SHARP HARDY-LERAY INEQUALITY FOR AXISYMMETRIC DIVERGENCE-FREE FIELDS

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ABSTRACT. We show that the sharp constant in the classical n-dimensional Hardy-Leray inequality can be improved for axisymmetric divergence-free fields, and find its optimal value. The same result is obtained for $n = 2$ without the axisymmetry assumption.

KEYWORDS: Hardy inequality, Leray inequality, Navier-Stokes equation, divergence-free fields.

1. INTRODUCTION

Let $\mathbf{u}$ denote a $C_0^\infty(\mathbb{R}^n)$ vector field in $\mathbb{R}^n$. The following n-dimensional generalization of the one-dimensional Hardy inequality [1],

$$\int_{\mathbb{R}^n} \frac{|\mathbf{u}|^2}{|x|^2} dx \le \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2 dx \tag{1.1}$$

appears for $n = 3$ in the pioneering Leray's paper on the Navier-Stokes equations [2]. The constant factor on the right-hand side is sharp. Since one frequently deals with divergence-free fields in hydrodynamics, it is natural to ask whether this restriction can improve the constant in (1.1).

We show in the present paper that this is the case indeed if $n > 2$ and the vector field $\mathbf{u}$ is axisymmetric by proving that the aforementioned constant can be replaced by the (smaller) optimal value

$$\frac{4}{(n-2)^2} \left(1 - \frac{8}{(n+2)^2}\right) \tag{1.2}$$

which, in particular, evaluates to 68/25 in three dimensions. This result is a special case of a more general one concerning a divergence-free improvement of the multi-dimensional sharp Hardy inequality

$$\int_{\mathbb{R}^n} |x|^{2\gamma-2} |\mathbf{u}|^2 dx \le \frac{4}{(2\gamma+n-2)^2} \int_{\mathbb{R}^n} |x|^{2\gamma} |\nabla \mathbf{u}|^2 dx \tag{1.3}$$

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Let $\boldsymbol{\phi}$ be a point on the $(n-2)$-dimensional unit sphere S^{n-2} with spherical coordinates $\{\theta_j\}_{j=1,\dots,n-3}$ and φ, where $\theta_j \in (0,\pi)$ and $\varphi \in [0,2\pi)$. A point $x \in \mathbb{R}^n$ is represented as a triple $(\rho, \theta, \boldsymbol{\phi})$, where $\rho > 0$ and $\theta \in [0,\pi]$. Correspondingly, we write $\mathbf{u} = (u_\rho, u_\theta, \mathbf{u}_{\boldsymbol{\phi}})$ with $\mathbf{u}_{\boldsymbol{\phi}} = (u_{\theta_{n-3}}, \dots, u_{\theta_1}, u_\varphi)$.

The *condition of axial symmetry* means that $\mathbf{u}$ depends only on ρ and θ.

For higher dimensions, our result is as follows.

Theorem 1. *Let $\gamma \neq 1 - n/2$, $n > 2$, and let $\mathbf{u}$ be an axisymmetric divergence-free vector field in $C_0^\infty(\mathbb{R}^n)$. We assume that $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ for $\gamma < 1 - n/2$. Then*

$$\int_{\mathbb{R}^n} |x|^{2\gamma-2} |\mathbf{u}|^2 dx \le C_{n,\gamma} \int_{\mathbb{R}^n} |x|^{2\gamma} |\nabla \mathbf{u}|^2 dx \tag{1.4}$$

with the best value of $C_{n,\gamma}$ given by

$$C_{n,\gamma} = \frac{4}{(2\gamma+n-2)^2} \left(1 - \frac{2}{n+1+(\gamma-n/2)^2} \right) \tag{1.5}$$

for $\gamma \le 1$, and by

$$C_{n,\gamma}^{-1} = \left(\frac{n}{2} + \gamma - 1 \right)^2 + \min \left\{ n-1,\ 2 + \min_{x \ge 0} \left(x + \frac{4(n-1)(\gamma-1)}{x+n-1+(\gamma-n/2)^2} \right) \right\} \tag{1.6}$$

for $\gamma > 1$.

The two minima in (1.6) can be calculated in closed form, but their expressions for arbitrary dimensions turn out to be unwieldy, and we omit them.

However, the formula for $C_{3,\gamma}$ is simple.

