On Borel summation and Stokes phenomena for rank one nonlinear systems of ODE's

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1 Introduction

In this paper we study analytic (linear or) nonlinear systems of ordinary differential equations, at an irregular singularity of rank one, under nonresonance conditions. It is shown that the formal asymptotic exponential series solutions (*transseries* solutions: countable linear combinations of formal power series multiplied by small exponentials) are Borel summable in a generalized sense along any direction in which the exponentials decay. Conversely, any solution that decreases along some direction is the Borel sum of a transseries.

The summation procedure introduced is an extension of Borel summation which is linear, multiplicative, commutes with differentiation and complex conjugation. The summation algorithm uses the formal solutions alone (and not the differential equation that they solve). Along singular (Stokes) directions, the functions reconstructed by summation are shown to be given by Laplace integrals along special paths, a subset of Écalle's median paths.

The one-to-one correspondence established between actual solutions and generalized Borel sums of transseries is constant between Stokes lines and changes if a Stokes line is crossed (local Stokes phenomenon). We analyze the connection between local and classical Stokes phenomena.

We study the analytic properties of the Borel (formal inverse Laplace) transform of the series contained in the transseries of the transseries and give a systematic description of their singularities. These Borel transforms satisfy a hierarchy of convolution equations, for which we give the general solution in a space of hyperfunctions. In addition, we show that they are *resurgent functions* in the sense of Écalle.

The summation procedure is not unique; we classify all proper extensions of Borel summation to transseries solutions of nonresonant systems.

We find formulas connecting the different series contained in the transseries among themselves (resurgence equations). Resurgence turns out to be closely linked to the local Stokes phenomenon.

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The connection to Berry's hyperasymptotics and applications to the classification of differential equations are briefly discussed.

1.1 General setting

We consider the differential system

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n \tag{1.1}$$

under the following *assumptions:*

(a1) The function $\mathbf{f}$ is analytic at $(\infty, 0)$.

(a2) Nonresonnance: the eigenvalues λ_i of the linearization

$$\hat{\Lambda} := -\left(\frac{\partial f_i}{\partial y_j}(\infty, 0)\right)_{i,j=1,2,\ldots n} \tag{1.2}$$

are linearly independent over $\mathbb{Z}$ (in particular nonzero) and such that the Stokes lines are distinct (a somewhat less restrictive condition is actually used, cf. §1.1.2).

Normalization. It is convenient to prepare (1.1) in the following way. Pulling out the inhomogeneous and the linear terms (relevant to leading order asymptotics) we get

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} - \frac{1}{x}\hat{B}\mathbf{y} + \mathbf{g}(x, \mathbf{y}) \tag{1.3}$$

Under the assumptions (a1) and (a2), by means of normal form calculations it is possible to arrange (1.3) so that ([11], [19])

(n1) $\hat{\Lambda} = \mathrm{diag}(\lambda_i)$ and

(n2) $\hat{B} = \mathrm{diag}(\beta_i)$

For convenience, we rescale x and reorder the components of $\mathbf{y}$ so that

(n3) $\lambda_1 = 1$, and, with $\phi_i = \arg(\lambda_i)$, we have $\phi_i < \phi_j$ if $i < j$. To simplify notations, we formulate some of our results relative to λ_1; they can be easily adapted to any other eigenvalue.

To unify the treatment we make, by taking $\mathbf{y} = \mathbf{y}_1 x^{-N}$ for some $N > 0$,

(n4) $\Re(\beta_j) < 0,\ j = 1, 2, \ldots, n$.

(there is an asymmetry at this point: the opposite inequality cannot be achieved, in general, as simply and without violating analyticity at infinity). Finally, through a transformation of the form $\mathbf{y} \leftrightarrow \mathbf{y} - \sum_{k=1}^{M} \mathbf{a}_k x^{-k}$ we arrange that

(n5) $\mathbf{f}_0 = O(x^{-M-1})$ and $\mathbf{g}(x, \mathbf{y}) = O(\mathbf{y}^2, x^{-M-1}\mathbf{y})$. We choose $M > 1 + \max_i \Re(-\beta_i)$.

Formal solutions. In prepared form, given (a1) and (a2), (1.3) admits an n–parameter family of formal exponential series solutions (transseries)

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\geq 0; |\mathbf{k}|>0} C_1^{k_1} \cdots C_n^{k_n} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}} \tag{1.4}$$

(see [11], [17],[21], and also § 2.4 below) where $m_i = 1 - \lfloor \beta_i \rfloor$, ($\lfloor\cdot\rfloor$ = integer part), $\mathbf{C} \in \mathbb{C}^n$ is an arbitrary vector of constants, and $\tilde{\mathbf{y}}_{\mathbf{k}} = x^{-\mathbf{k}(\boldsymbol{\beta}+\mathbf{m})} \sum_{l=0}^{\infty} \mathbf{a}_{\mathbf{k};l} x^{-l}$ are formal power series.

When x is large in some direction d in $\mathbb{C}$, an important role is played by the subset of transseries which are at the same time *asymptotic* expressions[1]: When there are infinitely many exponentials in (1.4) we ask that for all i with $C_i \neq 0$ we have $|\mathrm{e}^{-\lambda_i x}| \ll 1$ for large x in the given direction d in $\mathbb{C}$. Formally, agreeing to omit the terms with $C_i = 0$, with x in d, any *ascending* chain $\Re(-\mathbf{k}_1 \cdot \boldsymbol{\lambda} x) \leq \Re(-\mathbf{k}_2 \cdot \boldsymbol{\lambda} x) \leq \ldots$, $\mathbf{k}_i \neq \mathbf{k}_j$, in (1.4) must be *finite* (the terms of an asymptotic transseries are well-ordered with respect to $'' \ll ''$). Thus for x in some direction d we only consider those transseries that satisfy the condition:

(c1) $\xi + \phi_i := \arg(x) + \phi_i \in (-\pi/2, \pi/2)$ for all i such that $C_i \neq 0$. In other words, $C_i \neq 0$ implies that λ_i lies in a half-plane centered on $\overline{d}$, the complex conjugate direction to d.

From now on, $\boldsymbol{\lambda} = (\lambda_{i_1}, \ldots, \lambda_{i_{n_1}})$, $\boldsymbol{\beta} = (\beta_{i_1}, \ldots, \beta_{i_{n_1}})$, $\mathbf{m} = (m_{i_1}, \ldots, m_{i_{n_1}})$ and $\boldsymbol{\beta}' = \boldsymbol{\beta} + \mathbf{m}$ where the indices $i_1, \ldots, i_{n_1}$ satisfy (c1).

We will henceforth consider that (1.3) is presented in prepared form, and use the designation transseries only for those formal solutions satisfying (c1).

The series $\tilde{\mathbf{y}}_0$ is a formal solution of (1.3) while, for $\mathbf{k} \neq 0$, $\tilde{\mathbf{y}}_{\mathbf{k}}$ satisfy a hierarchy of linear differential equations [11] (see also § 2.4 for a brief exposition and notations). Generically all the series $\tilde{\mathbf{y}}_k$ are factorially divergent and there is no immediate way to uniquely associate actual functions to them. Neither can $\tilde{\mathbf{y}}$ be viewed as a *classical* asymptotic expansion since the $\tilde{\mathbf{y}}_{\mathbf{k}}$ are beyond all orders of each other (e.g., for $\mathbf{k} \neq 0$ and all $l \in \mathbb{N}$, $\mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{k}\cdot\mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}} = o(x^{-l})$).

One question is therefore to understand the relation between these (algorithmically obtained) formal solutions and the actual solutions of (1.3). In the present paper we show that a suitable generalization of Borel summation provides a one-to-one correspondence between transseries and actual solutions of (1.3):

$$\mathbf{y} \rightleftharpoons \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\geq 0; |\mathbf{k}|>0} C_1^{k_1} \cdots C_n^{k_n} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}} \quad (x \to \infty, \arg(x) = \xi) \tag{1.5}$$

Given $\mathbf{y}$, the value of C_i can change only when $\xi + \arg(\lambda_i - \mathbf{k} \cdot \boldsymbol{\lambda}) = 0$, $k_i \in \mathbb{N} \cup \{0\}$, i.e. when crossing one of the (finitely many by (c1)) Stokes lines. The correspondence (1.5) defines a summation method, in the sense that it is an extension of convergent summation which preserves its basic properties: linearity, multiplicativity, commutation with differentiation and with complex

[1]An asymptotic expansion of a function carries immediate information about behavior of the function near the expansion point (in contrast to antiasymptotic expansions, e.g. a convergent doubly infinite Laurent series)

conjugation. These properties are essential for obtaining true solutions out of transseries for nonlinear differential equations. Our procedure is similar to the medianization proposed by Écalle, but (due to the structure of (1.3)) requires substantially fewer analytic continuation paths. In addition we classify in the context of (1.3) all admissible summation methods (there is a one-parameter family of them, preserving the properties of usual summation). Summation recovers from transseries actual solutions of (1.3) without resorting to (1.3) in the process. In addition, the analysis reveals a rich analytic structure and formulas linking the various $\tilde{\mathbf{y}}_{\mathbf{k}}$ among themselves (resurgence relations). In [13] we studied this problem under further restrictions on the transseries (decay of the exponentials in a full half-plane) and on the differential equation. Removing those restrictions creates difficulties that required a new approach. New resurgence relations are found and in addition we provide a complete description, needed in applications, of the singularity structure of the Borel transforms of $\tilde{\mathbf{y}}_{\mathbf{k}}$.

1.1.1 Notes on Borel summation

The following is a very brief description; for more details on classical Borel summation see [9], [8] and for recent developments see [7] and especially [2].

If $\tilde{f} = \sum_{k=0}^{\infty} a_k x^{-k-r}$ is a formal series with $\Re(r) > 0$, its Borel transform is defined as the (still formal) series $\mathcal{B}\tilde{f} = \sum_{k=0}^{\infty} p^{k-1+r}/\Gamma(k+r+1)$, the term-by-term inverse Laplace transform of $\tilde{f}$. If $r \in \mathbb{N}^+$ and $\tilde{f}$ *converges* (to f), then $\mathcal{B}\tilde{f}$ converges in $\mathbb{C}$ to an analytic function which is Laplace ($\mathcal{L}$) transformable and $\mathcal{L}\mathcal{B}\tilde{f} = f$. A similar property holds more generally when $\Re(r) > 0$, with now f and $\mathcal{B}\tilde{f}$ ramified analytic functions. Even when $\tilde{f}$ is divergent (not faster than factorially), $\mathcal{B}\tilde{f}$ may have a nonzero radius of convergence and define a germ of an analytic function $F(p)$. If $F(p)$ can be analytically continued along a ray $\arg(p) = \phi$, and its growth is at most exponential, then $f = \mathcal{L}_\phi F = \int_{\mathrm{e}^{\mathrm{i}\phi}\mathbb{R}^+} F(p)\mathrm{e}^{-xp}\mathrm{d}p$ defines a function with the property $f \sim \tilde{f}$ as $x \to \infty$ with $\Re(x\mathrm{e}^{\mathrm{i}\phi}) > 0$. In general now $F(p)$ is singular (not only for $p = 0$), and $\mathcal{L}_\phi F$ (when it exists) will depend on ϕ; the usual convention is to choose ϕ so that

$$xp \in \mathbb{R}^+ \tag{1.6}$$

Thus, the Borel sum of $\tilde{f}$ in the direction x, if it exists, is defined as $\mathcal{L}_{\phi(x)}\mathcal{B}f$ with $-\phi(x) = \xi := \arg(x)$. However, when $\tilde{f}$ is a series with real coefficients, it is a common occurrence that $F(p)$ is singular for $p \in \mathbb{R}^+$ (because of conjugation symmetry), and then the *classical* Borel sum of $\tilde{f}$ along the real axis (the interesting direction in many cases) is undefined.

The difficulty is more serious than it may seem. Summation along paths that avoid the singularities from above or from below give different results and thus would lead to an ambiguous (or unnatural) procedure. More importantly, a "summation" procedure using such paths would not commute with complex-conjugation since the "sum" will be, in general, complex for real $\tilde{f}$ and would thus fail to be a (proper) summation method. Symmetry considerations suggest a first step towards *averaging*: summation along the half-sum of the two paths does commutes with complex conjugation. But this solves a problem only to create another one. The half-sum process fails to commute with multiplication (of series) and is thus not a summation method, either.

It turns out that there exist more sophisticated averages which have all the required properties to define a summation procedure. The technique of averaging, as well as the fundamental concepts of analyzable functions and transseries, were discovered and studied by Ecalle in his constructive approach to the Dulac conjecture (see [1], [2] and [3]). The concept of analyzable function (also discovered by Kruskal in the context of surreal analysis) is regarded as a very comprehensive generalization of analyticity/ quasianalyticity. The widely held belief is that all functions of “natural origin” must be analyzable. In particular, analyzable functions have uniquely associated transseries which are generalized-Borel summable, after a finite number of transformations [3]. We show that, in the particular case of (1.3), decreasing solutions are analyzable.

There is a wide class of admissible, all-purpose averaging methods ([4]). As yet there is no unique, natural average and the problem in its full generality is highly nontrivial. We obtain the *balanced average* directly from the study of the general solution of the inverse Laplace transform of (1.3). Its potential nonuniqueness is lifted, in our context, by imposing compatibility with *hyper-asymptotics* an important improvement in asymptotic calculations proposed by M. Berry ([22], [23], [24], [25]).

1.1.2 Nonresonance

(1) λ_i, $i = 1, ..., n_1$ are assumed $\mathbb{Z}$-linearly independent for any d. (2) Let $\theta \in [0, 2\pi)$ and $\tilde{\boldsymbol{\lambda}} = (\lambda_{i_1}, ..., \lambda_{i_p})$ where $\left|\arg \lambda_{i_j} - \theta\right| \in (-\pi/2, \pi/2)$ (those eigenvalues contained in the open half-plane H_θ centered along $e^{i\theta}$). We require that for any θ the complex numbers in the set $\{\tilde{\lambda}_i - \mathbf{k} \cdot \tilde{\boldsymbol{\lambda}} \in H_\theta : \mathbf{k} \in \mathbb{N}^p, \, i = 1, ..., p\}$ (note: the set is *finite*) have *distinct* directions. These are the Stokes lines $d_{i;\mathbf{k}}$.

That the set of $\boldsymbol{\lambda}$ which satisfy (1) and (2) has full measure follows from the fact that (1) and (2) follow from the condition:

$$\Big(\mathbf{m}, \mathbf{m}' \in \mathbb{Z}^n, \ \alpha \in \mathbb{R} \text{ and } (\mathbf{m} - \alpha\mathbf{m}') \cdot \boldsymbol{\lambda} = 0\Big) \ \Rightarrow \Big(\mathbf{m} = \alpha\mathbf{m}'\Big) \tag{1.7}$$

Indeed, if (1.7) fails, one of $\Re\lambda_j, \Im\lambda_j$ is a rational function with rational coefficients of the other $\Re\lambda_j$ and $\Im\lambda_j$, corresponding to a zero measure set in $\mathbb{R}^{2n}$.

1.2 Further notations and conventions

If y_1 and y_2 are inverse Laplace transformable functions, then in a neighborhood of the origin $\mathcal{L}^{-1}(y_1 y_2) = (\mathcal{L}^{-1} y_1) * (\mathcal{L}^{-1} y_2)$, where for $f, g \in L^1$ convolution is given by

$$f * g := p \mapsto \int_0^p f(s)g(p-s)\mathrm{d}s \tag{1.8}$$

We use the convention $\mathbb{N} \ni 0$. Let

$$\mathcal{W} = \{p \in \mathbb{C} : p \neq k\lambda_i \,, \forall k \in \mathbb{N}, i = 1, 2, \ldots, n\} \tag{1.9}$$

The directions $d_j = \{p : \arg(p) = \phi_j\}, j = 1, 2, \ldots, n$ (cf. (a2)) are the *Stokes lines* of $\tilde{\mathbf{y}}_0$ (note: sometimes known as *anti*-Stokes lines!). We construct over $\mathcal{W}$ a surface $\mathcal{R}$, consisting of homotopy classes of smooth curves in $\mathcal{W}$ starting at the origin, moving away from it, and crossing at most one Stokes line, at most once (see Fig. 1):

$$\mathcal{R} := \left\{\gamma : (0,1) \mapsto \mathcal{W} : \ \gamma(0_+) = 0; \ \frac{\mathrm{d}}{\mathrm{d}t}|\gamma(t)| > 0; \ \arg(\gamma(t)) \text{ monotonic}\right\} \text{ modulo homotopies} \tag{1.10}$$

Define $\mathcal{R}_1 \subset \mathcal{R}$ by (1.10) with the supplementary restriction $\arg(\gamma) \in (\psi_n - 2\pi, \psi_2)$ where $\psi_n = \max\{-\pi/2, \phi_n - 2\pi\}$ and $\psi_2 = \min\{\pi/2, \phi_2\}$. $\mathcal{R}_1$ may be viewed as the part of the covering $\mathcal{R}$, above a sector containing the real axis. Similarly we let $\mathcal{R}'_1 \subset \mathcal{R}_1$ with the restriction that the curves γ do not cross the Stokes directions $d_{i,\mathbf{k}}$ (cf. §1.1.2), other than $\mathbb{R}^+$, and we let $\psi_\pm = \pm\max(\pm\arg\gamma)$ with $\gamma \in \mathcal{R}'_1$.

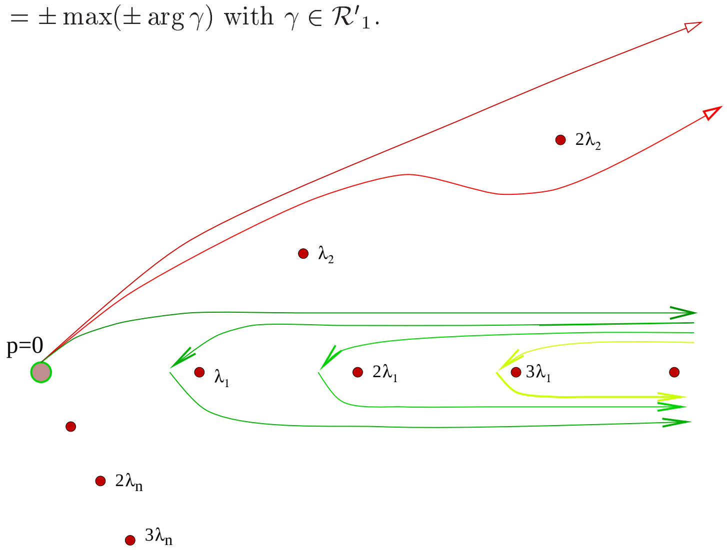

Fig 1. *The paths near λ_2 belong to $\mathcal{R}$.*
The paths near λ_1 relate to the balanced average

By $AC_\gamma(f)$ we denote the analytic continuation of f along a curve γ. For the analytic continuations near a Stokes line $d_{i;\mathbf{k}}$ we use symbols similar to Écalle's:

f^- is the branch of f along a path γ with $\arg(\gamma) < \phi_i$, while f^{-j+} denotes the branch along a path that crosses the Stokes line between $j\lambda_i$ and $(j+1)\lambda_i$ (see also [13]).

We use the notations $\mathcal{P}f$ for $\int_0^p f(s)\mathrm{d}s$ and $\mathcal{P}_\gamma f$ if integration is along the curve γ.

We write $\mathbf{k} \succeq \mathbf{k}'$ if $k_i \ge k_i'$ for all i and $\mathbf{k} \succ \mathbf{k}'$ if $\mathbf{k} \succeq \mathbf{k}'$ and $\mathbf{k} \neq \mathbf{k}'$. The relation $\succ$ is a well ordering on $\mathbb{N}^{n_1}$. We let $\mathbf{e}_j$ be the unit vector in the j^{th} direction in $\mathbb{N}^{n_1}$.

Formal expansions are denoted with a tilde, and capital letters $\mathbf{Y}, \mathbf{V}\ldots$ will usually denote Borel transforms or other functions naturally associated to Borel space. For notational convenience, we will not however distinguish between the series $\tilde{\mathbf{Y}}_k = \mathcal{B}\tilde{\mathbf{y}}_\mathbf{k}$, which in our case turn out to be convergent, and the sums $\mathbf{Y}_\mathbf{k}$ of these series as germs of ramified analytic functions.

By symmetry (renumbering the directions) it suffices to analyze the singularity structure of $\mathbf{Y}_0$ in $\mathcal{R}_1$ only. However, (c1) breaks this symmetry for $\mathbf{k} \neq 0$ and the properties of these $\mathbf{Y}_\mathbf{k}$ will be analyzed along some other directions as well.

$\mathcal{X}_A$ will denote the characteristic function of the set A. We write $|\mathbf{f}| := \max_i\{|f_i|\}$. We have

$$\mathbf{g}(x,\mathbf{y}) = \sum_{|\mathbf{l}|\ge 1} \mathbf{g}_\mathbf{l}(x)\mathbf{y}^\mathbf{l} = \sum_{s\ge 0; |\mathbf{l}|\ge 1} \mathbf{g}_{s,\mathbf{l}} x^{-s}\mathbf{y}^\mathbf{l} \quad (|x| > x_0, |\mathbf{y}| < y_0) \tag{1.11}$$

where $\mathbf{y}^\mathbf{l} = y_1^{l_1}\cdots y_n^{l_n}$ and $|\mathbf{l}| = l_1 + \cdots + l_n$. By construction $\mathbf{g}_{s,\mathbf{l}} = 0$ if $|\mathbf{l}| = 1$ and $s \le M$.

The formal inverse Laplace transform of $\mathbf{g}(x, \mathbf{y}(x))$ (formal since $\mathbf{y}$ is still unrestricted) is given by:

$$\mathcal{L}^{-1}\left(\sum_{|\mathbf{l}|\ge \mathbf{1}} \mathbf{y}(x)^\mathbf{l} \sum_{s\ge 0} \mathbf{g}_{s,\mathbf{l}} x^{-s}\right) = \sum_{|\mathbf{l}|\ge\mathbf{1}} \mathbf{G}_\mathbf{l} * \mathbf{Y}^{*\mathbf{l}} + \sum_{|\mathbf{l}|\ge\mathbf{2}} \mathbf{g}_{0,\mathbf{l}}\mathbf{Y}^{*\mathbf{l}} =: \mathcal{N}(\mathbf{Y}) \tag{1.12}$$

with $\mathbf{G}_\mathbf{l}(p) = \sum_{s=1}^\infty \mathbf{g}_{s,\mathbf{l}} p^{s-1}/s!$ and $(\mathbf{G}_\mathbf{l} * \mathbf{Y}^{*\mathbf{l}})_j := (\mathbf{G}_\mathbf{l})_j * Y_1^{*l_1} * .. * Y_n^{*l_n}$. By (n5), $\mathbf{G}_{1,\mathbf{1}}^{(l)}(0) = 0$ if $|\mathbf{l}| = 1$ and $l \le M$. The inverse Laplace transform of (1.3) is the convolution equation:

$$-p\mathbf{Y} = \mathbf{F}_0 - \hat{\Lambda}\mathbf{Y} - \hat{B}\mathcal{P}\mathbf{Y} + \mathcal{N}(\mathbf{Y}) \tag{1.13}$$

Let $\mathbf{d}_\mathbf{j}(x) := \sum_{\mathbf{l}\ge\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \mathbf{g}_\mathbf{l}(x) \tilde{\mathbf{y}}_0^{\mathbf{l}-\mathbf{j}}$. Straightforward calculation (see Appendix § 2.4; cf. also [13]) shows that the components $\tilde{\mathbf{y}}_\mathbf{k}$ of the transseries satisfy the hierarchy of differential equations

$$\mathbf{y}_{\mathbf{k}}' + \left(\hat{\Lambda} + \frac{1}{x}\left(\hat{B} + \mathbf{k}\cdot\mathbf{m}\right) - \mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{y}_k + \sum_{|\mathbf{j}|=1}\mathbf{d}_{\mathbf{j}}(x)(\mathbf{y}_{\mathbf{k}})^{\mathbf{j}} = \mathbf{t}_{\mathbf{k}} \tag{1.14}$$

where $\mathbf{t}_{\mathbf{k}} = \mathbf{t}_{\mathbf{k}}\left(\mathbf{y}_0, \{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right)$ is a *polynomial* in $\{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}$ and in $\{\mathbf{d}_{\mathbf{j}}\}_{\mathbf{j}\le\mathbf{k}}$ (see (2.158)), with $\mathbf{t}(\mathbf{y}_0, \emptyset) = 0$; $\mathbf{t}_{\mathbf{k}}$ satisfies the homogeneity relation

$$\mathbf{t}_{\mathbf{k}}\left(\mathbf{y}_0, \left\{C^{\mathbf{k}'}\mathbf{y}_{\mathbf{k}'}\right\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right) = C^{\mathbf{k}}\mathbf{t}_{\mathbf{k}}\left(\mathbf{y}_0, \{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right) \tag{1.15}$$

Taking $\mathcal{L}^{-1}$ in (1.14) we get, with $\mathbf{D}_{\mathbf{j}} = \sum_{\mathbf{l}\ge\mathbf{j}}\binom{\mathbf{l}}{\mathbf{j}}\left[\mathbf{G}_{\mathbf{l}} * \mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})} + \mathbf{g}_{0,\mathbf{l}} * \mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})}\right]$,

$$\left(-p + \hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{Y}_{\mathbf{k}} + \left(\hat{B} + \mathbf{k}\cdot\mathbf{m}\right)\mathcal{P}\mathbf{Y}_{\mathbf{k}} + \sum_{|\mathbf{j}|=1}\mathbf{D}_{\mathbf{j}} * \mathbf{Y}_{\mathbf{k}}^{*\mathbf{j}} = \mathbf{T}_{\mathbf{k}} \tag{1.16}$$

where $\mathbf{T}_{\mathbf{k}}$ is now a *convolution* polynomial, cf. (2.162).

1.3 Main results

(a) *Analytic structure.*

THEOREM 1 *(i)* $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$ *is analytic in* $\mathcal{R}\cup\{0\}$.

