

Analytic methods for obstruction to integrability in discrete dynamical systems

O Costin

Rutgers University
Piscataway, NJ 08854-8019

AND

M D Kruskal *Rutgers University*
Piscataway, NJ 08854-8019

Abstract

A unique analytic continuation result is proved for solutions of a relatively general class of difference equations, using techniques of generalized Borel summability.

This continuation allows for Painlevé property methods to be extended to difference equations.

It is shown that the Painlevé property (PP) induces, under relatively general assumptions, a dichotomy within first order difference equations: all equations with PP can be solved in closed form; on the contrary, absence of PP implies, under some further assumptions, that the local conserved quantities are strictly local in the sense that they develop singularity barriers on the boundary of some compact set.

The technique produces analytic formulas to describe fractal sets originating in polynomial iterations.

Contents

1	Introduction and main results	2
1.1	Setting	3
1.2	Analyzability: transseries and generalized Borel summability	4
	Transseries for difference equations	5
1.3	Uniqueness of continuation from \mathbb{N} to \mathbb{C}	5
1.4	Continuation of solutions of difference equations to the complex n plane	5
1.5	Continuability and singularities	5
1.6	Integrability	5
1.7	First order autonomous equations	6
1.8	Classification of equations of type (1.11) with respect to integrability	7
1.9	Failure of integrability test and barriers of singularities	7
1.10	Example: the logistic map	8
1.11	Application to the study of fractal sets	8

2	General remarks on integrability	9
3	Proofs	12
3.1	Proof of Theorem 1.1	12
	Outline	12
		12
		14
		14
	Completion of the proof of Theorem 1.1	14
3.2	Remarks on first order equations	15
	Relation to properties of the conjugation map	16
	Conserved quantities	17
3.3	End of proof of Lemma 1.4 (a)	17
3.4	Proof of Theorem 1.5	18
	Notations	18
	The “if” part of Theorem 1.5	18
	The “only if” part of Theorem 1.5	18
3.5	Proof of Theorem 1.8	19
3.6	Borel summability of formal invariant for logistic map when	
	$a = 1$	20
	Proof of Theorem 3.11	22
4	Julia sets for the map (1.13) for $a \in (0, 1)$	24
5	Behavior at the singularity barrier	26
	Analyticity	27
	End of the proof of Theorem 1.10 (iii)	28
	Appendix: Iterations of rational maps	28
	Further results used in the proofs	29
6.1	Proof of Proposition 1.7	30

1 Introduction and main results

Solvability of difference equations as well as chaotic behavior have stimulated extensive research. For differential equations the Painlevé test, which consists in checking whether all solutions of a given equation are free of movable non-isolated singularities provides a convenient and effective tool in detecting integrable cases (see §2).

A difficulty in applying Painlevé’s methods to difference equations resides in extending the solutions, which are defined on a discrete set, to

the complex plane of the independent variable in a natural and effective fashion, when, in the interesting cases, there is no explicit formula for them. A number of alternative approaches, but no genuine analog of the Painlevé test, have been proposed, see [1] [9] [26] [24] (a comparative discussion of the various approaches is presented in [1]).

The present paper proposes a natural way, based on generalized Borel summability, to extend the solutions in the complex plane (Theorem 1.1 below), allowing for a definition of a discrete Painlevé test. Subsequent analysis shows that the test is sharp in a class of first order difference equations: those passing the test are explicitly solvable (Theorem 1.5) while polynomial equations failing the test exhibit chaotic behavior and their local conserved quantities (see §1.9) develop barriers of singularities along fractal sets (Theorem 1.8).

The approach also allows for a detailed study of analytic properties near these singularity barriers as well as finding rapidly convergent series representing the corresponding fractal curves (Theorem 1.10).

1.1 Setting

We consider difference systems of equations which can be brought to the form

$$(1.1) \quad \mathbf{x}(n+1) = \hat{\Lambda} \left(I + \frac{1}{n} \hat{A} \right) \mathbf{x}(n) + \mathbf{g}(n, \mathbf{x}(n))$$

where $\hat{\Lambda}$ and \hat{A} are constant coefficient matrices, \mathbf{g} is convergently given for small \mathbf{x} by

$$(1.2) \quad \mathbf{g}(n, \mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}^m} \mathbf{g}_{\mathbf{k}}(n) \mathbf{x}^{\mathbf{k}}$$

with $\mathbf{g}_{\mathbf{k}}(n)$ analytic in n at infinity and

$$(1.3) \quad \mathbf{g}_{\mathbf{k}}(n) = O(n^{-2}) \text{ as } n \rightarrow \infty, \text{ if } \sum_{j=1}^m k_j \leq 1$$

under nonresonance conditions: Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$ and $\mathbf{a} = (a_1, \dots, a_m)$ where $e^{-\mu_k}$ are the eigenvalues of $\hat{\Lambda}$ and the a_k are the eigenvalues of \hat{A} . Then the nonresonance condition is

$$(1.4) \quad (\mathbf{k} \cdot \boldsymbol{\mu} = 0 \pmod{2\pi i} \text{ with } \mathbf{k} \in \mathbb{Z}^{m_1}) \Leftrightarrow \mathbf{k} = \mathbf{0}.$$

We consider the solutions of (1.1) which are small as n becomes large.

1.2 Analyzability: transseries and generalized Borel summability

These concepts were introduced by Écalle in the fundamental work [14]. Analyzability of difference equations was shown in [7, 14]. We give below a brief description of the concepts effectively used in the present paper and refer to [10, 7] for a general theory. An expression of the form

$$(1.5) \quad \tilde{\mathbf{x}}(t) := \sum_{\mathbf{k} \in \mathbb{N}^m} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} t} t^{\mathbf{k} \cdot \mathbf{a}} \tilde{\mathbf{x}}_{\mathbf{k}}(t)$$

where $\tilde{\mathbf{x}}_{\mathbf{k}}(t)$ are formal power series in powers of t^{-1} is an exponential power series; it is a transseries as $t \rightarrow +\infty$ if $\Re(\mu_j) > 0$ for all j with $1 \leq j \leq m$. Such a transseries is Borel summable as $t \rightarrow +\infty$ if there exist constants $A, \nu > 0$ and a family of functions

$$(1.6) \quad \mathbf{X}_{\mathbf{k}} \text{ analytic in a sectorial neighborhood } \mathcal{S} \text{ of } \mathbb{R}^+, \text{ satisfying} \\ \sup_{p \in \mathcal{S}, \mathbf{k} \in \mathbb{N}^m} \left| A^{|\mathbf{k}|} e^{-\nu|p|} \mathbf{X}_{\mathbf{k}} \right| < \infty$$

such that the functions $\mathbf{x}_{\mathbf{k}}$ defined by

$$(1.7) \quad \mathbf{x}_{\mathbf{k}}(t) = \int_0^\infty e^{-tp} \mathbf{X}_{\mathbf{k}}(p) dp$$

are asymptotic to the series $\tilde{\mathbf{x}}_{\mathbf{k}}$ i.e.

$$(1.8) \quad \mathbf{x}_{\mathbf{k}}(t) \sim \tilde{\mathbf{x}}_{\mathbf{k}}(t) \quad (t \rightarrow +\infty)$$

It is then easy to check that condition (1.6) implies that the sum

$$(1.9) \quad \mathbf{x}(t) = \sum_{\mathbf{k} \in \mathbb{N}^{n_0}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} t} t^{\mathbf{k} \cdot \mathbf{a}} \mathbf{x}_{\mathbf{k}}(t)$$

is convergent in the half plane $\mathbb{H} = \{t : \Re(t) > t_0\}$, for t_0 large enough. The function \mathbf{x} in (1.9) is by definition the Borel sum of the transseries $\tilde{\mathbf{x}}$ in (1.5). Generalized Borel summability allows for singularities of $\mathbf{X}_{\mathbf{k}}$ of certain types along \mathbb{R}^+ . The transseries $\tilde{\mathbf{x}}$ is (generalized) Borel summable in the direction $e^{i\varphi} \mathbb{R}^+$ if $\tilde{\mathbf{x}}(\cdot e^{-i\varphi})$ is (generalized) Borel summable. (Generalized) Borel summation is known to be an extended isomorphism between transseries and their sums, see [14], [15], [10].

Transseries for difference equations

Braaksma [7] showed that the recurrences (1.1) possess l -parameter transseries solutions of the form (1.5) with $t = n$ where $\tilde{\mathbf{x}}_{\mathbf{k}}(n)$ are formal power series in powers of n^{-1} and $l \leq m$ is chosen such that, after reordering the indices, we have $\Re(\mu_j) > 0$ for $1 \leq j \leq l$.

It is shown in [7] and [19] that these transseries are generalized Borel summable in any direction and Borel summable in all except m of them and that

$$(1.10) \quad \mathbf{x}(n) = \sum_{\mathbf{k} \in \mathbb{N}^l} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} n} n^{\mathbf{k} \cdot \mathbf{a}} \mathbf{x}_{\mathbf{k}}(n)$$

is a solution of (1.1), if $n > y_0$, t_0 large enough.

1.3 Uniqueness of continuation from \mathbb{N} to \mathbb{C}

The values of \mathbf{x} on the integers uniquely determine \mathbf{x} .

THEOREM 1.1 *In the assumptions in §1.1 and 1.2, define the continuation of $\mathbf{x}_{\mathbf{k}}(n)$ in the half plane $\{t : \Re(t) > t_0\}$ by $\mathbf{x}(t)$, cf. (1.6)–(1.9).*

The following uniqueness property holds. If in the assumptions (1.6)–(1.9) we have $\mathbf{x}(n) = 0$ for all except possibly finitely many $n \in \mathbb{N}$, then $\mathbf{x}(t) = 0$ for all $t \in \mathbb{C}$, $\Re(t) > t_0$.

