RIGOROUS BOUNDS OF STOKES CONSTANTS FOR SOME NONLINEAR ODES AT RANK ONE IRREGULAR SINGULARITIES

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Abstract. A rigorous way to obtain sharp bounds for Stokes constants is introduced and illustrated on a concrete problem arising in applications.
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1. Introduction

Stokes constants (multipliers) relate to the change in asymptotic behavior of a solution of a differential equation as the direction toward an irregular singularity is changed (cf. §2). If the constants are nonzero, then the change in behavior of the solution is nontrivial and this fact plays a very important role in a number of problems.

Many interesting results are known for linear ordinary differential equations, see [1, 13, 15, 16] and references therein; papers [8, 20] use hyperasymptotic methods to express Stokes constants as convergent series.

Stokes multipliers have been evaluated in closed form for a wide class of *integrable systems* sometimes using difficult and subtle arguments, see e.g. [9, 12, 14, 18, 19] and references therein. Integrability however is non-generic, and is thought to play a crucial role in any *explicit* evaluation of Stokes constants.

A theory of wide applicability of extended Borel summation, Borel plane singularities and their relation witgh Stokes phenomena and was introduced by Écalle, [10]. For generic nonlinear systems complete asymptotic expansions (transseries) of solutions and their Borel summability, singularities in Borel plane and formulas linking them to nonlinear Stokes transitions are rigorously obtained in [6]. The paper [7] finds the link between the structure of singularities of solutions in the Borel plane and summation to the least term, as well as with the behavior of the coefficients of the asymptotic series in generic nonlinear systems.

In applications it often only matters whether the Stokes constants are nonzero, while their exact value is not relevant. The results in [6] and [7] provide a rather straightforward way of obtaining rigorous and accurate *estimates* of the Stokes multipliers for a large class of linear or nonlinear differential systems; in principle any prescribed precision can be obtained, as well as the information that a constant does not vanish if such is the case.

One goal of the present paper is to present such a method and at the same time complete an argument in the proof by Tanveer and Xie [21] for the nonexistence of steady fingers with width less than $\frac{1}{2}$ when small nonzero surface tension is taken into account. Their argument relies on a conjectured nonzero value of a Stokes constant of the differential equation

$$2v'' - t + \frac{1}{v^2} = 0 \tag{1.1}$$

Eq. (1.1) appears as an "inner-equation" arising in the context of steady Hele-Shaw cell fingers (see also [2]). It convenient to illustrate our general technique through this particular equation. We show that two Stokes constants for (1.1) are given by

$$S_1 = ib\frac{\pi^{3/2}2^{13/14}}{\Gamma(1/7)\Gamma(3/7)} \tag{1.2}$$

$$S_2 = ie^{\frac{i\pi}{14}}b\frac{\pi^{3/2}2^{13/14}}{\Gamma(1/7)\Gamma(3/7)} \tag{1.3}$$

with

$$1 \le b \le 1 + \frac{12}{37} \tag{1.4}$$

In particular it follows from (1.4) that indeed S_1 and S_2 are nonzero[1].

It will become apparent that the method introduced here applies to generic systems of equations whose irregular singularity has rank one (this is the most frequent type of irregular singularities in applications). It relies on a detailed relation established in [7] between the Stokes constants and the behavior for large index of the coefficients of the asymptotic series solutions, followed by inductive proof of bounds on the solution of the recurrence relation defining them.

2. Stokes constants and exponential asymptotics

Consider a system of differential system of the form

$$\mathbf{y}' = \left(-\boldsymbol{\Lambda} - \frac{1}{x}\mathbf{B}\right)\mathbf{y} + \mathbf{f_0}(x) + \mathbf{g}(x, \mathbf{y}) \tag{2.5}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}\boldsymbol{\lambda}$, $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n)$, $\mathbf{B} = \mathrm{diag}\boldsymbol{\beta}$, $\boldsymbol{\beta} = (\beta_1, ..., \beta_n)$, $\mathbf{f_0} = O(x^{-2})$, $\mathbf{g} = O(x^{-2}; |\mathbf{y}|^2; |x^{-2}\mathbf{y}|)$ (as $|x| \to \infty$), with $\mathbf{f_0}$ analytic at ∞ on a half-line d and $\mathbf{g}$ analytic at $(\infty, 0)$ under *nonresonance assumptions* [6] (a slightly weaker condition than the linear independence of the eigenvalues λ_j over the rationals). The system (2.5) has then a *rank one irregular singularity* at infinity.

