NONLINEAR STOKES PHENOMENA IN FIRST OR SECOND ORDER DIFFERENTIAL EQUATIONS

DEDICATED TO PROFESSOR KAWAI ON THE OCCASION OF HIS 60TH BIRTHDAY

O. COSTIN

ABSTRACT. We study singularity formation in nonlinear differential equations of order $m \leqslant 2$, $y^{(m)} = A(x^{-1}, y)$. We assume A is analytic at $(0,0)$ and $\partial_y A(0,0) = \lambda \neq 0$ (say, $\lambda = (-1)^m$). If $m = 1$ we assume $A(0, \cdot)$ is meromorphic and nonlinear. If $m = 2$, we assume $A(0, \cdot)$ is analytic except for isolated singularities, and also that $\int_{s_0}^{\infty} |\Phi(s)|^{-1/2} d|s| < \infty$ along some path avoiding the zeros and singularities of Φ, where $\Phi(s) = \int_0^s A(0, \tau) d\tau$. Let $H_\alpha = \{z : |z| > a > 0, \arg(z) \in (-\alpha, \alpha)\}$.

If the Stokes constant S^+ associated to $\mathbb{R}^+$ is nonzero, we show that *all* y such that $\lim_{x \to +\infty} y(x) = 0$ are singular at $2\pi i$-quasiperiodic arrays of points near $i\mathbb{R}^+$. The array location determines and is determined by S^+. Such settings include the Painlevé equations P_I and P_{II}. If $S^+ = 0$, then there is exactly one solution y_0 without singularities in $H_{2\pi - \epsilon}$, and y_0 is entire iff $y_0 = A(z, 0) \equiv 0$.

The singularities of $y(x)$ mirror the singularities of the Borel transform of its asymptotic expansion, $\mathcal{B}\tilde{y}$, a nonlinear analog of Stokes phenomena. If $m = 1$ and A is a nonlinear polynomial with $A(z, 0) \not\equiv 0$ a similar conclusion holds even if $A(0, \cdot)$ is linear. This follows from the property that if f is superexponentially small along $\mathbb{R}^+$ and analytic in H_π, then f is superexponentially unbounded in H_π, a consequence of decay estimates of Laplace transforms.

Compared to [2] this analysis is restricted to first and second order equations but shows that singularities *always* occur, and their type is calculated in the polynomial case. Connection to integrability and the Painlevé property are discussed.

1. INTRODUCTION

Stokes transitions play an important role in the study of differential equations (see [1], [4], [10] and references therein). In linear differential equations, solutions that are small in some sector usually become exponentially large in complementary ones.

There is an interesting analog for analytic nonlinear equations which we study for first and second order ones.

Assume that $A(z, y)$ is analytic at $(0,0)$, and that the following nondegeneracy condition holds: $\partial_y A(0, y) := \lambda \neq 0$. Let, for $m = 1, 2$,

$$(1) \qquad \qquad y^{(m)} = A(1/x, y)$$

By a simple change of variable $y \mapsto y + a + bx^{-1} + cx^{-2}$ we can assume, without loss of generality, $A(z, 0) = o(z^2)$. The conditions in [3] and [2] (which do not require

$m \leqslant 2$) then apply. By a change of independent variable we can make $\lambda = (-1)^m$. Let $A_1'(0) = \alpha$.

There exists a one parameter family of solutions of (1) which decay algebraically as $x \to \infty$ [12], [3]. All these solutions can be written as generalized Borel summed transseries [3] (the notation here is similar to that in [2])

$$(2) \qquad y(x) = \sum_{k \geq 0} C^k \mathrm{e}^{-kx} x^{\alpha k} y_k(x) = \sum_{k \geq 0} C^k \mathrm{e}^{-kx} x^{\alpha k} \mathcal{LB}\tilde{y}_k(x) \equiv \mathcal{LB}\tilde{y}(x)$$

where $\tilde{y}_k$ are formal series in integer powers of x^{-1}. The functions $y_k := \mathcal{LB}\tilde{y}_k$, the generalized Borel sums of $\tilde{y}_k$, are analytic for $\Re(x) > \nu > 0$ uniformly in k. The function series in (2) is absolutely and uniformly convergent for $\Re(x) > \nu$ and defines an analytic function there. Any decaying solution can be represented uniquely in terms of Laplace transforms in the upper half plane, $y_k^+ = \mathcal{L}(\mathcal{B}\tilde{y})^+$ where $^+$ indicates the upper half plane continuation or in terms of the lower half plane continuations [3]

$$(3) \qquad y(x) = \sum_{k \geq 0} C^{+k} e^{-kx} x^{\alpha k} y_k^+ = \sum_{k \geq 0} C^{-k} e^{-kx} x^{\alpha k} y_k^-$$

and we have ([3] Eq. (1.27))

$$(4) \qquad\qquad\qquad C^+ - C^- = S^+$$

where $S^+ = S_1$ in [3] is the Stokes constant associated to $\mathbb{R}^+$.