The proof is given in §3.1.

1.4 Continuation of solutions of difference equations to the complex n plane

The representation (1.10) and Theorem 1.1 make the following definition natural.

1.5 Continuability and singularities

The function \mathbf{x} is analytic in \mathbb{H} and has, in general, nontrivial singularities in $\mathbb{C} \setminus \mathbb{H}$. The results in [12], extended to difference equations in [7, 8, 19], give constructive methods to determine those singularities that arise near the boundary of \mathbb{H} ; these form, generically, nearly periodic arrays.

1.6 Integrability

In particular, Painlevé's test of integrability (absence of movable non-isolated singularities) extends then to difference equations.

As in the case of differential equations, fixed singularities are singular points whose location is the same for all solutions; they define a common Riemann surface. Other singularities (i.e., whose location depends on initial data) are called *movable*.

DEFINITION 1.2 *We say that a difference equation has the Painlevé property if its solutions are analyzable and their analytic continuations on a Riemann surface common to all solutions, have only isolated singularities.*

Note. We follow the usual convention that an isolated singular point of an analytic function f is a point z_0 such that f is analytic in some disk centered at z_0 except perhaps at z_0 itself. Branch points are thus not isolated singularities and neither are singularity barriers; it is worth noting, however, that for differential equations there exist equations sometimes considered integrable (the *Chazy equation*, a third order nonlinear one is the simplest known example) whose solutions exhibit singularity barriers.

1.7 First order autonomous equations

These are equations of the type

$$(1.11) \quad x_{n+1} = G(x_n) := ax_n + F(x_n)$$

Some analyticity assumptions on F are required for our method to apply. We define a class of single valued functions closed under all algebraic operations and composition (the latter is needed since x_n written in terms of x_0 involves repeated composition).

We need to allow for singular behavior in F , and meromorphic functions are obviously not closed under composition. The following definition formalizes an extension of meromorphic functions, often used informally in the theory of integrability.

DEFINITION 1.3 *We define the "mostly analytic functions" to be the class \mathcal{M} of functions analytic in the complement of a closed countable set (which may depend on the function).*

LEMMA 1.4 (a) *The class \mathcal{M} is closed under addition, multiplication and multiplication by scalars, and also under division and composition between (nonconstant) functions. It includes meromorphic functions.*

(b) *If $G \in \mathcal{M}$ is not a constant, then the equation $G(x) = y$ has solutions for all large enough y .*

(c). The class \mathcal{M}_0 of $G \in \mathcal{M}$, with G analytic at zero, $G(0) = 0$ and $0 < |G'(0)| < 1$ is closed under composition.

In particular, $G^{\circ m} \in \mathcal{M}$ for $m \geq 1$.

Proof. All properties in (a) are obvious except for closure under composition and division, proved in § 3.3; (b) follows from the proof of Lemma 6.9. (c) is easily shown using (a).

1.8 Classification of equations of type (1.11) with respect to integrability

THEOREM 1.5 *Assume $G \in \mathcal{M}$ has a stable fixed point (say at zero) where it is analytic. Then the difference equation (1.11) has the Painlevé property iff for some $a, b \in \mathbb{C}$ with $|a| < 1$,*

$$(1.12) \quad G(z) = \frac{az}{1+bz}$$

The proof is given in §3.4.

REMARK 1.6 *The Painlevé property is not sensitive to which attracting fixed point of G or its iterates is used in the analysis. This follows from the Proposition below.*

Assume p is another attracting fixed point of G and let $G_1(s) = G(p+s) - p$ (G_1 has an attracting fixed point at the origin).

PROPOSITION 1.7 *The difference equation (1.11) has the Painlevé property iff the difference equation $x_{n+1} = G_1(x_n)$ has the Painlevé property. Furthermore if G has an iterate $G^{\circ m}$ with an attracting fixed point where the conjugation map extends analytically to \mathbb{C} except for isolated singularities, then the same is true for any attracting fixed point of any iterate $G^{\circ k}$.*

This is shown in §6.1.

1.9 Failure of integrability test and barriers of singularities

Conserved quantities are naturally defined as functions $C(x; n)$ with the property

$$C(x_{n+1}; n+1) = C(x_n; n)$$

We now look at cases without the Painlevé property, when G is a polynomial map. We arrive at the striking conclusion that these equations

are not solvable in terms of functions extendible to the complex plane, or on Riemann surfaces. The conserved quantities will typically develop singularity barriers.

We use, in the formulation of the following theorem, a number of standard notions and results relevant to iterations of rational maps; these are briefly reviewed in the Appendix, §6.

THEOREM 1.8 *Assume G is a nonlinear polynomial with an attracting fixed point at the origin. Denote by \mathcal{K}_p the maximal connected component of the origin in the Fatou set of G . (It follows that \mathcal{K}_p is an open, bounded, and simply connected set).*

Then the domain of analyticity of Q (see (3.26)) is \mathcal{K}_p , and $\partial\mathcal{K}_p$ is a singularity barrier of Q .

This theorem is proved in §3.5.

1.10 Example: the logistic map

The discrete logistic map is defined by

$$(1.13) \quad x_{n+1} = ax_n(1 - x_n)$$

The following result was proved by the authors in [11].

PROPOSITION 1.9 *The recurrence (1.13) has the Painlevé property in Definition 1.2 iff $a \in \{-2, 0, 2, 4\}$ (in which cases it is explicitly solvable). If $a \notin \{-2, 0, 2, 4\}$ then the conserved quantity has barriers of singularities.*

1.11 Application to the study of fractal sets

The techniques also provide detailed information on the Julia sets of iterations of the interval.

THEOREM 1.10 *Consider the equation (1.13) for $a \in (0, 1/2)$.*

(i) There is a conserved quantity K , satisfying the functional relation

$$(1.14) \quad K(z)^2 = aK(z^2)(1 + K(z))$$

which is a conformal map of the open unit disk S_1 onto the interior \mathcal{K}_p of the Julia set of (1.13).

(ii) K is Lipschitz continuous of exponent $\log_2(2 - a)$ in $\overline{S_1}$ (the Lipschitz constant can be determined from the proof).

(iii) ∂S_1 is a barrier of singularities of K . Near $1 \in \partial S_1$ we have

$$(1.15) \quad K(z) = \Phi\left(\tau\Psi(\ln \tau(z))\right)$$

where

$$(1.16) \quad \begin{aligned} \tau(z) &= \ln(z^{-1})^{\log_2(2-a)} \\ \Phi &\text{ is analytic at zero, } \Phi(0) = \frac{a}{1-a}, \Phi'(0) = 1 \\ \Psi &\text{ is real analytic and periodic of period } \ln 2. \end{aligned}$$

With $t = 1 - z$ we have

$$(1.17) \quad K = \frac{a}{1-a} + \sum_{l \in \mathbb{Z}} \sum_{k, m \in \mathbb{N}} C_{l; k, m} t^{il \log_2(2\pi) + k \log_2(2-a) + m}$$

where the series converges (rapidly) if t and $|\arg t|$ are small.

This theorem is proved in §4.

NOTE 1.11 *The proof of Proposition 5.1 shows that the Lipschitz exponent is optimal. The theorem is valid for any $a < 1$, and the proof is similar.*

NOTE 1.12 *It follows from Theorem 1.10 (iii) and (1.14) that every binary rational is a cusp of angle $\arctan\left(\frac{\pi}{2} \log_2(2-a)\right)$, see also Fig. 1.*

2 General remarks on integrability

This problem has a long history, and the task of finding of *differential* equations solvable in terms of known functions was addressed as early as the works of Leibniz, Riccati, Bernoulli, Euler, Laplace, and Lagrange. “In the 18th century, Euler was defining a function as arising from the application of finitely or infinitely many algebraic operations (addition, multiplication, raising to integer or fractional powers, positive or negative) or analytic operations (differentiation, integration), in one or more variables” [6]. It was later found that some linear equations have solutions which, although not explicit by this standard, have “good” *global* properties and can be thought of as defining new functions. To address the question whether nonlinear equations can define new functions, Fuchs had the idea that a crucial feature now known as the *Painlevé property* (PP) is the absence of movable (meaning their position is solution-dependent,