2.1. Power series solutions and exponentially small terms.

2.1. Power series solutions and exponentially small terms. The general type of formal solutions of differential systems in the presence of irregular singular points was studied in detail by Fabry [11] and Cope [3] (see also [6], [7], [4]). For (2.5) (assumed nonresonant) the general formal solution for large x has the form

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\in\mathbb{N}^n\setminus\mathbf{0}} \mathbf{C}^{\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x}x^{\mathbf{k}\cdot\boldsymbol{\beta}}\tilde{\mathbf{s}}_{\mathbf{k}} \tag{2.6}$$

where $\tilde{\mathbf{y}}_0$ and $\tilde{\mathbf{s}}_{\mathbf{k}}$ are power series (generically divergent) and $\mathbf{C} = (C_1, ..., C_n)$ are free parameters.

The general formal solutions (2.6) can in fact be calculated algorithmically, in a way that that will be briefly sketched.

The power series solution $\tilde{\mathbf{y}}_0$ is unique. To determine terms beyond all orders of this series, a formal calculation is carried out to find formal solutions which are

[1] Numerical calculation gives $b = 1.1722\cdots$

small perturbations to $\tilde{\mathbf{y}}_0$; for instance one substitutes $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \boldsymbol{\delta}$ in (2.5), where $\boldsymbol{\delta} \ll x^{-q}$ for all $q \in \mathbb{R}_+$ (as $x \to \infty$ on some direction). Since $\tilde{\mathbf{y}}_0$ satisfies (2.5) we formally get

$$\boldsymbol{\delta}' \sim \left(-\boldsymbol{\Lambda} - \frac{1}{x}\mathbf{B} \right) \boldsymbol{\delta}$$

hence in a first approximation

$$\boldsymbol{\delta} \sim \mathrm{e}^{-\boldsymbol{\Lambda} x} x^{-\mathbf{B}} \mathbf{C} \tag{2.7}$$

where $\mathbf{C}$ is a vector of free parameters.

Order by order perturbation expansion around (2.7) first produces power series multiplying the exponential (2.7) and then smaller and smaller exponentials, eventually leading to (2.6).

2.2. **Transseries solutions.** From the point of view of correspondence of formal solutions to actual solutions it was recognized that in general only *asymptotic* expressions of the form (2.6) (*transseries*, as introduced by Écalle [10]) can be lifted to actual functions.

Transseries and their correspondence with functions constitute the subject of exponential asymptotics, a field which developed substantially in the eighties with the work of Berry (hyperasymptotics), Écalle (the theory of analyzable functions) and Kruskal (tower representations and nice functions).

Let $d_\theta = \mathrm{e}^{i\theta}\mathbb{R}_+$ be a direction in the complex x plane. A *transseries solution along* d_θ is, in our context, a formal solution (2.6) whose terms are well ordered with respect to the relation: $\gg$ as $x \to \infty$, $x \in d_\theta$. In particular, a formal solution (2.6) is a transseries along d_θ if constants $C_j = 0$ for all j such that $\mathrm{e}^{-\lambda_j x}$ is not going to zero for $x \in d_\theta$.

2.3. **Exact solutions associated to transseries solutions.** For any direction d_θ there exists a one-to-one correspondence between transseries solutions along d_θ and actual solutions that go to zero on this direction. The correspondence is natural, constructive and compatible with all operations with functions (respectively, transseries) [10].

Actual solutions $\mathbf{y}$ corresponding to a transseries (2.6) on a direction d_θ have the same classical asymptotic expansion $\tilde{\mathbf{y}}_0$ as $x \to \infty$, $x \in d_\theta$. The constants C_j multiply small exponentials in the transasymptotic expansion of the solutions $\mathbf{y}$, and are beyond all orders of the power series $\tilde{\mathbf{y}}_0$. They therefore cannot be defined using the classical Poincaré definition of asymptoticity.