The singularities of $\mathbf{Y}_0$ *(which are contained in the set* $\{l\lambda_j : l\in\mathbb{N}^+, j = 1,2,\ldots,n\}$*) are described as follows. For* $l\in\mathbb{N}^+$ *and small* z*, using the notations explained in* §*1.2,*

$$\mathbf{Y}_0^{\pm}(z + l\lambda_j) = \pm\left[(\pm S_j)^l(\ln z)^{0,1}\mathbf{Y}_{l\mathbf{e}_j}(z)\right]^{(lm_j)} + \mathbf{B}_{lj}(z) =$$
$$\left[z^{l\beta_j'-1}(\ln z)^{0,1}\,\mathbf{A}_{lj}(z)\right]^{(lm_j)} + \mathbf{B}_{lj}(z)\ \ (l = 1,2,\ldots) \tag{1.17}$$

where the power of $\ln z$ *is one iff* $l\beta_j\in\mathbb{Z}$*, and* $\mathbf{A}_{lj}, \mathbf{B}_{lj}$ *are analytic for small* z*. The functions* $\mathbf{Y}_{\mathbf{k}}$ *are, exceptionally, analytic at* $p = l\lambda_j$*,* $l\in\mathbb{N}^+$*, iff,*

$$S_j = r_j\Gamma(\beta_j')\left(\mathbf{A}_{1,j}\right)_j(0) = 0 \tag{1.18}$$

where $r_j = 1 - \mathrm{e}^{2\pi i(\beta_j'-1)}$ *if* $l\beta_j\notin\mathbb{Z}$ *and* $r_j = -2\pi i$ *otherwise. The* S_j *are Stokes constants, see Theorem 5.*

(ii) $\mathbf{Y}_{\mathbf{k}} = \mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$*,* $|\mathbf{k}| > 1$*, are analytic in* $\mathcal{R}\backslash\{-\mathbf{k}'\cdot\boldsymbol{\lambda} + \lambda_i : \mathbf{k}'\le\mathbf{k}, 1\le i\le n\}$*. For* $l\in\mathbb{N}$ *and* p *near* $l\lambda_j$*,* $j = 1,2,\ldots,n$ *there exist* $\mathbf{A} = \mathbf{A}_{\mathbf{k}jl}$ *and* $\mathbf{B} = \mathbf{B}_{\mathbf{k}jl}$ *analytic at zero so that (*z *is as above)*

$$\mathbf{Y}^{\pm}_{\mathbf{k}}(z+l\lambda_j)=\pm\Big[(\pm S_j)^l\binom{k_j+l}{l}(\ln z)^{0,1}\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(z)\Big]^{(lm_j)}+\mathbf{B}_{\mathbf{k}lj}(z)=$$

$$\Big[z^{\mathbf{k}\cdot\boldsymbol{\beta}'+l\beta'_j-1}(\ln z)^{0,1}\,\mathbf{A}_{\mathbf{k}lj}(z)\Big]^{(lm_j)}+\mathbf{B}_{\mathbf{k}lj}(z)\ (l=0,1,2,\ldots)\tag{1.19}$$

where the power of $\ln z$ *is* 0 *iff* $l=0$ *or* $\mathbf{k}\cdot\boldsymbol{\beta}+l\beta_j-1\notin\mathbb{Z}$ *and* $\mathbf{A}_{\mathbf{k}0j}=\mathbf{e}_j/\Gamma(\beta'_j)$. *Near* $p\in\{\lambda_i-\mathbf{k}'\cdot\boldsymbol{\lambda}:0\prec\mathbf{k}'\le\mathbf{k}\}$, *(where* $\mathbf{Y}_0$ *is analytic)* $\mathbf{Y}_{\mathbf{k}}$, $\mathbf{k}\neq 0$ *have convergent Puiseux series.*

Remark: The fact that the singular part of $\mathbf{Y}_{\mathbf{k}}(p+l\lambda_j)$ in (1.17) and (1.19) is a *multiple* of $\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(p)$ is the effect of *resurgence* and provides a way of determining the $\mathbf{Y}_{\mathbf{k}}$ given $\mathbf{Y}_0$ provided the S_j are nonzero. Since, generically, the S_j are nonzero this is a surprising upshot: given one formal solution, (generically) an n parameter family of solutions can be constructed out of it, without using (1.3) in the process; the differential equation itself is then recoverable ([16]).

By Theorem 1 the Borel transforms $\mathbf{Y}_{\mathbf{k}}=\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$ define germs of ramified analytic functions and are continuable on the surface $\mathcal{R}$. In order to be able to take Laplace transforms we need to define $\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$ along any direction d in $\mathcal{S}$. If $d\neq d_{j,\mathbf{k}}$ then $\mathbf{Y}_{\mathbf{k}}$ are analytically continuable along d and the continuations turn out to have all the properties that we need. But along Stokes lines $d_{j,\mathbf{k}}$ analytic continuation is impossible: in general the functions $\mathbf{Y}_{\mathbf{k}}$ have an infinite array of branch points (1.19). In addition, while both $\mathbf{Y}^+_{\mathbf{k}}$ and $\mathbf{Y}^-_{\mathbf{k}}$ turn out to be Laplace transformable (in distributions) along $d_{j,\mathbf{k}}$, $\mathcal{L}\mathbf{Y}^+_{\mathbf{k}}$ and $\mathcal{L}\mathbf{Y}^-_{\mathbf{k}}$ are generically *different*. Neither the upper nor the lower continuation would give rise to a definition of Borel summation which commutes with complex-conjugation, as discussed in the introduction. Other analytic continuations along paths γ that *cross* $d_{j,\mathbf{k}}$ have even worse problems, namely that $AC_\gamma(\mathbf{Y}_{\mathbf{k}}*\mathbf{Y}_{\mathbf{k}})\neq AC_\gamma(\mathbf{Y}_{\mathbf{k}})*_\gamma AC_\gamma(\mathbf{Y}_{\mathbf{k}})$, (see [13]; for notations only, see also §2.1.5). As $\mathcal{B}$ transforms differential equations into convolution equations, the implication is that with such a γ, $\mathcal{L}AC_\gamma(\mathbf{Y}_{\mathbf{k}})$ would *not* be, in general, solutions of their differential equations. Individual analytic continuations are thus inadequate for solving (1.3), but some *averages* of analytic continuations do satisfy all the requirements. Let $\alpha=1/2+\mathrm{i}\sigma$ with $\sigma\in\mathbb{R}$ and $\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$ be extended along $d_{j,\mathbf{k}}$ by the weighted average of analytic continuations

$$\mathcal{B}_\alpha\tilde{\mathbf{y}}_{\mathbf{k}}=\mathbf{Y}^\alpha_{\mathbf{k}}=\mathbf{Y}^+_{\mathbf{k}}+\sum_{j=1}^{\infty}\alpha^j\left(\mathbf{Y}^-_{\mathbf{k}}-\mathbf{Y}^{-(j-1)+}_{\mathbf{k}}\right)\tag{1.20}$$

Remark 2 *Relation (1.20) gives the most general reality preserving, linear operator mapping formal power series solutions of (1.3) to solutions of (1.13) in distributions (more precisely in* $\mathcal{D}'_{m,\nu}$*; see §2.1.2).*

This remark follows easily from Proposition 23 and Theorem 4 below.

The choice $\alpha = 1/2$ has special properties; we call $\mathcal{B}_{\frac{1}{2}}\tilde{\mathbf{y}}_{\mathbf{k}} = \mathbf{Y}_{\mathbf{k}}^{ba}$ the balanced average of $\mathbf{Y}_{\mathbf{k}}$. For this choice the expression (1.20) coincides with the one in which $+$ and $-$ are interchanged (Proposition 34), accounting for the reality-preserving property. Clearly, if $\mathbf{Y}_{\mathbf{k}}$ is analytic along $d_{j,\mathbf{k}}$, then the terms in the infinite sum vanish and $\mathbf{Y}_{\mathbf{k}}^{\alpha} = \mathbf{Y}_{\mathbf{k}}$; we also let $\mathbf{Y}_{\mathbf{k}}^{\alpha} = \mathbf{Y}_{\mathbf{k}}$ if $d \neq d_{j,\mathbf{k}}$, where again $\mathbf{Y}_{\mathbf{k}}$ is analytic. It follows from (1.20) and Theorem 3 below that the Laplace integral of $\mathbf{Y}_{\mathbf{k}}^{\alpha}$ along $\mathbb{R}^+$ can be deformed into contours as those depicted in Fig. 1, with weight $-(-\alpha)^k$ for a contour turning around $k\lambda_1$.

In addition to symmetry (the balanced average equals the half sum of the upper and lower continuations on $(0, 2\lambda_j)$, [16]), an asymptotic property uniquely picks $C = 1/2$. Namely, for $C = 1/2$ alone are the $\mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}$ always *summable to the least term* cf. § 2.1.6.

(b) *Connection with (1.3) and (1.15).* Generalized Borel summation coincides with the usual Borel summation when the transseries consists of only one term, the first series, when that series is classically Borel summable. This is clear from theorem 3 (ii) below. Furthermore, generalized summation is a map from a class of formal series to functions which is linear, multiplicative, commutes with differentiation and complex conjugation (cf. § 2.1.6, § 2.1.2) so it is a summation procedure, which furthermore, establishes along every direction a one to one correspondence between transseries and decaying actual solutions of (1.3) cf. § 2.1.6, Proposition 34 and Theorem 3 below.

For clarity we state the results for $x \in S_x$, a sector in the right half plane containing $\lambda_1 = 1$ in which (c1) holds and for p in the associated domain $\mathcal{R}'_1$, but λ_1 plays no special role as discussed in the introduction.

THEOREM 3 *(i) The branches of $(\mathbf{Y}_{\mathbf{k}})_\gamma$ in $\mathcal{R}'_1$ ($\mathcal{R}_1$ if $\mathbf{k} = 0$) have limits in a C^*-algebra of distributions, $\mathcal{D}'_{m,\nu}(\mathbb{R}^+) \subset \mathcal{D}'$ (cf. § 2.1.2). Their Laplace transforms in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ $\mathcal{L}(\mathbf{Y}_{\mathbf{k}})_\gamma$ exist simultaneously and with $x \in \mathcal{S}_x$ and for any $\delta > 0$ there is a constant K and an x_1 large enough, so that for $\Re(x) > x_1$ we have $|\mathcal{L}(\mathbf{Y}_{\mathbf{k}})_\gamma(x)| \leq K\delta^{|\mathbf{k}|}$.*

In addition, $\mathbf{Y}_{\mathbf{k}}(p\mathrm{e}^{\mathrm{i}\phi})$ are continuous in ϕ with respect to the $\mathcal{D}'_{m,\nu}$ topology, (separately) on $(\psi_-, 0]$ and $[0, \psi_+)$.

If $m > \max_i(m_i)$ and $l < \min_i|\lambda_i|$ then $\mathbf{Y}_0(p\mathrm{e}^{\mathrm{i}\phi})$ is continuous in $\phi \in [0, 2\pi]\backslash\{\phi_i : i \leq n\}$ in the $\mathcal{D}'_{m,\nu}(\mathbb{R}^+, l)$ topology and has (at most) jump discontinuities for $\phi = \phi_i$. For each $\mathbf{k}$, $|\mathbf{k}| \geq 1$ and any K there is an $l > 0$ and an m such that $\mathbf{Y}_{\mathbf{k}}(p\mathrm{e}^{\mathrm{i}\phi})$ are continuous in $\phi \in [0, 2\pi]\backslash\{\phi_i; -\mathbf{k}'\cdot\boldsymbol{\lambda} + \lambda_i : i \leq n, \mathbf{k}' \leq \mathbf{k}\}$ in the $\mathcal{D}'_{m,\nu}((0, K), l)$ topology and have (at most) jump discontinuities on the boundary.

(ii) The sum (1.20) converges in $\mathcal{D}'_{m,\nu}$ (and coincides with the analytic continuation of $\mathbf{Y}_{\mathbf{k}}$ when $\mathbf{Y}_{\mathbf{k}}$ is analytic along $\mathbb{R}^+$). For any δ there is a large enough x_1 independent of $\mathbf{k}$ *so that $\mathbf{Y}_{\mathbf{k}}^{ba}(p)$ with $p \in \mathcal{R}'_1$ are Laplace transformable in $\mathcal{D}'_{m,\nu}$ for $\Re(xp) > x_1$ and furthermore $|(\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba})(x)| \leq \delta^{|\mathbf{k}|}$. In addition, if $d \neq \mathbb{R}^+$, then for large ν, $\mathbf{Y}_{\mathbf{k}} \in L^1_\nu(d)$.*

The functions $\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba}$ are analytic for $\Re(xp) > x_1$. For any $\mathbf{C} \in \mathbb{C}^{n_1}$ there is *an $x_1(\mathbf{C})$ large enough so that the sum*

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^{ba} + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba} \tag{1.21}$$

converges uniformly for $\Re(xp) > x_1(\mathbf{C})$, and $\mathbf{y}$ is a solution of (1.3). When the direction of p is not the real axis then, by definition, $\mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}$, $\mathcal{L}$ is the usual Laplace transform and (1.21) becomes

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}} \tag{1.22}$$

In addition, $\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba} \sim \tilde{\mathbf{y}}_{\mathbf{k}}$ for large x in the half plane $\Re(xp) > x_1$, for all $\mathbf{k}$, uniformly.

iii) More generally, for any α and any solution $\mathbf{y}$ of (1.3) such that $\mathbf{y} \sim \tilde{\mathbf{y}}_0$ for large x along a ray in S_x there exists a constant vector $\mathbf{C} = \mathbf{C}_{\alpha;\mathbf{y}}$ so that

$$\mathbf{y} = \mathcal{L}\mathcal{B}_\alpha\tilde{\mathbf{y}}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathcal{B}_\alpha\tilde{\mathbf{y}}_{\mathbf{k}} \tag{1.23}$$

Given α the representation (1.23) of $\mathbf{y}$ is unique (see also § 1.1.1 above for the convention on the direction of Laplace integration).

Of special interest are the cases $\alpha = 1/2$, discussed above, and also $\alpha = 0, 1$ which give:

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^{\pm} + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{\pm} \tag{1.24}$$

(c) Resurgence properties; local Stokes phenomenon.

It turns out that the formal series $\tilde{\mathbf{y}}_{\mathbf{k}}$ are connected among each-other via their Borel transforms. Resurgence formulas link $\mathbf{Y}_{\mathbf{k}}$ to analytic continuations of $\mathbf{Y}_{\mathbf{k}'}$ with $\mathbf{k}' \prec \mathbf{k}$, in a way that, generically, $\mathbf{Y}_0$ contains enough information to compute all $\mathbf{Y}_{\mathbf{k}}$.

Various resurgence properties have been observed in different contexts, and the term resurgence has been used with slightly different interpretations. In the hyperasymptotic theory of M. Berry, it was discovered that the first asymptotic series reappears in various shapes in the process of computing higher terms of the expansions. J. Écalle, in his comprehensive theory of analyzable functions, has obtained a general resurgence principle, the *bridge equation* [2]. The common denominator of resurgence is the reappearance of "earlier" terms in the formulas of "later" ones. It turns out that, for our problem, resurgence is fundamentally linked to the *Stokes phenomenon.* In the following formulas we make the convention $\mathbf{Y}_{\mathbf{k}}(p-j) = 0$ for $p < j$ as an element of $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$. We again state the results is stated for $p \in \mathcal{R}'_1$ and $x \in S_x$ but hold in any sector where (c1) is valid.

THEOREM 4 *i) For all* $\mathbf{k}$ *and* $\Re(p) > j, \Im(p) > 0$ *as well as in* $\mathcal{D}'_{m,\nu}$ *we have*

$$\mathbf{Y}_{\mathbf{k}}^{\pm j\mp}(p) - \mathbf{Y}_{\mathbf{k}}^{\pm(j-1)\mp}(p) = (\pm S_1)^j \binom{k_1+j}{j} \left(\mathbf{Y}^{\mp}_{\mathbf{k}+j\mathbf{e}_1}(p-j)\right)^{(mj)} \tag{1.25}$$

and also,

$$\mathbf{Y}_{\mathbf{k}}^{\pm} = \mathbf{Y}_{\mathbf{k}}^{\mp} + \sum_{j\geq 1} \binom{j+k_1}{k_1} (\pm S_1)^j (\mathbf{Y}^{\mp}_{\mathbf{k}+j\mathbf{e}_1}(p-j))^{(mj)} \tag{1.26}$$

ii) Local Stokes transition.
Consider the expression of a fixed solution $\mathbf{y}$ *of (1.3) as a Borel summed trans-series (1.21). As* $\arg(x)$ *varies, (1.21) changes only through* $\mathbf{C}$*, and that change occurs when Stokes lines are crossed (cf. §1.1.2; the Stokes lines of* $\mathbf{Y}_0$ *are the directions of* λ_i*). We have, in the neighborhood of* $\mathbb{R}^+$*, with* S_1 *defined in (1.18):*

$$\mathbf{C}(\xi) = \begin{cases} \mathbf{C}^- = \mathbf{C}(-0) & \text{for } \xi < 0 \\ \mathbf{C}^0 = \mathbf{C}(-0) + \frac{1}{2}S_1\mathbf{e}_1 & \text{for } \xi = 0 \\ \mathbf{C}^+ = \mathbf{C}(-0) + S_1\mathbf{e}_1 & \text{for } \xi > 0 \end{cases} \tag{1.27}$$

(d) Classical Stokes phenomena and local Stokes transitions. Again we formulate the result below for λ_1 but with straightforward adjustments it holds relative to any other eigenvalue. Let $\mathbf{C}$ be of the form $C_1\mathbf{e}_1$. Along the imaginary axis, condition (c1) fails. The positive and negative imaginary are the *antistokes* lines corresponding to $\lambda_1 = 1$ (note: sometimes called Stokes lines!). If we choose paths in the right half plane approaching the positive/negative imaginary axis in such a way that $|x^{-\beta_1-l}\mathrm{e}^{-x}| \to K \neq 0$ along them, where $l+\beta \in (0, M)$, then $\mathbf{y} \sim C^{\pm}x^{-l-\beta_1}\mathrm{e}^{-x} + \tilde{\mathbf{y}}_0$ for large x and the term multiplied by K is now the *leading* behavior of $\mathbf{y}$. The particular choice of K and l within this range is rather arbitrary, the main point being that along such special curves, the constant $\mathbf{C}$ is definable in terms of *classical* asymptotics. Within the right half plane, it is only near the imaginary axis that this happens, since otherwise the exponential term is smaller than all terms of $\tilde{\mathbf{y}}_0$. On the other hand Borel summation makes possible the definition of $\mathbf{C}$ throughout the right half plane, and we now address the issue of the relation between classical asymptotics and exponential asymptotics.

THEOREM 5 *Let* $\gamma^{\pm}$ *be two paths in the right half plane, near the positive/ negative imaginary axis such that* $|x^{-\beta_1+1}e^{-x\lambda_1}| \to 1$ *as* $x \to \infty$ *along* $\gamma^{\pm}$*. Consider the solution* $\mathbf{y}$ *of (1.3) given in (1.21) with* $\mathbf{C} = C\mathbf{e}_1$ *and where the path of integration is* $p \in \mathbb{R}^+$*. Then*

$$\mathbf{y} = (C \pm \frac{1}{2}S_1)\mathbf{e}_1 x^{-\beta_1+1}e^{-x\lambda_1}(1+o(1)) \tag{1.28}$$

for large x *along* $\gamma^{\pm}$*, where* S_1 *is the same as in (1.18), (1.27).*

Classical asymptotics loses track of the value of $\mathbf{C}$ along any ray other than the imaginary directions, as the terms multiplied by $\mathbf{C}$ will be hidden "beyond all orders" of the classically divergent series $\tilde{\mathbf{y}}_0$. In contrast to the classical picture, we see that through generalized Borel summation the constant $\mathbf{C}$ is precisely defined throughout the positive half-plane and the question of where the change in $\mathbf{C}$ occurs is well defined.

Formula (1.27) is the exponential asymptotic expression of the Stokes phenomenon. It shows that the constant jumps as the Stokes line is crossed, (1.27), as originally predicted by Stokes himself [10]. Subsequently, the original ideas of Stokes, based on optimal truncation of series were greatly refined by M. Berry, leading to his theory of hyperasymptotic expansions and a description of Stokes transitions for saddle integrals [22].

If more than one component of $\mathbf{C}$ is nonzero, then in general there is no direction along which $\mathbf{C}$ can be defined through *classical* asymptotics. Part of the difficulty of studying nonlinear Stokes phenomena using classical tools stems from this fact.

Relation (1.27) expresses the evolution of $\mathbf{C}$ and the presence of a Stokes phenomenon beyond all orders of Poincaré asymptotics.

2 Proofs and further results

The layout of the proofs is as follows. We study the formal inverse Laplace transforms of (1.3) and (1.14) in a C^*-algebra of distributions that we introduce. Using a fixed point principle we find the general solution of these convolution equations, and then study their properties. We then show that generalized Laplace transforms of these distributions exist and have all the required properties. The resurgence formulas are obtained by comparing different expressions for the same solution of (1.3) near a Stokes line.

Since the proofs rely to a large extent on the detailed study of the convolution equations (1.13), (1.16), we start with a few heuristic remarks. In a convolution equation such as (1.13), the term $(-p+\hat{\Lambda})\mathbf{Y}$ plays a role similar to that of the highest derivative term in a differential equation. To illustrate this, assume a solution $\mathbf{Y}$ is already given on an interval, say $(0,a)$, and we wish to extend it to $(0,a+\epsilon)$. We look for such a solution in the form $\mathbf{Y}+\boldsymbol{\delta}$, where we take $\mathbf{Y}=0$ on $(a,a+\epsilon)$ and $\boldsymbol{\delta}=0$ on $(0,a)$. If ϵ is small, then $\boldsymbol{\delta}*\boldsymbol{\delta}=0$ and the equation in $\boldsymbol{\delta}$ is linear inhomogeneous. The terms that involve integrals of $\boldsymbol{\delta}$ are of order $O(\epsilon)\|\boldsymbol{\delta}\|$ as $\epsilon\to 0$, so that the dominant terms are the forcing term, together with $(-p+\hat{\Lambda})\boldsymbol{\delta}$ provided the coefficient is invertible. If in addition the forcing is non-singular then $\boldsymbol{\delta}$ can be found, e.g., by a convergent ϵ expansion; this is the analog of an ordinary point of a differential equation. $\boldsymbol{\delta}$ can be singular if either $\Delta_p=\det(p-\hat{\Lambda})=0$ or the forcing is singular. To understand the qualitative behavior near a zero of Δ_p one has to keep (at least) one more term, the leading term among those previously discarded (i.e. the second term in the notation (1.16)). In this approximation, $\boldsymbol{\delta}$ satisfies a differential equation.

In our problem, there are n roots of Δ_p but because of the nonlinearity and nonlocality of the equations, a singularity generates (when convolved with itself) a whole array of singularities affecting $\boldsymbol{\delta}$ through the forcing term. Through convolution, a nonintegrable singularity produces further singularities of even lower regularity. We introduce a distribution space, $\mathcal{D}'_{m,\nu,l}$, whose degree of regularity decreases with the distance from the origin, at a "convolution-like" rate; these distributions form a convolution Banach algebra (cf. § 2.1.2).

Technically, the proofs rely on suitable fixed-point theorems in spaces having some of the properties we want to prove, notably in terms of regularity and behavior at infinity. This is combined with a local analysis near noninvertibility points of the dominant term, which is done by treating the convolution equations as a perturbation of the approximating differential equation mentioned above, which splits the singularity thus making again possible the use of fixed point theorems. This analysis is used in order to find the resurgence properties, which in turn are used to prove (using among others Lemma 11 below) the sharper results on global analyticity and structure of singularities.

We start by introducing some useful functional spaces and derive specific fixed point theorems.

2.1 Technical constructions and results

2.1.1 Focusing spaces and algebras

We say that a family of norms $\|\|_\nu$ depending on a parameter $\nu \in \mathbb{R}^+$ is **focusing** if for any f with $\|f\|_{\nu_0} < \infty$

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{2.1}$$

Let $\mathcal{E}$ be a linear space and $\{\|\|_\nu\}$ a family of norms satisfying (2.1). For each ν we define a Banach space $\mathcal{B}_\nu$ as the completion of $\{f \in \mathcal{E} : \|f\|_\nu < \infty\}$. Enlarging $\mathcal{E}$ if needed, we may assume that $\mathcal{B}_\nu \subset \mathcal{E}$. For $\alpha < \beta$, (2.1) shows that the identity is an embedding of $\mathcal{B}_\alpha$ in $\mathcal{B}_\beta$. Let $\mathcal{F} \subset \mathcal{E}$ be the projective limit of the $\mathcal{B}_\nu$. That is to say

$$\mathcal{F} := \bigcup_{\nu>0} \mathcal{B}_\nu \tag{2.2}$$

is endowed with the topology in which a sequence is convergent if it converges in *some* $\mathcal{B}_\nu$. We call $\mathcal{F}$ a **focusing space**.

Consider now the case when $(\mathcal{B}_\nu, +, *, \|\|_\nu)$ are commutative Banach algebras. Then $\mathcal{F}$ inherits a structure of a commutative algebra, in which $*$ ("convolution") is continuous. We say that $(\mathcal{F}, *, \|\|_\nu)$ is a **focusing algebra**.

2.1.2 Examples

Let $K \in \mathbb{R}^+$ and $\mathcal{S} = \mathcal{S}_{K,\alpha_1,\alpha_2} = \{p : \arg(p) \in [\alpha_1, \alpha_2] \subset (-\pi/2, \pi/2), |p| \le K\}$ (or a finite union of such sectors) and $\mathcal{V}$ be a small neighborhood of the origin.

$\overline{\mathcal{V}}$ will be the closure of $\mathcal{V}$, cut along the negative axis, and together with these upper and lower cuts.

(1). $L^1{}_\nu(\mathcal{K})$. Let $\mathcal{K} = \mathcal{S}_{K,\phi,\phi}$. The space $L^1{}_\nu(\mathcal{K})$ with convolution (1.8) is a commutative Banach algebra under each of the (equivalent) norms

$$\|f\|_\nu = \int_0^K \mathrm{e}^{-\nu t}|f(t\exp(\mathrm{i}\phi))|\mathrm{d}t \tag{2.3}$$

Indeed, with $F(s) := f(s\mathrm{e}^{\mathrm{i}\phi})$ and $G(s) := g(s\mathrm{e}^{\mathrm{i}\phi})$ we have:

$$\begin{aligned}\int_0^K \mathrm{d}t\mathrm{e}^{-\nu t}\left|\int_0^t \mathrm{d}sF(s)G(t-s)\right| &\le \int_0^K \mathrm{d}t\mathrm{e}^{-\nu t}\int_0^t \mathrm{d}s|F(s)G(t-s)|\\ &= \int_0^K\int_0^{K-v} \mathrm{e}^{-\nu(u+v)}|F(v)||G(u)|\mathrm{d}u\mathrm{d}v\\ &\le \int_0^K\int_0^K \mathrm{e}^{-\nu(u+v)}|F(v)||G(u)|\mathrm{d}u\mathrm{d}v = \|f\|_\nu\|g\|_\nu\end{aligned} \tag{2.4}$$

By dominated convergence $\|f\|_\nu \downarrow 0$ as $\nu \uparrow \infty$ and thus $L^1(\mathcal{K})$ is a focusing algebra.

(2) If $K = \infty$ in example (1), then the norms (2.3) are not equivalent anymore for different ν, but convolution is still continuous in (2.3) and the projective limit of the $L^1_\nu(\mathbb{R}^+\mathrm{e}^{\mathrm{i}\phi})$, $\mathcal{F}(\mathbb{R}^+\mathrm{e}^{\mathrm{i}\phi}) \subset L^1_{loc}(\mathbb{R}^+\mathrm{e}^{\mathrm{i}\phi})$, is a focusing algebra.