Corollary 1. *For $n = 3$ inequality (1.4) holds with the best constant*

$$C_{3,\gamma} = \begin{cases} \dfrac{4}{(2\gamma+1)^2} \cdot \dfrac{2+(\gamma-3/2)^2}{4+(\gamma-3/2)^2}, & \text{for } \gamma \le 1 \\[2ex] \dfrac{4}{8+(1+2\gamma)^2}, & \text{for } \gamma > 1. \end{cases} \tag{1.7}$$

For $n = 2$, we obtain the sharp constant in (1.4) without axial symmetry of the vector field.

Theorem 2. *Let $\gamma \neq 0$, $n = 2$, and let $\mathbf{u}$ be a divergence-free vector field in $C_0^\infty(\mathbb{R}^2)$. We assume that $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ for $\gamma < 0$. Then inequality (1.4) holds with the best constant*

$$C_{2,\gamma} = \begin{cases} \gamma^{-2} \dfrac{1+(1-\gamma)^2}{3+(1-\gamma)^2} & \text{for } \gamma \in [-\sqrt{3}-1, \sqrt{3}-1] \\[2ex] (\gamma^2+1)^{-1} & \text{otherwise.} \end{cases} \tag{1.8}$$

2. Proof of Theorem 1

In the spherical coordinates introduced above, we have

$$\text{div } \mathbf{u} = \rho^{1-n}\frac{\partial}{\partial \rho}\left(\rho^{n-1}u_\rho\right) + \rho^{-1}(\sin\theta)^{2-n}\frac{\partial}{\partial\theta}\left((\sin\theta)^{n-2}u_\theta\right) + \sum_{k=1}^{n-3}(\rho\sin\theta\sin\theta_{n-3}\cdots\sin\theta_{k+1})^{-1}(\sin\theta_k)^{-k}\frac{\partial}{\partial\theta_k}\left((\sin\theta_k)^k u_{\theta_k}\right) + (\rho\sin\theta\sin\theta_{n-3}\cdots\sin\theta_1)^{-1}\frac{\partial u_\varphi}{\partial\varphi} \tag{2.9}$$

Since the components u_φ and u_{θ_k}, $k = 1, ..., n-3$, depend only on ρ and θ, (2.9) becomes

$$\text{div } \mathbf{u} = \rho^{1-n}\frac{\partial}{\partial \rho}\left(\rho^{n-1}u_\rho(\rho,\theta)\right) + \rho^{-1}(\sin\theta)^{2-n}\frac{\partial}{\partial\theta}\left((\sin\theta)^{n-2}u_\theta(\rho,\theta)\right) + \sum_{k=1}^{n-3}k(\sin\theta_{n-3}\cdots\sin\theta_{k+1})^{-1}\cot\theta_k\frac{u_{\theta_k}(\rho,\theta)}{\rho\sin\theta} \tag{2.10}$$

By the linear independence of the functions

$$1,\ \ (\sin\theta_{n-3}\cdots\sin\theta_{k+1})^{-1}\cot\theta_k,\quad k = 1, ..., n-3$$

the divergence-free condition is equivalent to the collection of $n-2$ identities

$$\rho\frac{\partial u_\rho}{\partial\rho} + (n-1)u_\rho + \left(\frac{\partial}{\partial\theta} + (n-2)\cot\theta\right)u_\theta = 0 \tag{2.11}$$

$$u_{\theta_k} = 0, \quad k = 1, ..., n-3 \tag{2.12}$$

If the right-hand side of (1.4) diverges, there is nothing to prove. Otherwise, the matrix $\nabla\mathbf{u}$ is $O(|x|^m)$, with $m > -\gamma - n/2$, as $x \to 0$. Since $\mathbf{u}(\mathbf{0}) = \mathbf{0}$, we have $\mathbf{u}(x) = O(|x|^{m+1})$ ensuring the convergence of the integral on the left-hand side of (1.4). We introduce the vector field

$$\mathbf{v}(x) = \mathbf{u}(x)|x|^{\gamma-1+n/2} \tag{2.13}$$

The inequality (1.4) becomes

$$\left(\frac{1}{C_{n,\gamma}} - \left(\frac{n}{2}+\gamma-1\right)^2\right)\int_{\mathbb{R}^n}\frac{|\mathbf{v}|^2}{|x|^n}dx \le \int_{\mathbb{R}^n}\frac{|\nabla\mathbf{v}|^2}{|x|^{n-2}}dx \tag{2.14}$$