In this paper we prove, under some further assumptions, that all solutions, except at most one, that go to zero along some direction form arrays of singularities near $i\mathbb{R}$. After normalization of the equation, the array location is given by

$$(5) \qquad x_n = 2n\pi i + \alpha \ln(2n\pi i) + \ln C^{\pm} - \ln \xi_s + o(1) \quad (\mathbb{N} \ni n \to \pm\infty)$$

for some $\xi_s \neq 0$. (If $C^- = 0$ ($C^+ = 0$), then y has no singularities in the lower (upper, respectively) half plane.) In [2], Eq. (28) we showed that such arrays occur provided a solution of an associated equation, (6) below, is singular. Here we show that the solution is singular indeed, under our assumptions.

There is a correspondence between singularities in inverse Laplace space and singularities in the original space, a nonlinear analog of Stokes phenomena: if we change variables to make $\partial_y A(0,0) = \lambda$, then n becomes n/λ. Singularities in Borel space are spaced by (exactly) $|\lambda|$, see [3], while in the x plane their mirror singularities are, up to log corrections, as we see from (5), regularly spaced by $|\lambda|^{-1}$.

Part of the results rely on a general property that if an analytic function is superexponentially small along a ray, then it becomes superexponentially large along a complementary ray.

2. Main results

In [2] it is shown that there is a unique analytic solution F_0 of the associated equation

$$(6) \qquad\qquad\qquad \sum_{k=1}^{m} \xi^k F_0^{(k)} = A(0, F_0)$$

with $\xi^{-1} F_0(\xi) \to 1$ as $\xi \to 0$. F_0 has the property that

$$(7) \qquad\qquad\qquad y(x) = F_0(x^\alpha e^{-x}) + O(x^{-1})$$

as $x \to \infty$ if $\xi = x^\alpha e^{-x}$ stays bounded, and if ξ_s is a singular point of F_0, then y is singular along the array (5). From (7) and the argument principle, if $F_0(\xi)$ has a pole or an algebraic or logarithmic branch point at ξ_s then the actual solution y of (1) has a quasiperiodic array of singularities, to leading order of the same type as that of F_0. The origin of the singularities of $y(x)$ is the fact that in (2) the exponentials e^{-kx} become $O(1)$ in the antistokes direction, and they can "pile up" to create blow-up of the solution. The singularities form $2\pi i$-quasiperiodic arrays since they are governed by the size of $x^\alpha e^{-x}$. If y^- is the solution analytic in the lower half plane, then $C^- = 0$ and S^+ can be calculated from (5) and (4).

Assumptions. We recall A is analytic at $(0,0)$. Let $H = A(0, \cdot)$ and $\Phi(s) = \int_0^s H(\tau)d\tau$. If $m = 1$ assume H is meromorphic and nonlinear. If $m = 2$ assume H is analytic except for isolated singularities and such that $\int_a^\infty |\Phi(s)|^{-1/2}ds < \infty$ along a path P avoiding the zeros and singularities of Φ. This includes the Painlevé equations P_I and P_{II}, where H is a nonlinear polynomial [2]. Consider the one-parameter family $\mathcal{F}$ of solutions y of (1) such that $\lim_{x \to +\infty} y(x) = 0$.

Theorem 1. *(i) Under the assumptions above, all functions in $\mathcal{F}$ except at most one are singular in the upper or lower half–plane. The singularities form arrays given by (5). If H is polynomial of degree $N > 1$, then the local behavior of y near a singular point x_s is to leading order $(x - x_s)^{m/(1-N)}$.*

(ii) If there exists an $y_0 \in \mathcal{F}$ analytic in a sector of opening more than π centered on $\mathbb{R}^+$, then there is no other such y_0, and $S^+ = 0$; y_0 is entire iff $y_0 \equiv A(x,0) \equiv 0$.