(3a) $\mathcal{T}_\beta(\mathcal{S}\cup\overline{\mathcal{V}})$. For $\Re(\beta) > 0$ and $\phi_1 \ne \phi_2$, this space is given by $\{f : f(p) = p^\beta F(p)\}$, where F is analytic in the interior of $\mathcal{S}\cup\mathcal{V}$ and continuous in its closure. We take the family of (equivalent) norms

$$\|f\|_{\nu,\beta} = K\sup_{s\in\mathcal{S}\cup\overline{\mathcal{V}}}\left|\mathrm{e}^{-\nu p}f(p)\right| \tag{2.5}$$

It is clear that convergence of f in $\|\|_{\nu,\beta}$ implies uniform convergence of F on compact sets in $\mathcal{S}\cup\mathcal{V}$ (for p near zero, this follows from Cauchy's formula). $\mathcal{T}_\beta$ are thus Banach spaces and focusing spaces by (2.5). The spaces $\{\mathcal{T}_\beta\}_\beta$ are isomorphic to each-other. Taking $s = pt$ in (1.8) we find that

$$p^{-\beta_1-\beta_2-1}(f_1 * f_2)(p) = \int_0^1 t^{\beta_1}F_1(pt)(1-t)^{\beta_2}F_2(p(1-t))\mathrm{d}t = F(p) \tag{2.6}$$

where F is manifestly analytic, and that the application

$$(\cdot * \cdot) : \mathcal{T}_{\beta_1}\times\mathcal{T}_{\beta_2} \mapsto \mathcal{T}_{\beta_1+\beta_2+1} \tag{2.7}$$

is continuous:

$$
\begin{aligned}
\|f_1 * f_2\|_{\nu,\beta_1+\beta_2+1} &= K \sup_p \left| \mathrm{e}^{-\nu p} \int_0^p s^{\beta_1} F_1(s)(p-s)^{\beta_2} F_2(p-s)\mathrm{d}s \right| \\
&\leq K^{-1} \sup_p \int_0^p \left| K\mathrm{e}^{-\nu s} s^{\beta_1} F_1(s) K\mathrm{e}^{-\nu(p-s)} (p-s)^{\beta_2} F_2(p-s) \right| \mathrm{d}|s| \\
&\leq \|f_1\|_{\nu,\beta_1} \|f_2\|_{\nu,\beta_2}
\end{aligned}
\tag{2.8}
$$

A natural generalization of $\mathcal{T}_\beta$ is obtained taking $\beta_1, \ldots, \beta_N \in \mathbb{C}$ with positive real parts, no two of them differing by an integer. If $f_\beta = \sum_{i=1}^k p^{\beta_i} A_i(p)$ with A_i analytic, then $f_\beta \equiv 0$ iff $A_i \equiv 0$ for all i (e.g., by a Puiseux series argument). It is then natural to identify the space $\mathcal{T}_{\{\beta_1,\ldots,\beta_k\}}$ of functions of the form f_β with $\oplus_{i=1}^k \mathcal{T}_{\beta_i}$. Convolution with analytic functions is defined on $\mathcal{T}_{\{\beta_1,\ldots,\beta_k\}}$ while convolution of two functions in $\mathcal{T}_{\{\beta_1,\ldots,\beta_k\}}$ takes values in $\mathcal{T}_{\{\beta_i+\beta_j \bmod 1\}}$. We write $\mathcal{T}_{\{\cdot\}}$ when the concrete values of $\beta_1, \ldots, \beta_k$ do not matter.

(3b) A particular case of the preceding example is $\mathcal{A}_{z,l}(\mathcal{S} \cup \mathcal{V})$ consisting of analytic functions in the interior of $\mathcal{S} \cup \mathcal{V}$, continuous on its closure, and vanishing at the origin together with the first l derivatives. $\mathcal{A}_{z,l}$ can be identified with $\mathcal{T}_l$.

(4) $\mathcal{D}'_{m,\nu}$, the "staircase distributions". *Proofs* of the properties stated in this paragraph and more details are given in § 2.3.1. Let $\mathcal{D}(0,x)$ be the test functions on $(0,x)$ and $\mathcal{D} = \mathcal{D}(0,\infty)$. Let $\mathcal{D}'_m \subset \mathcal{D}'$ be the distributions f for which $f = F_k^{(km)}$ on $\mathcal{D}(0,k+1)$ with $F_k \in L^1[0,k+1]$. There is a uniquely associated "staircase decomposition", a sequence $\{\Delta_i(f)\}_{i\in\mathbb{N}} = \{\Delta_i\}_{i\in\mathbb{N}}$ such that $\Delta_i \in L^1(\mathbb{R}^+)$, $\Delta_i = \Delta_i \chi_{[i,i+1]}$ and

$$
f = \sum_{i=0}^{\infty} \Delta_i^{(mi)} \tag{2.9}
$$

Convolution is well defined on $\mathcal{D}'_m$ by

$$
f * \tilde{f} := (F_k * \tilde{F}_k)^{(2km)} \quad \text{in } \mathcal{D}'(0,k+1) \tag{2.10}
$$

and $(\mathcal{D}'_m, +, *)$ is a commutative algebra. We define, for $f \in \mathcal{D}'_m$,

$$
\|f\|_{\nu,m} := c_m \sum_{i=0}^{\infty} \nu^{im} \|\Delta_i\|_\nu \tag{2.11}
$$

where c_m is defined in Lemma 39, and $\|\Delta\|_\nu$ is computed from (2.3) with $K = \infty$. Then (2.11) is a norm on $\mathcal{D}'_m$ and $\mathcal{D}'_{m,\nu} = (\mathcal{D}'_m, +, *, \|\|_{m,\nu})$ is a Banach algebra. With respect to the family of norms $\|\|_{m,\nu}$, the projective limit of the $\mathcal{D}'_{m,\nu}$, $\mathcal{F}_m$ is a focusing algebra.

For any $f \in L^1_{\nu_0}(\mathbb{R}^+)$ there is a constant $C(\nu,\nu_0)$ such that $f \in \mathcal{D}'_{m,\nu}$ for all $\nu > \nu_0$ and

$$\|f\|_{\mathcal{D}'_{m,\nu}} \le C(\nu_0,\nu)\|f\|_{L^1_{\nu_0}} \tag{2.12}$$

and formula (2.10) is equivalent to (1.8), in this case.

The operator $f(p)\mapsto pf(p)$ is continuous from $\mathcal{D}'_{m,\nu}$ to $\mathcal{D}'_{m,\nu+\delta}$ for any $\delta>0$.

For $a\in\mathbb{R}^+$, $\mathcal{D}'_{m,\nu}(a,\infty)=\{f\in\mathcal{D}'_{m,\nu}:\Delta_i(x)=0 \text{ for } x<a\}$ is a closed ideal in $\mathcal{D}'_{m,\nu}$ (isomorphic to the restriction $\mathcal{D}'_{m,\nu}(a,\infty)$ of $\mathcal{D}'_{m,\nu}$ to $\mathcal{D}(a,\infty)$). The restrictions $\mathcal{D}'_{m,\nu}(a,b)$ of $\mathcal{D}'_{m,\nu}$ to $\mathcal{D}(a,b)$ are for $0<a<b<\infty$ Banach spaces with respect to the norm (2.11) restricted to (a,b).

The functions in $\mathcal{D}\left(\mathbb{R}^+\backslash\mathbb{N}\right)$ are dense in $\mathcal{D}'_{m,\nu}$, with respect to the norm (2.11) (Lemma 41).

If we choose a different interval length $l>0$ instead of $l=1$ in the partition associated to (2.9), we then write $\mathcal{D}'_{m,\nu}(l)$. Obviously, dilation gives a natural isomorphism between these structures. If $d=\{t\mathrm{e}^{\mathrm{i}\phi}:t\in\mathbb{R}^+\}$ is any ray, $\mathcal{D}'_{m,\nu}(d)$ and $\mathcal{F}_{m;\phi}$ are defined in an analogous way and have the same properties as their real counterpart.

Laplace transforms are naturally defined in $\mathcal{D}'_{m,\nu}$.

Lemma 6 *Laplace transform extends continuously from $\mathcal{D}(\mathbb{R}^+\backslash\mathbb{N})$ to $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ by the formula*

$$(\mathcal{L}f)(x) := \sum_{k=0}^{\infty} x^{mk}\int_0^\infty \mathrm{e}^{-sx}\Delta_k(s)\mathrm{d}s \tag{2.13}$$

In particular, with $f,g,h'\in\mathcal{D}'_{m,\nu}$ we have

$$\begin{aligned}&\mathcal{L}(f*g)=\mathcal{L}(f)\mathcal{L}(g)\\&\mathcal{L}(h')=x\mathcal{L}(h)-h(0)\\&\mathcal{L}(pf)=-(\mathcal{L}(f))'\end{aligned} \tag{2.14}$$

For $x\in S_\nu=\{x:\Re(x)>\nu\}$ the sum (2.13) converges absolutely. Laplace transform is, for fixed $x\in S_\nu$, a continuous functional (of norm less than one) on $\mathcal{D}'_{m,\nu}$.

$(\mathcal{L}f)(x)$ is analytic in S_ν.

Furthermore, $\mathcal{L}$ is injective *in $\mathcal{D}'_{m,\nu}$.*

The proof is given in § 2.3.4. We conclude this section with a few remarks.

Remark 7 *Let $\mathcal{U}$ be one of the spaces considered in the examples and ν be large:*

i) if g is analytic, (and if $g\in\mathcal{U}$ in Examples (2) and (4)) then $L_g:=f\mapsto f*g$ is a bounded operator and $\|L_g\|=O(\nu^{-1})$ ($\mathcal{P}$ is such an operator, since $\mathcal{P}f=f*1$);

ii) replacing $*$ by $*_\phi$ defined as $f*_\phi g:=\mathrm{e}^{\mathrm{i}\phi}(f*g)$ leads to an isomorphic structure;

iii) if $g\in\mathcal{U}$ is a real valued nonnegative function and $|f|\le g$ ($|\mathcal{P}^{km}f|\le\mathcal{P}^{km}g$ for all k, on $(0,k+1)$ if the space is $\mathcal{D}'_{m,\nu}$) then $\|f\|\le\|g\|$.

(i) In L^1_ν and $\mathcal{D}'_{m,\nu}$ this follows from the continuity of convolution and (2.12). In the examples (3), the natural inclusion $\mathcal{T}_{\beta+\mathbb{N}} \subset \mathcal{T}_\beta$ together with (2.7) and (2.8) makes convolution with an analytic function continuous in $\mathcal{T}_\beta$ and the claim follows from the estimate $\|f * g\| \le \max|g| \, \|f\| \, \|\mathcal{P}|e^{\nu p}|\|$.

(ii) The isomorphism is given by $f \mapsto e^{-i\phi} f$.

(iii) Since $\mathcal{P}$ is positivity preserving, writing $|f| \le g$ as $-g \le (\Re, \Im) f \le g$ the property is obvious when f, g are functions, while for $\mathcal{D}'_{m,\nu}$ it follows from equation (2.98) below.

□

2.1.3 Vectorial convolution and focusing spaces

We endow $\mathcal{B}^n_\nu$ with a Banach space structure by identifying it with $\mathcal{B}_\nu \oplus \cdots \oplus \mathcal{B}_\nu$ (n times). The projective limit of the $\mathcal{B}_\nu$, $\mathcal{F}^n$ is, clearly, a focusing space. We define a convolution on $(\cdot * \cdot) : \mathcal{F}^n \mapsto \mathcal{F}$ (*not* $\mathcal{F}^n \mapsto \mathcal{F}^n$) by

$$\mathbf{V} * \mathbf{W} := \sum_{i,j=1}^{n} V_i * W_j \tag{2.15}$$

We write $\mathbf{V}^{*\mathbf{l}} := V_1^{*l_1} * V_2^{*l_2} * \cdots * V_n^{*l_n}$ with the conventions $V^{*1} = V$ and that the factors with $l_i = 0$ are omitted.

2.1.4 A fixed point property

Lemma 8 *Let $\mathcal{F}$ be a focusing space and $\mathcal{N}$ be a (linear or nonlinear) operator defined on $\mathcal{F}$. Equivalently, in view of (2.1), let $\mathcal{N}$ be defined on $\bigcup_{\nu>\nu_0} B_\nu(\delta)$ with $B_\nu(\delta) = \{f : \|f\|_\nu \le \delta\}$ for some $\delta > 0$. Assume $\mathcal{N}$ is* **eventually contractive** *in the following sense. There exist ν_0, $\epsilon > 0$, $\alpha < 1$, so that if $\nu \ge \nu_0$ and $\|f\|_\nu + \|g\|_\nu \le \epsilon$ then*

$$\|\mathcal{N}(f+g) - \mathcal{N}(g)\|_\nu \le \alpha \|g\|_\nu \tag{2.16}$$

Then $\mathcal{N}$ has a unique fixed point $f_0 \in \mathcal{F}$.

If $\mathcal{N}$ depends continuously (in the strong topology) on a parameter ϕ for $\nu > \nu_0$ and if the constants ν_0, α and ϵ above do not depend on ϕ, then the fixed point f_ϕ is also continuous in ϕ. Furthermore, $\lim_{\nu\to\infty} \sup_\phi \|f_\phi\|_\nu = 0$.

The proof is straightforward. To show existence, take $\nu > \nu_0$ large enough so that, by (2.1) $\|\mathcal{N}(0)\|_\nu < (1-\alpha)\epsilon$. Then the closed ball $B_\nu(\epsilon)$ is mapped by $\mathcal{N}$ into itself for any ϕ by (2.16) and $\mathcal{N}$ is contractive there. The fixed point obtained, for instance, as the limit of the (convergent, uniformly in ϕ) iteration $\phi_{n+1} = \mathcal{N}(\phi_n); \phi_0 = 0$ is continuous in ϕ since $\mathcal{N}$ is. By construction $\|f_0\|_\nu \le \epsilon$, for all ϕ.

For uniqueness, let f_0 and f_1 be fixed points of $\mathcal{N}$; by (2.1) there is a $\nu > \nu_0$ so that $\|f_{0,1}\|_\nu < \epsilon$. Then by (2.16), $\|f_0 - f_1\|_\nu \le \alpha \|f_0 - f_1\|_\nu$ and thus $f_0 = f_1$. □ .

Let $\nu_0 > 0$ and let $\{\mathbf{M}\}_{\mathbf{l}}$ be a sequence of linear operators $\mathbf{M}_{\mathbf{l}} : \mathcal{B}_\nu \mapsto \mathcal{B}_\nu^n$ for $\nu \geq \nu_0$. Assume that for some κ and all $\mathbf{l}$, $|\mathbf{l}| \geq 1$,

$$\|\mathbf{M}_{\mathbf{l}}\|_\nu \leq C_\nu \kappa^{|\mathbf{l}|} \quad \text{and } C_{1,\nu} := \lim_{\nu\to\infty} \max_{|\mathbf{l}|=1} \|\mathbf{M}_{\mathbf{l}}\|_\nu = 0 \tag{2.17}$$

Let $\mathbf{F}_0 \in \mathcal{F}^n$ and $\mathbf{M} : \mathcal{F}^n \mapsto \mathcal{F}^n$ be defined by

$$\mathbf{M}(\mathbf{Y}) := \mathbf{F}_0 + \sum_{|\mathbf{l}|\geq \mathbf{1}} \mathbf{M}_{\mathbf{l}}\left(\mathbf{Y}^{*\mathbf{l}}\right) \tag{2.18}$$

Lemma 9 $\mathbf{M}$ *satisfies the assumptions of Lemma 8.* $\mathbf{M}$ *has therefore a unique fixed point in* $\mathcal{F}^n$.

Proof. We first need the following estimate.

Remark 10 *Let* $\mathbf{V}, \mathbf{W} \in \mathcal{F}^n$. *For* $|\mathbf{l}| > 0$ *and any* ν *we have, with* $\|\| = \|\|_\nu$,

$$\|\mathbf{W}_{\mathbf{l}}\| := \|(\mathbf{V}+\mathbf{W})^{*\mathbf{l}} - \mathbf{V}^{*\mathbf{l}}\| \leq |\mathbf{l}| \left(\|\mathbf{V}\| + \|\mathbf{W}\|\right)^{|\mathbf{l}|-1} \|\mathbf{W}\| \tag{2.19}$$

This inequality is obtained by induction on $\mathbf{l}$, with respect to $\prec$. For $|\mathbf{l}| = 1$, (2.19) is trivial. Assume (2.19) holds for all $\mathbf{l} \prec \mathbf{l}_1$; without loss of generality we may consider that $\mathbf{l}_1 = \mathbf{l}_0 + \mathbf{e}_1$. We have:

$$\begin{aligned}
&\|(\mathbf{V}+\mathbf{W})^{*\mathbf{l}_1} - \mathbf{V}^{*\mathbf{l}_1}\| = \|(\mathbf{V}+\mathbf{W})^{*\mathbf{l}_0} * (\mathbf{V}_1 + \mathbf{W}_1) - \mathbf{V}^{*\mathbf{l}_1}\| \\
&= \|(\mathbf{V}^{*\mathbf{l}_0} + \mathbf{W}_{\mathbf{l}_0}) * (V_1 + W_1) - \mathbf{V}^{*\mathbf{l}_1}\| = \|\mathbf{V}^{*\mathbf{l}_0} * W_1 + \mathbf{W}_{\mathbf{l}_0} * V_1 + \mathbf{W}_{\mathbf{l}_0} * W_1\| \\
&\leq \|\mathbf{V}\|^{|\mathbf{l}_0|}\|\mathbf{W}\| + \|\mathbf{W}_{\mathbf{l}_0}\|\|\mathbf{V}\| + \|\mathbf{W}_{\mathbf{l}_0}\|\|\mathbf{W}\| \\
&\leq \|\mathbf{W}\|\left(\|\mathbf{V}\|^{|\mathbf{l}_0|} + |\mathbf{l}_0|(\|\mathbf{V}\| + \|\mathbf{W}\|)^{|\mathbf{l}_0|}\right) \leq \|\mathbf{W}\|(|\mathbf{l}_0|+1)(\|\mathbf{V}\| + \|\mathbf{W}\|)^{|\mathbf{l}_0|}
\end{aligned}$$

and (2.19) is proven.

For the sum in $\mathbf{M}$ to converge in $\|\|_\nu$ it suffices to choose ν such that $\|\mathbf{V}\|_\nu < \kappa^{-1}$. Let $\epsilon < \kappa^{-1}$ and $\mathbf{V}, \mathbf{W}$ be such that $\|\mathbf{V}\| + \|\mathbf{W}\| < \epsilon$. We have

$$\|\mathbf{M}(\mathbf{V}+\mathbf{W}) - \mathbf{M}(\mathbf{V})\|_\nu \leq \left(nC_{1,\nu} + C_\nu \sum_{|\mathbf{l}|\geq \mathbf{2}} |\mathbf{l}|(\kappa\epsilon)^{|\mathbf{l}|-1}\right)\|\mathbf{W}\|_\nu$$

$$\leq \left(nC_{1,\nu} + n2^n C_\nu \kappa\epsilon + nC_\nu \frac{(2-\kappa\epsilon)(\kappa\epsilon)^{1+2(n-1)}}{(1-\kappa\epsilon)^{n+1}}\right)\|\mathbf{W}\|_\nu = K_\nu \|\mathbf{W}\|_\nu \tag{2.20}$$

where we separated out the terms with $|\mathbf{l}| = 2, l_i = 0$ or 1 and for the rest of the terms used the identity $\sum_{l_i\geq 2} |\mathbf{l}| e^{-\gamma|\mathbf{l}|} = -\frac{\mathrm{d}}{\mathrm{d}\gamma}\left(\sum_{l\geq 2} e^{-\gamma l}\right)^n$. We see that, in fact, $\lim_{\nu\to\infty} K_\nu = 0$. $\square$

2.1.5 Two lemmas on analytic structure

LEMMA 11 *Let f be analytic in the unit disc cut along the positive axis and let $0 < g(x) \in C^1[0,1]$. Assume that $\lim_{\epsilon\downarrow 0} f(x \pm i\epsilon g(x)) = f^{\pm}(x)$ in $L^1[0,1]$ and*

$$f^{+}(x) - f^{-}(x) = f_\delta(x) = x^r A(x) \tag{2.21}$$

with $\Re(r) > -1$, where $A(\xi)$ extends to an analytic function for $|\xi| < a \leq 1$. Then there exists a function B analytic in $|\xi| < a$ so that

$$\begin{aligned} f(\xi) &= \frac{1}{1-\mathrm{e}^{2\pi i r}}\xi^r A(\xi) + B(\xi) \quad (r \notin \mathbb{N}) \\ f(\xi) &= \frac{i}{2\pi}\ln(\xi)\xi^r A(\xi) + B(\xi) \quad (r \in \mathbb{N}) \end{aligned} \tag{2.22}$$

If $f^+(x) - f^-(x)$ is a linear combination $\sum_{i=1}^N x^{r_i} A_i(x)$ (under the same assumptions on r_i and A_i), then f is given by the corresponding superposition of terms of the form (2.22).

The proof is given in § 2.3.4.

In the following, $\gamma : \mathbb{R}^+ \mapsto \mathbb{C}$ will denote smooth curves in $\mathcal{R}'_1$, γ_k denotes a curve that crosses through the interval $(k, k+1)$, $\gamma_\epsilon = \Re(\gamma) + i\epsilon\Im(\gamma)$ (cf.§ 1.2). Let $a, b \in (0, \pi/2)$ and $\mathcal{S}_0 = \{p : \arg(p) \in (\psi_-, 0) \cup (0, \psi_+)\}$. Let f be a function analytic in $\mathcal{R}'_1$ so that $f \circ \gamma_\epsilon \in \mathcal{D}'_{m,\nu}$ has limits in $\mathcal{D}'_{m,\nu}$ as $\epsilon \downarrow 0$. We denote the space of such functions by $\mathcal{D}'_{m,\nu}(\mathcal{R}_1)$. Let $f_0^{\pm} = f^{\pm}$, $F_j = \mathcal{P}^{(mj)} f$, and for $j > 0$

$$f_j^+(z-j) = F_j^{-j+}(z) - F_j^{-(j-1)+}(z);\ \ f_j^-(z-j) = F_j^{+j-}(z) - F_j^{+(j-1)-}(z) \tag{2.23}$$

By construction the f_j^+ are in $L^1[0, 1-\epsilon)$, analytic in a sectorial neighborhood of $z = 0$ and can be extended analytically for $\Im(z) > 0, \Re(z) > 0$ (this last property motivates the choice $+$ for superscript, while the right shift is chosen in view of our application). Also by construction $f_j^{\pm}(z) = 0$ for $z < 0$; it is convenient to extend $f_j^{\pm}$ by zero throughout $\Re(z) < 0$. We have the "telescopic" decomposition

$$f^{\mp j\,\pm}(z) = \sum_{i=0}^{j} (f_i^{\pm})^{(mi)}(z-i) \tag{2.24}$$

Relation (2.24) holds in $\mathcal{D}'$ along the real axis, *and* as an equality of analytic functions for $\Re(z) > j, \Im(z) > 0$. For instance, for $f = (z-2)^{-1} \in \mathcal{D}'_{1\nu}$ we have $f_1^+ = 0$, $f_2^+ = -2\pi i z$ for $\Re(z) > 0$, and $f_2^+ = 0$ otherwise.

Conversely, a decomposition of the form (2.24) together with analyticity in $\mathcal{S}_0$ implies analyticity in $\mathcal{R}'_1$. More precisely, assume that $f(t\exp(i\phi)) \in$

$\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ for $\phi \in (\psi_-, 0) \cup (0, \psi_+)$ and that $\lim_{\phi\downarrow 0} f(\cdot\, \exp(\pm i\phi) = f^{\pm}$ in $\mathcal{D}'_{m,\nu}$. Assume in addition that there exists the decomposition

$$f^{\pm} = \sum_{k=0}^{\infty} \left[f_k^{\mp}(p-k)\chi_{[k,\infty]}\right]^{(mk)} \tag{2.25}$$

where for each k, $f_k^{\mp} \in \mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ (note: the $f_k^{\mp}\chi$ are uniquely determined, cf. Remark 43). Assume in addition that $f_k^{\mp}$ extend analytically to $\mathcal{S}_0^{\mp}$ in the following sense: there exist $g_k^{\mp}$ analytic in $\mathcal{S}_0^{\mp}$, with $g_k^{\mp}(\mp t\exp(i\phi)) \in \mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ and such that $\lim_{\phi\downarrow 0} g_k^{\mp}(\mp t \exp(i\phi)) = f_k^{\mp}$ in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$.

LEMMA 12 *(i) Under the above conditions, f extends analytically to $\mathcal{R}'_1$.*
(ii) If for small argument $f_k \in \mathcal{T}_\beta$ then

$$f^{\pm}(p+k) = \frac{1}{1-e^{2\pi i\beta}}(f_k^{\mp}(p))^{(mk)} + a(p) \ \text{ or } \ \frac{i}{2\pi}(f_k^{\mp}(p)\ln p)^{(mk)} + a(p) \tag{2.26}$$

according to whether $\beta \notin \mathbb{Z}$ or $\beta \in \mathbb{Z}$ respectively, where a is analytic at zero. As in Lemma 11, if $f_k = \sum_{i\le j} f_{ki}$ with $f_{ki} \in \mathcal{T}_{\beta_i}$ then $f^{\pm}(p+k)$ is the corresponding superposition of terms of the form (2.26).

The proof is given in § 2.3.4.

2.1.6 Convolutions, analyticity and averaging

Define $\mathcal{F}(\mathcal{R}'_1)$ as the functions in $\mathcal{D}'_{m,\nu}(\mathcal{R}'_1)$ such that, in the decomposition (2.24) $\|(f_j(p-j))^{(mj)}\| \le K_f(\nu)^j$ where $\lim_{\nu\to\infty} K_f(\nu) = 0$.

If γ is a straight line in $\mathcal{R}'_1$ then $AC_\gamma(f*g) = AC_\gamma(f)_\gamma * AC_\gamma(g)$ (if γ is *not* equivalent to a straight line, this equality is generally *false* [13]). Convolution does however commute with suitable averages of analytic continuations, as Proposition 13 below shows. In view of symmetry, we only need to look at the properties of the + decomposition.

For $\alpha \in \mathbb{C}$, consider the operator $\mathcal{A}_\alpha : \mathcal{F}(\mathcal{R}'_1) \mapsto \mathcal{F}_m(\mathbb{R}^+)$ given by

$$\mathcal{A}_\alpha(f) := f^{[\alpha]}(p) = \sum_{i=0}^{\infty} \alpha^i \left(f_i^+(p-j)\right)^{(mi)} \tag{2.27}$$

In our assumptions, convergence is ensured in $\mathcal{D}'_{m,\nu}$ for large enough ν. An important case is the balanced average, $\alpha = 1/2$. The operator $\mathcal{A}_{\frac{1}{2}}$ is similar to Écalle's medianization, and is designed to substitute for analytic continuation along the singularity line $\mathbb{R}^+$ in a way compatible with the $*-$algebra structure. As mentioned before, it can be shown that under our assumptions on (1.3), only for the choice $\alpha = 1/2$ is the difference between the $f = \mathcal{L}\mathcal{A}_\alpha F$ and the optimally truncated asymptotic series of f always of the order of magnitude of the least term of the series [15]. Borel summability techniques and hyperasymptotic methods [22], [23] give, whenever they both apply, the same association between transseries and actual functions.

Proposition 13 *i) If $f,g\in\mathcal{F}(\mathcal{R}'_1)$ then $f*g$ defined for small argument by (1.8) extends to a function in $\mathcal{F}(\mathcal{R}'_1)$. We have*

$$(f*g)^{\pm}_j=\sum_{s=0}^{j} f^{\pm}_s * g^{\pm}_{j-s} \tag{2.28}$$

and $K_{f*g}(\nu)\le K_f(\nu)+K_g(\nu)$. If h is analytic and bounded in the right half plane and $f\in\mathcal{F}(\mathcal{R}'_1)$ then $hf\in\mathcal{F}(\mathcal{R}'_1)$.

ii) If h is analytic in $\mathcal{R}'_1\cup\mathbb{R}^+$ and $f,g\in\mathcal{F}(\mathcal{R}'_1)$, $a,b\in\mathbb{C}$, then

$$\begin{aligned}\mathcal{A}_\alpha(af+bg) &= a\mathcal{A}_\alpha(f)+b\mathcal{A}_\alpha(g)\\ \mathcal{A}_\alpha(hf) &= h\mathcal{A}_\alpha(f)\\ \mathcal{A}_\alpha(1) &= 1\\ \mathcal{A}_\alpha(f*g) &= \mathcal{A}_\alpha(f)*\mathcal{A}_\alpha(g)\end{aligned} \tag{2.29}$$

If $\mathbf{M}$ satisfies the hypothesis of Lemma 8 and in addition $\mathbf{M_l}(\mathcal{F}(\mathcal{R}'_1))\subset\mathcal{F}(\mathcal{R}'_1)$ and $\mathcal{A}_\alpha\mathbf{M_l}=\mathbf{M_l}\mathcal{A}_\alpha$ then

$$\mathcal{A}_\alpha\mathbf{M}=\mathbf{M}\mathcal{A}_\alpha \tag{2.30}$$

In particular, if $\mathbf{Y}$ is a fixed point of $\mathcal{M}$ then so is $\mathcal{A}_\alpha\mathbf{Y}$. An example is the case $\mathbf{M_l}(\mathbf{Y})=\mathbf{G_l}*\mathbf{Y}^{*\mathbf{l}}$ with $\mathbf{G_l}$ analytic in $\mathcal{R}'_1\cup\mathbb{R}^+$ and such that for some κ and all $\mathbf{l}$ we have $|\mathbf{G_l}(p)|\le\exp(\kappa p)$.

