the principal symbol of $\mathcal{P}(\partial_{\mathbf{x}})$. Then the following holds. (The precise conditions and results are given in [11].)

Theorem 47 (large $|\mathbf{x}|$ existence)**.** *Under the assumptions above, for any $T > 0$ (122) has a unique solution $\mathbf{u}$ that for $t \in [0, T]$ is $O(|\mathbf{x}|^{-1})$ and analytic in $\mathcal{S}$.*

Determining asymptotic properties of solutions of PDEs is substantially more difficult than the corresponding question for ODEs. Borel-Laplace techniques however provide a very efficient way to overcome this difficulty. The paper shows that formal series solutions are actually Borel summable, a fortiori asymptotic, to actual solutions.

Condition 48. *The functions $\mathbf{b_{q,k}}(\mathbf{x}, t)$ and $\mathbf{r}(\mathbf{x}, t)$ are analytic in $(x_1^{-\frac{1}{N_1}}, ..., x_d^{-\frac{1}{N_d}})$ for large $|\mathbf{x}|$ and some $N \in \mathbb{N}$.*

Theorem 49. *If Condition 48 and the assumptions of Theorem 47 are satisfied, then the unique solution $\mathbf{f}$ found there can be written as*

$$\mathbf{f}(\mathbf{x}, t) = \int_{\mathbb{R}^{+d}} e^{-\mathbf{p}\cdot\mathbf{x}^{\frac{n}{n-1}}} \mathbf{F}_1(\mathbf{p}, t) d\mathbf{p} \tag{124}$$

where $\mathbf{F}_1$ is (a) analytic at zero in $(p_1^{\frac{1}{nN_1}}, ..., p_d^{\frac{1}{nN_d}})$; (b) analytic in $\mathbf{p} \ne \mathbf{0}$ in the poly-sector $|\arg p_i| < \frac{n}{n-1}\varphi + \frac{\pi}{2(n-1)}$, $i \le d$; and (c) exponentially bounded in the latter poly-sector.

Existence and asymptoticity of the formal power series follow as a corollary, using Watson's Lemma.

Eariler, Borel summability has been shown for heat equation by Lutz, Miyake and Schäfke [21] and generalized to linear PDEs with constant coefficients by Balser [1] and in special classes of higher order nonlinear PDEs in [10].

3.4. **More general irregular singularities and multisummability.** The simple normalizing procedure described above does not always work; although this situation is in some sense nongeneric it plays an important role in some applications. In some instances mixed type transseries; for instance, by taking $e^{-x}Ei(x)$ which is a solution of a rank one differential equation as the rhs of (97) we get

$$f' + 2xf = e^{-x}Ei(x) \tag{125}$$

which can be brought to a second order meromorphic equation by using the equation of $e^{-x}Ei(x)$, i.e. by adding (125) to its derivative,

$$Df := f'' + (2x+1)f' + 2(1+x)f = \frac{1}{x} \tag{126}$$

The power series solution can be obtained contractively as in the previous examples. For the complete transseries solution, the substitution $f = e^w$ now yields

$$\frac{1}{2}w'^2 + xw' + x = 0$$

which has two solutions of level zero,

$$w_1 = -x^2 + x + \frac{1}{2}\ln x + K_1 - \frac{1}{2x} - ...$$
$$w_2 = -x - \frac{1}{2}\ln x + K_2 + \frac{1}{2x} + ... \tag{127}$$

and thus

$$f = f_1 + C_2 e^{-x^2+x} x^{\frac{1}{2}} f_2 + C_3 e^{-x} x^{-\frac{1}{2}} f_3 \tag{128}$$

with f_1, f_2, f_3 power series.