For nonresonant systems with a rank 1 irregular singular point this correspondence was established in [5] and [6]. It was shown that the series $\tilde{\mathbf{y}}_0$ is Borel summable (in a generalized sense); its inverse Laplace transform $\mathbf{Y}_0(p)$ is analytic at $p = 0$ and is (generically) singular at an array of points (which determine the Stokes directions). If we denote

$$\tilde{\mathbf{y}}_0(x) = \sum_{n=2}^{\infty} \frac{1}{x^n} \mathbf{y}_{0;n} \tag{2.8}$$

then we have

$$\mathbf{Y}_0(p) = \sum_{n=2}^{\infty} \frac{p^{n-1}}{\Gamma(n)} \mathbf{y}_{0;n} \tag{2.9}$$

The other power series $\tilde{\mathbf{s}}_{\mathbf{k}}$ ($|\mathbf{k}| > 0$) in a transseries solution (2.6) are also Borel summable in the generalized sense. It is then shown in [6] that the series of functions obtained from a transseries solution by replacing the power series with their Borel sums is convergent for x large enough on appropriate directions. The solution thus obtained is asymptotic to the power series $\tilde{\mathbf{y}}_0$ for large x on the direction on which summation was performed. The correspondence obtained between transseries solutions and actual solutions is one-to-one and is compatible with all operations.

2.4. **Stokes phenomenon.** Consider a solution $\mathbf{y}$ that goes to zero for $x \to \infty$ on a direction d_θ. To associate a transseries $\tilde{\mathbf{y}}$ to $\mathbf{y}$ along d_θ means in fact to specify the parameters C_j, hence C_j may be different for different values of θ.

It turns out, however, that the C_j, as functions of θ, are piecewise constant; the directions d_θ at which one of the C_j has a jump discontinuity are called *Stokes directions* (see [6] for more details).

The way Stokes multipliers S are related to the classical asymptotic behavior of the solutions is given in the following Proposition of [6]:

Proposition 1. *Consider eq. (2.5) under the assumptions given. Assume λ_1 is an eigenvalue of least modulus. Without loss of generality we can assume that $\lambda_1 = 1$ and $\Re\beta_j < 0$ for all j (these inequalities can be arranged in the normalization process, [6]). Let $\gamma^\pm$ be two paths in the right half plane, near the positive/ negative imaginary axis such that $|x^{-\beta_1+1}e^{-x\lambda_1}| \to 1$ as $x \to \infty$ along $\gamma^\pm$. Consider the solutions $\mathbf{y}$ of (2.5) which are small in any proper subsector of the right half plane.*

Then, along $\gamma^\pm$ we have, for some C,

$$\mathbf{y} = (C \pm \frac{1}{2}S_1)\mathbf{e}_1 x^{-\beta_1+1}e^{-x\lambda_1} + o(\mathbf{e}_1 x^{-\beta_1+1}e^{-x\lambda_1}) \tag{2.10}$$

for large x along $\gamma^\pm$ (where $\mathbf{e}_j$ is the j^{th} unit vector $\mathbf{e}_j = (0, ..., 0, \underbrace{1}_{j}, 0, ..., 0)$).

Stokes constants relate to the Maclaurin series of $\mathbf{Y}_0$ in the following way ([6] and [7][2]):

Theorem 2 ([7])**.** *Under the same assumptions as in Proposition 1 we have*

$$\mathbf{Y}_0^{(r)}(0) = \sum_{j;|\lambda_j|=1} \frac{\Gamma(r-\beta_j+1)}{2\pi i e^{i(r+1-\beta_j)\phi_j}} \left(S_j\mathbf{e}_j + \mathbf{h}_j(r) \right) \tag{2.11}$$

where $\mathbf{Y}_0(p)$ is the generalized inverse Laplace transform of $\tilde{\mathbf{y}}_0$ (see (2.8), (2.9)), $\phi_j = \arg\lambda_j$ (ordered increasingly, starting with $\lambda_1 = 1$, $\phi_1 = 0$), $\mathbf{h}_j(r) \sim r^{-1}\mathbf{h}_{j;0}$ for large r.

3. Main Results

Normalization. To apply Theorem 2 to solutions of (1.1), the equation has to be shown to be amenable to the normal form (2.5). The substitution

$$v(t) = t^{-1/2} \left(1 + u(x)\right) \quad \text{where } x = \frac{4}{7}t^{7/4} \tag{3.12}$$

transforms (1.1) to

$$u'' = u + \frac{1}{7}\frac{1}{x}u' - \frac{12}{49}\frac{1}{x^2}u - \frac{3u^2+2u^3}{2(1+u)^2} - \frac{12}{49}\frac{1}{x^2} \tag{3.13}$$

[2]There is a typo in formula (2.1) of [7]: β' should read β (as obtained in its proof).