(iii) For $m = 1$, if $A(z,y)$ is polynomial of degree $N > 1$ in y and $A(z,0) \not\equiv 0$, all solutions in $\mathcal{F}$ except at most one have infinitely many singularities in $H_{\pi+\epsilon}$, even if $A(0, \cdot)$ is linear. The singularity type is as in (i).

2.1. **General decay estimates.** We state separately some useful and relatively basic estimates, helpful in proving Theorem 1 (iii).[1]

Lemma 2. *If the function $f \not\equiv 0$ is analytic in a half plane $\{z : \Re(z) > \beta\}$ and for all $a > 0$ we have $f(t) = O(e^{-at})$ for $t \in \mathbb{R}^+$, $t \to +\infty$, then for any $\epsilon \geqslant 0$ small enough $f(z)e^{-bz^{1-\epsilon}}$ is unbounded in the closed right half plane for all $b > 0$.*

Lemma 3. *Assume $F \in L^1(\mathbb{R}^+)$ and for some $\epsilon > 0$ we have*

$$(8) \qquad\qquad \mathcal{L}F(x) = O(e^{-\epsilon x}) \quad as \quad x \to +\infty$$

Then $F = 0$ a.e. on $[0, \epsilon]$. (The result is sharp as discussed after the proof.)

Corollary 4. *Assume $F \in L^1$ and $\mathcal{L}F = O(e^{-ax})$ as $x \to +\infty$ for all $a > 0$. Then $F = 0$ a.e. on $\mathbb{R}^+$.*

Corollary 5. *Assume $\alpha > \frac{1}{2}, f \not\equiv 0$ is analytic in $S = \{z : |z| > R, 2|\arg z| \leqslant \pi/\alpha\}$ and for any $a > 0$ we have $f(t) = o(e^{-at^\alpha})$ as $t \to +\infty$. Then for any $b, p > 0$ and $0 < \alpha' < \alpha$, the function $f(z)e^{-bz^{\alpha'}}$ is unbounded in S.*

3. PROOFS

3.1. **Proof of Theorem 1.** (i) Let first $m = 1$. We show by contradiction that F_0 in (6) is not entire. Assume $g(t) := F_0(e^t)$ was entire. We have $g' = H(g)$. First,

---

[1]Lemma 2 resembles Carlson's Lemma [11] pp. 185, but does not appear to imply it or to follow from it. Also, the referee points out that similar estimating methods appear in [5].

H must have roots or else $G = 1/H$ would be entire and then the function Q given by

$$(9) \qquad Q(g) = \int G(g)dg + C = t$$

would be entire, and injective since Q has an (entire) inverse. Thus Q is linear.

Assume, without loss of generality, that $H(0) = 0$. Next we show that if $H(\tau) = 0$ then $g(t_0) \neq \tau$ for all $t_0 \in \mathbb{C}$. Indeed, if not, $g'(t_0) = 0$ and we would have

$$(10) \qquad \frac{(g(t) - \tau)'}{g(t) - \tau} = \frac{H(g(t)) - H(\tau)}{g(t) - \tau} \to H'(\tau) \ \text{ as } t \to t_0$$

But this is a contradiction, since t_0 is a zero of finite order of $g(t) - \tau$ and the left side of (10) tends to infinity. Since $g(t) \neq 0$ for all $t \in \mathbb{C}$, we have $g = e^h$ with h entire, and since g is nontrivial it cannot avoid any other value, thus 0 is the only root of H. Then $h' = e^{-h}H(e^h)$ where now the right hand side is meromorphic and everywhere nonzero, a case we have already analyzed.

Let now $m = 2$, $g(t) = F_0(e^{-t})$. We have $g'' = H(g)$. Then, multiplying by g' and integrating once, we get $g'^2 = \Phi(g) + C$. Using the condition $g \sim e^{-t}$ for large t and the analyticity of H for small g we get $C = 0$. Choosing a determination of the log and a t_0 such that $g(t_0)$ is not a zero or a singularity of Φ we get

$$(11) \qquad J(g(t_0), g(t)) := \int_{g(t_0)}^{g(t)} \Phi(s)^{-1/2} ds = t - t_0$$

We take the path P towards infinity. We have that $J = \lim_{g \to \infty} J(g(t_0), g)$, $g \in P$, is finite and $t_s := t_0 + J$ is clearly a singular point of g.