The proof and further details are given in Appendix 2.3.4.

Let now $\mathcal{F}_r(\mathcal{R}'_1)\subset\mathcal{F}(\mathcal{R}'_1)$ consist in functions f whose only singularities are regular, in the sense that the elements f_j (cf. (2.23)) are of the form $(\sum_{i=1}^{N_j} p^{a_{i,j}}A_{i,j})^{(mj)}$ where $A_{i,j}$ are analytic near $p=0$.

Remark 14 *$\mathcal{F}_r(\mathcal{R}'_1)$ is stable with respect to convolution.*

By construction, (Proposition 13) and (2.28), for small p, $(f_1*f_2)_j$ is a sum of terms of the form

$$p^{a_1}A_1*p^{a_2}A_2$$

and the proof follows without any difficulty from (2.6).

2.2 Main proofs

PROPOSITION 15 *i) For any $\kappa > \max\{x_0, y_0^{-1}\}$, cf. (1.11), there is a constant $K > 0$ such that for all $\mathbf{l} \succ 0$*

$$\sup_{p\in\mathbb{C}} \mathrm{e}^{-\kappa|p|}|\mathbf{G}_{\mathbf{l}}(p)| \le K\kappa^{|\mathbf{l}|} \tag{2.31}$$

(cf. (1.13)).

ii) Let $\mathcal{F}$ be one of the focusing algebras in § 2.1.2 and $\mathbf{Y} \in \mathcal{F}$. Let

$$\mathbf{D}_{\mathbf{j}} = \sum_{\mathbf{l}\ge\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \left[\mathbf{G}_{\mathbf{l}} * \mathbf{Y}^{*(\mathbf{l}-\mathbf{j})} + \mathbf{g}_{0,\mathbf{l}} * \mathbf{Y}^{*(\mathbf{l}-\mathbf{j})}\right]$$

Then for large ν and some $\kappa_1 > 0$, $\|\mathbf{D}_{\mathbf{j}}\| \le \kappa_1^{|\mathbf{j}|}$ while for $|\mathbf{j}| = 1$, $\|\mathbf{D}_{\mathbf{j}}\| = o(\nu^{-M})$.

Proof.

(i) From the analyticity of $\mathbf{g}$ it follows that

$$|\mathbf{g}_{m,\mathbf{l}}| < \text{const. } \kappa^{m+|\mathbf{l}|} \tag{2.32}$$

where the constant is independent on m and $\mathbf{l}$. Then, by (1.12),

$$|\mathbf{G}_{\mathbf{l}}(p)| < \text{const. } \kappa^{|\mathbf{l}|+1}\frac{\mathrm{e}^{\kappa|p|}-1}{\kappa|p|} < \text{const. } \kappa^{|\mathbf{l}|+1}\mathrm{e}^{\kappa|p|} \tag{2.33}$$

The last claim is a direct consequence of (n5).

(ii) Note first that $\sum_{|\mathbf{l}|=l} 1 \le l^n \le 2^{nl}$ (since $l_i \le l$). Also, $\binom{l_i}{j_i} \le 2^{l_i}$ so that $\binom{\mathbf{l}}{\mathbf{j}} \le 2^{n|\mathbf{l}|}$. By (n5) we have $\mathbf{g}_{0,\mathbf{l}} = 0$ if $|\mathbf{l}| \le 1$. Choosing $\delta < 2^{2nl+1}\kappa$ and ν such that $\|\mathbf{Y}\| \le \delta$, $\|\mathbf{D}_{\mathbf{j}}\|$ is estimated by

$$\sum_{l\ge j} 2^{2nl}\delta^{l-j}\kappa_l^l$$

where $j = |\mathbf{j}|, \kappa_l = \|\mathbf{G}_{\mathbf{l}}\| + |\mathbf{g}_{0,\mathbf{l}}| \le 2\kappa$ if $l > 1$ and $\kappa_l = o(\nu^{-M})$ for $l = 1$, by (i), and the result follows.

□

Without loss of generality, we analyze (1.13) in a neighborhood of d_1, the Stokes line corresponding to $\lambda_1 = 1$. (For the equations (1.16) we will need, in addition, to study a direction where $p - \lambda_i + \mathbf{k}\cdot\boldsymbol{\lambda} = 0$.)
Let ϵ and c_0 be small and positive, $\mathcal{V} = \{p : |p| < 1\}$,

$$\mathcal{S}_c = \{p : \arg(p) \in [\psi_n - 2\pi + c, -c] \cup [c, \psi_2 - c]\} \tag{2.34}$$

$\mathcal{S}_0 = \cup_{0<c<c_0}\mathcal{S}_c$, $\mathcal{S}_c^{\pm} = \mathcal{S}_c \cap \{p : \pm\arg(p) > 0\}$, and let $\mathcal{S}_0', \mathcal{S}_c', \mathcal{S}_c^{\pm\prime}$ be defined correspondingly, with ψ_- and ψ_+ replacing $\psi_n - 2\pi$ and ψ_2, respectively.

LEMMA 16 *i) For any ray $d \subset \mathcal{S}_0$, there is a unique solution of (1.13) in $\mathcal{D}'_m(d,l)$ for any $m \in \mathbb{N}$, $l \in (0,1)$. This solution, $\mathbf{Y}_0$, is in fact analytic in $\mathcal{S}_0 \cup \mathcal{V}$ and, for large enough $\nu(d)$, $\mathbf{Y}_0 \in L^1_\nu(d)$.*

ii) The function $\mathbf{Y}_0(\cdot\,\mathrm{e}^{\mathrm{i}\phi})$ is continuous in $\phi \in (\psi_n, 0]$ and (separately in) $\phi \in [0,\psi_2)$ in the $\mathcal{D}'_{\nu,m}(\mathbb{R}^+,l)$ topology and $\sup_{\phi\in[\psi_1,\psi_2]} \|\mathbf{Y}_0(p\mathrm{e}^{\mathrm{i}\phi})\|_{m,\nu} \to 0$ as $\nu \to \infty$.

iii) The description (1.17) holds for $j = 1, l = 1$.

iv) For $a > 1$, there is a one-parameter family of solutions of (1.13) in $\mathcal{D}'_m(0,a)$.

Note: the hyperfunctions $\mathbf{Y}_0(p\mathrm{e}^{\pm 0i}) = \mathbf{Y}_0^{\pm}$ are different, in general.

COROLLARY 17 *For $k \in \mathbb{N} \cup \{0\}$, the function $\mathcal{P}^{mk}\mathbf{Y}_0(p\mathrm{e}^{\mathrm{i}\phi})$ is continuous in $S_k^- := \{p : 0 \le |p| < k+1; \arg(p) \in (\psi_n, 0]\}$ and in $S_k^+ := \{p : 0 \le |p| < k+1; \arg(p) \in [0,\psi_2)\}$ (and, of course, analytic in S_0).*

Proof of Lemma 16.

We write (1.13) in the form:

$$\mathbf{Y} = \left(\hat{\Lambda} - p\right)^{-1}\left(\mathbf{F}_0 - \hat{B}\mathcal{P}\mathbf{Y} + \mathcal{N}(\mathbf{Y})\right) = \mathcal{M}(\mathbf{Y}) \tag{2.35}$$

Let d_K be an initial segment of d of length $K < \infty$. As the matrix $\hat{\Lambda} - p$ is invertible in $\mathcal{S}_c$, it is easy to see that the operator $\mathcal{M}$ in (2.35) satisfies the conditions of Lemma 9, in the spaces $L^1_\nu(d)$, $\mathcal{A}_{z,l}(\mathcal{S})$ $(l \le M)$ (cf. Examples (1) through (4) in §2.1.2 and Remark 15). The conditions are in addition also satisfied in $\mathcal{D}'_{m,\nu}(d)$; due to the special structure of this space, the proof the boundedness of the operator $U = (\Lambda - p)^{-1}$ is more delicate, and is given in Lemma 42.

Thus, $\mathcal{M}$ has a unique fixed point in each of these spaces. The obvious inclusions between these spaces complete the proof of part (i). For the rest of Lemma 16 we need more results.

PROPOSITION 18 *The properties stated in Lemma 16 hold in $\mathcal{S}_0 \cap \{p : |p| < 1+\epsilon\}$*

The proof is given in § 2.2.2. □

PROPOSITION 19 *Let $\mathbf{W}_0$ be a solution of (1.13) in $\mathcal{D}'_m(0,a)$ with $a > 1$. For $b \ge a$ there exists a unique solution of (1.13) in $\mathcal{D}'_m(0,b)$, which agrees with $\mathbf{W}_0$ on $\mathcal{D}(0,a)$.*

We use the decomposition $\mathcal{D}'_m(0,b) = \mathcal{D}'_m(0,a) \bigoplus \mathcal{D}'_m(a,b)$, $(a < b \le \infty)$ We identify $\mathbf{W}_0$ with an element of $\mathcal{D}'_m(0,b)$ by extending it with zero and define $\mathcal{M}_1$ on $\mathcal{F}_m(a,b) = \cup_{\nu>\nu_0}\mathcal{D}'_{m,\nu}(a,b)$ by

$$\mathcal{M}_1(\mathbf{W}_1) = \mathcal{M}(\mathbf{W}_0 + \mathbf{W}_1) - \mathcal{M}(\mathbf{W}_0) \tag{2.36}$$

and (2.35) becomes

$$\mathbf{W}_1 = \mathcal{M}_1(\mathbf{W}_1) \tag{2.37}$$

By Lemma 8, $\mathcal{M}$ is eventually contractive and by (2.36), clearly, so is $\mathcal{M}_1$. By Lemma 8 then, $\mathcal{M}_1$ has a unique fixed point in $\mathcal{F}_m(a,b)$ ($b \le \infty$). In view of the inclusions between $\mathcal{F}_m(a,\infty)$ and $\mathcal{F}_m(a,b)$, the proof is complete. $\square$.

Proof of Lemma 16 (ii)

Let $\phi \in [0,\epsilon]$ with ϵ small. We regard the space $\mathcal{F}_m$ as fixed and $\mathcal{M}_1$ as being dependent on ϕ through p and $*_\phi$ (cf. Remark 7). Firstly, $\mathbf{W}_0(\phi)$ is continuous in $\phi \in [0,\epsilon]$ and $\|\mathbf{W}_0(p\mathrm{e}^{\mathrm{i}\phi})\|_\nu = O(1/\nu)$ uniformly in ϕ as follows from Proposition 18. Then the infinite sum in the definition of $\mathcal{M}_1$ is uniformly convergent in $\mathcal{D}'_{m,\nu}(a,\infty)$ for ν large enough by (2.31). The operator $U = (p\mathrm{e}^{\mathrm{i}\phi} - \hat{\Lambda})^{-1} = \mathrm{e}^{-\mathrm{i}\phi}(p - \mathrm{e}^{-\mathrm{i}\phi}\hat{\Lambda})^{-1}$ is strongly continuous in ϕ in $\mathcal{F}_m(a,\infty)$, $a > 1$, cf. Lemma 42. By Remark 7, $\mathcal{M}_1$ is ϕ-continuous and Lemma 8 applies. $\square$

We now study the convolution equations associated to the higher terms in the transseries, (1.16).

LEMMA 20 *i) Given the vector of constants $\mathbf{C} \in \mathbb{C}^{n_1}$ and and in addition given $\mathbf{Y}_0$ for $d = \mathbb{R}^+$, there is a unique solution of (1.16) in $\mathcal{F}_m$ with the (singular) initial condition*

$$\mathbf{Y}_{\mathbf{e}_j}(p) = C_j \Gamma(\beta'_j)^{-1} p^{\beta'_j}(\mathbf{e}_j + o(1)) \qquad (p \to 0,\ j = 1,2,\ldots,n_1) \tag{2.38}$$

The general solution of (2.156), (1.16) is

$$\mathbf{C}^{\mathbf{k}}\mathbf{Y}_{\mathbf{k}},\ \mathbf{C} \in \mathbb{C}^{n_1} \tag{2.39}$$

where $\mathbf{Y}_{\mathbf{k}}$ is the solution for $\mathbf{C} = (1,1,\ldots,1)$.

ii) In a neighborhood of $p = 0$ we have

$$\mathbf{Y}_{\mathbf{k}}(p) = p^{\mathbf{k}\boldsymbol{\beta}'-1}\mathbf{A}_{\mathbf{k}}(p)$$

with $\mathbf{A}_{\mathbf{k}}$ analytic near the origin.

iii) The functions $\mathbf{Y}_{\mathbf{k}}$, $\mathbf{k} \succeq 0$ are analytic in $\mathcal{R}'_1$ and $\mathbf{Y}_{\mathbf{k}}(x\mathrm{e}^{\mathrm{i}\phi})$ are continuous in ϕ with respect to the $\mathcal{D}'_{m,\nu}$ topology for $\phi \in (\psi_-, 0]$ and for $\phi \in [0, \psi_+)$.

iv) Each $\mathbf{Y}_{\mathbf{k}}$ is in $\mathcal{F}(\mathcal{R}'_1)$ (cf. § 2.1.6). Furthermore, there is a constant K and a function $\delta(\nu)$ such that $\lim_{\nu\to\infty}\delta(\nu) = 0$ and in the decomposition (2.24) of $\mathbf{Y}_{\mathbf{k}}$ we have $\mathbf{Y}_{\mathbf{k};j} \in \mathcal{T}_{(\mathbf{k}+j\mathbf{e}_1)\boldsymbol{\beta}'-1}$ and

$$\sup_{\phi,\mathbf{k}} \delta(\nu)^{-|\mathbf{k}|}\|\mathbf{Y}_{\mathbf{k}}\|_{\mathcal{D}'_{m,\nu}(\mathrm{e}^{\mathrm{i}\phi}\mathbb{R}^+)} < K$$

$$\sup_{\phi,\mathbf{k},j} \delta(\nu)^{-|\mathbf{k}|+j}\|\mathbf{Y}_{\mathbf{k};j}\|_{\mathcal{D}'_{m,\nu}(\mathrm{e}^{\mathrm{i}\phi}\mathbb{R}^+)} < K \tag{2.40}$$

where ϕ runs in (ψ_-,ψ_+).

v) The functions $\mathbf{Y}_{\mathbf{k}}(\cdot\,\mathrm{e}^{\mathrm{i}\phi})$, $\phi\in(\psi_-,\psi_+)$, are simultaneously Laplace transformable in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ and their Laplace transforms are solutions of (1.14). The expression

$$\mathbf{y}^{\pm} = \mathcal{L}\mathbf{Y}_0^{\pm} + \sum_{\mathbf{k}\succ 0} x^{\mathbf{m}\cdot\mathbf{k}}\mathbf{C}^{\mathbf{k}}\mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{\pm} \tag{2.41}$$

is uniformly convergent for large enough $x(\mathbf{C})$ in some open sector. In addition, (2.41) is a solution of (1.3) *and $\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{\pm}\sim\tilde{\mathbf{y}}_{\mathbf{k}}$ for large x in the half plane $\Re(xp)>0$.*

Without loss of generality, we analyze (1.16) in $\mathcal{T}_{\{\cdot\}}(\mathcal{S}_c^{+\prime})$, $\mathcal{D}'_{m,\nu}(d)$, with $d\in\mathcal{S}_c^{+\prime}$, and in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$. We denote all the corresponding norms by $\|\|_\nu$.

Remark 21 *Assume that for $\mathbf{k}'\prec\mathbf{k}$ we have $\mathbf{Y}_{\mathbf{k}'}\in\mathcal{T}_{\mathbf{k}'\boldsymbol{\beta}'-1}$. Then, in (2.162), we have $\mathbf{T}_{\mathbf{k}}(\mathbf{Y}_0,\{\mathbf{Y}_{\mathbf{k}'}\})\in\mathcal{T}_{\mathbf{k}\boldsymbol{\beta}'-1}$.*

This follows immediately from Equations (2.6) and (2.7) and from the homogeneity of $\mathcal{T}$ implicit in the sum $\sum_{(\mathbf{i}_{mp};\mathbf{k})}$ (the notation is explained after (2.158)). □

For $|\mathbf{k}|>1$ we take $\mathbf{W}_{\mathbf{k}}:=\mathbf{Y}_{\mathbf{k}}$ and $\mathbf{R}_{\mathbf{k}}:=\mathbf{T}_{\mathbf{k}}$ and write (1.16) as

$$(1+J_{\mathbf{k}})\mathbf{W}_{\mathbf{k}} = \hat{Q}_{\mathbf{k}}^{-1}\mathbf{R}_{\mathbf{k}} \tag{2.42}$$

with $\hat{Q}_{\mathbf{k}}:=(-\hat{\Lambda}+p+\mathbf{k}\cdot\boldsymbol{\lambda})$ (notice that for $|\mathbf{k}|>1$ and $p\in\mathcal{S}_0'$ we have $\det\hat{Q}_{\mathbf{k}}(p)\neq 0$).

$$(J_{\mathbf{k}}\mathbf{W})(p) := \hat{Q}_{\mathbf{k}}^{-1}\left(\left(\hat{B}+\mathbf{m}\cdot\mathbf{k}\right)\int_0^p \mathbf{W}(s)\mathrm{d}s - \sum_{j=1}^{n}\int_0^p W_j(s)\mathbf{D}_j(p-s)\mathrm{d}s\right) \tag{2.43}$$

The case $|\mathbf{k}|=1$ is special in that $p=0$ is a singularity. The corresponding statements of Lemma 20 for $|p|<\epsilon$ with ϵ small are proven in Proposition 32: let $\mathbf{W}_{\mathbf{k}}^0$ be the functions provided there. For the analytic part of Lemma 20 we need to show that $\mathbf{W}_{\mathbf{k}}^0$ extend analytically to solutions of (1.16). To unify the treatment we derive equations of the form (2.42) for these continuations. Let $\delta\in\mathcal{S}_c^{+\prime},|\delta|<\epsilon$. Using Proposition 32 and standard analyticity arguments, it suffices to show that $\mathbf{W}_{\mathbf{k}}^0$ extend analytically in any sector $\subset\mathcal{S}_c^{+\prime}$ centered at δ, and that the corresponding convolution equation is satisfied along the ray $d_\delta\ni\delta$. We let $\mathbf{a}=\mathbf{W}_{\mathbf{k}}^0(\delta)$, and with $\mathbf{W}_{\mathbf{k}}^1=\mathbf{W}_{\mathbf{k}}^0$ for $|p|<|\delta|$ along d_δ and zero otherwise, we write $\mathbf{Y_k}(p) = \mathbf{W^1_k}(p) + \mathbf{a} + \mathbf{W_k}(p-\delta)$. For $p \in d_\delta$ we find that $\mathbf{W_k}$ must satisfy (2.42) where $\hat{Q}_\mathbf{k} := (-\hat{\Lambda} + p + \mathbf{k}\cdot\boldsymbol{\lambda} + \delta)$ and $\mathbf{R_k}(p)$ is given by

$$\left(\mathbf{m}\cdot\mathbf{k} + \hat{B}\right)\left(\int_0^\delta \mathbf{W^0_k}(s)\mathrm{d}s - \mathbf{a}p\right) + \sum_{j=1}^n \int_0^\delta (\mathbf{W^0_k})_j(s)\mathbf{D}_{\mathbf{e}_j}(p-s)\mathrm{d}s - \mathbf{a}\mathbf{D}_{\mathbf{e}_j}(p)$$

$\mathbf{R_k}(p)$ is manifestly analytic in $\mathcal{S}^{+\prime}$. Since $\mathbf{W_k}(p) = \mathbf{W^0_k}(p+\delta) - \mathbf{a}$ is already a solution of (2.42) for small p, and in this case the left side of (2.42) vanishes for $p = 0$, we have $\mathbf{R_k} \in \mathcal{T}_1$, for $|\mathbf{k}| = 1$.

Combined with Remark 21 and induction on $\mathbf{k}$, the following result completes the proof of Lemma 20, parts (i) and (ii).

PROPOSITION 22 *i) For large ν and constants K_1 and $K_2(\nu)$ independent of $\mathbf{k}$, with $K_2(\nu) = O(\nu^{-1})$ we have $\|Q_\mathbf{k}^{-1}\| \le \frac{K_1}{|\mathbf{k}|}$ and*

$$\|J_\mathbf{k}\| \le K_2(\nu) \tag{2.44}$$

ii) For large ν, the operators $(1 + J_\mathbf{k})$ defined in $\mathcal{D}'_{m,\nu}$, and also in $\mathcal{T}_{\mathbf{k}\boldsymbol{\beta}'-1}$ for $|\mathbf{k}| > 1$ and in $\mathcal{T}_1$ for $|\mathbf{k}| = 1$ are simultaneously invertible. Given $\mathbf{Y}_0$ and $\mathbf{C}$, the $\mathbf{W_k}$, $|\mathbf{k}| \ge 1$ are uniquely determined. For any $\delta > 0$ there is a large enough ν, so that

$$\|\mathbf{W_k}\| \le \delta^{|\mathbf{k}|},\ \ k = 0, 1, .. \tag{2.45}$$

(in the $\mathcal{D}'_{m,\nu}$ topology, (2.45) hold uniformly in $\phi \in [\psi_- + \epsilon, 0]$ and $\phi \in [0, \psi_+ - \epsilon]$ for any small $\epsilon > 0$).

□

Proof.

(i) For $\mathcal{T}_{\mathbf{k}\boldsymbol{\beta}'-1}$, this follows immediately from Remark 7, and the constants will depend on the parameter c in $\mathcal{S}^+_c \cap \mathcal{S}_\mathbb{R}$. Given $\mathbf{Y}_0$, the estimates (2.44) are also true in $\mathcal{D}'_{m,\nu;\phi}$ this time *uniformly* in ϕ, down to $\phi = 0$. The proof of (2.44) in this case is given in Appendix 2.2.2, in Lemma 42, from which the continuity of $J_\mathbf{k}$ in ϕ also follows.

(ii) From (2.42) and (i) we get, for some K and $j \ge 1$ $\|\mathbf{W_k}\| \le K\|\mathbf{R_k}\|$. We first show inductively that the $\mathbf{W_k}$ are bounded. Choosing a suitably large $\nu(\epsilon)$ we can make $\max_{|\mathbf{k}|\le 1}\|\mathbf{W_k}\|_\nu \le \epsilon$ for any positive ϵ (uniformly in ϕ). We show by induction that $\|\mathbf{W_k}\|_\nu \le \epsilon$ for all k. Using (2.45), (1.16), (2.160), Proposition 15 and the crude estimate $\binom{a}{b} \le 2^a$ we get

$$\|\mathbf{W_k}\|_\nu \le K\|\mathbf{R_k}\|_\nu \le \sum_{\mathbf{l}\le\mathbf{k}} \kappa_1^{|\mathbf{l}|}\epsilon^{|\mathbf{k}|} \sum_{(\mathbf{i}_{mp})} 1 \le \epsilon^{|\mathbf{k}|}\sum_{s=0}^{|\mathbf{k}|}\kappa_1^s 2^{n_1(|\mathbf{k}|+s)}2^{s+n_1} \le (C_1\epsilon)^{|\mathbf{k}|} \tag{2.46}$$

where C_1 does not depend on $\epsilon, \mathbf{k}$. Choosing ϵ so that $\epsilon < C_1^{-2}$ we have, for $|\mathbf{k}| \geq 2$ $(C_1\epsilon)^{|\mathbf{k}|} < \epsilon$ completing the induction step . But as we now know that $\|\mathbf{W}_\mathbf{k}\|_\nu \leq \epsilon$, the same inequalities (2.46) show that in fact $\|\mathbf{W}_\mathbf{k}\|_\nu \leq (C_1\epsilon)^{|\mathbf{k}|}$. Choosing ϵ small enough, the first part of Proposition 22, (ii) follows. □

PROPOSITION 23 *i) Let $\mathbf{Y}_0$ be given by Lemma 16. We have, in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$*

$$\mathbf{Y}_0^{\pm} = \mathbf{Y}_0^{\mp} + \sum_{k=1}^{\infty} (\pm S_1)^k \left(\mathbf{Y}^{\mp}_{k\mathbf{e}_1}(p-k)\mathcal{X}_{[k,\infty]}\right)^{(mk)} \tag{2.47}$$

(cf. (1.18)) and $\mathbf{Y}_0$ is analytic in $\mathcal{R}_1$.

ii) The general solution of (1.13) in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ is

$$\mathbf{Y}_0^{+} + \sum_{k=1}^{\infty} C^k \left(\mathbf{Y}^{+}_{k\mathbf{e}_1}(p-k)\mathcal{X}_{[k,\infty]}\right)^{(mk)} \tag{2.48}$$

with arbitrary C (a similar statement holds with $\mathbf{Y}_0^-$ replacing $\mathbf{Y}_0^+$).

Proof.

We start with (ii). Assuming first (2.48) is indeed a solution of (1.13), to see that there are no others, it suffices by Proposition 19 to check that (2.48) is the general solution on $[0, 1+\epsilon)$. The latter part is immediate from Remark 31 and Proposition 32 below. Now $\mathbf{Y}_0^+$ is a solution of (1.13), by Lemma 16 (ii); the sum (2.48) is convergent in $\mathcal{D}'_{m,\nu}$ by (2.46). Since $\left(\mathbf{Y}_{k\mathbf{e}_1}(p-k)\mathcal{X}_{[k,\infty]}\right)^{(mk)} \in \mathcal{D}'_{m,\nu}(k,\infty)$, to show that (2.48) is a solution on $\mathbb{R}^+$, we check inductively on j that $\mathbf{H}_j = \sum_{k=0}^{j} \left(\mathbf{Y}_{k\mathbf{e}_1}(p-k)\mathcal{X}_{[k,\infty]}\right)^{(mk)}$ solves (1.13) in $\mathcal{D}'[0, j+1)$. Assuming this for $j' < j$ and looking for a solution on $[0, j+1)$ in the form $\mathbf{H}_{j+1} = \mathbf{H}_j + \left(\tilde{\mathbf{Y}}_j(p-j)\mathcal{X}_{[j,\infty]}\right)^{(mj)}$ we obtain, by a straightforward calculation, using the induction hypothesis and (2.166)

$$\begin{aligned}(\hat{\Lambda} - p)\tilde{\mathbf{Y}}_j^{(mj)} + \hat{B}\mathcal{P}\tilde{\mathbf{Y}}_j^{(mj)} - \sum_{j=1}^{n} (\tilde{\mathbf{Y}}_j)_j^{(mj)} * \mathbf{D}_{\mathbf{j}\mathbf{e}_j} \\ = \sum_{|\mathbf{l}|>1} \mathbf{d}_\mathbf{l} * \sum_{\Sigma s = j} * \prod_{i=1}^{n} * \prod_{j=1}^{l_i} (\tilde{\mathbf{Y}}_{s_{i,j}})_i^{(s_{i,j})} =: \mathbf{R}_j(p)\end{aligned} \tag{2.49}$$

which integrated (mj) times is exactly the equation for $\mathbf{Y}_{\mathbf{j}\mathbf{e}_j}$, cf. also § 2.4. The claim now follows from Lemma 20, (iii). For (i), we note as before that $\mathbf{Y}_0^{\pm}$ are indeed solutions of (1.13). Applying (ii), we only need to identify C for which purpose we compare the left side with the right side on $(1, 1+\epsilon)$, where all the terms except for $k=1$ vanish and Remark 33 below applies. Lemma 12 completes the proof.