Substitution (3.12) is natural and a general procedure for finding normalizing substitutions was described in [4].

Equation (3.13) can be written as a system

$$\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{7x} \end{pmatrix}\mathbf{u} + \begin{pmatrix} 0 \\ -\frac{12}{49} \end{pmatrix}\frac{1}{x^2} + \begin{pmatrix} 0 \\ h(x,\mathbf{u}) \end{pmatrix} \tag{3.14}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad , \quad h(x,\mathbf{u}) = -\frac{12}{49}\frac{1}{x^2}u_1 - \frac{3u_1^2 + 2u_1^3}{2(1+u_1)^2}$$

The dominant linear part of (3.14) is diagonalized by substituting

$$\mathbf{u}(x) = S(x)\mathbf{y}(x) \quad \text{with} \quad S(x) = \begin{pmatrix} 1 & 1 \\ -1 + \frac{1}{14x} & 1 + \frac{1}{14x} \end{pmatrix} \tag{3.15}$$

which gives the normal form (2.5) with $n = 2$ and

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \mathbf{B} = \begin{pmatrix} -\frac{1}{14} & 0 \\ 0 & -\frac{1}{14} \end{pmatrix} \tag{3.16}$$

$$\mathbf{f_0}(x) = \frac{1}{x^2}\mathbf{f_0} \quad \text{, with } \mathbf{f_0} = \frac{6}{49}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.17}$$

$$\mathbf{g}(x,\mathbf{y}) = \frac{15}{392}\frac{1}{x^2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{y} + \frac{1}{2}h\left(x, S\mathbf{y}\right)\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.18}$$

Equation (2.5) with (3.16), (3.17), (3.18) has a rank 1 irregular singularity at $x = \infty$ and it is written in normal form. The eigenvalues of the matrix $\mathbf{\Lambda}$ are $\boldsymbol{\lambda} = (1, -1)$ and $\beta_1 = \beta_2 = -1/14$.

The dominant power series in the transseries solution $\tilde{\mathbf{y}}_0 = (\tilde{y}_{0;1}, \tilde{y}_{0;2})$ is

$$\tilde{y}_{0;1+\sigma}(x) = \frac{6}{49}x^{-2} + (-1)^\sigma\frac{87}{343}x^{-3} + \frac{2028}{2401}x^{-4} + (-1)^\sigma\frac{57798}{16807}x^{-5} + ...$$

$(\sigma \in \{0, 1\})$.

Transseries solutions. For eq. (2.5), (3.16–3.18) the transseries solutions along d_θ with $\theta \in (-\pi/2, \pi/2)$ must have $C_2 = 0$ (while C_1 is arbitrary); transseries along directions with $\theta \in (\pi/2, 3\pi/2)$ must have $C_1 = 0$ (and C_2 is arbitrary).

Stokes phenomena. The problem of physical interest depends on the solutions $\mathbf{y}$ which are classically asymptotic to the power series $\tilde{\mathbf{y}}_0$ for $|x| \to \infty$ in the sector $\arg x \in [0, 5\pi/8]$ [21]. The fact that the asymptoticity is required on a large enough sector implies that this solution is unique and its transseries on any direction of argument in $(0, 5\pi/8]$ are expansions (2.6) with $C_1 = C_2 = 0$.

Indeed, a transseries (2.6) for $\arg x \in [0, \pi/2]$ must have $C_2 = 0$. Since $i\mathbb{R}_+$ is not a Stokes direction however, C_2 remains zero in all directions with $\arg x \in (\pi/2, 5\pi/8]$; but in these directions a transseries also has $C_1 = 0$ and since $i\mathbb{R}_+$ is not a Stokes direction, $C_1 = 0$ also for $\arg x \in (0, \pi/2]$.

Since $\mathbb{R}_+$ is a Stokes direction, C_1 may become nonzero here, and its value is the Stokes constant S_1.

Relation (2.11) for equation (2.5), (3.16-3.18) is

$$\mathbf{Y}_0^{(r)}(0) = \frac{\Gamma\left(r + \frac{15}{14}\right)}{2\pi i}\left(S_1\mathbf{e}_1 + S_2 \mathrm{e}^{-i\left(r + \frac{15}{14}\right)\pi}\mathbf{e}_2 + \mathbf{h}(r)\right) \tag{3.19}$$

with $\mathbf{h}(r) \sim r^{-1}\mathbf{h}_0$ for large r.