(ii) It follows from (4) that unless $S^+ = 0$ every solution develops singularities in the upper or lower half plane (in both, except when $C_- = 0$ or $C^+ = 0$). Conversely, if $S^+ = 0$, it follows from [3] that $\mathcal{B}\tilde{y}$ is analytic in the $\mathbb{C} \setminus \mathbb{R}^-$ (Theorem 1) and exponentially bounded in distributions in $H_{2\pi - \epsilon}$ (Theorem 2). Thus $\mathcal{L}\mathcal{B}\tilde{y}$ is a solution of the original differential equation with the asymptotic expansion $\tilde{y}$ in any sector of opening less than $2\pi + \pi/2$ centered on $\mathbb{R}^+$. In particular, this solution cannot be entire unless it is identically zero, and then $A_0 \equiv 0$. The type of singularity follows from (11) and the discussion at the beginning of §2.

For part (iii) we need the following result.

Lemma 6. *Assume y solves*

$$(12) \qquad y' = \sum_{k=0}^{N} B_k(x) y^k$$

$N > 1$, in a sector S where the analytic coefficients B_k are $O(x^{q_k})$ and $B_N \not\equiv 0$. Assume furthermore that for a sequence $x_j \to \infty$ in S, $y(x_j)$ grows faster than polynomially in x_j. Then for all j large enough, within a distance $O(x_j^{-q_N} y(x_j)^{1-N})$ of x_j there is a singularity of y.

Note. Heuristically, near a large x_j, the dominant balance is of the form $y' \sim Cy^N x^{q_N}$, $C \neq 0$, which forms a singularity as described.

Proof. Let $x = x_j + \sigma$ and $\eta = 1/y$. The equation for σ is

$$(13) \qquad \frac{d\sigma}{d\eta} = -\frac{\eta^{N-2}}{\sum_{k=0}^{N} \eta^k B_{N-k}(x_j + \sigma)} = G(\sigma, \eta)$$

with the initial condition $\sigma(1/y(x_j)) = 0$. It is straightforward to check that in our assumptions, for large enough x_j, $G(\sigma, \eta)$ is analytic in the polydisk $\{(\sigma, \eta) : |\sigma| < 1, |\eta| < 2/|y(x_j)|\}$ where furthermore $|G(\sigma, \eta)| = O(\eta^{N-2} x_j^{-q_N})$. Then, in the disk $\{\eta : |\eta| < 2/|y(x_j)|\}$ there exists a unique analytic solution σ and $\sigma = O(\eta^{N-1} x_j^{-q_N})$. It is clear that $x_j + \sigma(0)$ is a zero of η and thus a singularity of y. □

Proof of Theorem 1 (iii) Assume that there is a solution y_0 of such that $y_0 \to 0$ as $x \to +\infty$ with finitely many singularities in a sector S of opening $\pi + 2\epsilon$ centered on $\mathbb{R}^+$. (There may be an exceptional solution, for instance when the Stokes constant is zero and the asymptotic series converges.) Let y_2 be another solution of $y' = A(x^{-1}, y)$ such that $y_2 \to 0$ as $x \to \infty$ and let $\delta = y_2 - y_1$. The equation for δ is of the form (12) with $B_0 = 0$. From the general theory of differential equations [12] or from [3] it follows that $\delta = O(e^{-x})$ as $x \to \infty$. By Lemma 2 there is a sequence of $x_j \to \infty$ in S so that $\delta \geq const. \exp(|x_j|^{1-\epsilon})$. Near every x_j with j large enough there is, by Lemma 6, a singularity of δ and thus of y_2.

3.2. **Connection with integrability.** It is seen that if H is a polynomial of degree $N > 2$ if $m = 1$ or $N > 3$ if $m = 2$ a one parameter family of solutions forms arrays of branch-point singularities in $\mathbb{C}$ and the Painlevé test fails. For other values of p, the corrections $F_j, j > 1$ [2] can be calculated to determine the exact type of singularity. More can be done however. Assuming we are in the case where singularities are branch points, in view of the exact description of type and location of singularities it is possible to calculate the monodromy of solutions along a curve in $\mathbb{C}$ winding among sufficiently many singularities to show dense branching, a concept proposed by Kruskal, which can be used to show absence of continuous first integrals as in [7], [8]. This will be the subject of a different paper.