Clearly, no change of independent variable brings (128) to the form (58). On the other hand, neither of variable x nor $x^2 - x$ (equivalently, x^2) can be used for Borel summation:

(i) Borel transform in the variable x^2 (or $x^2 - x$) yields a function that has still a *divergent* power series at the origin. Indeed the the coefficients c_n of the power series f_1, which satisfy the recurrence relation

$$c_{n+1} = (n-1)\Big[c_n + \frac{1}{2}c_{n-1} - \frac{1}{2}(n-2)c_{n-2}\Big] \tag{129}$$

(with $c_0 = 0 = c_1 = 0, c_2 = \frac{1}{2}$) grow like $n!$, while a Borel transform in x^2 only decreases the growth of the coefficients by a factor of, roughly, $(n!)^{-1/2}$. At a closer look it can be shown seen that the factorial growth of the coefficients and the presence of the term e^{-x} are interrelated.

(ii) It can be checked that in the variable x, the Borel transform F of the power series solution f of (126) is not Laplace transformable because of faster than exponential growth. Indeed, F satisfies the equation

$$2F' - pF = \frac{1}{1-p}; \quad F(0) = 0 \tag{130}$$

The superexponential growth, in turn, can be related to the presence of e^{-x^2+x} in the transseries.

3.4.1. This suggests decomposing the formal power series solution into two parts, one summable in the variable x, the other one in the variable x^2. We can adjust the parameters a and b so that

(i) The initial condition $c_0 = 0, c_1 = -a/2, c_3 = -\frac{a}{4} + \frac{b}{2}$ in the recurrence (129) of the equation $Df = -a - bx^{-2}$ implies at most $(n!)^{-1/2}$ growth of c_n and at the same time,

(ii) The Borel transform of equation $Df = a + x^{-1} + bx^{-2}$ has a solution with subexponential growth.

The formal series solution of $Df = x^{-1}$ is thus decomposed in a sum of separately Borel summable series in the variables x^2 and x respectively. That this (rather ad-hoc) procedure works is a reflection of general theorems in multisummability.

3.4.2. *Multisummability.* A powerful and general technique, that of *acceleration and multisummability* introduced by Écalle, see [14] and [15], adequately deals with mixed divergences in very wide generality. The procedure is universal and works in the same way for very general systems.

In the case of solutions of meromorphic differential equations, mixed types of diveregence relate to the presence of exponential terms with exponents of different powers. Then multisummation consists in Borel transform with respect to the lowest power of x in the exponents of the transseries, usual summation $\mathcal{S}$, a sequence of transformations called accelerations (which mirror in Borel space the passage from one power in the exponent to the immediately larger one) followed by a final Laplace transform in the largest power of x. More precisely ([15]):

$$\mathcal{L}_{k_1} \circ \mathcal{A}_{k_2/k_1} \circ \cdots \circ \mathcal{A}_{k_q/k_{q-1}} \mathcal{S}\mathcal{B}_{k_q} \tag{131}$$

where $(\mathcal{L}_k f)(x) = (\mathcal{L}f)(x^k)$, $\mathcal{B}_k$ is the formal inverse of $\mathcal{L}_k$, $\alpha_i \in (0,1)$ and the acceleration operator $\mathcal{A}_\alpha$ is formal the image, in Borel space, of the change of variable from x^α to x and is defined as

$$\mathcal{A}_\alpha \varphi = \int_0^\infty C_\alpha(\cdot, s)\varphi(s)ds \tag{132}$$

and where, for $\alpha \in (0,1)$, the kernel C_α is defined as

$$C_\alpha(\zeta_1,\zeta_2) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\zeta_2 z - \zeta_1 z^\alpha} dz \tag{133}$$

where we adapted the notations in [1] to the fact that the formal variable is large. In our example, $q = 2, k_2 = 1, k_1 = 2$.

In [4] Braaksma proved of multisummability of series solutions of general non-linear meromorphic ODEs using Borel transforms in the spirit of Ecalle'e theory.

Note 50. *(i) Multisummability of type (131) can be equivalently characterized by decomposition of the series into terms which are ordinarily summable after changes of independent variable of the form $x \to x^\alpha$. This is shown in* [1] *where it is used to give an alternative proof of multisummability of series solutions of meromorphic ODEs, closer to the cohomological point of view of Ramis and Sibuya.*

(ii) More general multisummability is described by Ecalle [15]*, allowing, among others, for stronger than power-like acceleration. This is relevant to more general transseries equations.*

(ii) We expect that the finite generation property of transseries allows for the cohomological approach (i) combined with (ii) to prove multisummability of transseries solutions of more general contractive equations, in the sense of Theorem 15. This would give a rigorous proof of closure of analyzable functions under all natural operations in the sense of Ecalle.