From (3.15) we have $u(x) = y_1(x) + y_2(x)$. Hence, using (3.19), we get:

$$U_0^{(r)}(0) = Y_{0;1}^{(r)}(0)+Y_{0;2}^{(r)}(0) = \frac{\Gamma\left(r+\frac{15}{14}\right)}{2\pi i}\left(S_1 + S_2 \mathrm{e}^{-i\left(r+\frac{15}{14}\right)\pi} + O(r^{-1})\right) \tag{3.20}$$

Though obvious, it is worth pointing out that the inverse Laplace transform $U(p)$ of $u(x)$ has the convergent series expansion at $p = 0$

$$U_0(p) = \sum_{n\geq 1} \frac{u_{2n}}{\Gamma(2n)} p^{2n-1} \tag{3.21}$$

hence $U_0^{(r)}(0) = 0$ if r is even and $U_0^{(r)}(0) = u_{r+1} > 0$ if r is odd. Then from (3.20) for r even it follows that

$$S_2 = -S_1 \mathrm{e}^{\frac{15}{14} i\pi} \tag{3.22}$$

Using (3.20) for $r = 2n-1$, (3.22) and (4.24) we get the formulas (1.2) and (1.3) where b satisfies (4.25).

4. Estimating the constant b

The constant b in (4.24) below can be estimated within any prescribed accuracy by the procedure described in this section. For the problem at hand however, the inequalities (1.4) obtained here are more than enough.

Proposition 3. *Any solution of equation (3.13) that goes to zero along $\mathbb{R}_+$ has the power series expansion*

$$u(x) \sim \sum_{n\geq 1} x^{-2n} u_{2n} \qquad (x \to +\infty) \tag{4.23}$$

where $u_{2n} > 0$ and

$$u_{2n} = b\frac{\sqrt{\pi}}{\Gamma(1/7)\Gamma(3/7)} n^{-13/14}\Gamma(2n+1)\left(1 + O(n^{-1})\right) \tag{4.24}$$

with b satisfying

$$1 \leq b \leq 1.324 \tag{4.25}$$

Solutions of (1.1) satisfying $v(t) \sim t^{-1/2}$ as $t \to +\infty$ have the asymptotic expansion

$$v(t) = \sum_{k=0}^{\infty} \frac{c_k}{t^{\frac{7}{2}k+\frac{1}{2}}} \ , \ c_0 = 1 \tag{4.26}$$

where the c_n satisfy the recurrence relation:

$$c_n = \frac{(7n-6)(7n-4)}{4} c_{n-1}$$

$$+\frac{1}{2}\sum_{k=1}^{n-1} \frac{(7n-7k-6)(7n-7k-4)}{2} c_{n-k-1} \sum_{i=0}^{k} c_i c_{k-i} - \frac{1}{2}\sum_{k=1}^{n-1} c_k c_{n-k} \tag{4.27}$$

Let d_n be the sequence satisfying the recurrence

$$d_n = \frac{(7n-6)(7n-4)}{4} d_{n-1} \quad (n \geq 1), \quad d_0 = 1 \tag{4.28}$$

Clearly, $d_n > 0$ and in fact

$$d_n = \left(\frac{49}{4}\right)^n \frac{\Gamma(n+\frac{1}{7})\Gamma(n+\frac{3}{7})}{\Gamma(\frac{1}{7})\Gamma(\frac{3}{7})} \tag{4.29}$$

Denote

$$b_n = \frac{c_n}{d_n} \tag{4.30}$$

From (4.27), (4.28) the recurrence for b_n is

$$b_n = b_{n-1} + Q_n \quad , \quad b_0 = 1 \tag{4.31}$$

where $Q_n = Q_n^+ - Q_n^-$ with

$$Q_n^+ = \frac{1}{2d_n} \sum_{k=1}^{n-1} \frac{(7n-7k-6)(7n-7k-4)}{2} b_{n-k-1} c_{n-k-1} T_k \tag{4.32}$$

$$Q_n^- = \frac{1}{2d_n} T_n' \quad , \quad T_k = 2d_k b_k + T_k' \tag{4.33}$$

$$T_k' = \sum_{i=1}^{k-1} b_i b_{k-i} d_i d_{k-i} \tag{4.34}$$

Proposition 4. *The sequence b_n converges and its limit b satisfies the estimate (4.25). Therefore*

$$c_n \sim \Gamma(2n+1) \left(\frac{49}{16}\right)^n n^{-\frac{13}{14}} \frac{b\sqrt{\pi}}{\Gamma(\frac{1}{7})\Gamma(\frac{3}{7})} \qquad (n \to +\infty) \tag{4.35}$$