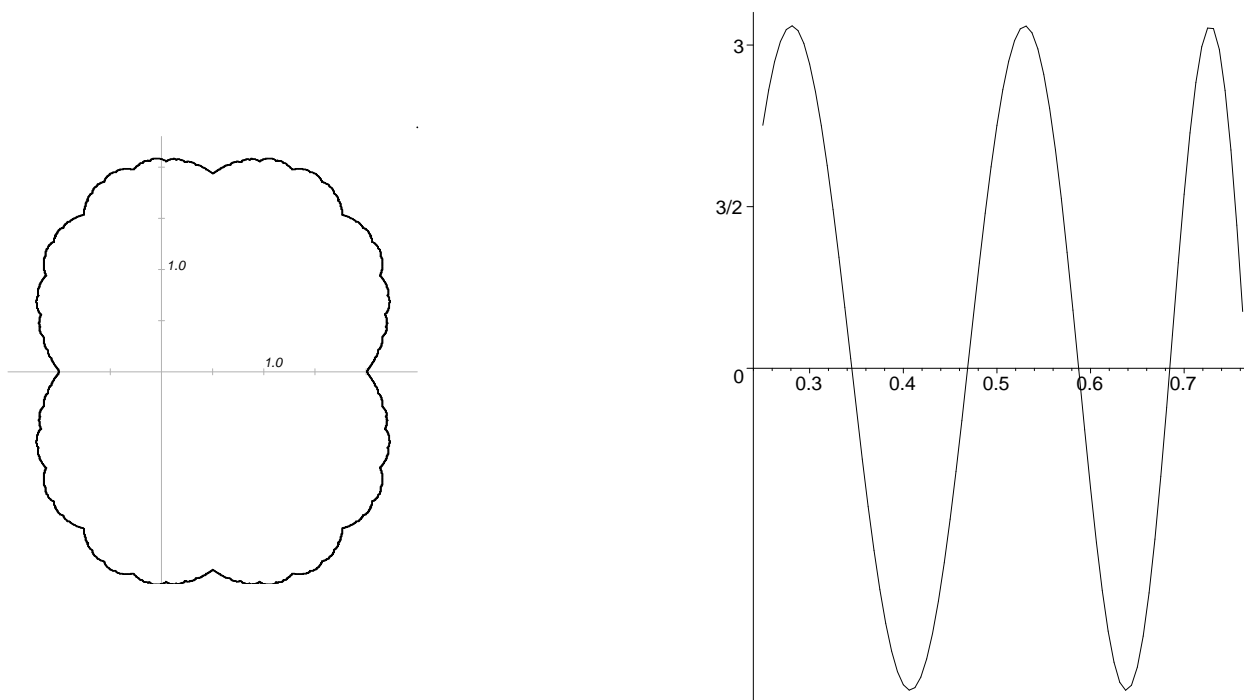


Figure 1.1. (a) Julia set for $G = \frac{1}{2}x(1-x)$. The set \mathcal{K}_p is the interior of the curve. (b) The function $10^9(\Psi(\ln \ln z_0) + c)$ for $a = \frac{1}{2}$, $c = .079324389476$ (the plot relies on (5.8), $N = 300$).

cf. §1.6) essential singularities, primarily branch-points, see [17]. First order equations were classified with respect to the PP by Fuchs, Briot and Bouquet, and Painlevé by 1888, and it was concluded that they give rise to no new functions. Painlevé took this analysis to second order, looking for all equations of the form $u'' = F(u', u, z)$, with F rational in u' , algebraic in u , and analytic in z , having the PP [29, 30]. His analysis, revised and completed by Gambier and Fuchs, found some fifty types with this property and succeeded to solve all but six of them in terms of previously known functions. The remaining six types are now known as the Painlevé equations, and their solutions, called the Painlevé transcendents, play a fundamental role in many areas of pure and applied mathematics. Beginning in the 1980's, almost a century after their discovery, these problems were solved, using their striking relation to linear problems¹, by various methods including the powerful techniques of isomonodromic deformation and reduction to Riemann-Hilbert problems [13], [16], [25].

Sophie Kovalevskaya searched for cases of the spinning top having the PP. She found a previously unknown integrable case and solved it in terms of hyperelliptic functions. Her work [20], [21] was so outstanding that not only did she receive the 1886 Bordin Prize of the Paris Academy of Sciences, but the associated financial award was almost doubled.

The method pioneered by Kovalevskaya to identify integrable equations using the Painlevé property is now known as the *Painlevé test*. Part of the power of the Painlevé test stems from the remarkable phenomenon that equations passing it can generally be solved by some method. This phenomenon is not completely understood. At an intuitive level, however, if for example all solutions of an equation are meromorphic, then by solving the equation “backwards,” these solutions and their derivatives can be written in terms of the initial conditions. This gives rise to sufficiently many integrals of motion with good regularity properties *globally* in the complex plane.

The Painlevé test has some drawbacks, notably lack of invariance under transformations. To overcome them, [22] introduced the poly-Painlevé test.

¹ Some linear problems conducive to Painlevé equations were known already at the beginning of last century. In 1905 Fuchs found a linear isomonodromic problem leading to P_{VI}.

3 Proofs

3.1 Proof of Theorem 1.1

Outline

The idea of the proof is to use the convergence of (1.9) and its asymptotic properties to show that all terms $\mathbf{x}_{\mathbf{k}}$ vanish.

We start with some preparatory results.

REMARK 3.1 *If $\mathbf{x}_{\mathbf{k}} \neq 0$ then also $\mathbf{X}_{\mathbf{k}} \neq \mathbf{0}$ (see (1.7)) so for small p we have $\mathbf{X}_{\mathbf{k}} = \sum_{j=L_{\mathbf{k}}}^{\infty} \mathbf{c}_j p^j$ with $\mathbf{c}_{L_{\mathbf{k}}} \neq 0$ for some $L_{\mathbf{k}} \geq 0$. By Watson's Lemma [5], for large z in the right half plane we have*

$$(3.1) \quad \mathbf{x}_{\mathbf{k}} \sim \sum_{j=L_{\mathbf{k}}}^{\infty} \frac{\mathbf{c}_j j!}{z^{j+1}}; \quad (\mathbf{c}_{L_{\mathbf{k}}} \neq 0)$$

REMARK 3.2 *Since $\Re(\mu_i) > 0$ we have $\Re(\boldsymbol{\mu} \cdot \mathbf{k}) \rightarrow \infty$ as $\mathbf{k} \rightarrow \infty$. Therefore for any K , the sets of the form*

$$(3.2) \quad \{\mathbf{k} \in \mathbb{N}^{m_1} : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) < K\}; \quad \{\mathbf{k} \in \mathbb{N}^{m_1} : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) = K\}$$

are finite.

DEFINITION 3.3 *We define $S = \{\mathbf{k} : \mathbf{X}_{\mathbf{k}} \neq 0\}$. We define inductively the finite sets T_i (cf. Remark 3.2) and the numbers M_i as follows:*

$$(3.3) \quad \begin{aligned} T_0 &= \left\{ \mathbf{k} \in S : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) = \min_{\mathbf{k} \in S} \Re(\boldsymbol{\mu} \cdot \mathbf{k}) =: M_0 \right\} \\ T_1 &= \left\{ \mathbf{k} \in S \setminus T_0 : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) = \min_{\mathbf{k} \in S \setminus T_0} \Re(\boldsymbol{\mu} \cdot \mathbf{k}) =: M_1 \right\} \\ &\dots \\ T_j &= \left\{ \mathbf{k} \in S \setminus T_0 \dots \setminus T_{j-1} : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) = \min_{\mathbf{k} \in S \setminus T_0 \dots \setminus T_{j-1}} \Re(\boldsymbol{\mu} \cdot \mathbf{k}) =: M_j \right\} \\ &\dots \end{aligned}$$

Let also

$$(3.4) \quad r_j = \max_{\mathbf{k} \in T_j} \Re(\mathbf{a} \cdot \mathbf{k})$$

Note also that for some $\alpha > 0$ we have

$$(3.5) \quad r_j \leq \alpha M_j$$

Applying Remark 3.2 again we see that

$$(3.6) \quad \bigcup_{j=0}^{\infty} T_j = S$$

LEMMA 3.4 We have (see (1.9)),

$$(3.7) \quad \mathbf{x}(z) = \sum_{\mathbf{k} \in T_0} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} z} z^{\mathbf{k} \cdot \mathbf{a}} \mathbf{x}_{\mathbf{k}}(z) + O(e^{-M_1 z} z^{r_1}) \quad (z \rightarrow +\infty)$$

PROOF: We write

$$(3.8) \quad \mathbf{x}(z) = \sum_{\mathbf{k} \in T_0} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} z} z^{\mathbf{k} \cdot \mathbf{a}} \mathbf{x}_{\mathbf{k}}(z) + \sum_{\mathbf{k} \in S \setminus T_0} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} z} z^{\mathbf{k} \cdot \mathbf{a}} \mathbf{x}_{\mathbf{k}}(z)$$

The second series is uniformly and absolutely convergent for large enough $z \in \mathbb{R}^+$ since it is bounded by the sub-sum of a (derivative of) a multi-geometric series

$$(3.9) \quad \sum_{\mathbf{k} \in S \setminus T_0} |\mathbf{A}^{\mathbf{k}} \mathbf{C}^{\mathbf{k}} z^{\mathbf{k} \cdot \mathbf{a}}| e^{-\mathbf{k} \cdot \Re(\boldsymbol{\mu}) z}$$

Since (3.9) is absolutely convergent it can be thus be convergently rearranged as

$$(3.10) \quad \sum_{j=1}^{\infty} e^{-M_j z} \sum_{\mathbf{k} \in T_j} |\mathbf{A}^{\mathbf{k}} \mathbf{C}^{\mathbf{k}} z^{\mathbf{k} \cdot \mathbf{a}}| = \sum_{j=1}^{\infty} e^{-M_j z} z^{r_j} D_j(z)$$

(see again Definition 3.3 and Remark 3.2). It is easy to see that $D_j(z)$ are nonincreasing in $z \in \mathbb{R}^+$ and for large enough $z > 0$ all products $z^{r_j} e^{-M_j z}$ are decreasing (cf. also (3.5)). Therefore the convergent series

$$(3.11) \quad \sum_{j=1}^{\infty} e^{-(M_j - M_1) z} z^{r_j - r_1} D_j(z)$$

is decreasing in $z > 0$ and so

$$\sum_{j=1}^{\infty} e^{-M_j z} z^{r_j} D_j(z) \leq Const. e^{-M_1 z} z^{r_1}$$

■

Note. A similar strategy could be also be used to show the classical Weierstrass preparation theorem.

