Optimal Uniform Estimates and Rigorous Asymptotics Beyond all Orders for a Class of Ordinary Differential Equations *

Ovidiu Costin †and Martin D. Kruskal ‡

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†Rutgers University, Mathematics Department, Hill Center , New Brunswick NJ 08903; e-mail: costin@maxwell.rutgers.edu

‡Rutgers University, Mathematics Department, Hill Center, New Brunswick, NJ 08903; e-mail: kruskal@math.rutgers.edu

Abstract

For first order differential equations of the form $y' = \sum_{p=0}^{P} F_p(x)y^p$ and second order homogeneous linear differential equations $y'' + a(x)y' + b(x)y = 0$ with locally integrable coefficients having asymptotic (possibly divergent) power series when $|x| \to \infty$ on a ray $\arg(x)$ =const, under some further assumptions, it is shown that, on the given ray, there is a one-to-one correspondence between true solutions and (complete) formal solutions. The correspondence is based on asymptotic inequalities which are required to be uniform in x and optimal with respect to certain weights.

1 Introduction and Main Results

The main purpose of the present paper is to give, in terms of uniform asymptotic estimates, a precise meaning to *complete* asymptotic expansions (e.g., as power series followed by exponentially small terms) of solutions of a class of differential equations in a neighborhood of an irregular singular point (chosen to be infinity). The study of the exponentially small terms in asymptotic expansions has known a rapid development in the last years, especially after the pioneering works of Ecalle (1981) and Berry (1989, 1990).

For first order polynomially nonlinear (or linear) and second order homogeneous linear ordinary differential equations we give, using asymptotic inequalities, a one-to-one correspondence between formal solutions and true solutions. The representation of a given function y turns out to be (under some assumptions) independent of the differential equation(s) of which y is a solution.

The key ingredient is the concept of uniform optimal asymptotic inequalities which we illustrate in the following.

Consider a formal power series $\tilde{f} = \sum_{k=0}^{\infty} f_k x^{-k}$, where f_k are complex numbers and x is thought of being a large variable. We say that a function f is uniformly asymptotic to the series $\tilde{f}$ with respect to the weight w, along a given ray in the complex plane, say $\mathcal{R} = x > x_0 > 0$ iff

$$\left| f(x) - \sum_{k=0}^{n-1} f_k x^{-k} \right| < w(n) x^{-n} \quad \forall x \in \mathcal{R} \ \text{ and } \forall n \in I\!N \tag{1.1}$$

Any function f that is asymptotic to the series $\tilde{f}$ is uniformly asymptotic with respect to *some* w. The minimal such w is obviously

$$w(f,\tilde{f};n) \equiv w_f(n) := \sup_{x>x_0} x^n \left| f(x) - \sum_{k=0}^{n-1} f_k x^{-k} \right| \tag{1.2}$$

(we might think of w_f as a transform of f with respect to $\tilde{f}$). With no restrictions on w we thus obtain the Poincaré asymptoticity. With more conditions on w we obtain sharper asymptoticity classes: e.g. if $f_k \sim \alpha^n (n!)^\beta$ and we consider the functions f for which $w_f(k) \le \gamma^n (n!)^\beta$, $\gamma > \alpha$ we obtain the familiar Gevrey or Gevrey-Roumieux classes (Ramis, 1978).

It is natural to take w_f as a measure of the separation between a function f and a formal series, $\tilde{f}$. We will say that f is closer to $\tilde{f}$ than g iff

$$w_f(n) \le w_g(n) \ \text{ for all large } n \tag{1.3}$$

or, on occasions if a weaker condition holds:

$$\limsup_{n\to\infty} \frac{w_f(n)}{w_g(n)} \le 1 \tag{1.4}$$

Given a formal series $\tilde{f}$, a ray $\mathcal{R}$ and a class of functions $\mathcal{F}$, it is also natural to ask what are the sharpest bounds compatible with it, i.e. what is the "greatest lower bound" of the weights w such that $w = w_f$ for *some* f.

We say that a function f is optimally asymptotic to $\tilde{f}$ with respect to $\mathcal{F}$ along $\mathcal{R}$, written

$$f \simeq \sum_{k=0}^{\infty} f_k x^{-k} \text{ as } x \to \infty \quad \text{on } \mathcal{R} \tag{1.5}$$

and correspondingly that w_f is an optimal weight for $\tilde{f}$ iff (1.3) (or sometimes (1.4)) holds for *all* $g \in \mathcal{F}$.

As a first example, note that if $\tilde{f}_0$ is a *convergent* series and $x_0 > \rho^{-1} > R^{-1}$ where R is the radius of convergence, and $\mathcal{F}$ is the set of all functions defined for $x > x_0$ then

$$f \simeq \tilde{f}_0 \Longleftrightarrow f = f_0 = \sum_{k=0}^{\infty} f_{0;k} x^{-k}$$

Indeed, $w_{f_0}(n) \le C\rho^n \left(1 - \rho x_0^{-1}\right)^{-1} \to 0$ as $n \to \infty$ and therefore uniform asymptoticity of a function f with respect to w_0 implies the convergence of the series $\tilde{f}_0$ to f.

Next, take $\tilde{f}_1 = \sum_{k=0}^{\infty} k! x^{-k-1}$. We will see that an optimal weight for this series, in the sense (1.4), along the ray $x > x_0 > 0$ behaves like $w(n) = (a_* \sqrt{n} + O(1))n!$ for large n, where a_* is defined in (1.16). An optimally asymptotic function is $f(x) = e^{-x} Ei(x) := e^{-x} P \int_{-\infty}^{x} t^{-1} e^t dt$.

It turns out that if $f_1,\ f_2$ are two functions optimally asymptotic to $\tilde{f}_1$ then $f_1 - f_2 = o(e^{-x})$ for large x. It is then meaningful to extend the notion of optimal asymptoticity to more general structures e.g., power series followed by exponentially small terms: in our example we can *define*

$$f(x) \simeq \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} + Ce^{-x} \Longleftrightarrow f(x) - Ce^{-x} \simeq \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} \tag{1.6}$$

since the second relation can hold for at most one value of C. On the other hand $e^{-x} Ei(x) + Ce^{-x}$ is the general solution of the equation $f' + f = x^{-1}$ whereas $\tilde{f} + Ce^{-x}$ is the general formal solution of that equation. It follows that the relation $f(x) \simeq \tilde{f} + Ce^{-x}$ establishes a one-to -one correspondence between true and formal solutions for this equation.

The main goal of the present paper is to show that by letting $\mathcal{F}$ to contain solutions of a class of first and second order differential equations, along some ray, optimal asymptoticity (in either the stronger form (1.3) or the weaker one (1.4)) always gives a one-to one correspondence between true and formal solutions.

As a manifestation of the Stokes phenomenon, the complete expansion of a given solution of a differential equation will depend on the ray along which the asymptotic estimates are considered. In our example, we have

$$e^{-x}\mathcal{E}i(x) \simeq \sum_{j=0}^{\infty} \frac{j!}{x^{j+1}} + \begin{cases} 0 & \text{if } \arg(x) = 0 \\ \pi i \, e^{-x} & \text{if } \arg(x) \in (0, \pi) \\ -\pi i \, e^{-x} & \text{if } \arg(x) \in (-\pi, 0) \end{cases} \tag{1.7}$$

For a complete study of the Stokes one would need to consider besides rays, parabolas (of the form $\arg(x) = \lambda \, |x|^{-1/2}$ for (1.7)); then the constant beyond all orders changes smoothly in a narrow region near the Stokes line (it is a smooth function of λ in the example above) see (Berry 1989).

However in this paper we are not making analyticity assumptions on the coefficients of the differential equations and we will be consequently mainly concerned with the behavior of solutions along a fixed ray.

A function is characterized by its generalized expansion with a precision comparable to that of truncation at the least term of the complete asymptotic expansion. The relation with this technique is explored in §2.

In this respect we mention the recent paper of McLeod (1992) in which it is shown that truncation at the least term is can be used to measure the terms beyond all orders for second order linear homogeneous differential equations with coefficients analytic at infinity. The corresponding results in our paper show that the same is true for equations with not necessarily analytic coefficients (and as such only defined on a ray) having possibly divergent series at infinity (divergent no faster than factorially) and first order polynomially nonlinear equations. Removing the analyticity and linearity assumptions raises difficulties requiring new techinques for the proofs. In particular we provide a method of estimating the growth of the coefficients of formal series solutions based on the recurrence relation that they satisfy.

We also discuss the possibility of extending our results to more general systems of differential equations.

We postpone further discussions and examples until after Theorem 1.1 below. In §2 we study, in some generality, the connection between optimal asymptoticity and the method of truncation near the least term. §3 is devoted to the proofs and in §4 we discuss, in the context of a particular family of differential equations, the question of typicality of the divergence of the asymptotic representations and give a decomposition formula which provides a different perspective on complete asymptotic expansions.

Consider the second order linear differential equation

$$y'' + a(x)y' + b(x)y = 0 \tag{1.8}$$

given on the ray

$$\mathcal{R}_\theta = \left\{x : e^{-i\theta}x > x_0\right\}$$

The coefficients $a(x)$ and $b(x)$ are assumed to be in $L^1_{loc}(\mathcal{R})$ nd to have asymptotic power series at infinity of the form

$$a(x) \sim \sum_{k=0}^{\infty} \frac{a_k}{x^k}; \quad b(x) \sim \sum_{k=0}^{\infty} \frac{b_k}{x^k} \quad (x\, e^{-i\theta} \to +\infty) \tag{1.9}$$

Moreover, we require that the functions $a(x)$ and $b(x)$ satisfy a Gevrey-like condition (Ramis 1978) on the given ray namely, for some $\kappa < |a_0^2 - 4b_0|^{-\frac{1}{2}}$

$$\begin{aligned} a_n, b_n &< \text{const}\kappa^n n! \\ |a(x) - \sum_{k=0}^{n-1} \frac{a_k}{x^k}| &< \text{const}\kappa^n n!\, |x|^{-n} \\ |b(x) - \sum_{k=0}^{n-1} \frac{b_k}{x^k}| &< \text{const}\kappa^n n!\, |x|^{-n} \end{aligned} \tag{1.10}$$

uniformly in $x \in \mathcal{R}_\theta, n \in I\!N$.

If the conditions (1.10) hold, then the rate of divergence of a formal series solution depends only on the first few terms in the asymptotic series of $a(x)$ and $b(x)$. The results below (not merely the proofs) depend on this assumption.

The polynomial

$$\lambda^2 + a_0\, \lambda + b_0 \tag{1.11}$$

is assumed to have distinct roots, λ_1, λ_2 (if the roots coincide then there are no terms beyond all orders to worry about). The following expression is a *formal* solution:

$$\tilde{S} = C_1\, e^{\lambda_1 x}\, x^{r_1}\, \tilde{S}_1(x) + C_2\, e^{\lambda_2 x}\, x^{r_2}\, \tilde{S}_2(x) \tag{1.12}$$

where for $i = 1, 2$

$$r_i = -\frac{a_1\, \lambda_i + b_1}{a_0 + 2\lambda_i}$$

and

$$\tilde{S}_i(x) = \sum_{k=0}^{\infty} \frac{s_{i;k}}{x^k}$$

are formal power series. The asymptotic behavior for large n of the coefficients of the power series, as follows from the Proposition 3.2 below, has the form

$$s_{1;n} \sim \left(R_1 + \sum_{m=1}^{\infty} \frac{R_{1,m}}{n^m} \right) \frac{\Gamma(n + r_2 - r_1)}{(\lambda_1 - \lambda_2)^n} \tag{1.13}$$

The expression for $s_{2,n}$ is obtained from the one above by interchanging the indices 1 and 2. Choosing

$$s_{1;0} = s_{2;0} = 1$$

the coefficients of the series are uniquely determined.