The condition div $\mathbf{u} = 0$ is equivalent to

$$\rho \text{ div } \mathbf{v} = \left(\frac{n-2}{2}+\gamma\right)v_\rho \tag{2.15}$$

To simplify the exposition, we assume first that $\mathbf{v}_{\boldsymbol{\phi}} = \mathbf{0}$. Now, (2.15) can be written as

$$\rho \frac{\partial v_\rho}{\partial \rho} + \left(\frac{n}{2} - \gamma\right) v_\rho + \mathcal{D} v_\theta = 0 \tag{2.16}$$

where

$$\mathcal{D} := \frac{\partial}{\partial \theta} + (n-2)\cot\theta \tag{2.17}$$

Note that $\mathcal{D}$ is the adjoint of $-\partial/\partial\theta$ with respect to the scalar product

$$\int_0^\pi f(\theta)\overline{g(\theta)}(\sin\theta)^{n-2} d\theta$$

A straightforward though lengthy calculation yields

$$\rho^2 |\nabla \mathbf{v}|^2 = \rho^2\Big(\frac{\partial v_\rho}{\partial \rho}\Big)^2 + \rho^2\Big(\frac{\partial v_\theta}{\partial \rho}\Big)^2 + \Big(\frac{\partial v_\rho}{\partial \theta}\Big)^2 + \Big(\frac{\partial v_\theta}{\partial \theta}\Big)^2 \\ + v_\theta^2 + (n-1)v_\rho^2 + (n-2)(\cot\theta)^2 v_\theta^2 + 2\left(v_\rho \mathcal{D} v_\theta - v_\theta \frac{\partial v_\rho}{\partial \theta}\right) \tag{2.18}$$

Hence

$$\rho^2 \int_{S^{n-1}} |\nabla \mathbf{v}|^2 ds = \int_{S^{n-1}} \Big\{\rho^2\Big(\frac{\partial v_\rho}{\partial \rho}\Big)^2 + \Big(\frac{\partial v_\theta}{\partial \theta}\Big)^2 + \rho^2\Big(\frac{\partial v_\theta}{\partial \rho}\Big)^2 + \Big(\frac{\partial v_\rho}{\partial \theta}\Big)^2 \\ + v_\theta^2 + (n-1)v_\rho^2 + (n-2)(\cot\theta)^2 v_\theta^2 + 4 v_\rho \mathcal{D} v_\theta\Big\} ds \tag{2.19}$$

Changing the variable ρ to $t = \log\rho$, and applying the Fourier transform with respect to t,

$$\mathbf{v}(t,\theta) \mapsto \mathbf{w}(\lambda,\theta)$$

we derive

$$\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}} dx \\ = \int_{\mathbb{R}} \int_{S^{n-1}} \Big\{(\lambda^2 + n - 1)|w_\rho|^2 + (\lambda^2 - n + 3)|w_\theta|^2 \\ + \Big|\frac{\partial w_\rho}{\partial \theta}\Big|^2 + \Big|\frac{\partial w_\theta}{\partial \theta}\Big|^2 + (n-2)(\sin\theta)^{-2}|w_\theta|^2 + 4\mathrm{Re}(\overline{w}_\rho \mathcal{D} w_\theta)\Big\} ds d\lambda \tag{2.20}$$

and

$$\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n} dx = \int_{\mathbb{R}} \int_{S^{n-1}} |\mathbf{w}|^2 ds d\lambda \tag{2.21}$$

From (2.15), we obtain

$$w_\rho = -\frac{\mathcal{D} w_\theta}{i\lambda + n/2 - \gamma} \tag{2.22}$$

which implies

$$|w_\rho|^2 = \frac{|\mathcal{D}w_\theta|^2}{\lambda^2 + (n/2-\gamma)^2} \tag{2.23}$$

and

$$\mathrm{Re}(\overline{w}_\rho \mathcal{D}w_\theta) = -\frac{(n/2-\gamma)|\mathcal{D}w_\theta|^2}{\lambda^2 + (n/2-\gamma)^2} \tag{2.24}$$