Assume first, to get a contradiction, that we have $\mathbf{x}_0 \neq 0$ and so $\mathbf{X}_0 \neq 0$ so for small p we have $\mathbf{X}_{\mathbf{k}} = \sum_{j=m_0}^{\infty} \mathbf{c}_j p^j$ with $\mathbf{c}_{m_0} \neq 0$. Then, since

$$\mathbf{x}(n) = \mathbf{x}_0(n) + O(e^{-M_1 n} n^{r_1})$$

and by Remark 3.1

$$(3.12) \quad \lim_{n \rightarrow \infty} n^{-m_0-1} \mathbf{x}_0 = (m_0 + 1)! \mathbf{c}_{m_0} \neq 0$$

which contradicts $\mathbf{x}(n) = 0$ for $n \in \mathbb{N}$.

Let now

$$(3.13) \quad R_0 = \max \{ \Re(\mathbf{k} \cdot \mathbf{a} - L_{\mathbf{k}} - 1) : \mathbf{k} \in T_0 \}$$

and

$$(3.14) \quad T'_0 = \{ \mathbf{k} \in T_0 : \Re(\mathbf{k} \cdot \mathbf{a} - L_{\mathbf{k}} - 1) = R_0 \}$$

LEMMA 3.5 *We have*

$$(3.15) \quad \mathbf{x}(z) = \sum_{\mathbf{k} \in T'_0} \mathbf{C}^{\mathbf{k}} c_{L_{\mathbf{k}}} L_{\mathbf{k}}! z^{\mathbf{k} \cdot \mathbf{a} - L_{\mathbf{k}} - 1} e^{-\mathbf{k} \cdot \boldsymbol{\mu} z} + o\left(z^{R_0} e^{-M_0 z}\right) \quad \text{for } (z \rightarrow +\infty)$$

PROOF: This is an immediate consequence of Remark 3.1, Lemma 3.4, and (3.13) and (3.14). \blacksquare

Completion of the proof of Theorem 1.1

The proof now follows, by reductio ad impossibile, from (3.15), the assumption that $\mathbf{x}(n) = 0$ for all large enough $n \in \mathbb{N}$, the fact that by construction all $c_{L_{\mathbf{k}}}$ are nonzero and the following Lemma.

LEMMA 3.6 *Let $d_{\mathbf{k}} \in \mathbb{C}$. Then*

$$\sum_{\mathbf{k} \in T'_0} d_{\mathbf{k}} n^{\mathbf{k} \cdot \mathbf{a} - M_{\mathbf{k}}} e^{-\mathbf{k} \cdot \boldsymbol{\mu} n} = o\left(n^{R_0} e^{-K_1 n}\right) \quad (\text{as } n \rightarrow \infty, n \in \mathbb{N})$$

iff all $d_{\mathbf{k}}$ are zero.

PROOF: We now take $n_0 = \text{card}(T'_0)$, n large enough and note that $(n+j)^b = n^b(1+o(n^{-1}))$ if $j \leq n_0$. Then a simple estimate shows that to prove the Lemma it suffices to show that the following equation cannot hold for all $0 \leq l \leq n_0 - 1$

$$(3.16) \quad \sum_{\mathbf{k} \in T'_0} d_{\mathbf{k}} e^{-(n+l)\mathbf{k} \cdot \boldsymbol{\mu}} = q_l$$

where

$$(3.17) \quad q_l = o(e^{-nM_0}) \quad (\text{as } n \rightarrow \infty, n \in \mathbb{N})$$

If $n_0 = 1$ this is immediate. Otherwise, we may think of (3.16) for $0 \leq l \leq n_0 - 1$ as a system of equations for the $d_{\mathbf{k}}$ with $\mathbf{k} \in T'_0$. The determinant Δ of the system is a number of absolute value e^{-nlM_0} times the Vandermonde determinant of the quantities $\{e^{-\mathbf{k} \cdot \boldsymbol{\mu}}\}_{\mathbf{k} \in T'_0}$. In particular, for some $C > 0$ independent of n we have that $e^{-nlM_0} |\Delta|$ is independent of n ,

$$(3.18) \quad e^{-nlM_0} |\Delta| = C \left| \prod_{\mathbf{k}_1 \neq \mathbf{k}_2 \in T'_0} (e^{-(\mathbf{k}_1 - \mathbf{k}_2) \cdot \boldsymbol{\mu}} - 1) \right|$$

and nonzero by (1.4). Similarly, the minor $\Delta_{\mathbf{k}}$ of any $d_{\mathbf{k}}$ is bounded by $D_{\mathbf{k}} e^{-n(l-1)M_0}$ with $D_{\mathbf{k}}$ independent of n . We get $d_{\mathbf{k}} = o(1)$ for large n for all $\mathbf{k} \in T'_0$, and so $d_{\mathbf{k}} = 0$. ■

3.2 Remarks on first order equations

It turns out [27] that for *first order autonomous* equations near an attracting fixed point, the series \tilde{x}_k of (1.5) are mere constants and the transseries (1.5) are *classically convergent* for large enough n to actual solutions of the equation. This is a consequence of the Poincaré equivalence theorem, see [27].

NOTE 3.7 *If $|a| = 1$ factorially divergent series do occur. In §3.6 we show how to use Borel summation instead of usual convergence when $a = 1$.*

Assume for now that in (1.11) $G \in \mathcal{M}$ is analytic at zero, $F(0) = F'(0) = 0$ and $0 < |a| < 1$. As we mentioned, there is a one-parameter family of solutions presented as simple transseries of the form

$$(3.19) \quad x_n = x_n(C) = \sum_{k=1}^{\infty} e^{nk \ln a} C^k D_k$$

with D_k independent of C , which converge for large n . By definition their continuation to complex n is

$$(3.20) \quad x(z) = x(z; C) = \sum_{k=1}^{\infty} e^{zk \ln a} C^k D_k,$$

which is analytic for large enough z . To test for the Painlevé property, we proceed to find the properties of $x(z)$ for those values of z where (3.20) is no longer convergent, and then find the singular points of $x(z)$.

Note. In general, although (3.20) represents a continuous one-parameter family of solutions, there may be more solutions. We also examine this issue.

Relation to properties of the conjugation map

We can alternatively, and it turns out equivalently, define a continuation as follows. By the Poincaré theorem [2] p. 99 there exists a unique map φ with the properties

$$(3.21) \quad \varphi(0) = 0, \quad \varphi'(0) = 1 \quad \text{and } \varphi \text{ analytic at } 0$$

and such that

$$(3.22) \quad \varphi(az) = G(\varphi(z)) = a\varphi(z) + F(\varphi(z))$$

The map φ is a *conjugation map* between (1.11) and its linearization

$$(3.23) \quad X_{n+1} = aX_n$$

since, in view of (3.22),

$$(3.24) \quad x_n = \varphi(Ca^n)$$

for given C and n large enough, x_n is a solution of the recurrence (1.11).

We obtain a continuation of x from \mathbb{N} to \mathbb{C} through

$$(3.25) \quad x(z) = \varphi(Ca^z)$$

LEMMA 3.8 (i) *For equations of type (1.11), the continuations (3.20) and (3.25) agree.*

(ii) *$x(z; C)$ defined by (3.20) has only isolated movable singularities iff φ has only isolated singularities in \mathbb{C} .*

PROOF: Indeed, φ is analytic at the origin, and a power series expansion for large n of $\varphi(Ca^n)$ leads to a solution of the form (3.19), which obviously solves (1.11). If n_0 is large enough, it is clear that (3.19) can be inverted for C in terms of x_{n_0} and we can also find C' so that $x_{n_0} = \varphi(C'a^{n_0})$. On the other hand x_{n_0} uniquely determines all x_n with $n > n_0$. For equations of type (1.11), writing $x(z) = \varphi(Ca^z)$ is thus tantamount to making the substitution $n = z$ in (3.19). Note that, a^z is entire and φ is analytic at zero, and the presence of a singularity of φ which is not isolated is equivalent to the presence of a similar but *movable* singularity of $x(z) = \varphi(Ca^z)$ since its position depends on C . ■

Conserved quantities

The connection between C and the equivalence map is seen as follows. Near an attracting fixed point, say 0, we have a continuous one-parameter family of solutions of (1.11) in the form (3.24).

On the other hand the conjugation map φ is invertible for small argument by (3.21). We may then write

$$(3.26) \quad C = C(n, x_n) = \varphi^{-1}(x_n)a^{-n} =: Q(x_n)a^{-n}$$

where we see that $C(n, x_n)$ is a conserved quantity of (1.11), and $Q = \varphi^{-1}$ is analytic near zero. Clearly any equation near a stable fixed point is, in the sense of (3.26), *locally* solvable. Definition 1.2 requires however global properties.

Note first that, from the properties of φ (or from the constancy of C), Q satisfies the functional equation

$$(3.27) \quad Q(z) = a^{-1}Q(G(z))$$

3.3 End of proof of Lemma 1.4 (a)

PROOF: For $i = 1, 2$, let $G_i \in M$, analytic in $\mathbb{C} \setminus E_i$ and let $\mathbb{C} \setminus E$ be the set of analyticity of $G_1 \circ G_2$. Then $E \subset \tilde{E} := E_2 \cup G_2^{-1}(E_1)$ is closed since the set of analyticity of any analytic function is open. It remains to show E is countable. Since G_2 is not identically constant, for $x \notin E_2$ there is a least $k = k(x)$ such that $G_2^{(k)}(x) \neq 0$ and then G_2 has multiplicity exactly k in a small disk D_x around x . Then $G_2^{-1}(E_1) \cap D_x$ is countable. Since for every x there is an open set D_x such that $\tilde{E} \cap D_x$ is countable it follows that \tilde{E} , thus E , is also countable. In the same way, for any $a \notin E_i$ we have that $G_i^{-1}(a)$ is countable. For division, note that $1/G$ is defined

wherever G is defined and nonzero. Since G is not a constant the same argument as above shows that $G^{-1}(0)$ is countable. ■

3.4 Proof of Theorem 1.5

Notations

In the following we will write $D_r(z_0)$ for the disk $\{z \in \mathbb{C} : |z - z_0| < r\}$, D_r will denote $D_r(0)$, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

A number of notations, definitions and results in iterations of rational maps used in the proof are reviewed in §6.