The expression (1.12) containing two arbitrary constants is the general formal solution of our equation, in the differential algebra generated by power series and exponentials, see Cope (1934), Ritt (1950), Ecalle (1993).

The notion of optimal asymptoticity can be extended in a natural way to asymptotic structures of the form (1.12). We say that f is uniformly asymptotic to $\tilde{S}$ for a weight (w_1, w_2) along the ray $\mathcal{R}_\theta$ iff

$$\begin{aligned} &\left| y(x) - C_1\, x^{r_1}\, e^{\lambda_1 x} \sum_{j=0}^{k_1-1} \frac{s_{1;j}}{x^j} - C_2\, x^{r_2}\, e^{\lambda_2 x} \sum_{j=0}^{k_2-1} \frac{s_{2;j}}{x^j} \right| \\ &\qquad \le |x^{r_1} e^{\lambda_1 x}| w_1(k_1) + |x^{r_2} e^{\lambda_2 x}| w_2(k_2) \quad \forall x \in \mathcal{R}_\theta \text{ and } k_1, k_2 \in I\!N \end{aligned} \tag{1.14}$$

We say that y is optimally asymptotic to $\tilde{S}$ along the ray $\mathcal{R}_\theta$ and we write

$$y \simeq C_1\, x^{r_1}\, e^{\lambda_1 x} \sum_{j=0}^{\infty} \frac{s_{1;j}}{x^j} + C_2\, x^{r_2}\, e^{\lambda_2 x} \sum_{j=0}^{\infty} \frac{s_{2;j}}{x^j} \tag{1.15}$$

if there is a weight $(w_1(y; n), w_2(y; n))$ such that (1.14) holds and moreover for any $f \in \mathcal{F}$, $w_1(y; n) \le w_1(f; n)$ and $w_2(y; n) \le w_2(f; n)$ for all large n.

Theorem 1.1 below states that for each formal solution there is, on the given ray, a unique true solution of the equation which is optimally asymptotic to it relative to the space of solutions of the same differential equation. The theorem can be easily extended to encompass differential equations that can be brought to the form that is treated here by an algebraic change of independent variable and linear changes of the dependent variable; see also Remark 4 below.

Let B be a (large enough, positive) constant, $r := r_2 - r_1$, $\rho = \Re(r)$, and $\Delta = \lambda_2 - \lambda_1$.

Theorem 1.1 *For any* C_1, C_2 *there is a* unique *true solution* $y(x)$ *of the differential equation (1.8), under the assumptions following it, with the property (1.14) for the weight*

with
$$w_i(y;k_i) = |C_i|(A_i\sqrt{k_i}+B)\Gamma(k_i-(-1)^i\rho)|\Delta|^{-k_i} \quad i=1,2$$

$$A_1 = \begin{cases} a_*|R_1| & \text{if } e^{i\theta}(\lambda_1-\lambda_2)\in \mathbb{R}^+ \\ 0 & \text{otherwise} \end{cases} \tag{1.16}$$

where
$$a_* := \sqrt{2}\max_{x\geq 0} e^{-x^2}\int_0^x e^{t^2}dt = 0.765151\cdots \tag{1.17}$$

and A_2 *is obtained from (1.16) by interchanging the indices* 1 *and* 2.

Comments.

1) Let $y(x)$ be the unique solution provided by the theorem and choose the special truncation orders $k_1 = k_2 = [|x\Delta|]$, which correspond generically to "truncation at the least term" of the series. It will follow from the proof that the remainder (left side of (1.14)) is in this case less than

$$\text{const}\,\frac{1}{\sqrt{|x|}}e^{-|x\Delta|}\max\left\{|x^{r_2}e^{\lambda_1 x}|, |x^{r_1}e^{\lambda_2 x}|\right\} \tag{1.18}$$

i.e., asymptotically smaller than any (nonzero) solution of the equation by a factor of $\sqrt{|x|}$ on the Stokes line $\Im(x\Delta)=0$ and by an exponential factor in the generic case $\Im(x\Delta)\neq 0$, indicating why there is uniqueness of the representation. There is, in effect, a connection between the shape of the optimal weight and the precision of the least term truncation method, as will become clear in the following section. Moreover, the solution that is best approximated by the simultaneous least term truncation of the two component series in (1.15) is precisely the one having the optimal weight. There are differences between the two approaches but the terms beyond all orders that they predict agree for this class of differential equations.

2) The values of the constants A_1 and A_2 in (ii) are crucial for the result. If say, $\Re(e^{i\theta}\lambda_2) < \Re(e^{i\theta}\lambda_1)$ and $R_1 \neq 0$ then there will be no solutions satisfying the required inequalities with $A_1 < a_*|R_1|$ whereas if $A_1 > a_*|R_1|$ there will be infinitely many such solutions. In contrast, the theorem is true for any large enough B. We also want to stress that uniqueness is *relative* to the space of solutions of the differential equations treated here.

3) As a manifestation of the Stokes phenomenon (Stokes 1857, Sibuya 1975, Wasow 1968, Berry 1989), the constants C_i in the asymptotic representation of a given solution depend on the direction θ of the ray considered (provided, of course, that the differential equation satisfies our hypothesis on more than one ray at infinity). The example (1.7) given at the beginning of the section illustrates this point.

The derivation of (1.7) is given at the end of §2. We also mention at this point that a more detailed analysis is possible along the same lines and it shows that the term beyond all orders varies smoothly on a scale of the order $\arg(x) = O\left(x^{-1/2}\right)$ in a way which agrees with the results obtained by the hyperasymptotic technique of Berry (1989, 1990). We will however not pursue this issue here.

4) One can in principle allow for more general formal structures than (1.12), for instance those obtained by substitutions and formal algebraic operations on (1.12) (for a discussion on what formal structures are relevant to solving differential equations see (Ecalle, 1993) to allow for asymptotic representations of the solutions of equations that are not of the form required by the

theorem, but can be brought to that form; the asymptotic inequalities are well suited for simple algebraic operations. Consider for instance the Airy equation

$$y'' - x\,y = 0 \tag{1.19}$$

After the substitutions $y(x) = \exp(\frac{2}{3}\,x^{\frac{3}{2}})g(x)$ and $x = s^{2/3}$ we get

$$g'' + (\frac{4}{3} + \frac{1}{3s})g' + \frac{2}{9s}\,g = 0 \tag{1.20}$$

to which Theorem 1.1 applies, and we obtain, after undoing the transformations, the following asymptotic representation of the general solution of the Airy equation (1.19):

$$y(x) \simeq C_1 e^{\frac{2}{3}x^{3/2}} x^{-1/4} \sum_{k=0}^{\infty} \frac{s_k}{x^{3k/2}} + C_2 e^{-\frac{2}{3}x^{3k/2}} x^{-1/4} \sum_{k=0}^{\infty} \frac{(-)^k s_k}{x^{3k/2}} \tag{1.21}$$

when x becomes large along a given ray in the complex plane. For a fixed solution C_1, C_2 will depend on the ray. A brief derivation of (1.21) and the expressions of A_1, A_2 and s_k are given at the end of §2.

5) Whenever $e^{i\theta}\Delta \notin \mathbb{R}^-$ all the terms of the expansion (1.12) are simultaneously visible to the inequalities, as will follow from the proof of the theorem. If on the other hand $e^{i\theta}\Delta < 0$, i.e., on the Stokes line, besides the first series only the first term of the second series is caught by the inequalities. However this first term is enough in order to establish a one-to-one (linear) correspondence between formal and true solutions, so that we keep, by courtesy, all the other terms of the second series.

6) To obtain complete asymptotic representations by means of optimal inequalities for higher order differential equations or systems of equations it seems necessary to use, inductively, asymptotic comparisons with solutions of lower order ODE's.

The result below shows that a given function cannot be a solution of two essentially different second order differential equations that satisfy our assumptions. We say that a differential equation of the type considered in the theorem is in canonical form if

$$\theta = 0;\ \lambda_1 = 0;\ |a_0| = 1;\ r_1 = 0;\ b_0 = b_1 = 0 \tag{1.22}$$

This can be always achieved by the substitution $y(x) = \exp(\lambda_1\,x)x^{r_1}\tilde{y}(x)$ followed by a change of independent variable $x = e^{i\theta}|\lambda_1 - \lambda_2|^{-1}\tilde{x}$. The differential equation is said to be formal if the coefficients $a(x), b(x)$ are replaced by their formal series, $\tilde{a}(x), \tilde{b}(x)$.

Condition (1.10) plays a crucial role in the result below.

Proposition 1.1 *Let $\tilde{S} = \sum_{n=0}^{\infty} s_n x^{-n}$ be a formal power series such that s_n have the asymptotic behavior (1.13) with $R_1 \neq 0$. Then $\tilde{S}$ formally solves at most one second-order* canonical *formal differential equation of the form (1.8), where the series $\tilde{a}(x)$ and $\tilde{b}(x)$ satisfy the first condition in (1.10).*

It follows for instance that within the class of differential equations we are concerned with, there is a one-to-one correspondence between generic formal solutions and true solutions. Indeed, a true solution is associated with a unique representation within the solutions of the differential equation it originates in. But then, the representation determines uniquely the power series of the coefficients of the equation and by this determines uniquely the differential equation itself. Conversely, a formal expansion *solution* is associated with a unique differential equation and within that differential equation with a unique true solution. In this sense we also obtain a summation method. One can attempt to continue this construction inductively, considering at the n-th step differential equations whose coefficients are solutions of the equations gotten at the step $n-1$ and obtain more general representation spaces and summation methods.

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Similar results hold for first order nonlinear differential equations of the form

$$y' = F(y,x) := \sum_{p=0}^{P} F_p(x) y^p \tag{1.23}$$

defined on the ray $\mathcal{R}_\theta$.

We require that for $p = 0,..,P$, $F_p(\cdot) \in C[R_\theta]$, $F(0,\infty) \equiv f_{0,0} = 0$, $F_y(0,\infty) =: \alpha = -\,f_{1,0} \neq 0$, and we impose the Gevrey-like condition

$$\begin{aligned} \left| F_p(x) - \sum_{k=0}^{n-1} \frac{f_{p,k}}{x^k} \right| &< C_F \kappa^n n! |x|^{-n} \\ |f_{p,n}| &< C_F \kappa^n \, n! \end{aligned} \tag{1.24}$$

(where C_F is a constant) uniformly in $x \in \mathcal{R}_\theta, n \in I\!N$, where $\kappa < |\alpha|^{-1}$.