Introducing this into (2.20), we arrive at the identity

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}} dx = \int_0^\infty \int_0^\pi \Bigg\{ (\lambda^2+n-1)\frac{|\mathcal{D}w_\theta|^2}{\lambda^2+(n/2-\gamma)^2} + (\lambda^2-n+3)|w_\theta|^2 \\ + \left|\frac{\partial w_\theta}{\partial\theta}\right|^2 + (n-2)(\sin\theta)^{-2}|w_\theta|^2 + \frac{1}{\lambda^2+(n/2-\gamma)^2}\left|\frac{\partial}{\partial\theta}\mathcal{D}w_\theta\right|^2 \\ - 4\left(\frac{n}{2}-\gamma\right)\frac{|\mathcal{D}w_\theta|^2}{\lambda^2+(n/2-\gamma)^2}\Bigg\} d\theta d\lambda \end{aligned} \tag{2.25}$$

We simplify the right-hand side of (2.25) to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}} dx = \int_0^\infty \int_0^\pi \Bigg\{ \left(\frac{-n-1+\lambda^2+4\gamma}{\lambda^2+(n/2-\gamma)^2}+1\right)|\mathcal{D}w_\theta|^2 \\ + (\lambda^2-n+3)|w_\theta|^2 + \frac{1}{\lambda^2+(n/2-\gamma)^2}\left|\frac{\partial}{\partial\theta}\mathcal{D}w_\theta\right|^2 \Bigg\} d\theta d\lambda \end{aligned} \tag{2.26}$$

On the other hand, by (2.21) and (2.22)

$$\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^{n-2}} dx = \int_0^\infty \int_0^\pi \left(\frac{|\mathcal{D}w_\theta|^2}{\lambda^2+(n/2-\gamma)^2} + |w_\theta|^2\right) d\theta d\lambda \tag{2.27}$$

Defining the self-adjoint operator

$$T := -\frac{\partial}{\partial\theta}\mathcal{D} \tag{2.28}$$

or, equivalently,

$$T = -\delta_\theta + \frac{n-2}{(\sin\theta)^2} \tag{2.29}$$

where δ_θ is the θ-part of the Laplace-Beltrami operator on S^{n-1}, we write (2.26) and (2.27) as

$$\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}} dx = \int_{\mathbb{R}} \int_{S^{n-1}} Q(\lambda, w_\theta) ds d\lambda \tag{2.30}$$

and

$$\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n} dx = \int_{\mathbb{R}} \int_{S^{n-1}} q(\lambda, w_\theta) ds d\lambda \tag{2.31}$$

respectively, where Q and q are sesquilinear forms in w_θ, defined by

$$\begin{aligned} Q(\lambda, w_\theta) = \left(\frac{-n-1+\lambda^2+4\gamma}{\lambda^2+(n/2-\gamma)^2} + 1 \right) T w_\theta \cdot \overline{w_\theta} \\ + (\lambda^2 - n + 3)|w_\theta|^2 + \frac{1}{\lambda^2+(n/2-\gamma)^2} |T w_\theta|^2 \end{aligned} \tag{2.32}$$

and

$$q(\lambda, w_\theta) = \frac{T w_\theta \cdot \overline{w_\theta}}{\lambda^2+(n/2-\gamma)^2} + |w_\theta|^2 \tag{2.33}$$

The eigenvalues of T are $\alpha_\nu = \nu(\nu+n-2)$, $\nu \in \mathbb{Z}^+$. Representing w_θ as an expansion in eigenfunctions of T, we find

$$\begin{aligned} \inf_{w_\theta} \frac{\displaystyle\int_{\mathbb{R}} \int_{S^{n-1}} Q(\lambda, w_\theta) ds d\lambda}{\displaystyle\int_{\mathbb{R}} \int_{S^{n-1}} q(\lambda, w_\theta) ds d\lambda} \\ = \inf_{\lambda \in \mathbb{R}} \inf_{\nu \in \mathbb{N}} \frac{\left(\frac{-n-1+\lambda^2+4\gamma}{\lambda^2+(n/2-\gamma)^2} + 1 \right) \alpha_\nu + \lambda^2 - n + 3 + \frac{\alpha_\nu^2}{\lambda^2+(n/2-\gamma)^2}}{\frac{\alpha_\nu}{\lambda^2+(n/2-\gamma)^2+1}} \end{aligned} \tag{2.34}$$

Thus our minimization problem reduces to finding

$$\inf_{x \geq 0} \inf_{\nu \in \mathbb{N}} f(x, \alpha_\nu, \gamma) \tag{2.35}$$

where

$$f(x, \alpha_\nu, \gamma) = x - n + 3 + \alpha_\nu \left(1 - \frac{16(1-\gamma)}{4x + 4\alpha_\nu + (n-2\gamma)^2} \right) \tag{2.36}$$