$\square$

Proof of Lemma 20, (iv). We let $\mathbf{Y}_j = \mathbf{Y}_{j\mathbf{e}_j}$. By Lemma 12, $\mathbf{D}_{j\mathbf{e}_j} \in \mathcal{F}(\mathcal{R}'_1)$ since

$$\mathbf{D}^+_{\mathbf{j}} = \sum_{\mathbf{l}\geq\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \mathbf{G}_{\mathbf{l}} * (\mathbf{Y}^+_0)^{*(\mathbf{l}-\mathbf{j})} = \sum_{\mathbf{l}\geq\mathbf{j}} \mathbf{G}_{\mathbf{l}} * \Bigg[\mathbf{Y}^-_0 + \sum_{s\geq 1} S^s(\mathbf{Y}^-_s(p-s))^{(ms)}\Bigg]^{*(\mathbf{l}-\mathbf{j})} = \sum_{\mathbf{k}\succeq 0}\left(\sum_{s\geq 1} S^s(\mathbf{Y}^-_s)^{(ms)}\right)^{*\mathbf{k}} \mathbf{Q}_{\mathbf{kj}} = \sum_{l=0}^{\infty} S^l(\mathbf{D}^-_{\mathbf{j};l})^{(ml)} \tag{2.50}$$

where $\mathbf{Q}_{\mathbf{kj}} = \sum_{\mathbf{l}\geq\mathbf{j}+\mathbf{k}} \binom{\mathbf{l}}{\mathbf{j}}\binom{\mathbf{l}-\mathbf{j}}{\mathbf{k}} \mathbf{G}_{\mathbf{l}}(\mathbf{Y}^-_0)^{*(\mathbf{l}-\mathbf{k}-\mathbf{j})}$ and

$$(\mathbf{D}^-_{\mathbf{j};l})^{(ml)} = \sum_{\substack{\in\mathbb{N}^n\\ |\mathbf{k}|\in[0,l]}} \mathbf{Q}_{\mathbf{kj}} * \sum_{(i_{rs}:l)} {}^*\prod_{r=1}^{n} {}^*\prod_{s=1}^{k_r} (\mathbf{Y}^-_{i_{rs}})_r^{(mi_{rs})} = \left(\sum_{\substack{\in\mathbb{N}^n\\ |\mathbf{k}|\in[0,l]}} \mathbf{Q}_{\mathbf{kj}} * \sum_{(i_{rs}:l)} {}^*\prod_{r=1}^{n} {}^*\prod_{s=1}^{k_r} (\mathbf{Y}^-_{i_{rs}})_r\right)^{(ml)} \tag{2.51}$$

and the notation $\sum_{(i_{rs}:l)}$ is explained after Eq. (2.158)); in particular, (1) there are only finitely many terms in (2.51) and (2) by homogeneity, $\mathbf{D}_{\mathbf{j};l} \in \mathcal{T}_{l\beta'_1-1}$. In addition, it follows, as in (2.46), (noting that we only need the $\mathbf{j}$ with $|\mathbf{j}| = 1$) that $\|\mathbf{Q}_{\mathbf{k};\mathbf{j}}\| \leq K(4\kappa)^{|\mathbf{k}|}$ and $\|\mathbf{D}_{\mathbf{j};l}\| \leq K(2^{2n+3}\kappa C_1\epsilon)^l$. If we look for $\mathbf{Y}^+_{\mathbf{k}}$ in the form $\mathbf{Y}^-_{\mathbf{k}} + \sum_{l=1}^{\infty}(\mathbf{Y}^-_{\mathbf{k};l}(p-l))^{(ml)}$ then the equation for $\mathbf{Y}^-_{\mathbf{k};l}$, $l \geq 1$, reads

$$\left(-p + \hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda} - l\right)\mathbf{Y}^-_{\mathbf{k};l} + \left(\hat{B} + \mathbf{m}\cdot\mathbf{k} + m_1 l\right)\mathcal{P}\mathbf{Y}^-_{\mathbf{k};l} + \sum_{|\mathbf{j}|=1} \mathbf{D}^-_{\mathbf{j};0} * \left(\mathbf{Y}^-_{\mathbf{k};l}\right)^{\mathbf{j}} = \left(\mathbf{T}_{\mathbf{k}}(\mathbf{Y}^-_{\cdot;l})\right) - \sum_{s=1}^{l-1}\sum_{|\mathbf{j}|=1} \mathbf{D}_{\mathbf{j};s} * \left(\mathbf{Y}^-_{\mathbf{k};l-s}\right)^{\mathbf{j}} \tag{2.52}$$

By induction, exactly as in (iii), it follows that $\mathbf{Y}_{\mathbf{k};l}(\cdot\, \mathrm{e}^{\mathrm{i}\phi}) \in \mathcal{T}_{\mathbf{k}\boldsymbol{\beta}'+l\beta'_1-1}$, are $\phi-$continuous in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ and that, with ν large enough independent of $\mathbf{k}, l, \phi$, $\|\mathbf{Y}_{\mathbf{k};l}\|_{\mathcal{D}'_{m,\nu}} \leq K\delta^{|\mathbf{k}|+l}$. Analyticity in $\mathcal{R}'_1$ follows now from Lemma 12.

(v) Laplace transformability as well as the fact that $\mathbf{y}_{\mathbf{k}}$ solve (1.14) follow immediately from (2.45) and Lemma 6. Uniform convergence follows from (2.40) and Lemma 6. Let $\mathbf{y}^{\pm}_{\mathbf{k}} = \mathcal{L}\mathbf{Y}^{\pm}_{\mathbf{k}}$. Now since $\mathbf{y}^{\pm}_0 + \sum x^{\mathbf{m}\cdot\mathbf{k}}\mathbf{C}^{\mathbf{k}}\mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}\mathbf{y}^{\pm}_k$ formally solves (1.3) (by the very construction of (1.14), see §2.4) and is a uniformly convergent function series, the conclusion follows together with the fact that $\mathcal{L}\mathbf{Y_k} \sim \tilde{\mathbf{y}}_\mathbf{k}$ since by (ii), and (iv) $\mathcal{L}\mathbf{Y_k}$ have power series asymptotics which by construction must be formal solutions of (1.14).

□

Proof of Theorem 1, (ii) Modulo relabeling of the spatial directions and rescaling, the proofs of the properties of the solution $\mathbf{Y}_0$ of (1.13) are valid in a sector centered around any eigenvalue λ_i. The above proofs of the analytic properties work with virtually no change along any direction so that $p + \mathbf{k}' \cdot \boldsymbol{\lambda} - \lambda_i \neq 0$ and the same is true with respect to the $\mathcal{D}'_{m,\nu}$ properties of $\mathbf{Y_k}$ restricted to *compact sets* in $\mathbb{C}$. The analysis in $\mathcal{D}'_{m,\nu}(e^{i\phi}\mathbb{R}^+)$ (i.e., along the infinite ray) and the associated *norm* estimates are *not* valid along directions so that $\Re(p - \lambda_i + \mathbf{k}' \cdot \boldsymbol{\lambda}) \leq 0$ (because of possible presence of small denominators in $(1 + J_\mathbf{k})^{-1}$, the only difference in this case, but a significant one). However, we do not take Laplace transforms along such directions (cf. (c1)) and the associated infinite ray norms are not needed for our purposes. For the analytic properties in the neighborhood of $-\mathbf{k}' \cdot \boldsymbol{\lambda} + \lambda_i$, see Remark 28. □

2.2.1 Higher resurgence relations

Proposition 24 *i) Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be solutions of (1.3) so that $\mathbf{y}_{1,2} \sim \tilde{\mathbf{y}}_0$ for large x in an open sector S (or in some direction d); then $\mathbf{y}_1 - \mathbf{y}_2 = \sum_j C_j \mathrm{e}^{-\lambda_{i_j} x} x^{-\beta_{i_j}} (\mathbf{e}_{i_j} + o(1))$ for some constants C_j, where the indices run over the eigenvalues λ_{i_j} with the property $\Re(\lambda_{i_j} x) > 0$ in S (or d). If $\mathbf{y}_1 - \mathbf{y}_2 = o(\mathrm{e}^{-\lambda_{i_j} x} x^{-\beta_{i_j}})$ for all j, then $\mathbf{y}_1 = \mathbf{y}_2$.*

ii) Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be solutions of (1.3) and assume that $\mathbf{y}_1 - \mathbf{y}_2$ has differentiable asymptotics of the form $\mathbf{K} a \exp(-ax) x^b (1 + o(1))$ with $\Re(ax) > 0$ and $\mathbf{K} \neq 0$, for large x. Then $a = \lambda_i$ for some i.

iii) Let $\mathbf{U_k} \in \mathcal{T}_{\{\cdot\}}$ for all $\mathbf{k}$, $|\mathbf{k}| > 1$. Assume in addition that for large ν there is a function $\delta(\nu)$ vanishing as $\nu \to \infty$ such that

$$\sup_{\mathbf{k}} \delta^{-|\mathbf{k}|} \int_d \left| \mathbf{U_k}(p) \mathrm{e}^{-\nu p} \right| \mathrm{d}|p| < K < \infty \tag{2.53}$$

Then, if $\mathbf{y}_1, \mathbf{y}_2$ are solutions of (1.3) in S where in addition

$$\mathbf{y}_1 - \mathbf{y}_2 = \sum_{|\mathbf{k}|>1} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{m}\cdot\mathbf{k}} \int_d \mathbf{U_k}(p) \exp(-xp) \mathrm{d}p \tag{2.54}$$

where $\boldsymbol{\lambda}, x$ are as in (c1), then $\mathbf{y}_1 = \mathbf{y}_2$, and $\mathbf{U_k} = 0$ for all $\mathbf{k}$, $|\mathbf{k}| > 1$.

Proof. (i) is a classical result (see [21] for the general treatment and [11] for a brief presentation of special cases and further references). However, what is actually needed for our purposes can be reduced to the more familiar *linear* asymptotic theory in the following way. Let d be a direction in the complex plane and let $\mathbf{y}_0$, $\mathbf{y}_1$ be solutions of (1.3) such that $\mathbf{y}_{0,1} \sim \tilde{\mathbf{y}}_0$ for large x along d. Then, by (n5), $\mathbf{y}_{0,1}(x) = O(x^{-M})$ and for any j, $\mathbf{g}^{(\mathbf{e}_j)}(x, \mathbf{y}_{0,1}(x)) = O(x^{-M})$. If $\boldsymbol{\delta} = \mathbf{y}_1 - \mathbf{y}_0$ then by hypothesis $\boldsymbol{\delta}(x) = o(x^{-l})$ along d, for all l. The function $\boldsymbol{\delta}$ is locally analytic and satisfies the equation

$$\boldsymbol{\delta}' = -\hat{\Lambda}\boldsymbol{\delta} - \frac{1}{x}\hat{B}\boldsymbol{\delta} + \sum_{|\mathbf{k}|=1} \mathbf{g}^{(\mathbf{k})}(x, \mathbf{y}_0)\boldsymbol{\delta}^{\mathbf{k}} + \sum_{|\mathbf{k}|>1} \mathbf{g}^{(\mathbf{k})}(x, \mathbf{y}_0)\boldsymbol{\delta}^{\mathbf{k}} =$$

$$- \hat{\Lambda}\boldsymbol{\delta} - \frac{1}{x}\hat{B}\boldsymbol{\delta} + \frac{1}{x^M}\sum_{j=1}^{n}(\boldsymbol{\delta})_j \mathbf{h}_{\mathbf{e}_j}(x) \quad (2.55)$$

where $\mathbf{h}_{\mathbf{k}}(x)$ are bounded along d. Obviously, because of the link between $\boldsymbol{\delta}$ and $\mathbf{h}_{\mathbf{k}}$, the $\boldsymbol{\delta}$ we started with might be the only solution of (2.55) which is also a difference of solutions of (1.3). The asymptotic characterization we need holds nevertheless for *all* decaying solutions solutions of (2.55): since no two eigenvalues are equal, there exists by the well-known linear asymptotic theory [11] a fundamental set $\{\boldsymbol{\delta}_i\}_{1\le i\le n}$ of solutions of (2.55) such that $\boldsymbol{\delta}_i \sim \mathrm{e}^{-\lambda_i x}x^{-\beta_i}(\mathbf{e}_i + o(1))$. Thus $\boldsymbol{\delta} = \sum_{i=1}^{n} C_i\boldsymbol{\delta}_i = \sum_{i=1}^{n} C_i \mathrm{e}^{-\lambda_i x}x^{-\beta_i}(\mathbf{e}_i + o(1))$. Since $\Re(-\beta_i) > 0$ and the λ_i are distinct we must have $C_i = 0$ for all i for which $\Re(-\lambda_i x) \ge 0$, otherwise $|\boldsymbol{\delta}(x)|$ would be unbounded for large x; the first part of (i) is proven. If on the other hand $\boldsymbol{\delta} = o(\mathrm{e}^{-\lambda_{i_j}x}x^{-\beta_{i_j}})$ for all j, again because the λ_i are independent, it follows that $C_i = 0$ for all $i = 1, 2, \ldots, n$, thus $\boldsymbol{\delta} = 0$.

(ii) is now obvious.

For (iii), note first that by (2.53) and (c1) the RHS of (2.54) converges uniformly for large x in some open sector. In addition, by an arbitrarily small change in $\xi = \arg(x)$, we can make the set $\{\Re(x\lambda_i)\}_i$ $\mathbb{Z}$-independent (the existence of $\mathbf{k}(\xi) \ne 0$ s.t. $\Re(\mathrm{e}^{\mathrm{i}\xi}\mathbf{k}\cdot\boldsymbol{\lambda}) = 0$ for ξ in an interval of would imply the existence of a *common* $\mathbf{k}$ for a set of ξ with an accumulation point, giving $\mathbf{k}\boldsymbol{\lambda} = 0$). We choose such a ξ. Assume now there exist $\mathbf{k}$ so that $\mathbf{U}_{\mathbf{k}} \ne 0$; among them let $\mathbf{k}_0$ have the least $\Re(x\mathbf{k}\cdot\boldsymbol{\lambda})$. By (2.53) for large x, $\mathbf{y}_1 - \mathbf{y}_2 \sim \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}_0 x}x^{\mathbf{m}\cdot\mathbf{k}_0}\mathcal{L}_\phi\mathbf{U}_{\mathbf{k}_0}(1+o(1))$. Because $\mathbf{U}_{\mathbf{k}_0} \in \mathcal{T}_{\{\cdot\}}$, and by (2.53), $\mathcal{L}_\phi\mathbf{U}_{\mathbf{k}_0}$ has a differentiable power series asymptotics which is the term-by-term Laplace transform of the Puiseux series at the origin of $\mathbf{U}_{\mathbf{k}_0}$, and thus nonzero. This contradicts (i) because with $|\mathbf{k}_0| > 1$ we have $\boldsymbol{\lambda}\cdot\mathbf{k}_0 \ne \lambda_j$ for all j ($\mathbb{Z}-$independence). Thus $\mathbf{U}_{\mathbf{k}} = 0$ for all $\mathbf{k}$.

□

We let $\mathbf{C} \in (\mathbb{C}\backslash\{0\})^{n_1}$ be an arbitrary constant vector.
For x large enough, $\mathbf{y}^+$ defined in (2.41) is a solution of (1.3) in an open sector containing x. We now use Lemma 20 to write (2.41) in terms of functions analytic in the lower half plane:

$$\mathbf{y}^+ = \mathcal{L}\mathbf{Y}_0^- + \sum_{j=1}^{\infty} x^{mj} \mathrm{e}^{-jx} \mathcal{L}\mathbf{Y}_{0;j}^- + \sum_{\mathbf{k}\succ 0, j\geq 0} x^{\mathbf{m}\cdot\mathbf{k}+mj} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda}+j\lambda_1)x} \mathcal{L}\mathbf{Y}_{\mathbf{k};j}^-$$

$$= \mathcal{L}\mathbf{Y}_0^- + \sum_{\mathbf{k}\succ 0} x^{\mathbf{mk}} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} \sum_{\mathbf{k}';j:\mathbf{k}'+j\mathbf{e}_1=\mathbf{k}} \mathbf{C}^{\mathbf{k}'} \mathcal{L}\mathbf{Y}_{\mathbf{k}';j}^- \quad (2.56)$$

where, by (2.47) we have $\mathbf{Y}_{0;j} = S_1^j \mathbf{Y}_{j\mathbf{e}_1}^-$. On the other hand, the expression

$$\mathbf{y}^- = \mathcal{L}\mathbf{Y}_0^- + \sum_{\mathbf{k}\succ 0} x^{\mathbf{m}\cdot\mathbf{k}} \tilde{\mathbf{C}}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^- \quad (2.57)$$

is, for any $\tilde{\mathbf{C}}$, a solution of (1.3) as well. Choosing $\tilde{C}_1 = C_1+S_1; \tilde{C}_i = C_i;\ (i > 1)$ all the terms with $|\mathbf{k}| \leq 1$ in (2.57) and (2.56) coincide and thus

$$\mathbf{y}^+ - \mathbf{y}^- = \sum_{|\mathbf{k}|>1} x^{\mathbf{m}\cdot\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} \mathcal{L}\mathbf{U}_{\mathbf{k}} \quad (2.58)$$

where

$$\mathbf{U}_{\mathbf{k}} = \sum_{\mathbf{k}'+j\mathbf{e}_1=\mathbf{k}} \mathbf{C}^{\mathbf{k}'} \mathbf{Y}_{\mathbf{k}';j}^- - \tilde{\mathbf{C}}^{\mathbf{k}} \mathbf{Y}_{\mathbf{k}}^- \quad (2.59)$$

so that applying Proposition 24 (ii)

$$\mathbf{U}_{\mathbf{k}} = 0 \quad (2.60)$$

Since $C_i \neq 0$ we have, for any C_1,

$$(C_1 + S_1)^{k_1} \mathbf{Y}_{\mathbf{k}}^- = \sum_{\mathbf{k}'+j\mathbf{e}_1=\mathbf{k}} C_1^{k_1'} \mathbf{Y}_{\mathbf{k}';j}^- \quad (2.61)$$

with arbitrary C_1 so that

$$\mathbf{Y}_{\mathbf{k};j}^- = \binom{k_1+j}{j} S_1^j \mathbf{Y}_{\mathbf{k}+j\mathbf{e}_1}^- \quad (2.62)$$

Combined with the definition of $\mathbf{Y}_{\mathbf{k};j}^-$ this gives (1.26).

Solving for $\mathbf{Y}_{\mathbf{k}+j\mathbf{e}_1}^-$, (2.62) determines later series in the transseries in terms of earlier ones. The same arguments work of course with $-/+$ and $+S_1/(-S_1)$ interchanged.

Theorem 3 part (iii) follows from the following.

Proposition 25 *Any solution* $\mathbf{y}$ *of (1.3) so that* $\mathbf{y} \sim \tilde{\mathbf{y}}_0$ *along some direction* $d \subset S_x$ *is of the form (2.41), for a unique* $\mathbf{C}^+$ *(a similar statement holds with* $+/-$ *interchanged). Alternatively, a solution* $\mathbf{y}$ *of (1.3) so that* $\mathbf{y} \sim \tilde{\mathbf{y}}_0$ *along some direction* $d \subset S_x$ *can be represented as (1.21) or more generally as (1.23) where Laplace integration is along* $\mathbb{R}^+$ *(in distributions), for a unique* $\mathbf{C}$.

Proof. Let $\mathbf{y}$ be an arbitrary solution of (1.3) so that $\mathbf{y} \sim \tilde{\mathbf{y}}_0$ along $d \subset S_x$. Then, by Proposition 24, $\mathbf{y} - \mathbf{y}^+ = \sum_j \overline{\mathbf{C}}_j \mathrm{e}^{-\lambda_{i_j} x} x^{-\beta_{i_j}} (\mathbf{e}_{i_j} + o(1))$ for some constant $\overline{\mathbf{C}}$. Therefore $\mathbf{y}_1$ defined as the "+" solution in (2.41) with $\mathbf{C} = \overline{\mathbf{C}}$ will have the property $\mathbf{y}_1 - \mathbf{y} = o(C_j \mathrm{e}^{-\lambda_{i_j} x} x^{-\beta_{i_j}})$ for all j, hence $\mathbf{y} = \mathbf{y}_1$. Formula (1.23). The last part, as well as the middle formula in (1.27), follow through a straightforward calculation from the first part, (2.62) using (2.40) to control convergence.

□

Proof of theorem 5. Let $\mathbf{y}^{\pm}$ be defined by (2.41) with $\mathbf{C} = (\pm\frac{1}{2}S_1 + C)\mathbf{e}_1$, respectively. The same arguments leading to (2.60) show that $\mathbf{y}^+ = \mathbf{y}^- =: \mathbf{y}$. All the exponentials in the transseries of $\mathbf{y}$ are generated by construction by $e^{-\lambda x}$. Choosing p in the path of integration above/below $\mathbb{R}^+$ and consequently the $+/-$ representation (2.41) of $\mathbf{y}$ we have by Lemma 20 that $\mathcal{L}\mathbf{Y}^{\pm}_{\mathbf{k}} \sim \tilde{\mathbf{y}}_{\mathbf{k}}$ in (2.41), in the half plane $\Re(xp) > 0$. By construction $\mathcal{L}\mathbf{Y}_{\mathbf{e}_1} = x^{-\beta_1'}(\mathbf{e}_1 + o(1))$ (cf. 2.89) while for $j > 1$ we have $\mathcal{L}\mathbf{Y}_{j\mathbf{e}_1} \sim x^{-j\beta_1'}$ by Lemma 20 (ii). The condition $|x^{-\beta_1+1}e^{-x\lambda_1}| \to 1$ together with Lemma 20 guarantee the uniform convergence of the series (2.41). The conclusion is immediate.

□

2.2.2 Local analysis near $p = 1$.

We treat (1.13) near $p = 1$ as a perturbation of a differential equation having the same type of singularity. The associated differential equation splits the singularity, and our convolution equation is a regular perturbation of it, which is then solved by fixed point methods. Let $\mathbf{Y}_0$ be the unique solution in $\mathcal{A}_{z,l}$ of (1.13) and let $\epsilon > 0$ be small. Define

$$\begin{aligned} \mathbf{H}(p) &:= \begin{cases} \mathbf{Y}_0(p) & \text{for } p \in \mathcal{S}_0\,, |p| < 1 - \epsilon \\ 0 & \text{otherwise} \end{cases} \\ \text{and }\ \mathbf{W}(1-p) &:= \mathbf{Y}_0(p) - \mathbf{H}(p) \end{aligned} \tag{2.63}$$

($\mathbf{Y}(p) - \mathbf{H}(p) = \mathbf{W}(p-1)$ would be more "natural", but would later complicate notations). In terms of $\mathbf{W}$, for real $z = 1 - p, z < \epsilon$, (1.13) becomes:

$$-(1-z)\mathbf{W}(z) = \mathbf{F}_1(z) - \hat{\Lambda}\mathbf{W}(z) + \hat{B}\int_{\epsilon}^{z} \mathbf{W}(s)\mathrm{d}s + \mathcal{N}(\mathbf{H} + \mathbf{W}) \tag{2.64}$$

where

$$\mathbf{F}_1(1-s) := \mathbf{F}_0(s) - \hat{B}\int_0^{1-\epsilon} \mathbf{H}(s)\mathrm{d}s$$

PROPOSITION 26 *i) For small ϵ, $\mathbf{H}^{*\mathbf{l}}(1+z)$ extends to an analytic function in the disk $\mathbb{D}_\epsilon := \{z : |z| < \epsilon\}$. Furthermore, for any δ there is an ϵ and a constant $K_1 := K_1(\delta, \epsilon)$ such that for $z \in \mathbb{D}_\epsilon$*

$$|\mathbf{H}^{*\mathbf{l}}(1+z)| < K_1 \delta^{|\mathbf{l}|} \tag{2.65}$$

ii) The equation (2.64) can be written as

$$-(1-z)\mathbf{W}(z) = \mathbf{F}(z) - \hat{\Lambda}\mathbf{W}(z) + \hat{B}\int_\epsilon^z \mathbf{W}(s)\mathrm{d}s - \sum_{k=1}^n \int_\epsilon^z h_j(s)\mathbf{D}_j(s-z)\mathrm{d}s \tag{2.66}$$

where

$$\mathbf{F}(z) := \mathcal{N}(\mathbf{H})(1-z) + \mathbf{F}_0(z) \tag{2.67}$$

$$\mathbf{D}_j = \sum_{|\mathbf{l}|\geq 1} l_j \mathbf{G}_{\mathbf{l}} * \mathbf{H}^{*\bar{\mathbf{l}}^j} + \sum_{|\mathbf{l}|\geq 2} l_j \mathbf{g}_{0,\mathbf{l}} \mathbf{H}^{*\bar{\mathbf{l}}^j};\ \bar{\mathbf{l}}^j := (l_1, l_2, ..(l_j - 1), ..l_n) \tag{2.68}$$

extend to analytic functions in $\mathbb{D}_\epsilon$. Moreover, if $\mathbf{H}$ is a vector in $L^1{}_\nu(\mathbb{R}^+)$ then, for large ν, $\mathbf{D}_j \in L^1{}_\nu(\mathbb{R}^+)$ and the functions $\mathbf{F}(z)$ and $\mathbf{D}_j$ extend to analytic functions in $\mathbb{D}_\epsilon$. Furthermore, $\mathbf{D}_j \in \mathcal{A}_{z,M}$.

iii) Near $p = 1$ we have (cf. Lemma 16)

$$\begin{aligned} \mathcal{P}^{m+1}\mathbf{Y}_0 &= (1-p)^{\beta'}\mathbf{A} + \mathbf{B} \quad (\beta \notin \mathbb{Z}) \\ \mathcal{P}^{m+1}\mathbf{Y}_0 &= (p-1)\left(\ln(p-1)\mathbf{A}(p) + \mathbf{B}(p)\right) \quad (\beta \in \mathbb{Z}) \end{aligned} \tag{2.69}$$

where $\mathbf{A}, \mathbf{B}$ analytic at $p = 1$.

Proof.

Parts (i) and (ii), except for the last claim, are proven in [13], Propositions 18 and 19. To see that $\mathbf{D}_j \in \mathcal{A}_m$ it is enough to remark again that $\mathcal{A}_{z,M}$ is a convolution ideal of $\mathcal{A}_{z,0}$ and that $\mathbf{G}_\mathbf{j} \in \mathcal{A}_{z,M}$ for $|\mathbf{j}| = 1$, by (n5).

For (iii), consider again equation (2.66). Let $\hat{\Gamma} = \hat{\Lambda} - (1-z)\hat{1}$, where $\hat{1}$ is the identity matrix. By construction $\hat{\Gamma}$ and $\hat{B}$ are diagonal, $\hat{\Gamma}_{11} = z$ and $\hat{B}_{11} = \beta_1 =: \beta$. We write this as $\hat{\Gamma} = z \oplus \hat{\Gamma}_c(z)$ and similarly, $\hat{B} = \beta \oplus \hat{B}_c$, where $\hat{\Gamma}_c$ and $\hat{B}_c$ are $(n-1)\times(n-1)$ diagonal matrices. $\hat{\Gamma}_c(z)$ and $\hat{\Gamma}_c^{-1}(z)$ are analytic in $\mathbb{D}_\epsilon$.