We are only interested in the case when the equation allows for exponentially *small* terms beyond all orders; the condition for that (as will become immediately clear) is

$$\arg(x\alpha) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{1.25}$$

Lemma 1.1 *There exists a formal series solution*

$$\tilde{S} := \sum_{k=1}^{\infty} \frac{s_k}{x^k} \tag{1.26}$$

of equation (1.23). The coefficients of the series have the asymptotic behavior

$$s_k \sim \left(R + \sum_{m=1}^{\infty} \frac{R_m}{k^m} \right) \alpha^{-k} \Gamma(k+r) \quad (\text{as } k \to \infty) \tag{1.27}$$

where

$$r = f_{1,1} + 2\, f_{2,0} f_{0,1}/\alpha$$

If we are now looking for nearby formal solutions $S_\delta := S + \delta$ we get

$$\delta' = \left(-\alpha + \frac{r}{x}\right) \delta + O(\delta^2, \delta/x^2) \tag{1.28}$$

so that for large $|x|$

$$\delta \sim C\, x^r e^{-\alpha x} \tag{1.29}$$

The analog of Theorem 1.1 in this case reads: consider equation (1.23) under the assumptions (1.24) and (1.25), and let $\rho = \Re(r)$. Let B be large enough.

There is a one-to-one asymptotic correspondence between true solutions which decay at infinity and formal solutions:

Theorem 1.2 *Given any constant C there exists a* unique *true solution $y(x)$ of the differential equation (1.23) such that*

$$\left| y(x) - \sum_{j=0}^{k-1} \frac{s_j}{x^j} - C\, x^r\, e^{-\alpha\, x} \right| < (A\sqrt{k} + B)\Gamma(k+\rho)|x|^{-k} \tag{1.30}$$

for all $x \in \mathcal{R}_\theta$ and $k \in I\!N$, where

$$A = \begin{cases} a_*|R| & \text{if } e^{i\theta}\alpha \in I\!R^+ \\ 0 & \text{otherwise} \end{cases} \tag{1.31}$$

Conversely, given a solution of (1.23) such that $xy(x)$ is bounded for large x, there is a (unique) constant C such that, for all $x \in \mathcal{R}_\theta$ and $k \in I\!N$, (1.30) holds.

In the sense of (1.30) we then write

$$y(x) \simeq \sum_{j=0}^{\infty} \frac{f_j}{x^j} + C\, x^r\, e^{-\alpha\, x} \tag{1.32}$$

The comments that follow Theorem 1.1, with obvious adaptations, also apply in this case.

We could actually continue the construction of the formal solution and consider the complete formal solutions or "transseries"(Ecalle, 1990, 1993) which in this case have the form

$$Y_0 + x^r e^{-\alpha\, x} Y_1 + x^{r_2} e^{-2\,\alpha\, x} Y_2 + ...$$

in which Y_i are formal power series. Y_1 is determined up to an arbitrary constant which, once given, determines completely all the following power series Y_i, $i \geq 2$. We will however not pursue this direction here but merely remark that the one-to-one correspondence between formal solutions and true solutions is pinned down by the first term of the second formal series and are content to control the asymptotics only to that level.

2 Optimal estimates for a class of divergent series

In this section we estimate the optimal bounds for series that diverge in a way typical for differential equations and make the connection between optimal uniform inequalities and the method of truncation at the least term. We also give results on the precision with which a function is represented by its optimal asymptotic expansion. We consider formal power series for large argument on a ray and establish the connection to the technique of optimal truncation of divergent series. Let

$$\tilde{S} := \sum_{k=0}^{\infty} \frac{c_k}{x^k} \tag{2.1}$$

where

$$c_k = (1+\epsilon_k)\, k^{i\phi} C^k \exp(G(k)) \text{ with } \phi \in I\!R \text{ and } \epsilon_k \to 0 \text{ as } k \to \infty \tag{2.2}$$

Since the term C^k can be always absorbed into the independent variable we will take $C = 1$.

In (2.2) ϕ is real and we make the following assumptions. $G(\cdot) \in C^3(I\!R^+)$ is a real-valued increasing, convex function with the properties $G''(x) \to 0$ and $x^2 G'''(x) \to -\gamma < 0$ as $x \to \infty$. Such a series is necessarily divergent for all values of x (convergent series are discussed in the introduction). An example satisfying the requirements would be $c_k = \Gamma^\beta(k+r)$.

The behavior of the optimal weight for a divergent series depends critically on the ray along which we consider the asymptotic series. In the setting (2.2) the optimal weight is larger by a factor of order $\sqrt{n}$ along the real positive axis than it is along any other ray. This is a manifestation of the Stokes phenomenon at the level of the asymptotic series themselves. Intuitively we can account for this behavior in the following way. A natural scale for studying the difference between a function and the n-th truncate of its asymptotic series is the $(n+1)$-th term of the series (indeed, the $(n+1)$-th term is meant to be a correction for the above mentioned difference). When x is very large, the successive terms of the series start by decreasing fast as does the error in approximating the function by its truncated series. The least error suggested by our rough guide, the next term of the series, reaches a minimum when

$$G'(s_x) = \ln(|x|) \tag{2.3}$$

The effective width of the minimum is of the order $\sqrt{s_x}$ and within this width, for $x > 0$, the ratio of two terms is approximately one. But then we realize that if at one point within this range the difference between the function and the truncate of the series is approximately equal to the next term there will be a point within the same region where the difference will be roughly $\sqrt{n}$ times as big. If x is not real the ratio of the successive terms is of the form $e^{-i\theta}$ so that the overall accumulation of errors is still of the order of a constant.

We want to stress that, as follows from Lemma 3.1 below and the estimates in its proof, the least-term truncation of a formal series solution is a good approximation only for *one* solution of the given differential equation; for all the other solutions there are exponential corrections that are much larger than the least term of the series and have to be removed before calculating the function from its series (to such an accuracy).

Consider the power series $\tilde{S}$, under the stated assumptions, for x on the ray $\mathcal{R}_\theta$.

Lemma 2.1 *i) For any θ there exist (smooth) functions that are optimally asymptotic to the series (2.2). An optimal weight is*

$$g_*(n) = a_*\gamma^{-\frac{1}{2}}(\sqrt{n}+B)|c_{n+1}| \quad \textit{for } \theta = 0$$

More precisely, assume that Φ is a function and η and B are constants such that for all $x \in \mathcal{R}_\theta$ and for all $n \in I\!N$ we have

$$\left|\Phi(x) - \sum_{j=0}^{n}\frac{c_j}{x^j}\right| < \eta\, a_*\gamma^{-\frac{1}{2}}(\sqrt{n}+B)\frac{|c_{n+1}|}{x^{n+1}} \tag{2.4}$$

Then, $\eta \geq 1$ and there exist functions satisfying (2.4) for $\eta = 1$, for some B.

ii) For $\theta \neq 0$ assume further that

$$\epsilon_k = o(k^{-1/2}) \tag{2.5}$$

Then the optimal weights can be taken

$$g_*(n) = (a(\theta) + Bn^{-1})|c_{n+1}|$$

where the constant $a(\theta)$ can be computed explicitly and for small θ has the behavior

$$a(\theta) \sim \theta^{-1} \tag{2.6}$$

The behavior (2.6) is actually present in the concrete estimates of the exponential integral, see (3.83).

The result below answers the existence part of Lemma 2.1 (i) and gives the connection with the technique of summation to the least term. Let n_x be defined, for x large enough by $n_x = [s_x]$, (cf. (2.3); [ ] denotes the integer part)

Proposition 2.1 *i) Let $\theta = 0$. A function Φ satisfies the uniform inequalities (2.4) with $\eta = 1$ iff*

$$\limsup_{x\to\infty} n_x^{-1/2}\frac{x^{n_x}}{|c_{n_x}|}\left|\Phi(x) - \sum_{j=0}^{n_x}\frac{c_j}{x^j}\right| = 0 \tag{2.7}$$

Instead, if the limit above is $\epsilon > 0$ and Φ satisfies (2.4) for some η then $\eta \geq 1+\epsilon$.

ii) If $\theta \neq 0$ and (2.5) holds, then a function Φ is uniformly asymptotic to the series (2.2) along the ray $\mathcal{R}_\theta$ with respect to a weight

$$w(n) = C|c_{n+1}|$$

for some C iff

$$\limsup_{|x|\to\infty}\left|\frac{x^{n_x}}{c_{n_x}}\right|\left|\Phi(x) - \sum_{j=0}^{n_x}\frac{c_j}{x^j}\right| < \infty \tag{2.8}$$

Corollary. If Φ_1, Φ_2 are two functions satisfying the hypothesis of part i) of Lemma 2.1 with $\eta = 1$ then

$$\limsup_{x\to\infty} n_x^{-1/2}\frac{x^{n_x}}{|c_{n_x}|}|\Phi_1(x) - \Phi_2(x)| = 0$$

In particular when $G(n) = \ln(\Gamma(n+n_0))$ then

$$\Phi_1(x) - \Phi_2(x) = o(e^{-x}\,x^{n_0}) \tag{2.9}$$

Correspondingly, in the case iii) we have

$$\limsup_{x\to\infty} n_x^{-1/2}\frac{x^{n_x}}{|c_{n_x}|}|\Phi_1(x) - \Phi_2(x)| = 0$$

and for $G(n) = \Gamma(n+n_0)$,

$$\Phi_1(x) - \Phi_2(x) = O\left(e^{-|x|}\,x^{n_0-1/2}\right) \tag{2.10}$$

Classically, two functions Φ_1 and Φ_2 have the same asymptotic series (provided they have one) if their difference is asymptotically less than any power of x. The corolarry shows that there is a definite precision gain when requiring optimal uniform estimates.

3 Proofs and further results

Before giving the general proof we mention as an illustration the particularly easy computation of the optimal weight for the exponential integral: see the arguments starting with eq. (3.86).

In this section we make the following convention: for $n > k$,

$$\sum_{j=n}^{k} a_j = -\sum_{j=k}^{n} a_j$$

For the proof of Lemma 2.1 we need the following elementary result. The notations and hypothesis are those that precede Proposition 2.1.

Proposition 3.1 *i)Let k_0 be large enough. Then for $\theta = 0$,*

$$\lim_{x\to\infty} \sup_{K\in\mathbb{N}} \left\{ (K+k_0)^{-1/2} x^K e^{-G(K)} \left| \sum_{j=n_x}^{K} \frac{c_j}{x^j} \right| \right\} = a_* \gamma^{-\frac{1}{2}} \tag{3.1}$$

The supremum in (3.1) is actually attained for

$$K \sim n_{\pm} := [s_x \pm \frac{1}{a_*}\sqrt{\frac{\gamma}{s_x}}]$$

ii) If $\theta \neq 0$ then

$$\lim_{x\to\infty} \sup_{K\in\mathbb{N}} \left\{ x^K e^{-G(K)} \left| \sum_{j=n_x}^{K} \frac{c_j}{x^j} \right| \right\} < \infty \tag{3.2}$$

Proof

The proof is essentially straightforward. Note first that from the assumptions it follows that $G''(x) \sim \gamma/x$ for large x.

Note also that we need only consider the case $\epsilon_j = 0$. Indeed, the presence of a (large enough) constant k_0 makes the proposition above insensitive to the behavior of c_j for small j. On the other hand,

$$\left| \sum_{j=n_x}^{K} \frac{c_j}{x^j} \right| = \left| \sum_{j=n_x}^{K} j^{i\phi} e^{-G(j)+j\,\ln(x)} \right| + O(\max_{K\le j\le n_x} |\epsilon_j|)) \sum_{j=n_x}^{K} e^{-G(j)+j\,\ln(|x|)} \tag{3.3}$$

and the maximum above approaches zero as $K \to \infty$.