References

[1] W Balser *From divergent power series to analytic functions, Springer-Verlag, 1994.*
[2] Berry, M. V.; Howls, C. J. 1990 Hyperasymptotics. *Proc. R. Soc. Lond. A* **430**, 653–668.
[3] Berry, M. V.; Keating, J. P. 1992 A new asymptotic representation for $\zeta(\frac{1}{2}+it)$ and quantum spectral determinants. *Proc. Roy. Soc. London Ser. A **437** (1992), no. 1899, 151–173.*

[4] B L J Braaksma, *Multisummability of formal power series solutions of nonlinear meromorphic differential equations*, Ann. Inst. Fourier, Grenoble,**42**, 3, 517-540 (1992).
[5] C M Bender and S A Orszag *Advanced mathematical methods for scientists and engineers I; Asymptotic methods and perturbation theory*, Mc Graw Hill (1978) and Springer (1999).
[6] E Borel *Leçons sur les fonctions monogènes, Gauthier-Villars*, Paris (1917).
[7] B L J Braaksma *Transseries for a class of nonlinear difference equations* J. Differ. Equations Appl. 7, no. 5, 717–750 (2001).
[8] B. L. J. Braaksma, R Kuik (submitted).
[9] J H Conway, *On numbers and games*, Academic Press (1976).
[10] O. Costin, S. Tanveer, Existence and uniqueness for a class of nonlinear higher-order partial differential equations in the complex plane, Comm. Pure Appl. Math, Vol. LIII, 1092—1117 (2000).
[11] O. Costin, S. Tanveer *On the existence and uniqueness of solutions of nonlinear evolution systems of PDEs in* $\mathbb{R}^+ \times \mathbb{C}^d$*, their asymptotic and Borel summability properties* preprint (2003).
[12] O Costin *On Borel summation and Stokes phenomena for rank one nonlinear systems of ODE's Duke Math. J. Vol. 93, No.2 pp. 289–344 (1998).*
[20] R Kuik *Transseries in Difference and Differential Equation* Thesis, University of Groningen (2003).
[13] O Costin and R D Costin *On the formation of singularities of solutions of nonlinear differential systems in antistokes directions* To appear in Inventiones Mathematicae.
[14] J Ecalle *Les fonctions resurgentes, vol. I, II and III*, Publ. Math. Orsay, 1985.
[15] J Écalle, *in Bifurcations and periodic orbits of vector fields, NATO ASI Series, Vol. 408, 1993*
[16] H Flashka and A C Newell *Monodromy and spectrum preserving transformations* Commun. Math. Phys. **76** pp. 65–116 (1980).
[17] J Ecalle, F Menous, *Well-behaved convolution averages and the non-accumulation theorem for limit-cycles.* The Stokes phenomenon and Hilbert's 16th problem, 71–101, World Sci. Publishing (1996)
[18] L Fuchs *Sur quelques équations différentielles linéaires du second ordre* C. R. Acad. Sci., Paris **141** pp. 555-558 (1905).
[19] G H Hardy. Orders of Infinity, Cambridge Univ. Press, (1954).
[20] R Kuik, Transseries in differential and difference equations, PhD Thesis, University of Groningen, ISBN 90-367-1771-x (2003).
[21] D. A. Lutz, M. Miyake and R. Schäfke On the Borel summability of divergent solutions of the heat equation, Nagoya Math. J. **154**, 1, (1999).
[22] L van den Dries, A Macintyre and D Marker *The elementary theory of restricted analytic elds with exponentiation* Ann. of Math. 140 (1994)
[23] W Wasow, *Asymptotic expansions for ordinary differential equations*, Interscience Publishers (1968).

Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019