PROPOSITION 3.9 *Let R be a rational function of degree $d \geq 2$. Then R has infinitely many distinct periodic points.*

PROOF: By definition, points of different period are distinct and by Lemma 6.6 there are periodic points for every $n \geq 4$. ■

The “if” part of Theorem 1.5

In this direction the proof is trivial. Indeed, if G is linear fractional, then the general nonidentically zero solution of the equation (1.11) can be obtained by substituting $x = 1/y$ in (1.11) which then becomes linear. We get

$$x_n = \left(Ca^{-n} + \frac{b}{a-1} \right)^{-1}$$

with the continuation $x(z) = \left(Ce^{-z \ln a} + (a-1)^{-1}b \right)^{-1}$, a meromorphic function.

The “only if” part of Theorem 1.5

For the proof we will show that if f has only isolated singularities, and $f(az) = G(f(z))$, then f itself is linear-fractional. Then G is also linear-fractional since $G(w) = f(af^{-1}(w))$.

LEMMA 3.10 *If f has only isolated singularities and f is not linear-fractional then for any large enough w , the equation $f(z) = w$ has at least two distinct roots.*

PROOF: If f is rational, then the property is immediate. Then assume that that f is not rational, thus f has at least one essential singularity, possibly at infinity [18]. If f has an essential singularity in \mathbb{C} ,

then it is isolated by hypothesis and then the property follows from Theorem 6.7. Then assume that f has no essential singularity in \mathbb{C} , thus infinity is the only essential singularity of f . If it is isolated then Theorem 6.7 applies again. Otherwise f has infinitely many poles accumulating at infinity. Since f maps a neighborhood of every pole into a full neighborhood of infinity, any sufficiently large value of f has multiplicity larger than one. ■

End of proof of Theorem 1.5. Let now G^{om} be defined on $\mathbb{C} \setminus E_m$ and let $E = \cup_{m=1}^{\infty} E_m$; then E is countable and G^{om} is defined on $\mathbb{C} \setminus E$ for any m .

Assume f is not linear-fractional and has only isolated singularities. We let z_1 and z_2 be in $\mathbb{C} \setminus E$ and such that $f(z_1) = f(z_2)$, cf. Lemma 3.10. Then $f(az_1) = G(f(z_1)) = G(f(z_2)) = f(az_2)$ and in general $f(a^n z_1) = f(a^n z_2)$. But since $a^n z_1 \rightarrow 0$ this contradicts (3.21).

3.5 Proof of Theorem 1.8

PROOF: The fact that \mathcal{K}_p is bounded for a nonlinear polynomial map follows from the fact that after the substitution $x = 1/y$, the map $y_{n+1} = 1/G(1/y_n)$ is attracting at $y = 0$. Thus, cf. [4] Theorem 5.2.3 p. 83, \mathcal{K}_p is simply connected. Let $a_1 \in (|a|, 1)$ and let D_ϵ be a disk such that $|G(z)| < a_1|z|$ for $z \in D_\epsilon$ and Q is analytic in D_ϵ .

By definition, for every $z_0 \in \mathcal{K}_p$ there exists $m(z_0)$ such that $G^{[m(z_0)]}(z_0) \in D_\epsilon$. Since $G^{[m(z_0)]}(z)$ is continuous in z , there is a disk $D_{\epsilon(z_0)}(z_0)$ such that $G^{[m(z_0)]}(D_{\epsilon(z_0)}(z_0)) \subset D_\epsilon$. It follows in particular that \mathcal{K}_p is open.

Since \mathcal{K}_p is open and connected, it is arcwise connected. Let z_0 be arbitrary in \mathcal{K}_p and let C be an arc connecting z_0 to $z = 0$. Since C is compact and

$$C \subset \bigcup_{z \in C} D_{\epsilon(z)}(z)$$

there is a finite subcovering

$$C \subset \mathcal{O}_C = \bigcup_{i=1}^N D_{\epsilon(z_i)}(z_i)$$

with $z_i \in C$. Let M be the largest of the $m(z_i), i = 1, \dots, N$. Then, by construction,

$$(3.28) \quad G^{[M]}(\mathcal{O}_C) \in D_\epsilon$$

We see from (3.27) that $aQ(z) = Q(G(z)) = a^{-1}Q(G(G(z)))$ and in general, for $n \in \mathbb{N}$,

$$(3.29) \quad Q(z) = a^{-n}Q(G^{[n]}(z))$$

We define $Q(z)$ in \mathcal{O}_C by $Q(z) = a^{-M}Q(G^{[M]}(z))$. By (3.28), and because (3.29) holds in \mathcal{D}_ϵ , this unambiguously defines an analytic continuation of Q from D_ϵ to $D_\epsilon \cup \mathcal{O}_C$. Since \mathcal{K}_p is open and simply connected and since Q is analytic near zero and can be continued analytically along any arc in \mathcal{K}_p , standard complex analytic results show that Q is (single valued and) analytic in \mathcal{K}_p .

For the last part, note that the boundary of \mathcal{K}_p lies in the Julia set J , which is the closure of repelling periodic points (see Appendix, Lemma 6.3). Assume that x_0 is a repelling periodic point of G of period n , and that x_0 is a point of analyticity of Q . Relation (3.29) implies that $Q(x_0) = 0$ and that $Q'(x_0) = a^{-n}(G^{[n]}'(x_0))Q'(x_0)$ but since $|a| < 1$ and $|(G^{[n]}'(x_0))| > 1$ this implies $Q'(x_0) = 0$. Inductively, in the same way we see that $Q^{(m)}(x_0) = 0$ for all m , which under the assumption of analyticity entails $Q \equiv 0$ which contradicts (3.21). ■

3.6 Borel summability of formal invariant for logistic map when $a = 1$

We now consider an example which cannot be reduced to the previous types, namely when $a = 1$, and when therefore the Poincaré equivalence theorem fails. In the recurrence

$$(3.30) \quad x_{n+1} = x_n(1 - x_n)$$

zero is a fixed point, and it can be shown in a rather straightforward way that there are no attracting fixed points of this map, or of any of its iterates. However, failure of the Painlevé property can be checked straightforwardly, and Borel summability makes it possible to analyze the properties of this equation rigorously.

A formal analysis of the Painlevé property is relatively straightforward using methods similar to those in [12]. We concentrate here on properties of the conserved quantities. The recurrence $a_{n+1} = a_n(1+a_n)^{-1}$ is exactly solvable and differs from the logistic map by $O(a_n^3)$ for small a_n . The exact solution is $n - a_n^{-1} = \text{Const}$, which suggests looking in the logistic map case for a constant of the iteration in the form of an expansion starting

with $C = n - a_n^{-1}$. This yields

$$(3.31) \quad C(n; v) \sim n - v^{-1} - \ln v - \frac{1}{2}v - \frac{1}{3}v^2 - \frac{13}{36}v^3 - \frac{113}{240}v^4 + \dots$$

which is indeed a formal invariant, but the associated series is factorially divergent as will appear clear shortly. Nevertheless we can show that the expansion is Borel summable to an actual conserved quantity in a sectorial neighborhood of $v = 0$.

THEOREM 3.11 *There is a conserved quantity C defined near the origin in $\mathbb{C} \setminus \mathbb{R}^-$, of the form*

$$C(n; v) = n - v^{-1} - \ln(v) - R(v)$$

where $R(v)$ has a Borel summable series at the origin in any direction in the open right half plane. $R(v)$ has a singularity barrier touching the origin tangentially along \mathbb{R}^- . This singularity barrier is exactly the boundary of the Leau domain of (3.30).

We let

$$(3.32) \quad C(n; v) := n - v^{-1} - \ln v - R(v)$$

and impose the condition that C is constant along trajectories. This yields

$$(3.33) \quad R(v) = R(v - v^2) + \frac{v}{1 - v} + \ln(1 - v)$$

where the RHS of (3.33) is $R(v - v^2) + O(v^2)$. The substitution

$$(3.34) \quad R(v) = h(v^{-1} - 2)$$

followed by $v = 1/(x + 1)$ yields

$$(3.35) \quad h(x - 1) = h\left(x + x^{-1}\right) + \frac{1}{x} + \ln\left(\frac{x}{x + 1}\right)$$

which by formal expansion in powers of x^{-1} becomes

$$(3.36) \quad h(x - 1) = \sum_{k=0}^{\infty} \frac{h^{(k)}(x)}{k!} x^{-k} + \frac{1}{x} + \ln\left(\frac{x}{x + 1}\right)$$

Proof of Theorem 3.11

PROPOSITION 3.12

i) $R(v) = h(v^{-1} - 2)$ has a Borel summable series at the origin along \mathbb{R}^+ . More precisely, $h(x)$ can be written in the form

$$(3.37) \quad h(x) = \int_0^\infty e^{-px} H(p) dp$$

and where $H(p)$ is analytic at zero and in the open right half plane $\mathbb{H} = \Re(p) > 0$ and has at most exponential growth along any ray towards infinity in \mathbb{H} .

ii) h is analytic in a region of the form $\{x : \arg(x) \neq \pi : |x| \geq \nu(\arg(x))\}$. The function ν is continuous in $(-\pi, \pi)$. (The expression of $\nu : (-\pi, \pi) \mapsto \mathbb{R}^+$ will follow from the proofs below.)

iii) By (3.37) and Watson's Lemma [5], h has an asymptotic power series for large x , $h(x) \sim \sum_{k=0}^\infty H^{(k)}(0)x^{-k}$, which is a formal solution of (3.36).

iv) The function $R(v)$ is analytic in a region near the origin, the origin excluded, of the form $\mathcal{V} = \{v : \arg(v) \neq \pi, 0 < |v| < \nu^{-1}(\arg(\varphi))\}$. By (iii) the relation (3.31) is an asymptotic expansion for small $v \in \mathcal{V}$, and from (3.37) the power series contained there is Borel summable.

v) The function R given by (3.34) satisfies (3.33).

vi) The function R is analytic in L_f , the Leau domain of f , and has a singularity barrier on the Julia set of f .