Since $\gamma \leq 1$, it is clear that f is increasing in x, so the value (2.35) is equal to

$$\inf_{\nu \in \mathbb{N}} f(0, \alpha_\nu, \gamma) = \inf_{\nu \in \mathbb{N}} \left(3 - n + \alpha_\nu \left(1 - \frac{16(1-\gamma)}{4\alpha_\nu + (n-2\gamma)^2} \right) \right) \tag{2.37}$$

We have

$$\frac{\partial}{\partial \alpha_\nu} f(0, \alpha_\nu, \gamma) = 1 - \frac{16(1-\gamma)(n-2\gamma)}{(4\alpha_\nu + (n-2\gamma)^2)^2} \tag{2.38}$$

Noting that

$$4\alpha_\nu + (n-2\gamma)^2 \geq 4(n-1) + (n-2\gamma)^2 \geq 4\sqrt{n-1}(n-2\gamma) \tag{2.39}$$

we see that

$$\frac{\partial}{\partial \alpha_\nu} f(0, \alpha_\nu, \gamma) \geq 1 - \frac{1-\gamma}{(n-1)(n-2\gamma)} > 0 \tag{2.40}$$

Thus the minimum of $f(0, \alpha_\nu, \gamma)$ is attained at $\alpha_1 = n-1$ and equals

$$3 - n + (n-1)\left(1 - \frac{16(1-\gamma)}{4(n-1) + (n-2\gamma)^2}\right) = \frac{2(\gamma - 1 + n/2)^2}{n - 1 + (\gamma - n/2)^2} \tag{2.41}$$

This completes the proof for the case $\mathbf{v}_{\boldsymbol{\phi}} = \mathbf{0}$.

If we drop the assumption $\mathbf{v}_{\boldsymbol{\phi}} = \mathbf{0}$, then, to the integrand on the right-hand side of (2.19), we should add the terms

$$\rho^2 \left(\frac{\partial v_\varphi}{\partial \rho}\right)^2 + \left(\frac{\partial v_\varphi}{\partial \theta}\right)^2 + (\sin\theta \sin\theta_{n-3} \cdots \sin\theta_1)^{-2} v_\varphi^2 \tag{2.42}$$

The expression in (2.42) equals

$$\rho^2 \left|\nabla(v_\varphi e^{i\varphi})\right|^2 \tag{2.43}$$

As a result, the right-hand side of (2.30) is augmented by

$$\int_{\mathbb{R}} \int_{S^{n-1}} R(\lambda, w_\varphi) ds d\lambda \tag{2.44}$$

where

$$R(\lambda, w_\varphi) = \lambda^2 |w_\varphi|^2 + |\nabla_\omega (w_\varphi e^{i\varphi})|^2 \tag{2.45}$$

with $\omega = (\theta, \theta_{n-3}, ..., \varphi)$. Hence,

$$\inf_{\mathbf{v}} \frac{\displaystyle\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}} dx}{\displaystyle\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n} dx} = \inf_{w_\theta, w_\varphi} \frac{\displaystyle\int_{\mathbb{R}} \int_{S^{n-1}} \Big(Q(\lambda, w_\theta) + R(\lambda, w_\varphi)\Big) ds d\lambda}{\displaystyle\int_{\mathbb{R}} \int_{S^{n-1}} \Big(q(\lambda, w_\theta) + |w_\varphi|^2\Big) ds d\lambda} \tag{2.46}$$

Using the fact that w_θ and w_φ are independent, the right-hand side is the lesser of (2.34) and

$$\inf_{w_\varphi} \frac{\displaystyle\int_{\mathbb{R}}\int_{S^{n-1}} R(\lambda, w_\varphi) ds d\lambda}{\displaystyle\int_{\mathbb{R}}\int_{S^{n-1}} |w_\varphi|^2 ds d\lambda} \tag{2.47}$$

Since $w_\varphi e^{i\varphi}$ is orthogonal to one on S^{n-1}, we have

$$\int_{S^{n-1}} \left|\nabla_\omega \left(w_\varphi e^{i\varphi}\right)\right|^2 ds \geq (n-1)\int_{S^{n-1}} |w_\varphi|^2 ds \tag{2.48}$$

Hence the infimum in (2.47) is at most $n-1$, which exceeds the value in (2.41). The result follows for $\gamma \leq 1$.