Let

$$\mathbf{Q} := \mathcal{P}_\epsilon^{m+1}\mathbf{W} \tag{2.70}$$

with $\mathcal{P}_\epsilon := \mathbf{W} \mapsto \left(z \mapsto \int_\epsilon^z \mathbf{W}(s)\mathrm{d}s\right)$. By Lemma 16, (i), $\mathbf{Q}$ is analytic in $\mathbb{D}_\epsilon \cap (\{z : z+1 \in \mathcal{S}_0\})$. From (2.66) we obtain

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{Q}^{(m+1)}(z) - (\beta \oplus \hat{B}_c)\mathbf{Q}^{(m)}(z) \\ = \mathbf{F}(z) - \sum_{j=1}^{n} \int_\epsilon^z \mathbf{D}_j(s-z)Q_j^{(m+1)}(s)\mathrm{d}s \quad (2.71)$$

or, after m integrations by parts in the r.h.s. of (2.71), by Proposition 26 (ii), we get

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{Q}^{(m+1)}(z) - (\beta \oplus \hat{B}_c)\mathbf{Q}^{(m)}(z) \\ = \mathbf{F}(z) - (-1)^m \sum_{j=1}^{n} \int_\epsilon^z \mathbf{D}_j^{(m)}(s-z)Q_j'(s)\mathrm{d}s \quad (2.72)$$

so that with $\beta' = \beta_1', \hat{B}_c' = m_1 + \hat{B}_c$,

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{Q}'(z) - (\beta' \oplus \hat{B}_c')\mathbf{Q}(z) \\ = \mathcal{P}_\epsilon^m \mathbf{F}(z) - (-\mathcal{P}_\epsilon)^m \sum_{j=1}^{n} \int_\epsilon^z \mathbf{D}_j^{(m)}(s-z)Q_j'(s)\mathrm{d}s \\ = \mathbf{P}(z) + \sum_{j=1}^{n} \int_\epsilon^z \mathbf{D}_j'(s-z)Q_j(s)\mathrm{d}s \quad (2.73)$$

where $\mathbf{P}(z) = \mathcal{P}_\epsilon^m \mathbf{F}(z)$. With the notation $(Q_1, \mathbf{Q}_\perp) := (Q_1, Q_2, .., Q_n)$ we write the system in the form

$$\begin{aligned} (z^{-\beta'}Q_1(z))' &= z^{-\beta'-1}\left(P_1(z) + \sum_{j=1}^{n} \int_\epsilon^z D_{1j}'(s-z)Q_j(s)\mathrm{d}s\right) \\ (\mathrm{e}^{\hat{C}(z)}\mathbf{Q}_\perp)' &= \mathrm{e}^{\hat{C}(z)}\hat{\Gamma}_c(z)^{-1}\left(\mathbf{P}_\perp + \sum_{j=1}^{n} \int_\epsilon^z \mathbf{D}_\perp'(s-z)Q_j(s)\mathrm{d}s\right) \\ \hat{C}(z) &:= -\int_0^z \hat{\Gamma}_c(s)^{-1}\hat{B}_c'(s)\mathrm{d}s \\ \mathbf{Q}(\epsilon) &= 0 \end{aligned} \quad (2.74)$$

After integration we get:

$$\begin{aligned} Q_1(z) &= R_1(z) + J_1(\mathbf{Q}) \\ \mathbf{Q}_\perp(z) &= \mathbf{R}_\perp(z) + J_\perp(\mathbf{Q}) \end{aligned} \quad (2.75)$$

with

$$
\begin{aligned}
J_1(\mathbf{Q}) &= z^{\beta'}\int_\epsilon^z t^{-\beta'-1}\sum_{j=1}^n\int_\epsilon^t Q_j(s)D'_{1j}(t-s)\mathrm{d}s\mathrm{d}t\\
J_\perp(\mathbf{Q})(z) &:= \mathrm{e}^{-\hat{C}(z)}\int_\epsilon^z \mathrm{e}^{\hat{C}(t)}\hat{\Gamma}_c(t)^{-1}\left(\sum_{j=1}^n\int_\epsilon^z \mathbf{D}'_\perp(s-z)Q_j(s)\mathrm{d}s\right)\mathrm{d}t\\
\mathbf{R}_\perp(z) &:= \mathrm{e}^{-\hat{C}(z)}\int_\epsilon^z \mathrm{e}^{\hat{C}(t)}\hat{\Gamma}_c(t)^{-1}\mathbf{F}_\perp(t)\mathrm{d}t\\
R_1(z) &= z^{\beta'}\int_\epsilon^z t^{-\beta'-1}P_1(t)\mathrm{d}t \qquad\qquad (\beta'\neq 1)\\
R_1(z) &= P_1(0)z\ln z + z\int_\epsilon^z \frac{P_1(s)-P_1(0)}{s}\mathrm{d}s \quad (\beta'=1)
\end{aligned}
\tag{2.76}
$$

Consider the space $\mathcal{U}_{\beta'}$ given by

$$
\mathcal{U}_{\beta'} = \left\{\mathbf{Q} \text{ analytic in } \{z: 0<|z|<\epsilon, \arg(z)\neq\pi\} : \mathbf{Q} = z^{\beta'}\mathbf{A}(z)+\mathbf{B}(z)\right\}
\tag{2.77}
$$

for $\beta'\neq 1$ and

$$
\mathcal{U}_1 = \left\{\mathbf{Q} \text{ analytic in } \{z: 0<|z|<\epsilon, \arg(z)\neq\pi\} : \mathbf{Q} = z\ln z\mathbf{A}(z)+z\mathbf{B}(z)\right\}
$$

where $\mathbf{A},\mathbf{B}$ are analytic in $\mathbb{D}_\epsilon$. (The decomposition of $\mathbf{Q}$ in (2.77) is unambiguous since $z^{\beta'}$ and $z\ln z$ are not meromorphic in $\mathbb{D}_\epsilon$.)

The norm

$$
\|\mathbf{Q}\| = \sup\{|\mathbf{A}(z)|, |\mathbf{B}(z)| : z\in\mathbb{D}_\epsilon\}
\tag{2.78}
$$

makes $\mathcal{U}_{\beta'}$ a Banach space.

Proposition 27 *The operator $J := (\mathbf{Q}\mapsto(J_1\mathbf{Q}, J_\perp\mathbf{Q})$ has norm $O(\epsilon)$, for small ϵ, in $\mathcal{U}_{\beta'}$ as well as in $L^1[-\epsilon,\epsilon]$. Along any segment d_ϵ originating at $z=\epsilon$ in the region $|z|<\epsilon, \arg(z)\neq\pi$, Equation (2.73) has a unique solution in $L^1_\nu(d_\epsilon)$. This solution belongs to $\mathcal{T}_\beta$.*

The proof uses the following elementary identities:

$$
\begin{aligned}
&\int_\epsilon^z A(s)s^r\mathrm{d}s = \text{const.} + z^{r+1}\int_0^1 A(zt)t^r\mathrm{d}t = \text{const.} + z^{r+1}\mathit{Analytic}(z)\\
&\int_0^z s^r\ln s\,A(s)\mathrm{d}s = z^{r+1}\ln z\int_0^1 A(zt)t^r\mathrm{d}t + z^{r+1}\int_0^1 A(zt)t^r\ln t\mathrm{d}t
\end{aligned}
\tag{2.79}
$$

where the second equality is obtained by differentiating with respect to r the first equality.

Using (2.79) it is straightforward to check that the r.h.s. of (2.75) extends to a linear inhomogeneous operator on $\mathcal{U}_{\beta'}$ with image in $\mathcal{U}_{\beta'}$ and that the norm of J is $O(\epsilon)$ for small ϵ. For instance, one of the terms in J for $\beta'=1$,

$$\begin{aligned} z\int_0^z t^{-2}\int_0^t & s\ln s\, A(s)D'(t-s)\mathrm{d}s \\ &= z^2\ln z\int_0^1\int_0^1 \sigma A(z\tau\sigma)D'(z\tau-z\tau\sigma)\mathrm{d}\sigma\mathrm{d}\tau \\ &+ z^2\int_0^1 \mathrm{d}\tau\ln\tau\int_0^1 \mathrm{d}\sigma\,\sigma(1+\ln\sigma)A(z\tau\sigma)D'(z\tau-z\tau\sigma) \end{aligned} \tag{2.80}$$

manifestly in $\mathcal{U}_{\beta'}$ if A is analytic in $\mathbb{D}_\epsilon$. Comparing with (2.77), the extra power of z accounts for a norm $O(\epsilon)$ for this term.

Therefore, in (2.74) $(1-J)$ is invertible and the solution $\mathbf{Q}\in\mathcal{U}_{\beta'}$. In view of the the uniqueness of the solution of (1.13) in $\mathcal{S}_0$, (Lemma 16, (i)) the rest of the proof of Proposition 26, (iii) is immediate.

□

A short calculation shows that:

Remark 28 *i) The equation for $\mathbf{Y_k}$ near $z=0$ where $z=-p-\mathbf{k}\cdot\boldsymbol{\lambda}+\lambda_i$ can be written in the form (2.66), for a different $\mathbf{F}$, and with $\hat{B}+\mathbf{k}\cdot\mathbf{m}$ instead of $\hat{B}$. Thus $\mathbf{Y_k}(z)\in\mathcal{T}_{\{\cdot\}}$.*

ii) The equation for $\mathbf{Y_k}$ near $z=0$ where $z=-p-\mathbf{k}'\cdot\boldsymbol{\lambda}+\lambda_i$ where $\mathbf{k}'\prec\mathbf{k}$ can be written as

$$(1+J_{\mathbf{k}})\mathbf{Y_k}(z)=\mathbf{R_k}(z) \tag{2.81}$$

where

$$J_{\mathbf{k}}\mathbf{Y}=(\hat{B}+\mathbf{m}\cdot\mathbf{k})\hat{M}^{-1}\mathcal{P}\mathbf{Y}+\hat{M}^{-1}\sum_{|\mathbf{j}|=1}\overline{\mathbf{D}}_{\mathbf{j}}*\mathbf{Y}^{\mathbf{j}},$$

$\hat{M}=z+\hat{\Lambda}-\lambda_i+(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\lambda}$, $\mathbf{R_k}=\hat{M}^{-1}\mathbf{T_k}$ and $\overline{\mathbf{D}}_{\mathbf{j}}$ analytic for small z. Thus, arguments virtually identical to those for (2.75) imply that $\mathbf{Y_k}\in\mathcal{T}_{\{\cdot\}}$ near these points.

2.2.3 The solutions of (2.72) on $[-\epsilon,\epsilon]$

Let $\mathbf{Q}_0$ be the solution given by Proposition 27, take ϵ small enough and denote by $\mathcal{O}_\epsilon$ a neighborhood in $\mathbb{C}$ of width ϵ of the interval $[0,1+\epsilon]$. We look for solutions of (2.73) in $L^1[-\epsilon,\epsilon]$. The main difference with respect to the previous section is that in integrating (2.73) to the analog of (2.75)for negative z, the constant of integration will now be undetermined leading to a one-parameter family of solutions. See also Remark 30 below.

Remark 29 . *As* $\phi \to \pm 0$, $\mathbf{Q}_0(ze^{i\phi}) \to \mathbf{Q}_0^{\pm}(z)$ *in the sense of* $L^1([0, 1+\epsilon])$ *and also in the sense of pointwise convergence for* $z \neq 0$, *where*

$$\mathbf{Q}_0^{\pm}(z) := \begin{cases} \mathbf{Q}_0(z) & z > 0 \\ |z|^{\beta'} e^{\mp i\pi(\beta')}\mathbf{a}_1(p) + \mathbf{a}_2(p) & z < 0 \end{cases} \quad (\beta \neq 1)$$

$$\mathbf{Q}_0^{\pm} := \begin{cases} \mathbf{Q}_0(z) & z > 0 \\ z(\ln(|z|) - \pi i)\mathbf{a}_1(z) + z\mathbf{a}_2(z) & z < 0 \end{cases} \quad (\beta' = 1) \tag{2.82}$$

Moreover, $\mathbf{Q}_0^{\pm}$ *are* L^1_{loc} *solutions of (27) on the interval* $[-\epsilon, \epsilon]$.

The proof is immediate from Propositions 26 and 27.

□

Remark 30 *For any* $\lambda \in \mathbb{C}$ *the combination* $\mathbf{Q}_\lambda = \lambda\mathbf{Q}_0^+ + (1-\lambda)\mathbf{Q}_0^-$ *is a solution of (2.72) in* $L^1[-\epsilon, \epsilon]$.

Proof. Follows from Remark 29 as (2.72) is linear.

□

*

Let now $\mathbf{Q}_0$ be any solution of (2.72) in $L^1[-\epsilon, \epsilon]$. We search for other solutions in the form $\mathbf{Q} = \mathbf{Q}_0 + \mathbf{q}$. Since (2.72) is linear and $\mathbf{Q}_0$ is already a solution we have

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{q}'(z) - (\beta' \oplus \hat{B}_c')\mathbf{q}(z) = \sum_{j=1}^{n} \int_{\epsilon}^{z} \mathbf{D}_j'(s-z)q_j(s)\mathrm{d}s \tag{2.83}$$

and, by the uniqueness of $\mathbf{Q}_0$ for $z > 0$ we have $\mathbf{q} = 0$ for $q < 0$ and the equation becomes

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{q}'(z) - (\beta' \oplus \hat{B}_c')\mathbf{q}(z) = \sum_{j=1}^{n} \int_{0}^{z} \mathbf{D}_j'(s-z)q_j(s)\mathrm{d}s \tag{2.84}$$

with the initial condition $\mathbf{q}(0) = 0$. Changing variables to $z = -p$ ($p > 0$ now corresponds to going beyond the singularity) and $\mathbf{q}(z) = \mathbf{Y}(-p)$ we have

$$(p \oplus \hat{\Gamma}_c(-p))\mathbf{Y}'(p) - (\beta' \oplus \hat{B}_c')\mathbf{Y}(p) + \sum_{j=1}^{n} \int_{0}^{p} \mathbf{D}_j'(p-t)Y_j(t)\mathrm{d}t = 0 \tag{2.85}$$

We recognize in (2.85) the equation for $\mathcal{P}\mathbf{Y}_{\mathbf{e}_1}$:

REMARK 31 *Equation (2.85) is at the same time the equation for* $\mathcal{P}\mathbf{Y}_{\mathbf{e}_1}$ *and for the difference* $\mathcal{P}^{m+1}(\mathbf{Y}_0^{[1]} - \mathbf{Y}_0^{[2]})$ *where* $\mathbf{Y}_0^{[1,2]}$ *are any solutions of (1.13).*

□

PROPOSITION 32 *Let* ϵ *be small. In* $\mathcal{T}_{\beta'}(\{|p| < \epsilon\})$ *as well as in* $\mathcal{D}'_{m,\nu}(0, \epsilon e^{i\phi})$ *for any* ϕ*, there is a unique solution of (2.85)* $\mathbf{W}_0$ *such that, for small* p*,* $\mathbf{W}_0 = \Gamma(\beta')^{-1}p^{\beta'}(\mathbf{e}_1 + o(1))$*. The general solution of (2.85) is* $\mathbf{Y} = C\mathbf{W}_0$*, with* $C \in \mathbb{C}$ *arbitrary.*

Notes: (1) Modulo relabeling of the spatial directions, the statement and proof hold for any of the equations for $\mathbf{Y}_{\mathbf{e}_j}$.

(2) The point $p = 0$ is singular, and so is the "initial condition" $\mathbf{W}_0 \sim \Gamma(\beta')^{-1}p^{\beta'}\mathbf{e}_1$.

Proof. We have

$$\begin{aligned}
&(p^{-\beta'}Y_1(z))' = -p^{-\beta'-1}\sum_{j=1}^{n}\int_0^p D'_{1j}(p-t)Y_j(t)\mathrm{d}t \\
&(e^{\hat{E}(p)}\mathbf{Y}_\perp)' = -e^{\hat{E}(p)}\hat{\Gamma}_c(-p)^{-1}\sum_{j=1}^{n}\int_0^p \mathbf{D}'_\perp(p-t)Y_j(t)\mathrm{d}t \\
&\hat{E}(p) := -\int_0^p \hat{\Gamma}_c(-t)^{-1}\hat{B}'_c(t)\mathrm{d}s \\
&\mathbf{Y}(0) = 0
\end{aligned} \tag{2.86}$$

After integration we get:

$$\begin{aligned}
(1 + J_1)\mathbf{Y}_1(z) = CR_1(p) \\
(1 + J_\perp)\mathbf{Y}_\perp(p) = 0
\end{aligned} \tag{2.87}$$

with $C \in \mathbb{C}$ arbitrary and

$$\begin{aligned}
&J_1\,\mathbf{V} = p^{\beta'}\int_0^p t^{-\beta'-1}\sum_{j=1}^{n}\int_0^t Y_j(s)D'_{1j}(t-s)\mathrm{d}s\mathrm{d}t \\
&J_\perp\,\mathbf{V} := e^{-\hat{E}(p)}\int_0^p e^{\hat{E}(t)}\hat{\Gamma}_c(-t)^{-1}\left(\sum_{j=1}^{n}\int_0^t \mathbf{D}'_\perp(t-s)Y_j(s)\mathrm{d}s\right)\mathrm{d}t \\
&R_1(p) = p^{\beta'}
\end{aligned} \tag{2.88}$$

In a region $|p| < \epsilon$, for small ϵ, the norm of the operator J defined on $\mathcal{T}_{\beta'}$ is $O(\epsilon)$, exactly as in Proposition 27. Given C the solution of the system (2.87) is unique and can be written as

$$\mathbf{Y} = C\mathbf{W}_0;\quad \mathbf{W}_0 := \Gamma(\beta')^{-1}(\hat{1}+J)^{-1}\mathbf{R} \neq 0 \tag{2.89}$$

(The prefactor $\Gamma(\beta')^{-1}$ was introduced so that the coefficient of the leading power in the asymptotic series of $\mathcal{L}\mathbf{W}_0$ is one).

The proof is essentially the same if we consider (2.87) in $L^1_\nu(0,\epsilon e^{i\phi})$, which coincides with $\mathcal{D}'_{m,\nu}(0,\epsilon e^{i\phi})$ for small ϵ.

Remark 33 *On* $(1,1+\epsilon)$, $(\mathbf{Y}_0^+ - \mathbf{Y}_0^-)(p) = S_1\mathbf{Y}_{\mathbf{e}_1}^{(m)}(p-1)$.

The existence of some S_1 is obvious from Remark 31 and Proposition 32. Its value follows by comparing (2.69), (2.62) and (2.89).

Proposition 34 *i) Let* $\mathbf{Y}_\mathbf{k}^+, \mathbf{k} \geq 0$ *solve (1.13), (1.14) in* $\mathcal{T}_{\{\cdot\}}(\mathcal{S}^{+\prime})$. *Then* $\mathbf{Y}_\mathbf{k}^{ba}$, *cf. (1.20) solve (1.13), (1.14) in* $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$

ii) For any of the functions $\mathbf{Y}_\mathbf{k}$, *interchanging* $+$ *with* $-$ *in (1.20) does not change the balanced average.*

Proof (i) The fact that $\mathbf{Y}_0^{ba}$ is a solution of (1.13) follows from Proposition 23. From Proposition 23 and Proposition 13 we see that $\mathbf{D}_\mathbf{j}^{ba}$ is obtained by simply replacing $\mathbf{Y}_0^+$ by $\mathbf{Y}_0^{ba}$ in (2.50) (notice that on any finite interval, there are finitely many terms in the expression of $\mathbf{D}_\mathbf{j}^{ba} - \mathbf{D}_\mathbf{j}^+$.) The rest of the proof merely consists in inductively applying $\mathcal{A}_\alpha$ to the equations (1.14), noting that each contains finitely many convolutions, and applying the commutation relation (2.29).

(ii) This is true for $\mathbf{Y}_0$ as an immediate verification shows that the $+$ and $-$ averages coincide on $(0,2)$ (where they consist in two terms). Thus by Proposition 19 they have to coincide on $\mathbb{R}^+$. With this, for the rest of the $\mathbf{Y}_\mathbf{k}$ the property follows by an obvious induction from Proposition 13.

2.3 Appendix

2.3.1 The C^*–algebra $\mathcal{D}'_{m,\nu}$

Let $\mathcal{D}$ be the space of test functions (compactly supported C^∞ functions on $(0,\infty)$) and $\mathcal{D}(0,x)$ be the test functions on $(0,x)$.

We say that $f \in \mathcal{D}'$ is a staircase distribution if for any $k = 0,1,2,...$ there is an L^1 function on $[0,k+1]$ so that $f = F_k^{(km)}$ (in the sense of distributions) when restricted to $\mathcal{D}(0,k+1)$ or

$$F_k := \mathcal{P}^{mk} f \in L_1(0,k+1) \tag{2.90}$$

(since $f \in L^1_{loc}[0,1-\epsilon]$ and by Remark 43, $\mathcal{P}f$ is well defined). With this choice we have

$$F_{k+1} = \mathcal{P}^m F_k \text{ on } [0,k] \text{ and } F_k^{(j)}(0) = 0 \text{ for } j \leq mk-1 \tag{2.91}$$

We denote these distributions by $\mathcal{D}'_m$ ($\mathcal{D}'_m(0,k)$ respectively, when restricted to $\mathcal{D}(0,k)$) and observe that $\bigcup_{m>0} \mathcal{D}'_m \supset S'$, the distributions of slow growth. The inclusion is strict since any element of S' is of finite order.

Let $f \in L^1$. Taking $F = \mathcal{P}^j f \in C^j$ we have, by integration by parts and noting that the boundary terms vanish,

$$(F * F)(p) = \int_0^p F(s)F(p-s)\mathrm{d}s = \int_0^p F^{(j)}(s)\mathcal{P}^j F(p-s) \tag{2.92}$$

so that $F * F \in C^{2j}$ and

$$(F * F)^{(2j)} = f * f \tag{2.93}$$

This motivates the following definition: for $f, \tilde{f} \in \mathcal{D}'_m$ let

$$f * \tilde{f} := (F_k * \tilde{F}_k)^{(2km)} \quad \text{in } \mathcal{D}'(0, k+1) \tag{2.94}$$

We first check that the definition is consistent in the sense that

$$(F_{k+1} * F_{k+1})^{(2m(k+1))} = (F_k * F_k)^{(2mk)}$$

on $\mathcal{D}(0, k+1)$. For $p < k+1$ integrating by parts and using (2.91) we obtain

$$\frac{\mathrm{d}^{2m(k+1)}}{\mathrm{d}p^{2m(k+1)}} \int_0^p F_k(s)\mathcal{P}^{2m}\tilde{F}_k(p-s)\mathrm{d}s = \frac{\mathrm{d}^{2mk}}{\mathrm{d}p^{2mk}} \int_0^p F_k(s)\tilde{F}_k(p-s)\mathrm{d}s \tag{2.95}$$

The same argument shows that the definition is compatible with the embedding of $\mathcal{D}'_m$ in $\mathcal{D}'_{m'}$ with $m' > m$. Convolution is commutative and associative: with $f, g, h \in \mathcal{D}'_m$ and identifying $(f * g)$ and h by the natural inclusion with elements in $\mathcal{D}'_{2m}$ we obtain $(f * g) * h = ((F * G) * H)^{(4mk)} = f * (g * h)$.

The following staircase decomposition exists in $\mathcal{D}'_m$.

Lemma 35 . *For each $f \in \mathcal{D}'_m$ there is a* unique *sequence $\{\Delta_i\}_{i=0,1,..}$ such that $\Delta_i \in L^1(\mathbb{R}^+)$, $\Delta_i = \Delta_i \chi_{[i,i+1]}$ and*

$$f = \sum_{i=0}^{\infty} \Delta_i^{(mi)} \tag{2.96}$$

Also (cf. (2.91)),

$$F_i = \sum_{j \le i} \mathcal{P}^{m(i-j)} \Delta_i \ \ on\ [0, i+1) \tag{2.97}$$

Note that the infinite sum is $\mathcal{D}'-$convergent since for a given test function only a finite number of distributions are nonzero.

Proof

We start by showing (2.97). For $i = 0$ we take $\Delta_0 = F_0\mathcal{X}[0,1]$ (where $F_0\mathcal{X}[0,1]$ $:= \phi \mapsto \int_0^1 F_0(s)\phi(s)\mathrm{d}s$). Assuming (2.97) holds for $i < n$ we simply note that

$$\Delta_n := \mathcal{X}_{[0,n+1]}\left(F_n - \sum_{j\le n-1}\mathcal{P}^{m(n-j)}\Delta_j\right)$$
$$= \mathcal{X}_{[0,n+1]}\left(F_n - \mathcal{P}^m(F_{n-1}\mathcal{X}_{[0,n]})\right) = \mathcal{X}_{[n,n+1]}\left(F_n - \mathcal{P}^m(F_{n-1}\mathcal{X}_{[0,n]})\right) \quad (2.98)$$

(with $\mathcal{X}_{[n,\infty]}F_n$ defined in the same way as $F_0\mathcal{X}[0,1]$ above) has, by the induction hypothesis and (2.91) the required properties. Relation (2.96) is immediate. It remains to show uniqueness. Assuming (2.96) holds for the sequences $\Delta_i, \tilde{\Delta}_i$ and restricting f to $\mathcal{D}(0,1)$ we see that $\Delta_0 = \tilde{\Delta}_0$. Assuming $\Delta_i = \tilde{\Delta}_i$ for $i < n$ we then have $\Delta_n^{(mn)} = \tilde{\Delta}_n^{(mn)}$ on $\mathcal{D}(0,n+1)$. It follows from Remark 43 that $\Delta_n(x) = \tilde{\Delta}_n(x) + P(x)$ on $[0,n+1)$ where P is a polynomial (of degree $< mn$). Since by definition $\Delta_n(x) = \tilde{\Delta}_n(x) = 0$ for $x < n$ we have $\Delta_n = \tilde{\Delta}_n(x)$. ☐

The expression (2.94) hints to decrease in regularity, but this is not the case. In fact, we check that the regularity of convolution is not worse than that of its arguments.

Remark 36

$$(\cdot * \cdot) : \mathcal{D}_n \mapsto \mathcal{D}_n \quad (2.99)$$

Since

$$\mathcal{X}_{[a,b]} * \mathcal{X}_{[a',b']} = \left(\mathcal{X}_{[a,b]} * \mathcal{X}_{[a',b']}\right)\mathcal{X}_{[a+a',b+b']} \quad (2.100)$$

we have

$$F * \tilde{F} = \sum_{j+k\le\lfloor p\rfloor}\mathcal{P}^{m(i-j)}\Delta_j * \mathcal{P}^{m(i-k)}\tilde{\Delta}_k = \sum_{j+k\le\lfloor p\rfloor}\Delta_j * \mathcal{P}^{m(2i-j-k)}\tilde{\Delta}_k \quad (2.101)$$

which is manifestly in $C^{2mi-m(j+k)}[0,p) \subset C^{2mi-m\lfloor p\rfloor}[0,p)$.

☐

2.3.2 Norms on $\mathcal{D}'_m$

For $f \in \mathcal{D}'_m$ define

$$\|f\|_{\nu,m} := c_m \sum_{i=0}^{\infty} \nu^{im} \|\Delta_i\|_{L^1_\nu} \tag{2.102}$$

(the constant c_m, immaterial for the moment, is defined in (2.112). When no confusion is possible we will simply write $\|f\|_\nu$ for $\|f\|_{\nu,m}$ and $\|\Delta\|_\nu$ for $\|\Delta_i\|_{L^1_\nu}$ (no other norm is used for the Δ's). Let $\mathcal{D}'{}_{m,\nu}$ be the distributions in $\mathcal{D}'_m$ such that $\|f\|_\nu < \infty$.