• i) Take first $\phi = 0$. Let $\beta \in (\frac{1}{2}, \frac{2}{3})$ and start with the range of K so that

$$|K - n_x| < n_x^{\beta}$$

which is actually the important range with respect to (3.1)

Here, the Euler-Maclaurin summation method is suited. For definiteness we take $K \geq n_x$ (the other case is very similar). We have

$$\begin{aligned} &\frac{x^{n_x}}{c_{n_x}} e^{G(j)-j\ln x} = (1+O(s_x^{3\beta-2}))e^{\frac{1}{2}G''(s_x)(j-s_x)^2} = \\ &(1+O(s_x^{3\beta-2}))\int_j^{j+1} e^{\frac{1}{2}G''(s_x)(j-s_x)^2}dt = \\ &(1+O(s_x^{3\beta-2}))\int_j^{j+1} e^{\frac{1}{2}G''(s_x)(t-s_x)^2}dt \end{aligned} \tag{3.4}$$

so that

$$
\begin{aligned}
&\max_{|K-n_x|<n_x^\beta}\left\{K^{-1/2}e^{-G(K)+K\ \ln\ x}\sum_{j=n_x}^{K} e^{G(j)-j\ \ln\ x}\right\}=\\
&\left(1+O(n_x^{3\beta-2})\right)\max_{|K-n_x|<n_x^\beta}\left\{K^{-1/2}e^{-\frac{1}{2}G''(s_x)(K-s_x)^2}\int_{n_x}^{K+1} e^{\frac{1}{2}G''(s_x)(t-s_x)^2}dt\right\}=\\
&\left(1+O(n_x^{3\beta-2})\right)a_*(K\,G''(s_x))^{-\frac{1}{2}}\to a_*\gamma^{-\frac{1}{2}}
\end{aligned}\tag{3.5}
$$

as $x\to\infty$.

• It remains to obtain an upper bound in the region $|K-n_x|\geq n_x^\beta$. Assume $K<n_x$ (the opposite case is treated similarly) and let $q=s_x-\frac{1}{2}s_x^\beta$. With $F(z):=G(z)-z\ \ln(x)$ we have in view of the estimates above,

$$
\sum_{j=K}^{n_x} e^{F(j)-F(K)}\leq\sum_{j=K}^{q} e^{F(j)-F(K)}+\mathrm{const}\sqrt{q}e^{F(q)-F(K)}\tag{3.6}
$$

On the other hand, from the definition of s_x,

$$
F(j)-F(K)=\int_K^j dz\int_{s_x}^{z}dt\,G''(t)\leq\mathrm{const}\int_K^j\ln(z/s_x)dz
$$

Throughout the domain of integration $s_x>z+s_x^\beta>z+z^\beta$ so that

$$
\ln(z/s_x)<\ln\left(1-\frac{z^\beta}{z+z^\beta}\right)<-\frac{1}{2}z^{\beta-1}
$$

and we get, for some positive constants C_1,C_2,

$$
\sum_{K}^{q}e^{F(j)-F(K)}<\sum_{n=0}^{q-K}e^{-C_1[(K+n)^\beta-K^\beta]}<C_2\,(K+k_0)^{1-\beta}<\frac{1}{2}\sqrt{\frac{(K+k_0)}{\gamma}}a_*
$$

In the same way, for some positive constants,

$$
e^{F(q)-F(K)}\leq e^{-C_1[q^\beta-K^\beta]}<e^{-C_2K^{\beta-1}(q-K)}<e^{-C_3q^\beta}<e^{-C_4x^{\beta/\gamma}}\tag{3.7}
$$

so that the second term in (3.6) vanishes as $x\to\infty$.

• If now $\phi\neq0$ the upper bounds are trivially obtained by taking absolute values in the sums. For a lower bound we note simply that

$$
\sum_{n_x}^{n_+}\frac{c_j}{x^j}=n_x^{i\phi}\sum_{j=n_x}^{K}e^{-G(j)+j\ \ln(x)}\left(1+O(n_x^{-1/2})\right)
$$

• ii) Let again $F(k)=G(k)-k\ln(|x|)-ik\theta$. Given a function $f(x)$ which is asymptotic to the series $\tilde{S}$ we write

$$
f(x)=:\sum_{k=0}^{n_x}\frac{c_k}{x^k}-\frac{e^{F(n_x)}}{1-e^{-i\theta}}\chi(x)\tag{3.8}
$$

and we will choose χ to make $f(x)$ optimally asymptotic to $\tilde{S}$. Consider first the region $|k-n_x| < n_x^\beta$ and define

$$\sigma_k := \frac{k^{i\phi}e^{F(k)}}{(e^{-i\theta}-1)}\left(1-\frac{\gamma e^{i(k-n_x)}}{n_x}\sum_{j=-n_x^\beta}^{K-n_x-1} je^{-ij\theta}\right) \tag{3.9}$$

It is easy to check that

$$\sigma_{k+1}-\sigma_k = \frac{c_k}{x^k}\left(1+o(n_x^{-1/2})\right) \tag{3.10}$$

(to get (3.9), it is simpler in this case to solve (3.10) directly by perturbation expansion than to use the Euler-Maclaurin summation formula). Using (3.8), (3.9) and (3.10) we get

$$E_n := \left|\frac{x^{n+1}}{c_{n+1}}\right| |f(x)-\sum_{k=0}^{n}\frac{c_k}{x^k}| =$$

$$\frac{1}{|1-e^{-i\theta}|}\left|1-e^{i(n_x-n)\theta}\left(e^{-i\theta h_x(n)}-\chi+o(1)\right)e^{-\gamma\frac{(n_x-n)^2}{2n_x}}\right| \tag{3.11}$$

where $h_x(n)=1$ if $n>n_x$ and is zero otherwise. For large x (3.11) can be visualised as distance between $z=1$ to the points on two spirals centered at $e^{-i\theta h_x(n)}-\chi$, and with diameter slowly decresaing as $|n-n_x|$ increases. It is not hard to see that there is in this region a best choice χ_* of $\chi(x)$ (which minimizes the maximal error in (3.11) since for any value of x there is only a finite set of n such that $|n-n_x|<n_x^\beta$. For θ incommensurate with π the geometry of the problem shows that

$$\chi_* = e^{-i\frac{\theta}{2}}\cos\frac{\theta}{2}+o(1) \tag{3.12}$$

for which we get

$$a(\theta) = \frac{1+|\sin\frac{\theta}{2}|}{|1-e^{-i\theta}|} = \frac{1}{2}\left(1+\frac{1}{|\sin\frac{\theta}{2}|}\right) \tag{3.13}$$

If θ is a rational multiple of π the computation can still be done explicitly, in a straightforward manner and $a(\theta)$ is slightly less than (3.13).

In the region $|k-n_x|>n_x^\beta$ a similar calculation shows that the error is bounded by

$$E_n \le |1-e^{F'(n)-i\theta}|^{-1}+o(1)|\chi(x)| \le a(\theta)+o(1)|\chi(x)|$$

which shows that χ_* is indeed the optimal global choice of χ, as it is not possible to decrease the error in this region without making it unbounded in the $n-n_x|<n_x^\beta$ region.

□

Proof of Lemma 2.1. For part i), if we assume there existed a function Φ satisfying the estimate (2.4) uniformly in x and n for some $\eta<1$ it would follow that

$$\pm\left(\Phi(x)-\sum_{j=0}^{n_\pm}\frac{c_j}{x^j}\right) \le \eta\, a_*\gamma^{-\frac{1}{2}}(\sqrt{n_\pm}+B)\frac{c_{n_\pm+1}}{x^{n_\pm+1}} \tag{3.14}$$

By adding the two inequalities above we arrive at an immediate contradiction with Proposition 3.1.

Part ii) and iii) are straightforward consequences of Proposition 2.1. Indeed, note first an easy example of a function satisfying (2.7) and (2.8):

$$\Phi_*(x) := \sum_{j=0}^{n_x} \frac{c_j}{x^j} \tag{3.15}$$

It is not dificult to smooth Φ_* and preserve (2.7), (2.8) since the size of a jump at a point of discontinuity of Φ_* is $|c_{n_x} x^{-n_x}|$.

On the other hand, Proposition 2.1 follows immediately from Proposition 3.1 by the triangle inequality. □

*

Proof of Theorem 1.1

There are no assumptions that would distinguish between the quantities with subscript "1" from those with subscript "2". The existence part of the theorem follows (trivially, using triangle inequalities) if we prove it for $C_1 = 1$ and $C_2 = 0$. This case will follow from Lemma 3.1 below.

We can assume that the differential equation is brought to its canonical form because the asymptotic inequalities are transformed in an obvious way in the substituions involved. Also we note the following inequality that we will use frequently and which is clear from the integral representation of the Gamma function:

$$|\Gamma(x)| \le \Gamma(\Re(x)) \text{ for } \Re(x) > -1$$

Let $r := r_2 = -a_1,\ \rho := \Re(r)$ and $\alpha := a_0$.

Lemma 3.1 *There is a solution Y of the differential equation (1.8) under the assumptions (1.9)-(1.22) which satisfies the inequalities:*

$$|Y - \sum_{k=0}^{n} \frac{s_{1;k}}{x^k}| < const\, e^{-x}\, x^{\rho-1/2} \quad for \quad x \in (n, n+1) \tag{3.16}$$

where const *does not depend on n, x*

Before we prove this Lemma we need some results on the formal solutions. Let $s_k = s_{1,k}$. The condition that $\sum_{k=0}^{\infty} \frac{s_k}{x^k}$ is a formal solution of the equation leads to the following recurrence relation for the coefficients of the series:

$$s_n = \frac{1}{\alpha}\left(n - 1 + r + \frac{a_1 + b_2}{n}\right) s_{n-1} + \sum_{n \ge j \ge 2} \frac{1}{n\,\alpha}\left((j-n)a_j + b_{j+1}\right) s_{n-j} \tag{3.17}$$

Proposition 3.2 *The behavior of s_n for large n is*

$$s_n \sim \left(R + \sum_{m=1}^{\infty} \frac{R_m}{n^m}\right) \alpha^{-n}\Gamma(n+r) \tag{3.18}$$

In the assumption of analyticity at infinity of the coefficients of the equation, the leading behavior of s_n was shown in (Jurkat, 1976)

Remark. Although in a typical case the constant R will be nonzero as it is easy to understand by examining the recurrence (3.20) below, massive cancellations are possible so that the series $R + \frac{R_1}{n} + ..$ could be zero to all orders in $1/n$; this case is important in its own right–see Section 3 for some examples and applications in this connection.

Proof

In order to avoid dealing with the poles of the Γ function which might occur for (uninteresting) small values of n we let

$$\Gamma_x := \begin{cases} \Gamma(x) & \text{if } \Re(x) > -1 \\ 1 & \text{otherwise} \end{cases} \tag{3.19}$$

It is convenient to pull out the leading behavior suggested by (3.17):

$$s_n =: \alpha^{-n} \Gamma_{n+r} c_n$$

We get for c_n

$$c_n = c_{n-1}\left(1 + \frac{A_n}{n^2}\right) + O(n^{-2})c_{n-2} + \sum_{n-n_0 \ge j \ge 3} C(n,j)c_{n-j} + o(n^{-2}) \tag{3.20}$$

where the term with $j = 3$ and $n_0 > \max\{\rho, 3\}$ terms with j near n in the sum were treated separately. Due to (1.10) the latter give a collective a contribution of $o(n^{-2})$ and also, the following estimate holds for $C(n,j)$:

$$|C(n,j)| < \text{const}\, \frac{(n-j)!j!}{n!} = O(n^{-3}) \tag{3.21}$$

in the range $n-3 \ge j \ge 3$ (the inverse of the binomial coefficient is convex in j).