PROOF: The formal inverse Laplace transform of (3.36) is the equation

$$(3.38) \quad (e^p - 1)H = \frac{1 - e^{-p} - p}{p} + \sum_{k=1}^\infty \frac{(-p)^k}{k!} H * 1^{*k}$$

where $*$ denotes the Laplace-type convolution

$$F * G = \int_0^p F(s)G(p-s)ds$$

and F^{*k} is the convolution of F with itself k times. We rewrite (3.38) in the form

$$(3.39) \quad H = \frac{1 - e^{-p} - p}{p(e^p - 1)} + \frac{1}{(e^p - 1)} \sum_{k=1}^\infty \frac{(-p)^k}{k!} H * 1^{*k} = H_0 + \mathfrak{A}H$$

where \mathfrak{A} is a linear operator. We show now that this equation is contractive in an appropriate space of functions. Let $\nu > 0$ and let \mathcal{A} be the space of functions F analytic in a neighborhood \mathcal{N} of $[0, \infty)$ in the complex plane, with $F(0) = 0$, in the norm $\|F\|_\nu := \sup_{\mathcal{N}} |e^{-\nu|p|} F(p)|$. We choose $a \in (0, 2\pi)$, ϵ small and

$$(3.40) \quad \mathcal{N} = \left\{ p : |p| \leq \epsilon \right\} \cup \left\{ p : \arg(p) \in \left(-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon \right) \right\}$$

Since the norm $\|\cdot\|_\nu$ restricted to compact sets is equivalent to the usual sup norm, it is easy to check that \mathcal{A} is a Banach space.

PROPOSITION 3.13 *For large enough ν , the equation (3.39) is contractive in \mathcal{A} in the norm $\|\cdot\|_\nu$.*

First, it is easy to see that $H_0 \in \mathcal{A}$. If $f \in \mathcal{A}$ then

$$(3.41) \quad \begin{aligned} \sum_{k=1}^{\infty} \frac{(-p)^k}{k!} f * 1^{*k} &= \sum_{k=1}^{\infty} \frac{(-p)^k}{k!} \int_0^p f(s) \frac{(p-s)^{k-1}}{(k-1)!} ds \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k p^{2k}}{k!(k-1)!} \int_0^1 f(pt)(1-t)^{k-1} dt = \sum_{k=1}^{\infty} \frac{(-1)^k p^{2k}}{k!(k-1)!} \int_0^1 f(p(1-t)) t^{k-1} dt \end{aligned}$$

It is immediate that if p is in a compact set \mathcal{K} and f is analytic in \mathcal{K} then the sum in (3.41) is uniformly convergent in \mathcal{K} and analytic in p . Furthermore the sum is $O(p^3)$ for small p since $f \in \mathcal{A}$. Now we see that

$$(3.42) \quad \begin{aligned} &\left| e^{-\nu|p|} \sum_{k=1}^{\infty} \frac{(-1)^k p^{2k}}{k!(k-1)!} \int_0^1 f(p(1-t)) t^{k-1} dt \right| \\ &= \left| \sum_{k=1}^{\infty} \frac{(-1)^k p^{2k}}{k!(k-1)!} \int_0^1 e^{-\nu|p|(1-t)} f(p(1-t)) t^{k-1} e^{-\nu|p|t} dt \right| \\ &\leq \|f\|_\nu \sum_{k=1}^{\infty} \frac{|p|^{2k}}{k!(k-1)!} \int_0^1 t^{k-1} e^{-\nu|p|t} dt \leq \|f\|_\nu \sum_{k=1}^{\infty} \frac{|p|^{2k}}{k!(k-1)!} \int_0^\infty t^{k-1} e^{-\nu|p|t} dt \\ &= \|f\|_\nu \sum_{k=1}^{\infty} \frac{|p|^k}{k! \nu^k} \leq \|f\|_\nu \frac{|p|}{\nu} e^{|p|/\nu} \end{aligned}$$

and thus

$$(3.43) \quad \|\mathfrak{A}\| \leq \text{Const } \nu^{-1}$$

for sufficiently large ν , where we took into account the exponential decrease of $(e^p - 1)^{-1}$ for large p in \mathcal{N} . Thus the equation has a unique

fixed point $H \in \mathcal{A}$. In particular the Laplace transform $h(x) = \mathcal{L}H = \int_0^\infty e^{-xp} H(p) dp$ is well defined and analytic in the half-plane $\Re(x) > \nu$. It is now immediate to check that $h(x)$ satisfies the equation (3.36).

4 Julia sets for the map (1.13) for $a \in (0, 1)$

It is convenient to analyze the superattracting fixed point at infinity; the substitution $x = 1/y$ transforms (1.13) into

$$(4.1) \quad y_{n+1} = -\frac{y_n^2}{a(1-y_n)}$$

For small y_0 , the leading order form of equation (4.1) is $y_{n+1} = -a^{-1}y_n^2$ whose solution is $-y_0^{2^n} a^{-2^{n-1}-1}$. It is then convenient to seek solutions of (4.1) in the form $y_n = -G(y_0^{2^n} a^{-2^n})$ whence the initial condition implies $G(0) = 0$, $G'(0) = a$. Denoting $y_0^{2^n} a^{-2^n} = z$, the functional relation satisfied by G is

$$(4.2) \quad G(z^2) = \frac{G(z)^2}{a(G(z) + 1)}; \quad G(0) = 0, \quad G'(0) = a$$

LEMMA 4.1 ([11]) *There exists a unique analytic function G in the neighborhood of the origin satisfying (4.2). This G has only isolated singularities in \mathbb{C} if and only if $a \in \{-2, 2, 4\}$. In the latter case, (1.13) can be solved explicitly.*

If $a \notin \{-2, 2, 4\}$ then the unit disk is a barrier of singularities of G .

LEMMA 4.2 *G is analytic in the open unit disk S_1 and Lipschitz continuous in $\overline{S_1}$.*

PROOF: Lemma 4.1 proved in [11] guarantees the existence of some disk S_r centered at zero, of radius $r \leq 1$, where G is analytic and it is shown that inside that disk we have (cf. also 4.2)

$$(4.3) \quad G(z) = U(G(z^2)); \quad 2U(s) := s + (a^2 s^2 + 4s)^{\frac{1}{2}}$$

(with the choice of branch consistent with $G(0) = 0, G'(0) = a$). If $r < 1$ then (4.3) provides analytic continuation in a disk of radius $r^{\frac{1}{2}} > r$ if $a^2 G(z)^2 + 4aG(z) \neq 0$ in S_r .

NOTE 4.3 *$G(z_0) = 0$ in S_1 iff $z_0 = 0$.*

Indeed, assume $0 \neq z_0 \in S_r$ and $G(z_0) = 0$. Then we find from (4.2) that $G(z_0^{2^n}) = 0$ which is impossible since G is analytic at zero and $G'(0) = a$.
 \square

We are left to examine the possibility $G(z_0) = -4a^{-1}$ with $z_0 \in S_r$.

NOTE 4.4 $R(x) = \frac{x^2}{a(x+1)}$ is well defined and increasing on the interval $(-\infty, -4a^{-1})$.

The assumption $G(z_0) = -4a^{-1}$ thus implies that the values $G(z_0^{2^n})$ are in \mathbb{R}^- and decrease in n , again impossible G is analytic at 0 and $G(0) = 0$.

We now show G is bounded in S_1 . Indeed, by (4.3) we have

$$(4.4) \quad |G(z)| \leq U(|G(z^2)|)$$

on the other hand, a calculation shows that

$$(4.5) \quad U(s) \leq \frac{a}{1-a} \text{ for } s \in \left[0, \frac{a}{1-a}\right]$$

Since $G(0) = 0$ and G is analytic in S_1 , (4.4) and (4.5) imply that

$$(4.6) \quad \sup_{z \in S_1} |G(z)| \leq \frac{a}{1-a}$$

We next prove that G is injective. As a first step we have the following:

NOTE 4.5 $G' \neq 0$ in S_1 .

Indeed, otherwise differentiating (4.2) shows there would exist a sequence $z_n \rightarrow 0$ such that $G'(z_n) = 0$. \square

Now, G is injective in a neighborhood of the origin since $G'(0) = a$. Let then $z_1 \in S_1$ be a point of smallest modulus such that there exists $z_2 \neq z_1 \in S_1$ with $G(z_1) = G(z_2)$. For z_1 to exist, we need, again by (4.2) that $z_1^2 = z_2^2$ and thus $z_1 = -z_2$. Since $G' \neq 0$, by the open mapping theorem, the image under G of arbitrarily small disks around z_1 and $-z_1$ overlap nontrivially. For some C and any ϵ there exist therefore infinitely many z_i with $|z_1 - z_i| < \epsilon$ such that $G(z_i) = G(z'_i)$ and $|z'_i - (-z_1)| < C\epsilon$. The same argument using (4.2) shows that $z'_i = -z_i$. But since $G(z) = G(-z)$ for infinitely many $z \in S_1$ accumulating at z_1 , then G would be even, which is not the case since $G'(0) = a$. We now need two lower bounds.