3.3. **Proof of Lemma 3.** We write

$$(14) \qquad \int_0^\infty e^{-px} F(p)dp = \int_0^\epsilon e^{-px} F(p)dp + \int_\epsilon^\infty e^{-px} F(p)dp$$

we note that

$$(15) \qquad \left| \int_\epsilon^\infty e^{-px} F(p)dp \right| \leqslant e^{-\epsilon x} \int_\epsilon^\infty |F(p)|dp \leqslant e^{-p\epsilon} \|F\|_1 = O(e^{-\epsilon x})$$

Therefore

$$(16) \qquad g(x) = \int_0^\epsilon e^{-px} F(p)dp = O(e^{-\epsilon x}) \quad \text{as} \quad x \to +\infty$$

The function g is manifestly entire. Let $h(x) = e^{\epsilon x} g(x)$. Then by assumption h is entire and uniformly bounded for $x \in \mathbb{R}$ (since by assumption, for some x_0 and all $x > x_0$ we have $|h| \leqslant C$ and by continuity $\max |h| < \infty$ on $[0, x_0]$). The function is bounded by $\|F\|_1$ for $x \in i\mathbb{R}$. By Phragmén-Lindelöf's theorem (first applied in the first quadrant and then in the fourth quadrant, with $\beta = 1, \alpha = 2$) h is bounded in the closed right half plane. Now, for $x = -s < 0$ we have

$$(17) \qquad e^{-s\epsilon} \int_0^\epsilon e^{sp} F(p)dp \leqslant \int_0^\epsilon |F(p)| \leqslant \|F\|_1$$

Again by Phragmén-Lindelöf (and again applied twice) h is bounded in the closed left half plane thus bounded in C, and it is therefore a constant. But, by the

Riemann-Lebesgue lemma, $h \to 0$ for $x = is$ when $s \to +\infty$. Thus $h \equiv 0$. Therefore, with χ_A the characteristic function of A,

$$(18) \qquad \int_0^\epsilon F(p)e^{-isp}dp = \hat{\mathcal{F}}(\chi_{[0,\epsilon]}F) = 0$$

for all $s \in \mathbb{R}$ entailing the conclusion.

Note. In the opposite direction, by Laplace's method it is easy to check that for any small $\epsilon > 0$ we have $\mathcal{L}e^{-p^{-\frac{1-\epsilon}{\epsilon}}} = o\left(e^{-x^{1-\epsilon}}\right)$ and for any n $\mathcal{L}\left(e^{-E_{n+1}(1/p)}\right) = o(e^{-x/L_n(x)})$ where E_n is the n-th composition of the exponential with itself and L_n is the n-th composition of the log with itself. $\square$

3.4. **Proof of Proposition 2.** By a change of variable we may assume that $\beta = -1$. Assume that for some $b > 0$ $e^{-bz^{1-\epsilon}}f(z)$ was bounded in the closed right half plane. Then $\psi(z) = (1+z)^{-2}e^{-bz^{1-\epsilon}}f(z)$ satisfies the assumptions of Lemma 8. But then $\psi(z) = \mathcal{L}\mathcal{L}^{-1}\psi(z)$ satisfies the assumptions of Corollary 4 and $\psi \equiv 0$.

Note 7. *There is indeed loss of exponential rate: the entire function $\Gamma(x)^{-1} = O(e^{-n \ln n})$ on $\mathbb{R}^+$ but is bounded by $O(e^{|x|\pi/2})$ in the closed right half plane.*

4. APPENDIX: OVERVIEW OF LAPLACE TRANSFORM PROPERTIES

For convenience we provide some standard results on Laplace transforms.