By induction on n we see that that $|c_n| < A \prod_{j \le n}(1 + \text{const}\, j^{-2})$ for some large enough A and const, whence the sequence c_n is bounded. But then it follows immediately from (3.20) that the sequence $\{c_n\}$ is convergent. Furthermore, taking $c_n = d_n \prod_{j \le n}(1 + A\, j^{-2})$ we see that

$$d_n = d_{n-1} + \frac{K_n}{n^2} \tag{3.22}$$

where K_n is a bounded sequence and thus, if R is the limit of the c_n we have

$$c_n = R + O(n^{-1}) \tag{3.23}$$

It is now easy to bootstrap the estimates for c_n in the recurrence (taking out explicitely more and more terms from the sum) to get $c_n = R + \frac{R_1}{n} + O(n^{-2})$ and so on.

$\square$

Let

$$U_n = \left(|R| + O(n^{-1})\right)\Gamma(n+\rho)$$

represent an upper bound of $|s_n|$. Note that if $R \ne 0$ then $|s_n| \sim U_n$ for large n.

Let

$$Y_n := \sum_{k=0}^{n} \frac{s_k}{x^k}$$

Take n to be large enough so that for $x > n$ the asymptotic estimates for $a(x)$, $b(x)$ hold and so that $n + \rho > 0$.

Proposition 3.3 *In the differential equation verified by* Y_n

$$Y_n'' + a(x)Y_n' + b(x)Y_n = R_n \tag{3.24}$$

the inhomogeneous term satisfies the estimate

$$R_n(x) = (R + O(n^{-1}))\alpha^{-n-2}\frac{\Gamma(n+r+2)}{x^{n+2}} \sim (\sqrt{2\pi}R + O(x^{-1}))\alpha^{-n-2}x^{r-\frac{1}{2}}e^{-x} \tag{3.25}$$

for $x \in (n, n+1)$.

Proof
We substitute $a(x) = \sum_0^n \frac{a_k}{x^k} + \frac{A_{n+1}}{x^{n+1}}$ (and the corresponding expression for $b(x)$) in the expansion of the LHS of (3.24). To simplify the notations we let throughout this proof $a_{n+1} = A_{n+1}(x)$; $b_{n+1} = B_{n+1}(x)$. The coefficients of x^{-j} with $j \le n+1$ vanish by the definition of Y_n. The coefficients of x^{-n-1} add up to $\alpha(n+1)s_{n+1}$. Indeed, $-\alpha(n+1)s_{n+1}$ is the only term of order $n+1$ which is missing in (3.25) with respect to the corresponding expression of Y_M, $M > n$. So,

$$R_n = \alpha\frac{(n+1)s_{n+1}}{x^{n+2}} + \sum_{m \ge n+3} x^{-m}T_m \tag{3.26}$$

with

$$T_m := \sum_{\substack{k+j=m \\ k,j \le n}} ((k-1)s_{k-1}a_j + s_k\, b_j) \tag{3.27}$$

An individual term in T_m can be bounded by:

$$|(k-1)s_{k-1}a_j + s_k\, b_j| \le \text{const}\, U_n$$

as it is easy to see using estimates for s_k, a_k, b_k. Thus

$$x^{-n-3}T_{n+3} < \text{const}\, n^{-n-1}U_n$$

and the same is true for $\sum_{m>n+3} x^{-m}T_m$, as shown by a gross majorization by the number of terms ($O(n)$ by (3.27)) times the maximal term. It follows that the modulus of the sum on the RHS of (3.26) is at most

$$O(n^{-1})\frac{\Gamma(n+r+1)}{x^{n+1}}$$

for $x \in (n, n+1)$. The rightmost estimate in (3.25) is just Stirling's formula. □
We return to Lemma 3.1.

Proof

The equivalent vectorial equation is now preferable:

$$\mathbf{F}' = A\,\mathbf{F} \tag{3.28}$$

where

$$A := \begin{pmatrix} 0 & 1 \\ -b(x) & -a(x) \end{pmatrix}; \quad \mathbf{F} := \begin{pmatrix} f \\ f' \end{pmatrix}$$

On the interval $[n, n+1]$ we look for solutions in the form

$$\mathbf{F} = \begin{pmatrix} Y_n \\ Y_n' \end{pmatrix} + \mathbf{H}(n,\cdot)$$

$\mathbf{H}$ has to satisfy the differential equation

$$\mathbf{H}' = A\,\mathbf{H} + \mathbf{T}_n; \quad \mathbf{T}_n = \begin{pmatrix} 0 \\ -R_n \end{pmatrix} \tag{3.29}$$

and the continuity condition (which is actually enough to ensure the smoothness of $\mathbf{Y}$)

$$\mathbf{H}(n;n_+) = \mathbf{H}(n-1;n_-) - \mathbf{E}_n \quad (n_\pm := n \pm 0) \tag{3.30}$$
$$\mathbf{E}_n := \frac{s_n}{n^n}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

A differential equation of the form (1.8) under the given assumptions admits always two special solutions which for large $|x|$ have the behavior (see Wasow, 1968).

$$y_1(x) \sim 1 + \frac{s_1}{x} + O(x^{-2}); \quad y_2(x) \sim x^r \exp(-\alpha\, x)(1+O(x^{-1})) \ \ (x \to \infty) \tag{3.31}$$

It is convenient to choose a particular fundamental matrix of the system (3.28) constructed with these two solutions:

$$M(x) := \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \sim \begin{pmatrix} 1+O(x^{-1}) & (1+O(x^{-1}))x^r\, e^{-\alpha\, x} \\ O(x^{-2}) & -\alpha(1+O(x^{-1}))x^r\, e^{-\alpha\, x} \end{pmatrix} \tag{3.32}$$

and write the solution of (3.29) in the form

$$\mathbf{H}(n,n+z) = M(n+z)\left(\int_0^z M^{-1}(n+t)\,\mathbf{T}_n(t)dt + M^{-1}(n)\,(\mathbf{H}(n-1,n_-)-\mathbf{E}_n)\right) \tag{3.33}$$

Taking $z=1$ in the relation above we get a recurrence relation for the $\{\mathbf{H}(k-1,k_-)\}_k$. With the substitution $M^{-1}(k+1)\mathbf{H}(k,(k+1)_-) = \mathbf{q}_k$ it reads

$$\mathbf{q}_n = \mathbf{q}_{n-1} + \int_0^1 M^{-1}(n+t)\,\mathbf{T}_n(t)dt - M^{-1}(n)\mathbf{E}_n = \mathbf{q}_{n-1} + M^{-1}(n)\mathbf{V}(n) \tag{3.34}$$

with

$$\mathbf{V}(n) := \int_0^1 M^{-1}(n;n+t)\,\mathbf{T}_n(t)dt - \mathbf{E}_n \tag{3.35}$$

and where $M(n;n+t)$ is the fundamental matrix specified by $M(n,n)=I$, whence

$$\mathbf{H}(k,(k+1)_-) = M(k+1)\left(\sum_{j=k_0+1}^{k} M^{-1}(j)\mathbf{V}(j) + \mathbf{q}_{k_0}\right) \tag{3.36}$$

For large n and $t \le 1$ we obtain in a straightforward manner,

$$M^{-1}(n,n+t) = \begin{pmatrix} 1 & \frac{1}{\alpha}(1-e^{\alpha\, t}) \\ 0 & e^{\alpha\, t} \end{pmatrix} + O(n^{-1}) \tag{3.37}$$

and from (3.25)

$$\mathbf{T}_n(n+t) = (1+O(n^{-1}))\begin{pmatrix} 0 \\ -e^{-t} \end{pmatrix} R_n(n) \tag{3.38}$$

Beginning with this point, most estimates will be different in the special case $\alpha = 1$ which corresponds to being on the Stokes line. Combining (3.38) and Propositions 3.2, 3.3 we get from (3.35)

$$\mathbf{V}(n) = R_n(n)\left(\begin{pmatrix} e^{-1} \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} + O(n^{-1})\right) = \begin{pmatrix} \text{const} \\ O(n^{-1}) \end{pmatrix} R_n(n) \quad (\alpha = 1) \tag{3.39}$$

and similarly

$$\mathbf{V}(n) = \mathbf{O}(1) R_n(n) \quad (\alpha \neq 1) \tag{3.40}$$

In view of (3.32) a direct calculation gives for $M^{-1}(n)$

$$\begin{pmatrix} 1 + O(n^{-1}) & \alpha^{-1} + O(n^{-1}) \\ O(n^{-2}) n^{-r} e^{\alpha n} & (-\alpha^{-1} + O(n^{-1})) n^{-r} e^{\alpha n} \end{pmatrix}$$

(note that the absence of exponential factors from the first row is not an effect of an approximation; in fact $\arg(\alpha)$ is arbitrary and nothing is assumed about the size of the exponentials). Thus

$$M^{-1}(n)\,\mathbf{V}(n) = \begin{pmatrix} O(1) \\ O(n^{-r} e^{n}) \end{pmatrix} R_n(n) = \begin{pmatrix} O(n^{r-1/2}\, e^{-n}) \\ O(n^{-1/2}\, e^{(\alpha-1)n}) \end{pmatrix} \quad (\alpha \neq 1)$$

and correspondingly

$$M^{-1}(n)\,\mathbf{V}(n) = \begin{pmatrix} \text{const} \\ \text{const}\, n^{-r-1} e^{n} \end{pmatrix} R_n(n) = \begin{pmatrix} O(n^{r-1/2}\, e^{-n}) \\ O(n^{-3/2}) \end{pmatrix} \quad (\alpha = 1)$$

so that the series $\sum_k M^{-1}(k)\mathbf{V}(k)$ converges and

$$\sum_{k=n}^{\infty} M^{-1}(n)\mathbf{V}(n) = \begin{pmatrix} O(n^{r-1/2}\, e^{-n}) \\ O(n^{-1/2}\, e^{(\alpha-1)n}) \end{pmatrix} \quad (\alpha \neq 1) \tag{3.41}$$

and

$$\begin{pmatrix} O(n^{-r-1/2}\, e^{-n}) \\ O(n^{-1/2}) \end{pmatrix} \quad (\alpha = 1) \tag{3.42}$$

If we make the choice

$$\mathbf{q}_{k_0} = - \sum_{k=k_0+1}^{\infty} M^{-1}(k)\mathbf{V}(k)$$

in (3.36) we get

$$\mathbf{H}(k, (k+1)_-) = -M(k+1) \sum_{k+1}^{\infty} M^{-1}(j)\mathbf{V}(j) \tag{3.43}$$

so that in view of (3.41), (3.42) and (3.32)

$$\mathbf{H}(n, (n+1)_-) = \mathbf{O}(n^{r-1/2} e^{-n}) \tag{3.44}$$

for all α with $|\alpha| = 1$ □

At this point we can prove the existence part of Theorem 1.1. For $R \neq 0$ it is a direct consequence of the Lemma 3.1, and of the Propositions 2.1 and 3.2. If $R = 0$ we proceed in essentially the same way. Taking n such that $x \in [n, n+1]$ we get for the special solution provided by Lemma 3.1,

$$|Y - \sum_{j=0}^{k} \frac{s_k}{x^k}| \leq |Y - Y_n| + |\sum_{j=k}^{n} \frac{s_k}{x^k}| \tag{3.45}$$

Since now we have $|s_n| < \text{const}\Gamma(n + \rho - 1)|$ the second sum in (3.45) can be estimated using Proposition 3.1 taking $c_j := \text{const}\Gamma(n + \rho - 1)$ (and $G(j) := \ln(c_j)$). It follows immediately from (3.45) that Y satisfies the inequalities (1.14) with $C_1 = 1$, $C_2 = 0$, $A_1 = 0$ and some B.