PROPOSITION 4.6 For $a \in \left(0, \frac{1}{2}\right)$

$$(1 - |z|)^{1 - \log_2(2-a)} G'(z)$$

is bounded in S_1 .

PROOF: The function $H = 1/G$ which, by Note 4.3, is analytic in $S_1 \setminus 0$ satisfies

$$(4.7) \quad H(z^2) = aH(z)(1 + H(z))$$

Let

$$m_n = \max\{|H(z)| : |z| \in [2^{-\frac{1}{2^n}}, 2^{-\frac{1}{2^{n+1}}}]\}$$

Eq. (4.7) gives

$$m_{n+1} \leq \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m_n}{a}}$$

and it easy to see that this implies

$$(4.8) \quad \limsup_{n \rightarrow \infty} m_n \leq 1 + a^{-1}$$

We have

$$(4.9) \quad G'(z) = 2az \frac{G'(z^2)(1 + G(z))^2}{G(z)(2 + G(z))}$$

so that

$$|G'(z)| \leq |G'(z^2)| \max_{(1-a)|y| \leq a} \left| \frac{2a(1+y)^2}{y(2+y)} \right| = \frac{2}{2-a} |G'(z^2)|$$

if $a \leq 1/2$ from which Proposition 4.6 follows immediately. ■

NOTE 4.7 *A straightforward way to extend the result for larger values of $a < 1$ is to replace (4.9) by a corresponding equality obtained from a higher order iterate of (4.7).*

LEMMA 4.8 *G is gives a conformal transformation of S_1 onto a bounded region \mathcal{K}_p , whose boundary $\partial\mathcal{K}_p$ is a Lipschitz continuous nowhere differentiable curve.*

5 Behavior at the singularity barrier

PROPOSITION 5.1 *There is $\delta > 0$, a real analytic function Ψ , periodic of period $\ln 2$ and an analytic function Φ , $\Phi'(0) = 1$ such that for $|\arg(1 - z)| < \delta$ (1.15) holds.*

PROOF: Let $\omega = 2\pi/\ln 2$, $\beta = \log_2(2 - a)$. With $z_0 \in (0, 1)$ and $z_n = z_0^{1/2^n}$, the sequence $G_n = G(z_n)$ is increasing and bounded by L , see (4.3). It follows immediately from (4.3) and (4.4) that

$$(5.1) \quad L - G_n := \delta_n \downarrow 0 \quad \text{as } n \rightarrow \infty \quad (L := \frac{a}{1-a})$$

From (4.8) we have, with $C_1 = \frac{(1-a)^3}{a(2-a)} + C_2$, $C_2 = 1 - a$

$$(5.2) \quad \delta_{n+1} = \frac{1}{2-a} \left[\frac{1 - C_2\delta_{n+1}}{1 - C_1\delta_{n+1}} \right] \delta_n$$

Eqs. (5.1) and (5.2) imply that for any $\epsilon > 0$ we have

$$(5.3) \quad \delta_n = o((2 - a - \epsilon)^{-n}) \quad \text{as } n \rightarrow \infty$$

Let

$$(5.4) \quad \delta_n = \ln^\beta(1/z_n)e^{\theta_n} = 2^{-n\beta} \ln^\beta(1/z_0)e^{\theta_n}$$

cf. (5.1). Now

$$\left| e^{\theta_n - \theta_{n+1}} - 1 \right| = \frac{(C_1 - C_2)\delta_{n+1}}{1 - C_2\delta_{n+1}} = O(\delta_n) \quad \text{as } n \rightarrow \infty$$

and by (5.3), θ_n is convergent, $\theta_n \rightarrow \Theta$. Since $\theta_{n+1} - \theta_n \rightarrow 0$ it follows that

$$(5.5) \quad \Theta(z_0^2) = \Theta(z_0)$$

Analyticity

We let $1 - z_1$ be sufficiently small so that

$$(5.6) \quad \delta_n \leq c\alpha^n$$

with $\alpha < 1$ and c small enough so that the term in square brackets is sufficiently close to one for all $n \geq 0$ and $|z_0 - z_1| \leq \epsilon_1$ (cf. (5.3)), this amounts to a shift in n). If ϵ_1 is small enough, then it is easy to check that equation (5.2) is a contractive mapping in the ball of radius c $S_{\epsilon_1} = \{\zeta : |\zeta| \leq \epsilon_1\}$ in Banach space $l_{\infty, \alpha}(\mathbb{N})$ of vectors $\mathbf{v}(n; \zeta)$ analytic in $\zeta = z_0 - z_1$ with respect to the norm

$$\|\mathbf{v}\| = \sup_{n \geq 1; |\zeta| \leq \epsilon_1} |\mathbf{v}(n, \zeta)\alpha^{-n}|$$

and local analyticity in a neighborhood of the interval $[z_0, \sqrt{z_0}]$. By periodicity, real analyticity follows immediately and relation (5.5) is preserved. ■

End of the proof of Theorem 1.10 (iii)

We use the information obtained in §5. Let $e^{\theta_n} = (1 + w_n)e^{\Theta}$; given $\delta > 0$ we choose n_0 large enough and ϵ_2 so that $|w_n(z_0)| < \delta$ if $|z - z_0| < \epsilon_2$ and $n \geq n_0$. We let $h = e^{2\Theta}$, $\epsilon_n = 2^{n\beta}$, $s = \ln^\beta(1/z_0)$, $c = C_1 - C_2$, $C = c - C_2$ and obtain

$$(5.7) \quad w_n = \frac{Ce^{2\Theta}s\epsilon_n}{1 - \epsilon_n C_2 e^{2\Theta}s} \\ + w_{n+1} \frac{1 + 2Ce^{2\Theta}s\epsilon_n - CC_2e^{4\Theta}s^2\epsilon_n^2 + w_{n+1}Cse^{2\Theta}\epsilon_n(1 - \epsilon_n C_2 se^{2\Theta})}{1 - 2\epsilon_n se^{2\Theta} + 2\epsilon_n^2 s^2 e^{4\Theta} - w_{n+1}C_2 se^{2\Theta}(1 - \epsilon_n C_2 se^{2\Theta})}$$

As in §5, a contractive mapping argument shows that $\mathbf{w} = (w_n, w_{n+1}, \dots)$ is analytic in $se^{2\Theta}$, if s is small enough. The conclusion now follows from the definition

$$G(z_0^{2^{-n_0}}) = L + s\delta_{n_0}$$

and (5.4), (5.5), §5 and the substitution $e^{2\Theta(\cdot)} = \Psi(\ln(\ln(\cdot)))$. Formula (1.17) follows immediately from (1.15). \blacksquare

NOTE 5.2 With $z_n = z_0^{1/2^n}$, $\tau_n = \tau(z_n)$ (cf. (1.15) and $g_n = G(z_n) - L$ we have

$$(5.8) \quad \Psi(\ln \ln z_0) = \lim_{N \rightarrow \infty} \frac{\frac{g_{N+1}}{\tau_{N+1}} - \frac{g_N}{\tau_N}}{\tau_{N+1} - \tau_N}$$

Appendix: Iterations of rational maps

We introduce a number of definitions and results for iterations of rational maps, which are treated in much more detail and generality in [32] and [4]. We shall illustrate the main concepts on the simple case $G = ax(1 - x)$. In Figure 1, the interior (in the complex plane) of the fractal curves is a set invariant under G and with the further property that starting with z_0 inside the m -th iterate of G at z_0 , $G^{\circ m}(z_0)$, converges to zero as $m \rightarrow \infty$. These are *stable fixed domains* of G .

Consider the polynomial map G . A Fatou domain of G is a stable fixed domain V of G characterized by the property that $G^{\circ n}$ converges *in the chordal metric* on the Riemann sphere \mathbb{C}_∞ to a fixed point of G , locally uniformly in V .

DEFINITION 6.1 ([4], p. 50) *Let G be a non-constant rational function. The **Fatou set** of G is the maximal open subset of \mathbb{C}_∞ on which $\{G^{\circ n}\}$ is equicontinuous and the **Julia set** of G is its complement in \mathbb{C}_∞ .*

A Fatou domain is a **Leau** domain (or a parabolic basin) if $x_0 \in \partial V$ and the multiplier of x_0 (the derivative at x_0) is $\lambda = 1^2$. In Figure 1 this happens for $a = 1$.

The Julia set can be characterized by the following property.

LEMMA 6.2 ([4], p. 148) *Let G be a rational map of degree d , (cf. Definition 6.4) where $d \geq 2$. Then J is the derived set³ of the periodic points of G .*

Under the assumptions above, we have

LEMMA 6.3 ([4], p. 148) *J is the closure of the repelling points of G .*

DEFINITION 6.4 ([4], p. 30.) *If $R = P/Q$ where P and Q are polynomials, then the degree of the rational function R is $\max\{\deg(P), \deg(Q)\}$.*

DEFINITION 6.5 ([4]) *If R is a rational function and $R^{\circ m} = R \circ R \circ \dots \circ R$ n times, then a periodic point of period n of R is a point z such that $R^{\circ m} z = z$ and $R^{\circ m} z \neq z$ if $m < n$. A periodic point of R is a point of some period $n \geq 1$.*

We also use the following result of I. N. Baker:

LEMMA 6.6 ([3], [4]) *Let R be a rational function of degree $d \geq 2$, and suppose that R has no periodic points of period n . Then (d, n) is one of the pairs*

$$(2, 2), (2, 3), (3, 2), (4, 2)$$

(moreover, each such pair does arise from some R in this way).