Remark 37 *$\|\cdot\|_\nu$ is a norm on $\mathcal{D}'{}_{m,\nu}$.*

If $\|f\|_\nu = 0$ for all i, then $\Delta_i = 0$ whence $f = 0$. In view of Lemma 35 we have $\|0\|_\nu = 0$. All the other properties are immediate.

Remark 38 *$\mathcal{D}'{}_{m,\nu}$ is a Banach space. The topology given by $\|\cdot\|_\nu$ on $\mathcal{D}'{}_{m,\nu}$ is stronger than the topology inherited from $\mathcal{D}'$.*

Proof. If we let $\mathcal{D}'_{m,\nu}(k,k+1)$ be the subset of $\mathcal{D}'{}_{m,\nu}$ where all $\Delta_i = 0$ except for $i = k$, with the norm (2.102), we have

$$\mathcal{D}'{}_{m,\nu} = \bigoplus_{k=0}^{\infty} \mathcal{D}'_{m,\nu}(k,k+1) \tag{2.103}$$

and we only need to check completeness of each $\mathcal{D}'_{m,\nu}(k,k+1)$ which is immediate: on $L^1[k,k+1]$, $\|\cdot\|_\nu$ is equivalent to the usual L^1 norm and thus if $f_n \in \mathcal{D}'_{m,\nu}(k,k+1)$ is a Cauchy sequence then $\Delta_{k,n} \overset{L_\nu}{\to} \Delta_k$ (whence weak convergence) and $f_n \overset{\mathcal{D}'_{m,\nu}(k,k+1)}{\to} f$ where $f = \Delta_k^{(mk)}$. □

Lemma 39 *The space $\mathcal{D}'{}_{m,\nu}$ is a C^* algebra with respect to convolution.*

Proof. Let $f, \tilde{f} \in \mathcal{D}'_{m,\nu}$ with

$$f = \sum_{i=0}^{\infty} \Delta_i^{(mi)} \quad , \quad \tilde{f} = \sum_{i=0}^{\infty} \tilde{\Delta}_i^{(mi)}$$

Then

$$f * \tilde{f} = \sum_{i,j=0}^{\infty} \Delta_i^{(mi)} * \tilde{\Delta}_j^{(mj)} = \sum_{i,j=0}^{\infty} \left(\Delta_i * \tilde{\Delta}_j\right)^{m(i+j)} \tag{2.104}$$

and the support of $\Delta_i * \tilde{\Delta}_j$ is in $[i+j, i+j+2]$ i.e. $\Delta_i * \tilde{\Delta}_j = \chi_{[i+j,i+j+2]} \Delta_i * \tilde{\Delta}_j$.

We first evaluate the norm in $\mathcal{D}'_{m,\nu}$ of the terms $\left(\Delta_i * \tilde{\Delta}_j\right)^{m(i+j)}$.

I. Decomposition formula. Let $f = F^{(mk)} \in \mathcal{D}'(\mathbb{R}_+)$, where $F \in L^1(\mathbb{R}_+)$, and F is supported in $[k, k+2]$ i.e., $F = \chi_{[k,k+2]}F$ $(k \geq 0)$. Then $f \in \mathcal{D}'_m$ and the decomposition of f (cf. (2.96)) has the terms:

$$\Delta_0 = \Delta_1 = ... = \Delta_{k-1} = 0 \quad , \quad \Delta_k = \chi_{[k,k+1]}F \tag{2.105}$$

and

$$\Delta_{k+n} = \chi_{[k+n,k+n+1]}G_n, \quad \text{where } G_n = \mathcal{P}^m\left(\chi_{[k+n,\infty)}G_{n-1}\right), \quad G_0 = F \tag{2.106}$$

Proof of Decomposition Formula. We use first line of (2.98) of the paper

$$\Delta_j = \chi_{[j,j+1]}\left(F_j - \sum_{i=0}^{j-1}\mathcal{P}^{m(j-i)}\Delta_i\right) \tag{2.107}$$

where, in our case, $F_k = F$, $F_{k+1} = \mathcal{P}^m F$, ..., $F_{k+n} = \mathcal{P}^{mn}F$, ...

The relations (2.105) follow directly from (2.107). Formula (2.106) is shown by induction on n. For $n = 1$ we have

$$\Delta_{k+1} = \chi_{[k+1,k+2]}\left(\mathcal{P}^m F - \mathcal{P}^m\Delta_k\right)$$

$$= \chi_{[k+1,k+2]}\mathcal{P}^m\left(\chi_{[k,\infty)}F - \chi_{[k,k+1]}F\right) = \chi_{[k+1,k+2]}\mathcal{P}^m\left(\chi_{[k+1,\infty)}F\right)$$

Assume (2.106) holds for Δ_{k+j}, $j \leq n-1$. Using (2.107), with $\chi = \chi_{[k+n,k+n+1]}$ we have

$$\Delta_{k+n} = \chi\left(\mathcal{P}^{mn}F - \sum_{i=k}^{n-1}\mathcal{P}^{m(n-i)}\Delta_i\right) = \chi\mathcal{P}^m\left(G_{n-1} - \Delta_{n-1}\right)$$

$$= \chi\mathcal{P}^m\left(\chi_{[k+n-1,\infty)}G_{n-1} - \chi_{[k+n-1,k+n]}G_{n-1}\right) = \chi\mathcal{P}^m\left(\chi_{[k+n,\infty)}G_{n-1}\right) \quad \square$$

II. Estimating Δ_{k+n}. For f as in **I**, we have

$$||\Delta_{k+1}||_\nu \leq \nu^{-m}||F||_\nu \quad , \quad ||\Delta_{k+2}||_\nu \leq \nu^{-2m}||F||_\nu \tag{2.108}$$

and, for $n \geq 3$

$$||\Delta_{k+n}||_\nu \leq e^{2\nu-n\nu}(n-1)^{nm-1}\frac{1}{(nm-1)!}||F||_\nu \tag{2.109}$$

Proof of estimates of Δ_{k+n}.
(A) Case $n = 1$.

$$||\Delta_{k+1}||_\nu \leq \int_{k+1}^{k+2} dt\, e^{-\nu t}\mathcal{P}^m\left(\chi_{[k+1,\infty)}|F|\right)(t)$$

$$= \int_{k+1}^{k+2} dt\, e^{-\nu t}\int_{k+1}^{t} ds_1 \int_{k+1}^{s_1} ds_2 \ldots \int_{k+1}^{s_{m-1}} ds_m|F(s_m)|$$

$$\leq \int_{k+1}^{k+2} ds_m|F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_2}^{\infty} ds_1 \int_{s_1}^{\infty} dt\, e^{-\nu t}$$

$$= \int_{k+1}^{k+2} ds_m|F(s_m)|e^{-\nu s_m}\nu^{-m} \leq \nu^{-m}||F||_\nu \quad (2.110)$$

(B) Case $n = 2$:

$$||\Delta_{k+1}||_\nu \leq \int_{k+2}^{k+3} dt\, e^{-\nu t}\mathcal{P}^m\left(\chi_{[k+2,\infty)}\mathcal{P}^m\left(\chi_{[k+1,\infty)}|F|\right)\right)$$

$$= \int_{k+2}^{k+3} dt\, e^{-\nu t}\int_{k+2}^{t} dt_1 \int_{k+2}^{t_1} dt_2 \ldots \int_{k+2}^{t_{m-1}} dt_m \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \ldots \int_{k+1}^{s_{m-1}} ds_m|F(s_m)|$$

$$\leq \int_{k+2}^{k+3} ds_m|F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_2}^{\infty} ds_1 \int_{\max\{s_1,k+2\}}^{\infty} dt_m \int_{t_m}^{\infty} dt_{m-1} \ldots \int_{t_1}^{\infty} dt\, e^{-\nu t}$$

$$= \int_{k+2}^{k+3} ds_m|F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_2}^{\infty} ds_1 e^{-\nu\max\{s_1,k+2\}}\nu^{-m-1}$$

$$\leq \int_{k+2}^{k+3} ds_m|F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_3}^{\infty} ds_2 e^{-\nu s_2}\nu^{-m-2} = \int_{k+2}^{k+3} ds_m|F(s_m)|e^{-\nu s_m}\nu^{-2m}$$

(C) Case $n \geq 3$. We first estimate $G_2, ..., G_n$:

$$|G_2(t)| \leq \mathcal{P}^m\left(\chi_{[k+2,\infty)}\mathcal{P}^m\left(\chi_{[k+1,\infty)}|F|\right)\right)(t)$$

$$= \int_{k+2}^{t} dt_1 \int_{k+2}^{t_1} dt_2 \ldots \int_{k+2}^{t_{m-1}} dt_m \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \ldots \int_{k+1}^{s_{m-1}} ds_m|F(s_m)|$$

and using the inequality

$$|F(s_m)| = |F(s_m)|\chi_{[k,k+2]}(s_m) \leq |F(s_m)|e^{-\nu s_m}e^{\nu(k+2)}$$

we get

$$|G_2(t)| \leq e^{\nu(k+2)}||F||_\nu \int_{k+1}^{t} dt_1 \int_{k+1}^{t_1} dt_2 \ldots \int_{k+1}^{t_{m-1}} dt_m \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \ldots \int_{k+1}^{s_{m-2}} ds_{m-1}$$

$$= e^{\nu(k+2)}||F||_\nu (t-k-1)^{2m-1}\frac{1}{(2m-1)!}$$

The estimate of G_2 is used for bounding G_3:

$$|G_3(t)| \le \mathcal{P}^m\left(\chi_{[k+3,\infty)}|G_2|\right) \le \mathcal{P}^m\left(\chi_{[k+1,\infty)}|G_2|\right)$$

$$\le e^{\nu(k+2)}||F||_\nu (t-k-1)^{3m-1}\frac{1}{(3m-1)!}$$

and similarly (by induction)

$$|G_n(t)| \le e^{\nu(k+2)}||F||_\nu (t-k-1)^{nm-1}\frac{1}{(nm-1)!}$$

Then

$$||\Delta_{k+n}||_\nu \le e^{\nu(k+2)}||F||_\nu \frac{1}{(nm-1)!}\int_{k+n}^{k+n+1} dt\, e^{-\nu t}(t-k-1)^{nm-1}$$

and, for $\nu \ge m$ the integrand is decreasing, and the inequality (2.109) follows.
III. Final Estimate. Let $\nu_0 > m$ be fixed. For f as in **I**, we have for any $\nu > \nu_0$,

$$||f|| \le c_m \nu^{km}||F||_\nu \tag{2.111}$$

for some c_m, if $\nu > \nu_0 > m$.
Proof of Final Estimate

$$||f|| = \sum_{n\ge 0}\nu^{km+kn}||\Delta_{k+n}||_\nu \le \nu^{km}||F||_\nu \left[3 + \sum_{n\ge 3}\nu^{nm}e^{2\nu-n\nu}\frac{(n-1)^{nm-1}}{(nm-1)!}\right]$$

and, using $n-1 \le (mn-1)/m$ and a crude Stirling estimate we obtain

$$||f|| \le \nu^{km}||F||_\nu \left[3 + me^{2\nu-1}\sum_{n\ge 3}\left(e^{m-\nu}\nu^m/m^m\right)^n\right] \le c_m\nu^{km}||F||_\nu \tag{2.112}$$

Thus (2.111) is proven for $\nu > \nu_0 > m$.
End of the proof. From (2.104) and (2.111) we get

$$||f * \tilde{f}|| \le \sum_{i,j=0}^{\infty} ||\left(\Delta_i * \tilde{\Delta}_j\right)^{m(i+j)}||$$

$$\le \sum_{i,j=0}^{\infty} c_m^2 \nu^{m(i+j)}||\Delta_i * \tilde{\Delta}_j||_\nu \le c_m^2 \sum_{i,j=0}^{\infty}\nu^{m(i+j)}||\Delta_i||_\nu\, ||\tilde{\Delta}_j||_\nu = c_m^2||f||\,||\tilde{f}||$$

□

Remark 40 *Let $f \in \mathcal{D}'_{m,\nu}$ for some $\nu > \nu_0$ where $\nu_0^m = \mathrm{e}^{\nu_0}$. Then $f \in \mathcal{D}'_{m,\nu'}$ for all $\nu' > \nu$ and furthermore,*

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{2.113}$$

Proof. We have

$$\nu^{mk} \int_k^{k+1} |\Delta_k(s)| \mathrm{e}^{-\nu s} \mathrm{d}s = (\nu^m \mathrm{e}^{-\nu})^k \int_0^1 |\Delta_k(s+k)| \mathrm{e}^{-\nu s} \mathrm{d}s \tag{2.114}$$

which is decreasing in ν. The rest follows from the monotone convergence theorem. □

2.3.3 Embedding of L^1_ν in $\mathcal{D}'_m$

Lemma 41 *i) Let $f \in L^1_{\nu_0}$ (cf. Remark 40). Then $f \in \mathcal{D}'_{m,\nu}$ for all $\nu > \nu_0$.*
ii) $\mathcal{D}(\mathbb{R}^+ \backslash \mathbb{N}) \cap L^1_\nu(\mathbb{R}^+)$ is dense in $\mathcal{D}_{m,\nu}$ with respect to the norm $\|\|_\nu$.

Proof.
Note that if for some ν_0 we have $f \in L^1_{\nu_0}(\mathbb{R}^+)$ then

$$\int_0^p |f(s)| \mathrm{d}s \le \mathrm{e}^{\nu_0 p} \int_0^p |f(s)| \mathrm{e}^{-\nu_0 s} \mathrm{d}s \le \mathrm{e}^{\nu_0 p} \|f\|_{\nu_0} \tag{2.115}$$

to which, application of $\mathcal{P}^{k-1}$ yields

$$\mathcal{P}^k |f| \le \nu_0^{-k+1} \mathrm{e}^{\nu_0 p} \|f\|_{\nu_0} \tag{2.116}$$

Also, $\mathcal{P} \chi_{[n,\infty)} \mathrm{e}^{\nu_0 p} \le {\nu_0}^{-1} \chi_{[n,\infty)} \mathrm{e}^{\nu_0 p}$ so that

$$\mathcal{P}^m \chi_{[n,\infty)} \mathrm{e}^{\nu_0 p} \le {\nu_0}^{-m} \chi_{[n,\infty)} \mathrm{e}^{\nu_0 p} \tag{2.117}$$

so that, by (2.98) (where now F_n and $\chi_{[n,\infty]} F_n$ are in $L^1_{loc}(0, n+1)$) we have for $n > 1$,

$$|\Delta_n| \le \|f\|_{\nu_0} \mathrm{e}^{\nu_0 p} {\nu_0}^{1-mn} \chi_{[n,n+1]} \tag{2.118}$$

Let now ν be large enough. We have

$$\sum_{n=2}^\infty \nu^{mn} \int_0^\infty |\Delta_n| \mathrm{e}^{-\nu p} \mathrm{d}p \le \nu_0 \|f\|_{\nu_0} \sum_{n=2}^\infty \int_n^{n+1} \mathrm{e}^{-(\nu-\nu_0)p} \left(\frac{\nu}{\nu_0}\right)^p \mathrm{d}p$$
$$= \frac{\mathrm{e}^{-2(\nu-\nu_0-\ln(\nu/\nu_0))}}{\nu - \nu_0 - \ln(\nu/\nu_0)} \nu_0 \|f\|_{\nu_0} \tag{2.119}$$

For $n = 0$ we simply have $\|\Delta_0\| \le \|f\|$, while for $n = 1$ we write

$$\|\Delta_1\|_\nu \le \|1^{*(m-1)} * |f|\|_\nu \le \nu^{m-1}\|f\|_\nu \tag{2.120}$$

Combining the estimates above, the proof of (i) is complete. To show (ii), let $f \in \mathcal{D}'_{m,\nu}$ and let k_ϵ be such that $c_m \sum_{i=k_\epsilon}^{\infty} \nu^{im}\|\Delta_i\|_\nu < \epsilon$. For each $i \le k_\epsilon$ we take a function δ_i in $\mathcal{D}(i, i+1)$ such that $\|\delta_i - \Delta_i\|_\nu < \epsilon 2^{-i}$. Then $\|f - \sum_{i=0}^{k_\epsilon} \delta_i^{(mi)}\|_{m,\nu} < 2\epsilon$. □

Proof of continuity of $f(p) \mapsto pf(p)$. If $f(p) = \sum_{k=0}^{\infty} \Delta_k^{(mk)}$ then $pf = \sum_{k=0}^{\infty}(p\Delta_k)^{(mk)} - \sum_{k=0}^{\infty} mk\mathcal{P}(\Delta_k^{(mk)}) = \sum_{k=0}^{\infty}(p\Delta_k^{(mk)}) - 1 * \sum_{k=0}^{\infty}(mk\Delta_k)^{(mk)}$. The rest is obvious from continuity of convolution, the embedding shown above and the definition of the norms.

2.3.4 Laplace transforms

Proof of Lemma 6 . Let $\nu > \nu_0$ (cf. Remark 40). Equation (2.14) follows most easily from the corresponding well property of Laplace transforms on $\mathcal{D}$, from the continuity of $\mathcal{L}$ and Lemma 41 (ii). For the second notice that by the definition of $\mathcal{D}'_m$, $f' \in \mathcal{D}'_m$ implies $f \in AC(0, 1-\epsilon)$ and the property follows by density from the $\mathcal{D}$ identity $\mathcal{L}(\mathcal{P}g) = x^{-1}\mathcal{L}(g)$. The third equality also follows by density. The rest of the properties, except injectivity, follow immediately from the definitions and the topology used.

To show injectivity assume that $\mathcal{L}d(x) = 0$ where $d \in \mathcal{D}'_{m,\nu}$, $\nu < x_0 < x \in \mathbb{R}^+$. By analyticity, $\mathcal{L}d(x) = 0$ in (say) $S_2 := \{z : |z| > 2x_0 : |\arg(z)| < \pi/4\}$. Using dominated convergence, assuming x_0 is large enough, we have

$$\left|\sum_{k=1}^{\infty} x^{m(k-1)} \mathrm{e}^{-(k-1)x} \int_0^1 \mathrm{e}^{-sx}|\Delta_k(s+k)|\mathrm{d}s\right| \le 1$$

in S_2. Thus $|f(x)| = \left|\int_0^1 \mathrm{e}^{px}\Delta_0(1-p)\mathrm{d}p\right| \le |x|^m$ in S_2. The function $f^{(m)}(x)$ is entire, of exponential order less than $1+\epsilon$ for any ϵ and, using the previous inequality in Cauchy's formula we see that $|f^{(m)}(x)| <$ const. in (x_0, ∞). Since for $\phi \in (\pi/2, 3\pi/2)$ we obviously also have $f^{(m)}(r\mathrm{e}^{\mathrm{i}\phi}) \to 0$ as $r \to \infty$, an elementary instance of the Phragmén-Lindelöf principle [28] implies that $f^{(m)}$ is bounded in $\mathbb{C}$, therefore constant, so f itself is a polynomial that decays in the left half plane, thus $f = 0$. Therefore $\int_0^1 \mathrm{e}^{-px}\Delta_0(p)\mathrm{d}p = \mathcal{L}\Delta_0 = 0$ so that $\Delta_0 = 0$. Inductively and in the same way, we see that $\Delta_k = 0$, $k \in \mathbb{N}$.

Proof of Lemma 11. Take first $r \notin \mathbb{Z}$. Choose a_1, a_2 so that $0 < a_1 < a_2 < a$ and consider the closed contour C going along the upper cut from $\xi = 0$ to $\xi = a_2$, continuing towards the lower cut anticlockwise along the circle $C(a_2)$ of radius a_2 centered at origin, and finally coming from $\xi = a_2$ back to $\xi = 0$ along the lower cut. For $|\xi| < a_1$ we have, by the assumptions of the lemma,

$$2\pi i f(\xi) = \oint_C \frac{f(s)}{s-\xi}\mathrm{d}s = \oint_{C(a_2)} \frac{f(s)}{s-\xi}\mathrm{d}s + \int_0^{a_2} \frac{s^r A(s)}{s-\xi}\mathrm{d}s$$

$$ \tag{2.121} $$

On the other hand, defining $z^r A(z)$ in the interior of $C(a)$ cut along the positive axis (with the usual convention $\arg(z) = 0$ on the upper cut), we have, for the same contour as above and $\xi \in \mathcal{V}_{a_1}$

$$ 2\pi \mathrm{i}\xi^r A(\xi) = \oint_{C(a_2)} \frac{f(s)}{s-\xi}\mathrm{d}s + \left(1 - \mathrm{e}^{2\pi i r}\right) \int_0^{a_2} \frac{s^r A(s)}{s-\xi}\mathrm{d}s \tag{2.122} $$

Comparing (2.121) to (2.122) we get:

$$ \begin{aligned} f(\xi) = {} & \frac{1}{1-\mathrm{e}^{2\pi i r}}\xi^r A(\xi) \\ & - \frac{1}{2\pi i(1-\mathrm{e}^{2\pi i r})} \oint_{C(a_2)} \frac{A(s)}{s-\xi}\mathrm{d}s + \frac{1}{2\pi i} \oint_{C(a_2)} \frac{f(s)}{s-\xi}\mathrm{d}s \end{aligned} \tag{2.123} $$

As integrals of analytic functions with respect to complex absolutely continuous measures ($A(s)\mathrm{d}s$ and $f(s)\mathrm{d}s$), the last two terms in (2.123) are analytic in ξ for $|\xi| < a_1$. Since a_1 can be chosen arbitrarily close to a, the case $r \notin \mathbb{Z}$ is proven. For $r \in \mathbb{Z}$ the argument is essentially the same, in terms of $A(\xi)\xi^r \ln \xi$ instead of $\xi^r A(\xi)$. The proof generalizes immediately to linear combinations of $\xi^r A(\xi)$. □

Proof of Lemma 12. On the interval $(k, k+1)$ we have $f^+ = \sum_{i=1}^k (f_i^-)^{(mi)}$ or

$$ \mathcal{P}^{mk+1} f^+ = \sum_{i=1}^{k} \mathcal{P}^{m(k-i)+1} f_i^- \tag{2.124} $$

Let ϵ be small and positive. Since $f(t\mathrm{e}^{\mathrm{i}\phi})$ and $g_i^-(t\mathrm{e}^{\mathrm{i}\phi})$ converge as $\phi \to 0$ in $\mathcal{D}'_{m,\nu}$ we have that $\mathcal{P}^{m(k-i)+1} f(t\mathrm{e}^{\mathrm{i}\phi})$ and $\mathcal{P}^{m(k-i)+1} g_i^-(\mathrm{e}^{\mathrm{i}\phi} t)$ converge on $[0, k+1-\epsilon]$ uniformly to $\mathcal{P}^{mk+1} f^+$ and $\mathcal{P}^{m(k-i)+1} f_i^-$ respectively. The left side of (2.124) is the limit on $I = [k+\epsilon, k+1-\epsilon]$ of a function analytic in a neighborhood in the *upper* half plane of I and continuous on I while the right side is the limit of a function analytic in a neighborhood in the *lower* half plane of I and continuous on I. The equality of their continuous limits on I implies in particular that $\mathcal{P}^{mk+1} f(t\mathrm{e}^{\mathrm{i}\phi})$ extends analytically through I in the lower half plane, and its continuation is analytic where $\sum_{i=1}^k \mathcal{P}^{m(k-i)+1} g_i^-(t\mathrm{e}^{\mathrm{i}\phi})$ is. A corresponding statement is true for the upper plane continuation of $\mathcal{P}^{mk+1} f(t\mathrm{e}^{-\mathrm{i}\phi})$ and (i) follows. Part (ii) now follows also, as an immediate application of Lemma 11.

□

Proof of Proposition 13.

The fact that multiplication by a bounded analytic function is well defined on $\mathcal{F}(\mathcal{R}_1')$ is immediate. Since

$$2f * g = (f+g) * (f+g) - f * f - g * g \tag{2.125}$$

we may take $f = g$. With $h = \mathcal{P}^{mk+1} f \in \mathcal{F}(\mathcal{R}_1')$ it suffices to show for every k that $h * h$ (defined near zero by (1.8) and which equals $\mathcal{P}^{2mk+2}(f * f)$ there) extends analytically to $\mathcal{R}_1'$ for $\Re(x) < k$. Since f is analytic in $\mathcal{R}_1'$ so is h. In particular h can be analytically continued along any ray $d \subset \mathcal{R}_1'$ other than the real line, and we have, by analyticity and with $*_d$ meaning convolution along d,

$$AC(h * h) = AC(h) *_d AC(h) \tag{2.126}$$

Also, by (2.24)

$$h^-(p) = h^+(p) + \sum_{j=1}^{\infty} (h_j^+(p-j))^{(mj)} \tag{2.127}$$

Let $H_0 = h^+$ and $H_j(p) = (h_j^+(p))^{(mj)}$. By construction H_j' have L^1 boundary values on $[0, k-j+1)$ as $\Re(z) > 0, \Im(z) \downarrow 0$ and so H_j extend *continuously* to the strip $0 < \Re(z) < k-j+1, \Im(z) \geq 0$. We have, by (2.24) and continuity

$$h^-(z) = \sum_{i=0}^{j} H_i(z-j) \tag{2.128}$$

for $\Re(z) \in [0, j)$ and $\Im(z) \geq 0$ since $H_i(x) = 0$ in the left half plane, by definition. For the same reason we have, with $p' = p - i - j$ and $i + j \geq 1$,

$$\begin{aligned} &\int_0^p H_i(x-i) H_j(p-x-j) \mathrm{d}x \\ &= \left\{ \begin{array}{ll} \displaystyle\int_0^{p'} H_i(x) H_j(p'-x) \mathrm{d}x = J_{i,j}(p') & (\Re(p)' > i+j) \\ 0 & (\Re(p') < i+j) \end{array} \right. \end{aligned} \tag{2.129}$$

As both H_i and H_j are analytic in an open strip $\mathcal{S}$ in the first quadrant and continuous on $[0, k+1-\epsilon]$ we see from (2.129) that $J_{ij}(p')$ are also analytic in $\mathcal{S}$ and continuous on $[0, k+1-\epsilon]$. For $p \in (0, l+1)$, $l \leq k$ we have

$$\big(H^- * H^-\big)(p) = \sum_{i=0}^{l} \sum_{j=0}^{i} J_{ij}(p-i) \tag{2.130}$$

Now, by (2.126) and using the continuity of H and of convolution, we note that the left side of (2.130) represents the continuous limit along $(l, l+1)$ of $(H*H)^-$, a function analytic in a domain in the lower half plane while the right side is the limit of a function analytic in the upper half plane and $(l, l+1)$ is contained in the common boundary. As in the proof of Lemma 12 we conclude that $h*h$, thus $f*f$ extend analytically in $\mathcal{R}'_1$.

Going back to the definition of H we get on $(0, l+1)$,

$$\left(f^- * f^-\right)(p) = \left(f^+ * f^+\right)(p) + \sum_{i=1}^{l}\sum_{j=0}^{i}(f_j^+)^{(mj)} * (f_{i-j}^+)^{m(i-j)}(p-i) \tag{2.131}$$

where $f_j * f_{i-j} = (H_j * H_{i-j})^{(2mk+2)}$ is the convolution in $\mathcal{D}'_{m,\nu}$ and in our case gives a function analytic in the open region $\mathcal{S}$. By comparing with (2.24) and (2.128) we get by induction $(f*f)_j = \sum_{s=0}^{j} f_s * f_{j-s}$ or, using (2.125) we get (2.24).