Uniqueness. We can assume without loss of generality (1.22) and $\Re(\alpha \leq 0)$. The case $\Re(\alpha) = 0$ is actually trivial since it reduces to a statement about asymptotics to all orders. We show that if y satisfies (1.14) then $y = C_1 y_1 + C_2 y_2$ where $y_{1,2}$ are the solutions corresponding to $(C_1, C_2) = (1, 0)$ and $(0, 1)$ respectively whose existence has already been proven.

We have to distinguish the case $A_1 \neq 0$ (which implies $\alpha = 1$ and $R \neq 0$, cf.(1.16)). Taking $k_2 = 0$ (which means that we only keep the leading term of the second power series) it follows from (1.14) that

$$\left| F(x) - C_1 \sum_{j=0}^{k} \frac{s_{1,j}}{x^j} - C_2\, x^r\, e^{-x} \right| <$$
$$|C_1|(A_1 \sqrt{K_1} + B)\ \left|x^{-k-1}\Gamma(k+r+1)\right| + \text{const}\,|C_2|\,\left|x^{r-1}\, e^{-x}\right|$$

The case $C_1 = 0$ is trivial because the only solutions that decay exponentially at infinity are multiples of y_2. With $C_1 \neq 0$ an elementary computation shows that one can choose a B' so that for all $x \in \mathcal{R}_\theta$ and $k \in I\!N$

$$|C_1|(A_1\sqrt{k} + B)\ \left|x^{-k-1}\Gamma(k+r+1)\right| + \text{const}\,|C_2|\,\left|x^{r-1}\, e^{-x}\right| <$$
$$|C_1|(A_1\sqrt{k} + B')\ \left|x^{-k-1}\Gamma(k+r+1)\right|$$

Consequently, the function $y(x) - C_2\, x^r\, e^{-x}$ is optimally asymptotic to the series $C_1\, S_1$ (since $R \neq 0$). Because $C_1\, y_1(x)$ has the same property, we have, in view of the corolarry to Proposition 2.1

$$|y(x) - C_2\, x^r\, e^{-x} - C_1\, y_1(x)| = o(x^r\, e^{-x}) = o(y_2)\ as\ x \to \infty$$

which means that $y(x) - C_1\, y_1(x) = C_2\, y_2(x)$.

If now $A_1 = 0$ we take $k_1 = k_2 = [x]$ in (1.14) and get in a straightforward way:

$$\left| y(x) - C_1 \sum_{j=0}^{[x]} \frac{s_{1,j}}{x^j} - C_2\, x^r\, e^{-x} \sum_{j=0}^{[x]} \frac{s_{2,j}}{x^j} \right| < \text{const}\ x^{r-1/2}\, e^{-x}$$

Since by Lemma 3.1 the same inequality is true with $y(x)$ replaced by $C_1 y_1(x) + C_2 y_2(x)$ it follows that $|y(x) - C_1 y_1(x) - C_2 y_2(x)|$ decays faster than $y_2(x)$ as $x \to \infty$ which is possible only if it vanishes identically. $\square$

Proof of Lemma 1.1.

For this purpose we only need the recurrence (3.17), the asymptotic behavior of its solutions (3.18) and the condition (1.10). Assume we have a formal power series $S := \sum_{k=0}^{\infty} s_k x^{-k}$ that solves a canonical differential equation of the type considered. The recurrence (3.17) provides us in a straightforward way with a set of equations for a_j, b_j, $j = 0, 1, 2, \ldots$:

$$-s_1 a_j + s_0 b_{j+2} = T_j \quad \text{for } j = 0, 1, 2, \ldots \tag{3.46}$$

where T_j only depend on the series S and on the coefficients a_i, b_{i+2} with $i < j$.

The equations (3.46) alone would not determine the a_i, b_i uniquely but the condition (1.10) binds a_i and b_i together. The best way to see this is to divide (3.17) by s_n, use (3.18), (1.10) (and the assumption $R \neq 0$) to write an asymptotic expansion to all orders in $\frac{1}{n}$ of the resulting equation and then equate the successive powers of $\frac{1}{n}$.

The upshot is the system:

$$a_0 a_j - b_j = \tilde{T}_j \quad \text{for } j = 2, 3 \ldots \tag{3.47}$$

where $\tilde{T}_j$ depend on the series S and on a_i, b_i with $i < j$. But a_0 and a_1 are determined from (3.18), namely $a_0 = \alpha$, $a_1 = -r$, so that (3.46), (3.47) determine uniquely the a_i, b_i. $\square$

*

Proof of Lemma 1.1.

For simplicity we make a change of variables so that

$$|\alpha| = 1; \quad x \in \mathbb{R}^+$$

The recurrence relation for the formal series solution is obtained in the usual way, by inserting the formal series in the differential equation, expanding everything out and identifying the powers of x:

$$-\sum_{k=2}^{\infty} \frac{(k-1)s_{k-1}}{x^k} =$$
$$\sum_{j=0}^{\infty} \frac{f_{0,j}}{x^j} + \sum_{j=0}^{\infty} \frac{f_{1,j}}{x^j} \sum_{j=1}^{\infty} \frac{s_j}{x^j} + .. + \sum_{j=0}^{\infty} \frac{f_{P,j}}{x^j} \left(\sum_{j=1}^{\infty} \frac{s_j}{x^j} \right)^P =$$
$$\sum_{k=1}^{\infty} \frac{1}{x^k} \sum_{J_0} f_{p,i} s_{k_1} \ldots s_{k_p} \tag{3.48}$$

where J_0 consists of all the integer tuples $(i, k_1, \ldots, k_p)$, $k_i \geq 1$ and $i \geq 0$, with $k_1 + k_2 + .. + k_p + i = k$. It follows,

$$s_k = \frac{1}{\alpha}(k + r - 1)s_{k-1} + \sum_{J_1} f_{p,i} s_{k_1} \ldots s_{k_p} \tag{3.49}$$

where $r = f_{1,1} + 2f_{2,0}s_1$ and the index set J_1 in the sum excludes the tuples with $k_i > k - 2$ from J_0. We make the substitution (suggested by solving the linearized version of the recurrence)

$$s_k = \alpha^{-k} \Gamma_{k+r} \eta_k$$

where $\Gamma_{k+r} = \Gamma(k + r)$ if $\Re(k + r) > -1$ and $\Gamma_{k+r} = 1$ otherwise (again, to avoid unpleasant poles). For large k,

$$\eta_k = \eta_{k-1} + \sum_{J_1} f_{p,i} \eta_{k_1} \ldots \eta_{k_p} \alpha^i \frac{\Gamma_{k_1+r} \ldots \Gamma_{k_p+r}}{\Gamma_{k+r}} \tag{3.50}$$

The existence of a formal solution of the differential equation follows from the existence of a solution to the recurrence relation (3.50) which is obvious. The first objective is then to show that the sequence $\{\eta_k\}_k$ converges; the only delicate step is to show that the sequence $\{|\eta_k|\}_k$ is bounded. To this end we compare the recurrence (3.50) with a suitable *linear* recurrence (3.60) below.

Taking out of the sum the terms with $k_i > k - n$ for a conveniently large n we get, for some constants $A_1, .., A_n$,

$$\eta_k = \eta_{k-1} + \frac{A_1}{k^2}\eta_{k-2} + .. + \frac{A_n}{k^n}\eta_{k-n} + \sum_{J_2} f_{p,i} \eta_{k_1} \ldots \eta_{k_p} \alpha^i \frac{\Gamma_{k_1+r} \ldots \Gamma_{k_p+r}}{\Gamma_{k+r}} \tag{3.51}$$

where the restriction J_2 in the sum now reads $k_1 + k_2 + .. + k_p + i = k; k_i < k - n$. Taking the initial condition $\overline{\eta_1} = |\eta_1|$ in the recurrence

$$\overline{\eta}_k = \overline{\eta}_{k-1} + \frac{|A_1|}{k^2}\overline{\eta}_{k-2} + .. + \frac{|A_n|}{k^n}\overline{\eta}_{k-n} + \sum_{J_2} f_{p,i}\overline{\eta}_{k_1}...\overline{\eta}_{k_p}\frac{\Gamma_{k_1+\rho}...\Gamma_{k_p+\rho}}{\Gamma_{k+\rho}} \ (\rho = \Re(r)) \tag{3.52}$$

we clearly get $\overline{\eta}_k \geq |\eta_k|$. Now the solution of (3.52) with positive initial conditions is increasing in k so that for $A > \max\{A_1, .., A_n\}$

$$\overline{\eta}_k \leq \left(1 + \frac{A}{k^2}\right)\overline{\eta}_{k-1} + \sum_{J_2} f_{p,i}\overline{\eta}_{k_1}...\overline{\eta}_{k_p}\frac{\Gamma_{k_1+\rho}...\Gamma_{k_p+\rho}}{\Gamma_{k+\rho}} \tag{3.53}$$

Taking $\overline{\eta}_k = \sigma_k \prod_1^k (1 + A/j^2)/\prod_1^k (1 - 1/j^2)$ we get for some constant C

$$\sigma_k \leq \left(1 - \frac{1}{k^2}\right)\sigma_{k-1} + C\sum_{J_2} f_{p,i}\sigma_{k_1}...\sigma_{k_p}\frac{\Gamma_{k_1+\rho}...\Gamma_{k_p+\rho}}{\Gamma_{k+\rho}} \tag{3.54}$$

Noting that card$(J_2)\leq P\,k^p$ we get

$$\sigma_k \leq \left(1 - \frac{1}{k^2}\right)\sigma_{k-1} + \frac{C\,P}{k^2}\max_{J_2}\left\{k^{p+2}\,f_{p,i}\sigma_{k_1}...\sigma_{k_p}\frac{\Gamma_{k_1+\rho}...\Gamma_{k_p+\rho}}{\Gamma_{k+\rho}}\right\} \tag{3.55}$$

It is easy to see by a straightforward calculation that one can choose k and n large enough so that

$$C\,P\,k^{p+2}f_{p,i}\frac{\Gamma_{k_1+\rho}...\Gamma_{k_p+\rho}}{\Gamma_{k+\rho}} < \left(\frac{k_1!..k_p!i!}{(k_1 + .. + k_p + i)!}\right)^{\frac{1}{2}} < \left(\frac{k_1!..k_p!}{(k_1 + .. + k_p)!}\right)^{\frac{1}{2}} \tag{3.56}$$

when $k_i < k - n$ (for the second inequality above note that the middle term is a decreasing function of its arguments). Using (1.24) we get

$$\sigma_k \leq \left(1 - \frac{1}{k^2}\right)\sigma_{k-1} + \frac{1}{k^2}\max_{J_3}\{\sigma_{k_1}...\sigma_{k_p}\left(\frac{k_1!..k_p!}{(k_1 + .. + k_p)!}\right)^{\frac{1}{2}}\} \tag{3.57}$$