Further results used in the proofs

THEOREM 6.7 (Big theorem of Picard, local formulation [31], [18]) *If f has an isolated singularity at a point z_0 and if there exists some neighborhood of z_0 where f omits two values, then z_0 is a removable singularity or a pole of f .*

THEOREM 6.8 (Picard-Borel, [28]) *If φ is any nonconstant function meromorphic in \mathbb{C} , then φ avoids at most two values (infinity included).*

² [32], p. 54

³ By definition the derived set of a set E consists exactly in the points z which are limits of sequences $\{z_n\}$ where the $z_n \in E$ are distinct.

All we need in the present paper is that at most two finite values are excluded. This is immediately reduced to the more familiar Picard theorem by noting that if λ is an excluded value of f then $1/(f - \lambda)$ is entire.

6.1 Proof of Proposition 1.7

By Theorem 1.5, (1.11) does not have the Painlevé property at some stable fixed point iff G is not linear-fractional, in which case (1.11) fails to have the Painlevé property at any other stable fixed point. More generally, Proposition 1.7 follows from the following result.

LEMMA 6.9 *If $G^{\circ m}$ is of the form (1.12) then G is of the form (1.12).*

PROOF: Since (1.12) is one to one, the conclusion follows from the remark that if G is not linear-fractional, then $G(z)$ has multiplicity greater than one for all sufficiently large z (and then the same holds for $G^{\circ m}(z)$). Indeed, assume that G is not linear-fractional. If G is rational, then the conclusion is obvious. If the set of singularities of G is finite, then they are all isolated and at least one is an essential singularity (otherwise G is rational [18]) and Theorem 6.7 applies.

So we may assume the set of singularities is infinite. Since by assumption this set is closed and countable, it contains infinitely many isolated points. (Indeed, a set which is closed and dense in itself, i.e. a *perfect set*, is either empty or else uncountable.) Then if G has an isolated essential singularity, Theorem 6.7 applies, and if not then there are infinitely many poles of G . In the latter situation any sufficiently large value of G has multiplicity larger than one since G maps a neighborhood of every pole into a full neighborhood of infinity. ■

Completion of proof of Proposition 3.12. Part (ii) merely follows from the formula (3.37) and elementary contour deformation in the integral. Parts (iii) and (iv) are straightforward.

After the transformation $v = -u + 1/2$ the iteration associated to our map f is equivalent to that of the quadratic map $q(u) = u^2 + 1/2$.

Part (vi) follows from the following Lemma.

LEMMA 6.10 ([32], p. 174) *The Leau domain of q , is the filled in (interior of the) Julia set \mathcal{K}_p of q .*

Proof of Proposition 3.12(vi). Let $H(v) = R(v) + v^{-1} + \ln v$, defined and analytic in \mathcal{V} . By definition we have $H(v_{n+1}) = H(v_n) + 1$ i.e.

$$(6.1) \quad H(v) = H(f(v)) - 1 = H(v - v^2) - 1$$

and clearly R and H have the same type of singularities in $\mathbb{C} \setminus \mathbb{R}^- \setminus \{0\}$.

If $z_0 \in L_f$ we have by definition $|z_n| = |f^{\circ m}(z_0)| \rightarrow 0$. Then, we choose ϵ small enough and N so that $|z_n| < \epsilon$ for $n > N$. Since we must have for some $n > N$ that $|z_{n+1}| < |z_n|$, then $|1 - z_n| < 1$ and thus $\arg(z_n) \in (-\pi/2, \pi/2)$. A direct calculation shows that then $|\arg(z_{n+1})| < |\arg(z_n)|$ and thus, if $m > n$, then $\arg(z_m) \in (-\pi/2, \pi/2)$. Thus by Proposition 3.12, (ii and iv), eventually $z_n \in \mathcal{V}$. We know that \mathcal{V} is a domain of analyticity of R . By (6.1), if H is analytic at $z_{n+1} = z_n - z_n^2$ then H is analytic at z_n and by induction H is analytic at z_0 . Since L_f is simply connected, we have that H , and thus R , is analytic in L_f , as in the proof of Theorem 1.8.

On the other hand, if we assume that $v \in \partial L$ is a periodic point of f , say of period N , and that R , thus H , is analytic there, relation (6.1) implies that H is analytic at any point on the orbit of v and furthermore $H(v) = H(v) - N$, a contradiction. Since the closure of the periodic points is ∂L , ∂L is a singularity barrier of H . Furthermore ∂L is in the exterior of \mathcal{V} and since $\partial \mathcal{V}$ touches the origin tangentially along \mathbb{R}^- , so does ∂L since $0 \in \partial L$. \square

■

Acknowledgment. The authors are very grateful to R D Costin for many useful discussions and comments. The authors would also like to thank R Conte, F Fauvet, N Joshi and D Sauzin for interesting discussions.

Bibliography

- [1] M J Ablowitz, R Halburd, and B Herbst *On the extension of the Painlevé property to difference equations* Nonlinearity **13** pp. 889–905 (2000).
- [2] D V Anosov and V I Arnold eds. *Dynamical Systems I* Springer-Verlag (1988).
- [3] I N Baker *Fixpoints of polynomials and rational functions* J. London Math. Soc. **39** pp. 615–622 (1964).
- [4] A F Beardon *Iteration of Rational Functions* Springer Verlag, New York (1991).
- [5] C M Bender and S A Orszag *Advanced mathematical methods for scientists and engineers I; Asymptotic methods and perturbation theory*, Mc Graw Hill (1978) and Springer (1999).
- [6] E Borel *Leçons sur les fonctions monogènes*, Gauthier-Villars, Paris (1917).
- [7] B L J Braaksma *Transseries for a class of nonlinear difference equations* J. Differ. Equations Appl. 7, no. 5, 717–750 (2001).
- [8] B. L. J. Braaksma, R Kuik (submitted).

- [9] R Conte and M Musette *Rules of discretization for Painlevé equations* Theory of Nonlinear Special Functions (Montreal 13–17 May 1996) ed. L Vinet and P Winternitz (Berlin:Springer).
- [10] O Costin *On Borel summation and Stokes phenomena for rank one nonlinear systems of ODE's* *Duke Math. J.* Vol. 93, No.2 pp. 289–344 (1998).
- [11] O. Costin and M. Kruskal *Movable singularities of solutions of difference equations in relation to solvability, and study of a superstable fixed point* *Theoretical and Mathematical Physics*, 133(2): 1453–1460 (2002).
- [12] O Costin and R D Costin *On the formation of singularities of solutions of nonlinear differential systems in antistokes directions* To appear in *Inventiones Mathematicae*.
- [13] P A Deift and X Zhou *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation* *Ann. of Math.* (2) 137 no 2 pp. 295–368 (1993).
- [14] J Écalle *Les fonctions resurgentes, vol. I, II and III*, Publ. Math. Orsay, 1985.
- [15] J Écalle, in *Bifurcations and periodic orbits of vector fields*, *NATO ASI Series, Vol. 408*, 1993
- [16] H Flashka and A C Newell *Monodromy and spectrum preserving transformations* *Commun. Math. Phys.* **76** pp. 65–116 (1980).
- [17] L Fuchs *Sur quelques équations différentielles linéaires du second ordre* *C. R. Acad. Sci., Paris* **141** pp. 555–558 (1905).
- [18] M O González *Complex Analysis: Selected Topics* Marcel Dekker Inc., NY, Basel, Hong Kong (1991).
- [19] R Kuik, *Transseries in differential and difference equations*, PhD Thesis, University of Groningen, ISBN 90-367-1771-x (2003).
- [20] S. Kowalevski *Sur le problème de la rotation d'un corps solide autour d'un point fixe*, *Acta Math.*, **12** H.2, pp. 177–232 (1889)
- [21] S. Kowalevski *Mémoire sur un cas particulier du problème d'un corps pesant autour d'un point fixe, où l'intégration s'effectue à l'aide de fonctions ultraelliptiques du temps* Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut National de France, Paris 31 pp. 1–62 (1890)
- [22] M D Kruskal and P A Clarkson *The Painlevé-Kowalevski and poly-Painlevé tests for integrability* *Stud. Appl. Math.* 86 no. 2, pp. 87–165 (1992).
- [23] B Gambier *Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes* *Acta Math.* **33** pp 1–55 (1910).
- [24] B Grammaticos and A Ramani *Discrete Painlevé equations: derivation and properties* In “Applications of Analytic and Geometric Methods to Nonlinear Differential Equations”, ed. P A Clarkson, NATO ASI Series C, pp 299–313 (1993).
- [25] A R Its, A S Fokas and A A Kapaev *On the asymptotic analysis of the Painlevé equations via the isomonodromy method* *Nonlinearity* 7 no. 5, pp. 1921–1325 (1994).
- [26] N Joshi *Irregular singular behaviour in the first discrete Painlevé equation*, in *Symmetries and Integrability of Difference Equations III*, D Levi and O Ragnisco (eds), pp. 237–243, CRM Proc. Lecture Notes, 25, Amer. Math. Soc., Providence, RI, (2000).

- [27] H Levy and F Lessman *Finite Difference Equations*, Dover Publications Inc. New York, 1992.
- [28] R Nevanlinna *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Chelsea Pub. Co., New York, (1974).
- [29] P Painlevé *Mémoire sur les équations différentielles dont l'intégrale générale est uniforme* Bull. Soc. Math. France **28** pp 201–261 (1900).
- [30] P Painlevé *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme* Acta Math. **25** pp. 1–85 (1902).
- [31] W Rudin *Real and Complex Analysis*, McGraw-Hill (1987).
- [32] N Steinmetz *Rational iteration. Complex Analytic Dynamical Systems* Walter de Gruyter, Berlin; New York (1993).

Received Month 199X.