Since by assumption f_s and g_s belong to $\mathcal{D}'_{m,\nu}$ and the sum in (2.28) only contains a finite number of terms, it follows that all analytic continuations of $(f*g)$ also belong to $\mathcal{D}'_{m,\nu}$. Furthermore, it follows immediately that $K(f*g,\nu) \le 2K(f,\nu)K(g,\nu)$.

Only the last equality in (2.29) needs a proof; we have

$$(\mathcal{A}_\alpha(f))^{*2} = \left(\sum_{i=0}^{\infty}\alpha^i (f_i(p-i))^{(mi)}\right)^{*2} = \sum_{k=0}^{\infty}\alpha^k \sum_{j=0}^{k}(f_j * f_{k-j})^{(mk)}(p-k)$$
$$= \sum_{k=0}^{\infty}\alpha^k((f*f)_k)^{(mk)} = \mathcal{A}_\alpha(f*f) \tag{2.132}$$

We have $\|\mathcal{A}_C(f)\|_{m,\nu} \le \|f\|_{m,\nu}\sum_{j=0}^{\infty} C^j K(f,\nu)^j = (1-KC)^{-1}\|f\|_{m,\nu}$ so that if $\mathbf{Y} \in \mathcal{F}_r$ then $\|\mathcal{A}_C(\mathbf{Y}^{*\mathbf{l}})\|_{m,\nu} = \|(\mathcal{A}_C\mathbf{Y})^{*\mathbf{l}}\|_{m,\nu} \le (\|\mathbf{Y}\|_{m,\nu}/(1-KC))^{|\mathbf{l}|}$ so that if ν is large enough the sum involved in the expression of M is uniformly convergent in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ and (2.28) follows.

$\square$

Lemma 42 *(i) Let $k_0 \ge 0$ and let λ be such that $\Re(\lambda) < \alpha_1 < k_0$ and $|\Im(\lambda)| < \alpha_2$. Alternatively, let $k_0 \ge 0$ and λ be such that $0 < \alpha_1 < |\Im(\lambda)| < \alpha_2$. There exists a constant $C(\alpha_1,\alpha_2)$ independent of k_0, ν and λ so that*

$$\|U\|_{\mathcal{D}'_{m,\nu}(k_0,\infty)\mapsto\mathcal{D}'_{m,\nu}(k_0,\infty)} \le C(\alpha_1,\alpha_2)(1+|\lambda|)^{-1} \tag{2.133}$$

(ii) In both cases in (i), U is strongly continuous in λ.

Proof

The impediments in the proof come on the one hand from having to estimate quotients of the form $\int |\mathcal{P}^n U f| / \int |\mathcal{P}^n f|$ and on the other hand from the nonlocal character of the action of U in our space.

In view of Eq. (2.103) it is enough to find a $k-$independent upper bound for the norms of the restrictions of U to $\mathcal{D}'_{m,\nu}(k, k+1)$, $U : \mathcal{D}'_{m,\nu}(k, k+1) \mapsto \mathcal{D}'_{m,\nu}$. We are interested in $\Re(\lambda) < 1$ in the cases (a) $b > |\Im(\lambda)| > a > 0$, (b) $\lambda < -a < 0$ real, (c) $\lambda \in \mathbb{R}^+$ or λ complex, $|\Im(\lambda)| < b$ but with $\text{supp}(f) \in (k_1, \infty)$ with $k_1 > a > \Re(\lambda)$, $k_1 \in (a, a+1)$. We let $k_1 = 0$ in (a) and (b).
We have the following identity

$$\frac{f^{(r)}}{p-\lambda} = \left(r(p-\lambda)^{r-1} \int_k^p \frac{f(s)}{(s-\lambda)^{r+1}} \mathrm{d}s + \frac{f(p)}{p-\lambda} \right)^{(r)} \tag{2.134}$$

which is proved by straightforward differentiation of the r.h.s. or by writing $f^{(r)} = (p-\lambda) g^{(r)} = (pg - \lambda g)^{(r)} - r g^{(r-1)}$ so that $f = (p-\lambda)(\mathcal{P}g)' - r\mathcal{P}g$ and solving for $\mathcal{P}g$. We take $k \in \mathbb{N}$ with $k+1 > k_1$, a distribution f with $\text{supp}(f) \in (k_0, k+1)$ where $k_0 = \max\{k, k_1\}$ and we let

$$c = \int_{k_0}^{k+1} \frac{f(s)}{(s-\lambda)^{r+1}} \mathrm{d}s \quad (r := mk) \tag{2.135}$$

For ϵ small (to be made zero in the end), we write the decomposition

$$\begin{aligned} &\frac{f^{(r)}}{p-\lambda} \\ &= \left(r(p-\lambda)^{r-1} \int_{k_0}^p \frac{f(s)}{(s-\lambda)^{r+1}} \mathrm{d}s + \frac{f(p)}{p-\lambda} - cr(p-\lambda)^{r-1} \chi_{[k+1-\epsilon, \infty]} \right)^{(r)} \\ &\qquad + \left(cr(p-\lambda)^{r-1} \chi_{[k+1-\epsilon,\infty]} \right)^{(r)} = f_1 + f_2 = f_1 + c\, r\, g_2^{(r)} \end{aligned} \tag{2.136}$$

where by construction $f_1 \in \mathcal{D}'_{m,\nu}(k, k+1)$ whence, for $f \in \mathcal{D}(k_0, k+1)$ we have

$$\begin{aligned} \left\| \frac{f^{(r)}}{p-\lambda} \right\|_{m,\nu} &\le \|f_1\|_{m,\nu} + \|f_2\|_{m,\nu} = \|f_1\|_{\mathcal{D}'_{m,\nu}(k,k+1)} + \|f_2\|_{m,\nu} \\ &\le \|f_1 - f_2\|_{\mathcal{D}'_{m,\nu}(k,k+1)} + 2\|f_2\|_{m,\nu} \end{aligned} \tag{2.137}$$

and then, for some C_1

$$\|f_1 - f_2\|_{\mathcal{D}'_{m,\nu}(k,k+1)}$$
$$\leq \nu^r \int_{k_0}^{k+1} \left| r(p-\lambda)^{r-1} \int_{k_0}^{p} \frac{f(s)}{(s-\lambda)^{r+1}} \mathrm{d}s + \frac{f(p)}{p-\lambda} \right| \mathrm{e}^{-\nu p} \mathrm{d}p$$
$$\leq \sup\nolimits_* |p-\lambda|^{-1} \|f\|_{m,\nu} + \sup\nolimits_* \left| \frac{(p-\lambda)^{r-1}}{(s-\lambda)^{r+1}} \right| \|f\|_{m,\nu}$$
$$\leq C_1 \sup\nolimits_* |p-\lambda|^{-1} \|f\|_{m,\nu} \quad (2.138)$$

where the supremum is taken over $\{k \in \mathbb{N}, p, s \in [k, k+1] \cap (k_0, \infty)\}$. For the constant c in (2.136) we have, for some C_2 depending on a and otherwise independent of λ, k the estimate

$$|c| \leq \frac{C_2 \mathrm{e}^{\nu(k+1)}}{|k_0-\lambda|^{r+1}} \int_{k_0}^{k+1} |f(s)| \mathrm{e}^{-\nu s} \mathrm{d}s = \nu^{-r} \frac{C_2 \mathrm{e}^{\nu(k+1)}}{|k_0-\lambda|^{r+1}} \|f\|_{m,\nu} \quad (2.139)$$

Let $k' = k+1-\epsilon$. For some $C_3 = C_3(a) \leq \exp\left[(k_0+1)|k_0-\lambda|^{-1}\right]$ we have

$$\|g_2\|_{m,\nu,k} = \nu^r \int_{k'}^{k+1} \mathrm{e}^{-\nu x} |x-\lambda|^{r-1} \mathrm{d}x \leq C_3 \nu^{r-1} |k_0-\lambda|^{r-1} \mathrm{e}^{-\nu k'}$$
$$\Rightarrow c\, r \, \|g_2\|_{m,\nu,k} \leq \nu^{-r} mk \frac{\mathrm{e}^{\nu(k+1)}}{|k_0-\lambda|^{r+1}} \|f\|_{m,\nu} C_2 C_3 \nu^{r-1} |k_0-\lambda|^{r-1} \mathrm{e}^{-\nu k'}$$
$$= \frac{C_4 mk \mathrm{e}^{\nu\epsilon}}{\nu |k_0-\lambda|^2} \|f\|_{m,\nu} \quad (2.140)$$

For $n \geq k+1$ we write (2.98) as

$$\Delta_n(g_2^{(r)}) = \mathcal{X}_{[n,n+1]} \mathcal{P}^m \left(\mathcal{X}_{[n,\infty)} \mathcal{P}^{m(n-k-1)} g_2 \right) \quad (2.141)$$

(cf. (2.98)). For λ complex we take $K_1(a) = \sup_{s \geq k_0} (s-\lambda)^{-1}(s+1+|\lambda|)$. We let $\tilde{\lambda} = \lambda$ if λ is real and $\tilde{\lambda} = -1-|\lambda|$ otherwise and write $q = m(n-k-1)$. Noting that $K_1^{mk_0+m-1} \leq C_5(a,b) = K$ we have

$$\Gamma(q) \mathcal{P}^q |g_2| \leq K \int_{k'}^{x} (x-s)^{q-1} (s-\tilde{\lambda})^{r-1} \mathrm{d}s$$
$$\leq K \int_0^x (x-s)^{q-1} (s-\tilde{\lambda})^{r-1} \mathrm{d}s = \frac{(x-\tilde{\lambda})^{q+r-1} \Gamma(q) \Gamma(r)}{\Gamma(q+r)} \quad (2.142)$$

The estimate above is true for $\tilde{\lambda} \leq 1$ but is "optimal" only when the maximum of the integrand is inside the region of integration i.e. when $\tilde{\lambda} > -(2+m^{-1})k(n-$

$k)^{-1} + m^{-1}$. If this is not the case we prefer to simply estimate the integral in terms of the maximum of the integrand over the region of integration. So for, say, $\tilde{\lambda} < -3k$ we use the inequality

$$\Gamma(q)\mathcal{P}^q|g_2| \le K(x-k')^q(k'-\tilde{\lambda})^{r-1} \tag{2.143}$$

Now, for $\tilde{\lambda} > -3k$, using (2.142) and (2.141)

$$\mathcal{P}^m\Big(\chi_{[n,\infty)}\mathcal{P}^{m(n-k-1)}|g_2|\Big) \le \frac{\Gamma(r)}{\Gamma(q+r)\Gamma(m)}\int_n^x (x-s)^{m-1}(s-\tilde{\lambda})^{q+r-1}\mathrm{d}s$$
$$\le \frac{\Gamma(r)}{\Gamma(q+r)\Gamma(m)}(x-n)^m(x-\tilde{\lambda})^{q+r-1} \tag{2.144}$$

(as m is fixed we do not lose too much by this evaluation which has the advantage of preserving the behavior near $x = n$). Further, we have

$$\int_n^{n+1} \mathrm{e}^{-\nu x}(x-n)^m(x-\tilde{\lambda})^{q+r-1}\mathrm{d}x \le (n+1-\tilde{\lambda})^{q+r-1}\int_n^{n+1} \mathrm{e}^{-\nu x}(x-n)^m\mathrm{d}x$$
$$\le \frac{\Gamma(m)}{\nu^m}(n+1-\tilde{\lambda})^{q+r-1} \tag{2.145}$$

and

$$\sum_{n=k+1}^{\infty} \nu^{mn}\|\Delta\|_{m,\nu} \le \Gamma(mk)K\sum_{n=k+1}^{\infty} \frac{\nu^{m(n-1)}\mathrm{e}^{-\nu n}(n+1-\tilde{\lambda})^{mn-1-m}}{\Gamma(m(n-1))} \tag{2.146}$$

The ratio of two successive terms s_{n+1}/s_n of the infinite series above is estimated by:

$$\nu^m\mathrm{e}^{-\nu}\mathrm{e}^{\frac{mn-1-m}{n+1-\tilde{\lambda}}}\left(\frac{n+2-\tilde{\lambda}}{mn-m}\right)^m \le \frac{1}{2} \tag{2.147}$$

when $\nu > C_1$ for some C_1 independent of $k, n, \tilde{\lambda}$ in the region $k > 1, n > k, \tilde{\lambda} \in (-3k, 1)$. This means that

$$\sum_{n=k+1}^{\infty} \nu^{mn}\|\Delta\|_{m,\nu} \le 2K\nu^{mk}\mathrm{e}^{-\nu(k+1)}(k+2-\tilde{\lambda})^{mk-1} \tag{2.148}$$

and combining with (2.139) and (2.137) we have

$$2\|f_2\| \le 4K\nu^{-mk}\frac{\mathrm{e}^{\nu(k+1)}}{(k-\tilde\lambda)^{mk+1}}\|f\|_{m,\nu}\nu^{mk}\mathrm{e}^{-\nu(k+1)}(k+2-\tilde\lambda)^{mk-1}$$
$$\le 4K\mathrm{e}^{2\frac{(mk+1)}{k-\tilde\lambda}}\frac{\|f\|_{m,\nu}}{(k-\tilde\lambda)^2} \le 4K\mathrm{e}^{\frac{4m}{k_0\tilde\lambda}}\frac{\|f\|_{m,\nu}}{(k_0-\tilde\lambda)^2} \le \frac{C_6(a,b)\|f\|_{m,\nu}}{|k_0-\lambda|^2} \quad (2.149)$$

If we started with (2.143) we would have obtained in the same way, for $\tilde\lambda < -3k$, instead of (2.144),

$$\frac{1}{\Gamma(q)\Gamma(m)}(x-n)^m(x-k')^q(k'-\tilde\lambda)^{r-1} \quad (2.150)$$

and the calculations are similar from this point on. Condition (2.147) is of the same type, with k' replacing $-\tilde\lambda$ and final estimate is

$$C_7\|f\|_{m,\nu}\frac{(k'-\tilde\lambda)^{r-1}}{(k-\tilde\lambda)^{r+1}} \le C_8\frac{\|f\|_{m,\nu}}{(k_0-\tilde\lambda)^2} \quad (2.151)$$

Finally we take the limit $\epsilon \to 0$ and noting that $K \to 0$ as $\lambda \to \infty$, (i) is proved.
For (ii), merely notice that $U(\lambda_2) - U(\lambda_1) = (\lambda_2-\lambda_1)U(\lambda_1)U(\lambda_2)$.

$\square$

Remark 43 *Let $\psi \in L^1[0,1]$ with the property $\int_0^1 \psi(t)\phi^{(m)}(t)\mathrm{d}t = 0$ for all $\phi \in \mathcal{D}(0,1)$. Then ψ is a polynomial of degree at most $m-1$.*

This is a well-known property. We sketch an elementary proof for $m = 1$ (for general m the proof is similar). Let $x \in (0,1)$, and consider a sequence χ_n in $\mathcal{D}(0,x)$ L^1–convergent to $\mathcal{X}_{[0,x]}$. Then $\phi_n(t) := \chi_n(t) - \kappa^{-1}\chi_n(\kappa(1-t))$ with $\kappa = x(1-x)^{-1}$ converges to $\mathcal{X}_{[0,x]} - \kappa\mathcal{X}_{[x,1]}$. Furthermore, since $\int_0^1 \phi_n(t)\mathrm{d}t = 0$ we have $\Phi_n(t) := \int_0^t \phi_n(s)\mathrm{d}s \in \mathcal{D}(0,1)$. Since $\Phi_n' + \kappa^{-1} \to (1+\kappa^{-1})\mathcal{X}_{[0,x]}$ it follows that $\int_0^x(\psi - C) = 0$, where $C = \int_0^1 \psi(t)\mathrm{d}t$. Thus $\psi = C$ *a.e.*

2.4 Derivation of the equations for the transseries.

Consider first the scalar equation

$$y' = f_0(x) - \lambda y - x^{-1}By + g(x,y) = -y + x^{-1}By + \sum_{k=1}^{\infty} g_k(x)y^k \quad (2.152)$$

For $x \to +\infty$ we take

$$y = \sum_{k=0}^{\infty} y_k \mathrm{e}^{-kx} \quad (2.153)$$

where y_k will be either formal series $x^{-s_k}\sum_{n=0}^{\infty} a_{kn}x^{-n}$, with $a_{k,0}\neq 0$ or actual functions with the condition that (2.153) converges uniformly. As a transseries, (2.153) can be also understood as a well ordered double sequence $t_{kn}=x^{p_{kn}}\mathrm{e}^{-kx}$, with $p_{k\,n+1}<p_{kn}$. (The order relation is $x^p\mathrm{e}^{-kx}\gg x^{p'}\mathrm{e}^{-k'x}$ as $x\to+\infty$ iff $k<k'$ or $k=k'$ and $p>p'$; thus a strictly *increasing* sequence of terms of a transseries necessarily terminates.) Power series are a special case of transseries, with $y_1=y_2=\ldots=0$. Two transseries $\sum_{k=0}^{\infty} y_k\mathrm{e}^{-kx}$ coincide iff all corresponding component power series y_k coincide. Transseries of this type are closed under addition, multiplication and infinite sums of the form involved in (2.152) (this last aspect will become clear in the calculation leading to (2.155) below). Note that well-ordering plays an important part in defining multiplication of transseries; in contrast, for the unrestricted formal expansion $S=\sum_{k=-\infty}^{\infty} x^k$, no immediate meaning can be given to S^2. Let y_0 be the first term in (2.153) and $\delta=y-y_0$. We have

$$y^k-y_0^k-ky_0^{k-1}\delta=\sum_{j=2}^{k}\binom{k}{j}y_0^{k-j}\delta^j=\sum_{j=2}^{k}\binom{k}{j}y_0^{k-j}\sum_{i_1,\ldots,i_j=1}^{\infty}\prod_{s=1}^{j}\left(y_{i_s}\mathrm{e}^{-i_sx}\right)$$
$$=\sum_{m=1}^{\infty}\mathrm{e}^{-mx}\sum_{j=2}^{k}\binom{k}{j}y_0^{k-j}\sum_{(i_s)}^{(m;j)}\prod_{s=1}^{j}y_{i_s}\qquad(2.154)$$

where $\sum_{(i_s)}^{(m;j)}$ means the sum over all positive integers $i_1,i_2,\ldots,i_j$ with the restriction $i_1+i_2+\cdots+i_j=m$. Let $d_1=\sum_{k\geq 1}kg_ky_0^{k-1}$. Introducing $y=y_0+\delta$ in (2.152) and equating the coefficients of e^{-lx} we get, by separating the terms containing y_l for $l\geq 1$ and interchanging the j,k orders of summation,

$$y_l'+(\lambda(1-l)+x^{-1}B-d_1(x))y_l=\sum_{j=2}^{\infty}\sum_{(i_s)}^{(l;j)}\prod_{s=1}^{j}y_{i_s}\sum_{k\geq\{2,j\}}\binom{k}{j}g_ky_0^{k-j}$$
$$=\sum_{j=2}^{l}\sum_{(i_s)}^{(l;j)}\prod_{s=1}^{j}y_{i_s}\sum_{k\geq\{2,j\}}\binom{k}{j}g_ky_0^{k-j}=:\sum_{j=2}^{l}d_j(x)\sum_{(i_s)}^{(l;j)}\prod_{s=1}^{j}y_{i_s}\qquad(2.155)$$

where for the middle equality we note that the infinite sum terminates because $i_s\geq 1$ and $\sum_{s=1}^{j}i_s=l$. The fact mentioned before that $\sum_{k=1}^{\infty}g_k(x)y_k$ is well defined when y_k are formal series is now visible: collecting the coefficient of $x^p\mathrm{e}^{-kx}$, only *finite* sums of coefficients appear.

For a vectorial equation like (1.3) we first write

$$\mathbf{y}'=\mathbf{f}_0(x)-\hat{\Lambda}\mathbf{y}+x^{-1}\hat{B}\mathbf{y}+\sum_{\mathbf{k}\succ 0}\mathbf{g}_{\mathbf{k}}(x)\mathbf{y}^{\mathbf{k}}\qquad(2.156)$$

with $\mathbf{y}^{\mathbf{k}}:=\prod_{i=1}^{n_1}(\mathbf{y})_i^{k_i}$. The formal operations and ordering extend naturally to the vectorial general transseries (1.4), under the restriction $\Re(\mathbf{k}\cdot\boldsymbol{\lambda}x)>0$

As with (2.155), we introduce the transseries (1.4) in (2.156) and equate the coefficients of $\exp(-\mathbf{k}\cdot\boldsymbol{\lambda}x)$. Let $\mathbf{v_k}=x^{-\mathbf{k}\cdot\mathbf{m}}\mathbf{y_k}$ and

$$\mathbf{d_j}(x)=\sum_{\mathbf{l}\geq\mathbf{j}}\binom{\mathbf{l}}{\mathbf{j}}\mathbf{g_l}(x)\mathbf{v}_0^{\mathbf{l}-\mathbf{j}} \tag{2.157}$$

Noting that, by assumption, $\mathbf{k}\cdot\boldsymbol{\lambda}=\mathbf{k}'\cdot\boldsymbol{\lambda}\Leftrightarrow\mathbf{k}=\mathbf{k}'$ we obtain, for $\mathbf{k}\in\mathbb{N}^{n_1}$, $\mathbf{k}\succ 0$

$$\begin{aligned}\mathbf{v}_\mathbf{k}'&+\left(\hat\Lambda-\mathbf{k}\cdot\boldsymbol{\lambda}\hat I+x^{-1}\hat B\right)\mathbf{v_k}+\sum_{|\mathbf{j}|=1}\mathbf{d_j}(x)(\mathbf{v_k})^{\mathbf{j}}\\&=\sum_{\substack{\mathbf{j}\leq\mathbf{k}\\|\mathbf{j}|\geq 2}}\mathbf{d_j}(x)\sum_{(\mathbf{i}_{mp}:\mathbf{k})}\prod_{m=1}^{n}\prod_{p=1}^{j_m}\left(\mathbf{v}_{\mathbf{i}_{mp}}\right)_m=\mathbf{t_k}(\mathbf{v})\end{aligned} \tag{2.158}$$

where $\binom{\mathbf{l}}{j}=\prod_{j=1}^n\binom{l_i}{j_i}$, $(\mathbf{v})_m$ means the component m of $\mathbf{v}$, and $\sum_{(\mathbf{i}_{mp}:\mathbf{k})}$ stands for the sum over all vectors $\mathbf{i}_{mp}\in\mathbb{N}^n$, with $p\leq j_m, m\leq n$, such that $\mathbf{i}_{mp}\succ 0$ and $\sum_{m=1}^n\sum_{p=1}^{j_m}\mathbf{i}_{mp}=\mathbf{k}$. We use the convention $\prod_\emptyset=1,\sum_\emptyset=0$. With $m_i=1-\lfloor\beta_i\rfloor$ we obtain for $\mathbf{y_k}$

$$\mathbf{y}_\mathbf{k}'+\left(\hat\Lambda-\mathbf{k}\cdot\boldsymbol{\lambda}\hat I+x^{-1}(\hat B+\mathbf{k}\cdot\mathbf{m})\right)\mathbf{y_k}+\sum_{|\mathbf{j}|=1}\mathbf{d_j}(x)(\mathbf{y_k})^{\mathbf{j}}=\mathbf{t_k}(\mathbf{y}) \tag{2.159}$$

There are clearly finitely many terms in $\mathbf{t_k}(\mathbf{y})$. To find a (not too unrealistic) upper bound for this number of terms, we compare with $\sum_{(\mathbf{i}_{mp})'}$ which stands for the same as $\sum_{(\mathbf{i}_{mp})}$ except with $\mathbf{i}\geq 0$ instead of $\mathbf{i}\succ 0$. Noting that $\binom{k+s-1}{s-1}=\sum_{a_1+\ldots+a_s=k}1$ is the number of ways k can be written as a sum of s integers, we have

$$\sum_{(\mathbf{i}_{mp})}1\leq\sum_{(\mathbf{i}_{mp})'}1=\prod_{l=1}^{n_1}\sum_{(\mathbf{i}_{mp})_l}1=\prod_{l=1}^{n_1}\binom{k_l+|\mathbf{j}|-1}{|\mathbf{j}|-1}\leq\binom{|\mathbf{k}|+|\mathbf{j}|-1}{|\mathbf{j}|-1}^{n_1} \tag{2.160}$$

Remark 44 *Equation (2.158) can be written in the form (1.15)*

Proof. The fact that only predecessors of $\mathbf{k}$ are involved in $\mathbf{t}(\mathbf{y}_0,\cdot)$ and the homogeneity property of $\mathbf{t}(\mathbf{y}_0,\cdot)$ follow immediately by combining the conditions $\sum\mathbf{i}_{mp}=\mathbf{k}$ and $\mathbf{i}_{mp}\succ 0$. □

The formal inverse Laplace transform of (2.159) is then

$$\left(-p+\hat\Lambda-\mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{Y_k}+\left(\hat B+\mathbf{k}\cdot\mathbf{m}\right)\mathcal{P}\mathbf{Y_k}+\sum_{|\mathbf{j}|=1}\mathbf{D_j}*(\mathbf{Y_k})^{\mathbf{j}}=\mathbf{T_k}(\mathbf{Y})$$

$$(2.161)$$

with

$$\mathbf{T_k}(\mathbf{Y}) = \mathbf{T}\left(\mathbf{Y}_0, \{\mathbf{Y_{k'}}\}_{0\prec\mathbf{k'}\prec\mathbf{k}}\right) = \sum_{\mathbf{j}\leq k;\ |\mathbf{j}|>1} \mathbf{D_j}(p) * \sum_{(\mathbf{i}_{mp};\mathbf{k})} {}^{*}\prod_{m=1}^{n_1} {}^{*}\prod_{p=1}^{j_m} \left(\mathbf{Y}_{\mathbf{i}_{mp}}\right)_m \tag{2.162}$$

and

$$\mathbf{D_j} = \sum_{\mathbf{l}\geq\mathbf{m}} \binom{\mathbf{l}}{\mathbf{m}} \mathbf{G_l} * \mathbf{Y}_0^{*(\mathbf{l}-\mathbf{m})} + \sum_{\mathbf{l}\geq\mathbf{m};|\mathbf{l}|\geq 2} \binom{\mathbf{l}}{\mathbf{m}} \mathbf{g}_{0,\mathbf{l}} \mathbf{Y}_0^{*(\mathbf{l}-\mathbf{m})} \tag{2.163}$$

2.5 Useful formulas

A straightforward computation shows that

$$\mathcal{B}(\frac{1}{x^n}) = \frac{p^{n-1}}{\Gamma(n)} \text{ or } \mathcal{L}(p^n) = \frac{\Gamma(n+1)}{x^{n+1}} \tag{2.164}$$

$$p^q * p^r = \frac{\Gamma(q+1)\Gamma(r+1)}{\Gamma(q+r+2)} p^{q+r+1} \tag{2.165}$$

Also, with $f_{1,2}(p) := p \mapsto \mathcal{H}(p-k_{1,2})g_{1,2}(p-k_{1,2})$ we have

$$\Big(f_1 * f_2\Big)(p) = \mathcal{H}(p-k_1-k_2)\Big(g_1 * g_2\Big)(p-k_1-k_2) \tag{2.166}$$

3 Acknowledgments

Special thanks are due to Professors Martin Kruskal and Jean Écalle for their many in-depth comments, and to Professor B. L. J. Braaksma for carefully reading the manuscript and pointing out an error in a formula. The author is grateful to Professors Michael Berry and Percy Deift for very interesting discussions.

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