Remark 1 (Uniqueness). *Assume $F \in L^1(\mathbb{R}^+)$ and $\mathcal{L}F = 0$ for a set of x with an accumulation point. Then $F = 0$ a.e.*

Proof. By analyticity, $\mathcal{L}F = 0$ in the open right half plane and by continuity, for $s \in \mathbb{R}$, $\mathcal{L}F(is) = 0 = \hat{\mathcal{F}}F$ where $\hat{\mathcal{F}}F$ is the Fourier transform of F (extended by zero for negative values of p). Since $F \in L^1$ and $0 = \hat{\mathcal{F}}F \in L^1$, by the known Fourier inversion formula [6], $F = 0$. $\square$

Lemma 8. *Assume that $c \geqslant 0$ and $f(z)$ is analytic in $H_c := \{z : \Re z \geqslant c\}$. Assume further that $g(t) := \sup_{c' \geqslant c}|f(c'+it)| \in L^1(\mathbb{R}, dt)$. Let*

$$(19) \qquad F(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}f(x)dx =: (\mathcal{L}^{-1}f)(p)$$

Then for any $x \in \{z : \Re z > c\}$ we have $\mathcal{L}F = \int_0^\infty e^{-px}F(p)dp = f(x)$

Note that for any $x' = x_1' + iy_1' \in \{z : \Re z > c\}$

$$(20) \qquad \int_0^\infty dp \int_{c-i\infty}^{c+i\infty}\left|e^{p(s-x')}f(s)\right|d|s| \leqslant \int_0^\infty dp e^{p(c-x_1')}\|g\|_1 \leqslant \frac{\|g\|_1}{x_1'-c}$$

and thus, by Fubini we can interchange the orders of integration:

$$(21) \quad U(x') = \int_0^\infty e^{-px'}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}f(x)dx$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dx f(x)\int_0^\infty dp e^{-px'+px} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{f(x)}{x'-x}dx$$

Since $g \in L^1$ there exist subsequences $\{\tau_n\}, \{-\tau'_n\}$ tending to infinity such that $|g(\tau_n)| \to 0$. Let $x' > \Re x = x_1$ and consider the box $B_n = \{z : \Re z \in [x_1, x'], \Im z \in [-\tau'_n, \tau_n]\}$ with positive orientation.

$$(22) \qquad \int_{B_n} \frac{f(s)}{x' - s} ds = -f(x')$$

while, by construction,

$$(23) \qquad \lim_{n \to \infty} \int_{B_n} \frac{f(s)}{x' - s} ds = \int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x' - s} ds - \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{x' - s} dx$$

On the other hand, by dominated convergence, we have

$$(24) \qquad \int_{x'-i\infty}^{x'+i\infty} \frac{f(s)}{x' - s} ds \to 0 \ \text{ as } \ x' \to \infty$$

REFERENCES

[1] T Aoki, T Kawai, S Shunsuke, A Shudo and Y Takei, *Virtual turning points and bifurcation of Stokes curves for higher order ordinary differential equations.* J. Phys. A 38 (2005), no. 15, 3317–3336.

[2] O. Costin, R. D. Costin *On the formation of singularities of solutions of nonlinear differential systems in antistokes directions* (Inventiones Mathematicae **145** 3, pp 425-485 (2001)).

[3] O Costin On Borel summation and Stokes phenomena for rank one nonlinear systems of ODE's, Duke Math. J. Vol. 93, No 2: 289–344, (1998).

[4] A Its, A R Fokas and A S Kapaev *On the asymptotic analysis of the Painlevé equations via the isomonodromy method.* Nonlinearity 7 (1994), no. 5, 1291–1325.

[5] R Paley, N Wiener *Fourier transforms in the complex domain* New York, American Mathematical Society (1934).

[6] Walter Rudin *Real and Complex Analysis, Third Edition, Mc-Graw-Hill (1987).*

[7] R D Costin *Integrability Properties of Nonlinearly Perturbed Euler Equations* Nonlinearity 10 No 4 pp. 905-924 (1997)

[8] R D Costin *Integrability Properties of a Generalized Lame Equation; Applications to the Henon-Heiles System* Meth.Appl.An., Vol 4 No 2, pp. 113-123 (1997)

[9] G G Stokes, *On the discontinuity of arbitrary constants which appear in divergent developments* Trans. Camb. Phil. Soc. **X** pp. 106–128 (1864).

[10] Y Sibuya *Global theory of a second order linear ordinary differential equation with a polynomial coefficient Amsterdam* North-Holland Pub. Co. ; New York : American Elsevier Pub. Co., (1975).

[11] E C Titchmarsh, *The theory of functions*, Oxford University Press (1964).

[12] W Wasow *Asymptotic expansions for ordinary differential equations*, Interscience Publishers (1968).

MATHEMATICS DEPARTMENT, THE OHIO STATE UNIVERSITY, COLUMBUS OH 43210, USA, COSTIN@MATH.OHIO-STATE.EDU