For $\gamma > 1$ the proof is similar. Differentiation of f in α_ν gives

$$1 + \frac{16(\gamma-1)((n-2\gamma)^2+4x)}{(4x+4\alpha_\nu+(n-2\gamma)^2)^2} \tag{2.49}$$

which is positive. Hence the role of the value (2.41) is played by the smallest value of $f(\cdot, n-1, \gamma)$ on $\mathbb{R}^+$. Therefore,

$$\inf_{\mathbf{v}} \frac{\displaystyle\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}} dx}{\displaystyle\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n} dx} = 2 + \min_{x\geq 0}\left(x + \frac{4(n-1)(\gamma-1)}{x+n-1+(\gamma-n/2)^2}\right) \tag{2.50}$$

The proof is complete.

Proof of Corollary 1. We need to consider only $\gamma > 1$. It follows directly from (1.6) that

$$C_{3,\gamma}^{-1} = \left(\frac{3}{2}+\gamma-1\right)^2 + 2$$

which gives the result.

Remark 1. Using (2.22), we see that a minimizing sequence $\{\mathbf{v}_k\}_{k\geq 1}$ which shows the sharpness of inequality (1.4) with the constant (1.5) can be obtained by taking $\mathbf{v}_k = (v_{\rho,k}, v_{\theta,k}, \mathbf{0})$ with the Fourier transform $\mathbf{w}_k = (w_{\rho,k}, w_{\theta,k}, \mathbf{0})$ chosen as follows:

$$w_{\theta,k}(\lambda,\theta) = h_k(\lambda)\sin\theta, \quad w_{\rho,k}(\lambda,\theta) = \frac{1-n}{i\lambda+n/2-\gamma} h_k(\lambda)\cos\theta \tag{2.51}$$

The sequence $\{|h_k|^2\}_{k\geq 1}$ converges in distributions to the delta function at $\lambda = 0$. The minimizing sequence that gives the value (1.7) of $C_{3,\gamma}$ is

$$w_{\theta,k}(\lambda, 0) = 0, \quad w_{\rho,k}(\lambda,\theta) = 0, \text{ and } \ w_{\phi,k}(\lambda,\theta) = h_k(\lambda)\sin\theta$$

where $\{|h_k|^2\}_{k\geq 1}$ is as above.

3. Proof of Theorem 2.

The calculations are similar but simpler than those in the previous section. We start with the substitution $\mathbf{v}(x) = \mathbf{u}(x)|x|^{2\gamma}$ and write (2.14) in the form

$$\frac{1}{C_{2,\gamma}} = \gamma^2 + \inf_{\mathbf{v}} \frac{\displaystyle\int_{\mathbb{R}^2} |\nabla \mathbf{v}|^2 dx}{\displaystyle\int_{\mathbb{R}^2} |\mathbf{v}|^2 |x|^{-2} dx} \tag{3.52}$$

In polar coordinates ρ and φ, with $\varphi \in [0, 2\pi)$, we have

$$\int_{\mathbb{R}^2} |\nabla \mathbf{v}|^2 dx = \int_{\mathbb{R}^2} \Big\{ |\nabla v_\rho|^2 + |\nabla v_\varphi|^2 + \rho^{-2} \left({v_\rho}^2 + {v_\varphi}^2 - 4 v_\rho (\partial_\varphi v_\varphi) \right) \Big\} dx \tag{3.53}$$

Changing the variable ρ to $t = \log \rho$, and applying the Fourier transform $\mathbf{v}(\rho, \varphi) \to \mathbf{w}(\lambda, \varphi)$, we obtain

$$\int_{\mathbb{R}} \int_0^{2\pi} \Big\{ (\lambda^2 + 1)(|w_\rho|^2 + |w_\varphi|^2) + |\partial_\varphi w_\varphi|^2 + |\partial_\varphi w_\rho|^2 - 4 \left(\partial_\varphi w_\varphi \right) \overline{w_\rho} \Big\} d\varphi d\lambda \tag{3.54}$$