J_3 consists of all the tuples $k_1, .., k_p$ with $k_1 + .. + k_p \leq k$. By construction the empty tuple contributes with const $\kappa^k \leq$ const. It follows easily that

$$\sigma_k \leq \max_{J_3}\{\sigma_{k-1}; \sigma_{k_1}...\sigma_{k_p}\left(\frac{k_1!..k_p!}{(k_1 + .. + k_p)!}\right)^{\frac{1}{2}}\} \tag{3.58}$$

from which, if σ_1 is large enough which we are allowed to assume since the solution is increasing in σ_1, it follows that σ_k is bounded by the solution of the recurrence:

$$\sigma_k = \max_{J_3}\{\sigma_{k_1}...\sigma_{k_p}\left(\frac{k_1!..k_p!}{(k_1 + .. + k_p)!}\right)^{1/2}\} \tag{3.59}$$

For each k, (3.59) determines a tuple $k_1^*, .., k_p^*,\ k_1^* + .. + k_p^* \leq k$ for which the maximum is realized. If we take $\sigma_k = \exp(\phi_k)$ we get a linear recurrence:

$$\phi_k = \phi_{k_1^*} + .. + \phi_{k_p^*} + \frac{1}{2}\ln\left(\frac{k_1^*!..k_p^*!}{(k_1^* + .. + k_p^*)!}\right) \tag{3.60}$$

Consider the associated homogeneous recurrence

$$F_k = F_{k_1^*} + ... + F_{k_p^*};\quad F_1 = 1 \tag{3.61}$$

Inductively we see that:

$$F_k \le k \tag{3.62}$$

We claim that

$$\phi_k \le \phi_1 \, F_k - \frac{1}{2}\ln(F_k!) \tag{3.63}$$

Indeed, the inequality above is true for $k = 1$ and assuming it holds for all $k' < k$ we have (cf. also the comment following (3.56))

$$\phi_k \le \phi_1(F_{k_1^*} + .. + F_{k_p^*}) - \frac{1}{2}\ln(F_{k_1^*}!..F_{k_p^*}!) + \frac{1}{2}\ln(\frac{F_{k_1^*}!..F_{k_p^*}!}{(F_{k_1^*} + .. + F_{k_p^*})!})$$

in view of (3.62) which gives (3.63).

But note that the function $g(x) = A\,x - B\,\ln(x!) : (1,\infty) \mapsto \mathbb{R}$ is bounded from above so that the solution of (3.50) is bounded.

Now, taking n large enough one can make the nonlinear term in (3.51) smaller than const k^{-m} (cf. also (3.56)) with m as large needed. The rest of the proof is obvious. □

We return to the proof of Theorem 1.2.

Proof. We start by showing the existence of a special solution corresponding to $C = 0$. We look for solutions of the equation (1.23) in the form

$$y(x) = Y_n(x) + V(n;x); \quad Y_n := \sum_{k=1}^{n} \frac{s_k}{x^k}; \quad (x \in [n, n+1]) \tag{3.64}$$

where $n = [x]$. The differential equation for V reads:

$$V' = \frac{d}{dy}F(Y_n, x)\,V + \sum_{p=2}^{P} \frac{d^p}{dy^p}F(Y_n, x)\,V^p + T_n \tag{3.65}$$

where

$$T_n = F(Y_n, x) - Y_n'(x)$$

to which we add the condition of continuity of y:

$$V(n, n_+) = V(n-1, n_-) - \frac{s_n}{n^n} \tag{3.66}$$

Lemma 3.2 *There exists a solution of the problem (3.65)–(3.66) with the property*

$$|V(n,x)| \le const \frac{\Gamma_{n+\rho}}{x^n} \quad (x \in [n, n+1])$$

Again we need first some estimates on the power series solution.

Proposition 3.4 *For $x \in [n, n+1]$*

$$T_n = (R + O(\frac{1}{n}))\alpha^{-n-1}\frac{\Gamma_{n+\rho+1}}{x^{n+1}}$$

Proof
For $p \in \{0,..,P\}$ we write

$$F_p(x) = \sum_{k=0}^{n} \frac{f_{p,k}}{x^k} + \frac{R_{p,n}(x)}{x^{n+1}} \quad \text{with} \quad R_{p,n}(x) < C_F \kappa^n n! \quad (x \in [n, n+1]) \tag{3.67}$$

Then T_n is given by

$$\sum_{p=0}^{P} Y_n^p \left(\sum_{k=0}^{n} \frac{F_{p,k}}{x^k} + \frac{R_{p,n}}{x^{n+1}} \right) - Y_n' = \tag{3.68}$$

$$\sum_Q \frac{1}{x^Q} \sum_{J_Q} f_{p,i} s_{k_1} ... s_{k_p} + \sum_{k=1}^{n} \frac{k\, s_k}{x^{k+1}} + \sum_{i=1}^{P} \frac{R_{p,n}}{x^{n+1}} Y_n^p$$

where the index set J_Q contains tuples $(i, k_1, .., k_p) \in \{0, .., n\}$ with $i + k_1 + .. + k_p = Q$. The recurrence relation for s_k is such that all the terms with $Q \le n$ have to compensate each other whereas for $Q = n+1$ they add up to $\alpha s_{n+1} x^{-n-1}$. We thus get for T_n

$$\left| T_n - \alpha \frac{s_{n+1}}{x^{n+1}} \right| \le \sum_{Q \ge n+2} \frac{1}{x^Q} \sum_{J_Q} |f_{p,i}||s_{k_1}|...|s_{k_p}| + \sum_{i=1}^{P} \frac{|R_{p,n}|}{x^{n+1}} |Y_n^p| \tag{3.69}$$

The last term on the rhs of (3.69) can be estimated by const $\Gamma_{n+\rho+1}\, x^{-n-2}$. For the first term we write

$$\left(\sum_{n+2 \le Q \le n+s} + \sum_{n+s \le Q \le (P+1)n} \right) \frac{1}{x^Q} \sum_{J_Q} |f_{p,i}||s_{k_1}|...|s_{k_p}| \tag{3.70}$$

For all n, $|s_n|$ and $|f_{p,n}|$ are bounded by const $\Gamma_{n+\rho}$. Note that the Γ function is log-convex so that the maximum of a product of the form $\Gamma(x_1)..\Gamma(x_m)$ over a convex domain, is reached on the boundary of the domain. Now, if n is large enough compared to $s + p$,

$$\sum_{\substack{k_1+..+k_p=n+s \\ k_i \le n}} \Gamma_{k_1+\rho}..\Gamma_{k_p+\rho} \le \text{const}\, P \sum_{j=0}^{p+s+1} n^{-j} \sum_{\substack{k_1+..+k_{p-1}=s+j \\ k_i \le n}} \Gamma_{k_1+\rho}..\Gamma_{k_{p-1}+\rho} +$$

$$\text{const} \binom{n+p+s}{p} n^{-p-s-2} < \text{const}(P, s)\Gamma_{n+\rho} \tag{3.71}$$

It follows that the leftmost sum in (3.70) is less than const $\Gamma_{n+\rho} x^{-n-2}$ for $x \in [n, n+1]$. In the second sum we write

$$Q = q\, n + m$$

with $q \le P, m \le n$. The log-convexity argument together with the restriction on the indices i, k_j give

$$|f_{p,i}||s_{k_1}|...|s_{k_p}| < \text{const}\, \kappa^i\, i! \Gamma_{k_1+\rho}..\Gamma_{k_p+\rho} \le$$

$$\text{const}\, (\Gamma_{n+\rho})^q \Gamma_{m+\rho} \tag{3.72}$$

and we crudely bound the second sum in (3.70) by the number of terms times the maximal term. An elementary application of Stirling's formula gives

$$\text{const}\, n^{P+2} \max_{n+s\le Q\le nP} \left\{ \frac{\Gamma^q_{n+\rho}\Gamma_{m+\rho}}{n^{qn+m}} \right\} \le \tag{3.73}$$

$$\text{const}\, n^{P+2} \exp\left(\max_{n+s\le Q\le nP} \left\{ (m+\rho-\frac{1}{2})\ln(m) + (q\rho - \frac{q}{2} - m)\ln(n) - m - qn \right\}\right)$$

The expression to be maximized is decreasing in q and is convex in m. It is not difficult to see that for n large enough the maximum is reached at $q=1, m=s$. For large n (3.73) is bounded by const $n^{r-s-1/2+P+2}\, e^{-n}$. Choosing $s > P+4$ we make the second sum in (3.70) less than $\Gamma_{n+\rho} n^{-n-2}$. The proof is completed by combining these inequalities with (3.69) □

We take now n large enough and first find a suitable solution of the linearized version of the equation (3.65) on each interval $[n, n+1]$. Consider the differential equations:

$$H' = F_y(Y_n, x)H + T_n \tag{3.74}$$

with the initial conditions at $x=n$ chosen so that $y(x)$ as defined in (3.64) is continuous namely,

$$H(n, n_+) = H(n-1, n_-) - \frac{s_n}{n^n} \tag{3.75}$$

under the same hypothesis on F as before.

Proposition 3.5 *There is a solution to the problem (3.74), (3.75) such that*

$$|H(n,x)| \le const\, \frac{\Gamma_{n+\rho}}{x^n} \quad (x \in [n, n+1]) \tag{3.76}$$

The proof is the one-dimensional projection of the proof of lemma 3.1 □.

It also follows that for $x \in [n, n+1]$

$$H(n,x) \le \text{const}\, x^{\rho-1/2}\, e^{-x} \tag{3.77}$$

We now turn to the proof of Lemma 3.2. We shall look for solutions of the equation (3.65) in the form $V = H + \delta$ where H is the function provided by the Proposition 3.5. Then δ is a smooth function and satisfies the differential equation

$$\delta' = \frac{d}{dy}F(Y_n, x)\,\delta + \sum_{p=2}^{P} \frac{d^p}{dy^p}F(Y_n, x)\,(H+\delta)^p \tag{3.78}$$

We are looking for a solution of (3.78) of the order $O(H^2)$; such a solution is constructed as a fixed point of a contractive mapping.

Take n large enough, $\beta > 2\rho - 1$ and let $\mathcal{S}_{n,\beta}$ be the space of continuous functions on (n, ∞) with the norm

$$||f|| := \sup_{t>n} \left| t^{-\beta} e^{2\,t}\, f(t)\right| \tag{3.79}$$

and the operator

$$(\mathcal{L}\, f)(x) := \int_x^\infty \frac{d}{dy}F(Y_n, t)\, f(t) + \sum_{p=2}^{P} \frac{d^p}{dy^p}F(Y_n,t)\,(H(t)+f(t))^p\, dt \tag{3.80}$$

In view of (3.77) for large n, $H^2 \in \mathcal{S}_{n,\beta}$. Actually we can make the norm of H^2 very small by choosing n large enough. $\mathcal{L}$ maps $\mathcal{S}_{n,\beta}$ into itself and for a given ϵ there is n_ϵ large enough so that $\mathcal{L}$ maps $\mathcal{B}_\epsilon$, the ball of radius ϵ, into itself. Actually, $\mathcal{L}$ is a contraction for large n. Indeed

$$\mathcal{L}f = \int_x^\infty \alpha f(t)dt + O(f/x, f^2, Hf, H^2) \tag{3.81}$$

where O is with respect to the norm (3.79). But if $f, g \in \mathcal{B}_\epsilon$ and $\epsilon' > 0$ it is straightforward to see that $\|\mathcal{L}f - \mathcal{L}g\| < \frac{1+\epsilon'}{2}\|f-g\|$ if n is large in which case the equation (3.78) has a solution in $\mathcal{S}_{n,\beta}$. It follows that the equation (3.65) has a solution with the required properties $\square$.