The divergence-free condition for u becomes

$$w_\rho = -\frac{\partial_\varphi w_\varphi}{i\lambda + 1 - \gamma} \tag{3.55}$$

which yields

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla \mathbf{v}|^2 dx = \int_{\mathbb{R}} \int_0^{2\pi} \Big\{ \Big(\frac{\lambda^2 + 4\gamma - 3}{\lambda^2 + (1-\gamma)^2} + 1 \Big) |\partial_\varphi w_\varphi|^2 & \\ + \frac{|\partial_\varphi^2 w_\varphi|^2}{\lambda^2 + (1-\gamma)^2} + (\lambda^2 + 1)|w_\varphi|^2 \Big\} d\varphi d\lambda & \end{aligned} \tag{3.56}$$

Analogously,

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbf{v}|^2 |x|^{-2} dx &= \int_{\mathbb{R}} \int_0^{2\pi} (|w_\rho|^2 + |w_\varphi|^2) d\varphi d\lambda \\ &= \int_{\mathbb{R}} \int_0^{2\pi} \Big(\frac{|\partial_\varphi w_\varphi|^2}{\lambda^2 + (1-\gamma)^2} + |w_\varphi|^2 \Big) d\varphi d\lambda \end{aligned} \tag{3.57}$$

Therefore, by (3.52)

$$\frac{1}{C_{2,\gamma}} = \gamma^2 + \inf_{x \geq 0} \inf_{\nu \in \mathbb{N} \cup 0} f(x, \nu, \gamma) \tag{3.58}$$

where

$$f(x,\nu,\gamma) = x + 1 + \nu\left(1 - \frac{4(1-\gamma)}{x+\nu+(1-\gamma)^2}\right) \tag{3.59}$$

Let first $\gamma \le 1$. Then f is increasing in x, which implies $f(x,\nu,\gamma) \ge f(0,\nu,\gamma)$. Since the derivative

$$\frac{\partial}{\partial\nu} f(0,\nu,\gamma) = 1 - \frac{4(1-\gamma)^3}{(\nu+(1-\gamma)^2)^2} \tag{3.60}$$

is positive for $\nu \ge 2$, we need to compare only the values $f(0,0,\gamma)$, $f(0,1,\gamma)$ and $f(0,2,\gamma)$. An elementary calculation shows that both $f(0,0,\gamma)$ and $f(0,2,\gamma)$ exceed $f(0,1,\gamma)$ for $\gamma \notin (-1-\sqrt{3}, -1+\sqrt{3})$.

Let now $\gamma > 1$. We have

$$\frac{\partial}{\partial\nu} f(x,\nu,\gamma) = 1 + \frac{4(\gamma-1)(x+(1-\gamma)^2)}{(x+\nu+(1-\gamma^2))^2} > 0 \tag{3.61}$$

and therefore $f(x,\nu,\gamma) \ge f(x,0,\gamma) = x+1 \ge 1$. The proof of Theorem 2 is complete.

Remark 2. Minimizing sequences which give $C_{2,\gamma}$ in (1.8) can be chosen as follows:

$$w_{\rho,k}(\lambda,\varphi) = 0, \qquad w_{\varphi,k}(\lambda,\varphi) = h_k(\lambda)$$

for $\gamma \notin (-1-\sqrt{3}, -1+\sqrt{3})$, and

$$w_{\rho,k} = \frac{h_k(\lambda)\sin(\varphi-\varphi_0)}{i\lambda+1-\gamma}, \qquad w_{\varphi,k} = h_k(\lambda)\cos(\varphi-\varphi_0)$$

when $\gamma \in (-1-\sqrt{3}, -1+\sqrt{3})$, for any constant φ_0. Here $\{|h_k|^2\}_{k\ge1}$ converges in distributions to the delta function at 0.

Corollary 2. *Let $\gamma \neq 0$. Denote by ψ a real-valued scalar function in $C_0^\infty(\mathbb{R}^2)$ and assume in addition that $\nabla\psi(\mathbf{0}) = \mathbf{0}$ if $\gamma < 0$. Then the sharp inequality*

$$\int_{\mathbb{R}^2} |\nabla\psi|^2 |x|^{2(\gamma-1)} dx \le C_{2,\gamma} \int_{\mathbb{R}^2} \left(\psi^2_{x_1x_1} + 2\psi^2_{x_1x_2} + \psi^2_{x_2x_2}\right)|x|^{2\gamma} dx \tag{3.62}$$

holds with $C_{2,\gamma}$ given in (1.8).

Indeed, for $n = 2$, inequality (1.4) becomes (3.62) if ψ is interpreted as a stream function of the vector field $\mathbf{u}$, *i.e.* $\mathbf{u} = \nabla \times \psi$.

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