Lemma 3.2 together with Proposition 2.1 prove the existence part of Theorem 1.2 for $C=0$. Given this, for $C \neq 0$ the existence result follows easily, since for any C there exists a unique solution of the differential equation (1.23) of the form $y(x) = V(x) + \epsilon(x)$ with $V(x)$ the solution given in Lemma 3.2 and $\epsilon(x) \sim Ce^{-\alpha x}(x^{r+1} + O(x^r))$ $(|x| \to \infty)$. This is a standard result (Wasow, 1968), (Coddington, 1955) ; in our case $\epsilon(x)$ could be also directly obtained using the contractive mapping arguments above.

The uniqueness proof is very similar to the one in the linear case since no two solutions can differ by less than const $e^{-\alpha x}x^{\rho+1}$ as $x \to \infty$ without being equal to each other. Finally, the information that a solution $y(x)$ is decaying at infinity is enough to guarantee that y is asymptotic to the series (1.26). The difference $y(x) - V(x)$ has the asymptotic behavior $Ce^{-\alpha x}(x^{r+1} + O(x^r))$ for some C. Let y_C be the special solution satisfying (1.30). Since $y(x) - V(x) \sim Ce^{-\alpha x}(x^{r+1} + O(x^r))$ as well, it follows that $y(x) - y_C(x) = o(e^{-\alpha x}x^{r+1})$ which means $y(x) = y_C(x)$ (Coddington, 1955); a simple direct proof uses contraction mapping (3.80)). The proof of Theorem 1.2 is complete. $\square$

*

Discussion of the examples.

i) Proof of Eq. (1.7). We are looking for a constant C, dependent on the ray in the complex plane, such that

$$E_n(x) := \mathcal{E}i(x) - e^x \sum_{k=1}^n \frac{(k-1)!}{x^k} = C + o(1) \ (|x| \to \infty, \ \arg(x) = \theta) \tag{3.82}$$

By Theorem 1.1 and Proposition 2.1, there exists such a constant. We might as well compute it for $|x| \to \infty$ along a subsequence, say $x_n = ne^{i\theta}$.

• Take first $\theta > 0$ ($\theta < 0$ is similar). It is easy to show that

$$E_n(x) = \pi i + n!n^{-n} \int_C dt \frac{e^{nt}}{t^{n+1}}$$

where C is a contour joining $-\infty$ to $e^{i\theta}$ above the real axis; we choose it to be the stationary phase (or, which is the same, the steepest ascent) contour for the integrand: $\Im(nt - (n+1)\ln(t)) = \text{const}$.

The leading behavior of the integral for large n is due to the contribution of a region, near the right end point of C, where $z := t - e^{i\theta}$ is of the order $O(1/n)$.

$$\int_C dt \frac{e^{nt}}{t^{n+1}} = (1+o(1))e^{ne^{i\theta} - (n+1)i\theta} \int_{-\infty}^0 dz e^{n(1-e^{-i\theta})z}$$

so that, for $\theta > 0$,

$$E_n - \pi i = (1+o(1))\sqrt{\frac{2\pi}{n}} \frac{1}{1-e^{-i\theta}} e^{n(e^{i\theta}-1)-(n+1)i\theta} = o(1) \tag{3.83}$$

- For $\theta = 0$ we have:

$$E_n(n) = n!n^{-n}\mathrm{PV}\int_{-\infty}^{1} dt\frac{e^{nt}}{t^{n+1}} = -n!n^{-n}e^n\Im\int_0^{\infty}\frac{e^{-iny}}{(1-iy)^{n+1}}dt$$

where the last expression is obtained after pushing the contour parallel to the imaginary axis and taking $t = 1 + iy$. Now, choosing $\beta \in (-\frac{1}{2}, -\frac{1}{3})$ we get

$$\begin{aligned}\int_0^{\infty}\frac{e^{-iny}}{(1-iy)^{n+1}}dt &= \int_0^{n^{\beta}}\frac{e^{-iny}}{(1-iy)^{n+1}}dt + O(\exp(-\frac{1}{2}n^{2\beta+1})) = \\ &\int_0^{\infty} e^{-n\frac{y^2}{2}}\left(1 + iy - \frac{i}{3}ny^3\right)dy + o(n^{-1}) = \sqrt{\frac{\pi}{2n}} + \frac{i}{3n} + o(n^{-1})\end{aligned} \tag{3.84}$$

so that, after taking the imaginary part,

$$E_n(n) = -\frac{1}{3}\sqrt{\frac{2\pi}{n}}(1 + o(1)) \tag{3.85}$$

□

For the exponential integral $\mathrm{Ei}(x)$, it is easy to evaluate the optimal weight from (3.85) and its differential equation. The arguments below can be easily made rigorous but we will not insist on that, since we are only aiming at an illustration. Let

$$R(x) = e^{-x}\mathrm{Ei}(x) - \sum_{k=0}^{n-2}\frac{k!}{x^{k+1}} \tag{3.86}$$

Then, $R(x)$ satisfies the differential equation

$$R' + R = \frac{(n-1)!}{x^n}$$

Taking $R(x) = \frac{(n-1)!}{x^n}g(x)$ we get for g the equation $g' + (1 - nx^{-1})g = 1$. The relevant region is $x = n + O(\sqrt{n})$ so it is convenient to change variables further to $x = n + s\sqrt{n}$ and $g(x) = \sqrt{n}\phi(s)$. We get

$$\phi' + \frac{s}{1 + sn^{-1/2}}\phi = 1$$

with the initial condition, coming from (3.85) $\phi(0) = O(n^{-1/2})$. The solution is given by

$$\phi(s) = e^{-\frac{s^2}{2}}\int_0^s e^{\frac{t^2}{2}}dt + O(n^{-1/2})$$

The maximum value of $|\phi(s)|$ is therefore equal to a_*.

ii) Derivation of (1.21). To make the calculation of the asymptotic series easier we take further $g(s) = s^{-1/6}h(s)$ (to make $r_1 = 0$) and finally $s = 3t/4$ (to have $\lambda_1 = 0; \lambda_2 = -1$). The resulting equation is

$$h'' + h' + \frac{5}{36t^2}h = 0$$

having the following general formal solution:

$$C_1 \sum_{j=0}^{\infty} \frac{h_j}{t^j} + C_2 e^{-t} \sum_{j=0}^{\infty} \frac{(-)^j h_j}{t^j}$$

The recurrence for h_j:

$$(j+1)h_{j+1} = (j+\frac{1}{6})(j+\frac{5}{6})h_j$$

with the choice $h_0 = 1$, gives

$$h_n = \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{2\pi\Gamma(n+1)} = \frac{1}{2\pi}\Gamma(n)\left(1+O(\frac{1}{n})\right) \quad \text{for large } n$$

so that with this choice we get $R_1 = R_2 = \frac{1}{2\pi}$ and $f_k = \left(\frac{4}{3}\right)^{\frac{2k}{3}} h_k$. The contents of (1.21) is given by Theorem 1.1, after making the appropriate substitutions and change of variable back into eq. (1.14).

4 Convergence of asymptotic series and a decomposition property

In this section we give an example of a case of non-generic convergence of the asymptotic series and discuss its relevance for asymptotics beyond all orders. The Proposition below could be easily generalized to a larger class of differential equations but we now look, for the sake of simplicity, at equations of the form

$$\psi' + \psi = R(x) \tag{4.1}$$

where $R(x)$ is a rational function. We consider (4.1) for large x on a ray $\arg(x) = \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. If $R(x)$ is a polynomial the equation has an explicit solution of the form *Polynomial(x)* $+Ce^{-x}$ so the interesting case is

$$R(x) = \sum_{k,j=1}^{m} R_{k,j}(x-r_j)^{-k} \tag{4.2}$$

Proposition 4.1 *i) Given $R(x)$ of the form (4.2) there exists a unique constant K such that the solution of the equation*

$$\psi' + \psi = R(x) - \frac{K}{x} \tag{4.3}$$

is holomorphic in a neighborhood of infinity. Consequently, any solution of the equation (4.1) has a decomposition:

$$\psi(x) = K\, e^{-x}\mathcal{E}i(x) + H\left(\frac{1}{x}\right) + C\, e^{-x} \tag{4.4}$$

where the constants K, C as well as the analytic function $H(\cdot)$ are uniquely determined.
ii) Let $\theta = 0$. A solution of (4.1) has (in the sense of Theorem 1.2) the asymptotic representation

$$\psi(x) \simeq \sum_{j=0}^{\infty} \frac{s_k}{x^k} + C\, e^{-x} \tag{4.5}$$

Moreover the constant C is the same as in (4.1) and the power series is (factorially) divergent unless $K = 0$.

Comments. On the one hand the proposition above indicates in what sense divergence of the asymptotic series is generic. On the other hand the decomposition (4.4) suggests another point of view on the problem of the terms beyond all orders for the equation (4.1). Since in any reasonable definition of asymptotic representations, a function which is analytic at infinity should be represented by its own (convergent) asymptotic series, once the terms beyond all orders for a particular function (the exponential intergal) are defined, they can be determined unambiguously for the solutions of (4.1) with any rational inhomogeneity R. Part ii) shows that the results obtained in this way are consistent with those obtained through asymptotic estimates.

Proof

We have

$$R(x) = \int_0^\infty e^{-tx} \sum_{k,j=1}^{m} R_{k,j} e^{r_j t} \frac{t^{k-1}}{(k-1)!} dt \tag{4.6}$$

Let

$$K = \sum_{k,j=1}^{m} R_{k,j} e^{r_j} \frac{1}{(k-1)!} \tag{4.7}$$

Then the function:

$$\tilde{R}(t) := \frac{1}{1-t} \left(\sum_{k,j=1}^{m} R_{k,j} e^{r_j t} \frac{t^{k-1}}{(k-1)!} - K \right) \tag{4.8}$$

is entire so that the function

$$z \int_0^\infty e^{-t} \tilde{R}(z\,t) dt \tag{4.9}$$

is analytic in z for $|z| < \frac{1}{\rho}$. Indeed, the integrand is analytic in z and, $|\tilde{R}(z\,t)| < z^m t^m \, e^{|z|\,\rho\, t}$ so that the integral is uniformly convergent for $|z\,\rho| < 1$. Furthermore, the function:

$$\frac{1}{x} \int_0^\infty e^{-t} \tilde{R}\left(\frac{t}{x}\right) dt + K\mathcal{E}i(x) \tag{4.10}$$

is a solution of the equation

$$\psi' + \psi = R(x) - \frac{K}{x} \tag{4.11}$$

as it can be easily checked. The rest of the proof of *i)* is immediate.

Let

$$H(x) = \sum_{k=0}^{\infty} \frac{h_k}{x^k}$$

Since clearly $\frac{h_n}{n!} \to 0$ as $n \to \infty$ the proof of *ii)* is an easy application of Proposition 2.1 and of formula (1.7).□

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