The proof of Proposition 4 relies on the following two Lemmas:

Lemma 5. *Let $n \geq 5$. Assume there exist $A_1, A_2 > 0$ such that*

$$A_1 \leq b_k \leq A_2 \quad \textit{for all} \quad k \quad \textit{with} \quad 0 \leq k \leq n-1$$

Then

$$|Q_n| \leq \frac{B}{n^2} \tag{4.36}$$

where

$$B \;=\; 0.6\, A_2^2 + 0.0144\, A_2^3 \tag{4.37}$$

The proof of Lemma 5 is given in §4.1.

Lemma 6. *For $A_1 = 1$ and $A_2 = 1.324$ we have*

$$A_1 \leq b_k < A_2 \quad \textit{for } k = 0, 1, 2, ..., 7 \tag{4.38}$$

and

$$A_1 + \frac{B}{k} \leq b_k \leq A_2 - \frac{B}{k} \quad \textit{where } B = 1.0787, \; \textit{for all } k \geq 8 \tag{4.39}$$

The proof of Lemma 6 is given in §4.2.

Proof of Proposition 4. This is an immediate consequence of Lemmas 5 and 6. Indeed, by Lemma 6 we have $1 \leq b_k < 1.324$ for all $k \geq 0$. Then by Lemma 5 we have $|Q_n| < Bn^{-2}$ for all $n \geq 5$, and by (4.31) the sequence b_n is Cauchy. The estimate (4.25) follows from Lemma 6 and the asymptotic behavior (4.35) of c_n follows from (4.30) and (4.29). Relation (4.35) follows from (4.29) and the Stirling formula.

It only remains to prove Lemmas 5 and 6.

4.1. **Proof of Lemma 5.** Note the following estimate for any $N \geq N_0 \geq 3$:

$$\frac{1}{d_N}\sum_{i=1}^{N-1} d_i d_{N-i} = 2\frac{d_1 d_{N-1}}{d_N} + \frac{1}{d_N}\sum_{i=2}^{N-2} d_i d_{N-i}$$

$$\leq 2\frac{d_1 d_{N-1}}{d_N} + (N-3)\frac{d_2 d_{N-2}}{d_N} \leq \frac{1}{N^2}E(N) \leq \frac{1}{N^2}E(N_0) \quad \text{for } N \geq N_0 \geq 3$$

where

$$E(N) = \frac{6}{49}\frac{N^2}{(N-\frac{6}{7})(N-\frac{4}{7})} + \frac{240}{49^2}\frac{N^2}{(N-\frac{4}{7})(N-\frac{11}{7})(N-\frac{13}{7})} \tag{4.40}$$

Therefore

$$D_N \equiv \frac{1}{d_N}\sum_{i=1}^{N-1} d_i d_{N-i} \leq \frac{1}{N^2}E(N_0) \quad \text{for } N \geq N_0 \geq 3 \tag{4.41}$$

It is easy to check that the estimate also holds for $N = N_0 = 2$. To estimate T_k' (see (4.34)) note that since it was assumed that $b_k \geq A_1 > 0$ for all $k \leq n-1$, then for $N_0 \leq k \leq n-1$, we have, using (4.41),

$$0 < T_k' \leq A_2^2 D_k d_k \leq A_2^2 E(N_0)\frac{1}{k^2} d_k \quad \text{for } N_0 \leq k \leq n-1 \tag{4.42}$$

Let $N_0 = 5$; we have $E(5) < 0.24$. For $k = 2, 3, 4$ to estimate (4.42) further note that $D_k d_k = k^2 D_k \frac{d_k}{k^2} < 0.22\frac{d_k}{k^2}$. Then from (4.42) we get

$$0 < T_k' < 0.24\, A_2^2 \frac{1}{k^2} d_k \leq 0.06\, A_2^2\, d_k \quad \text{for } 2 \leq k \leq n-1 \tag{4.43}$$

It follows that (see (4.33))

$$0 < T_k \leq \alpha d_k, \quad \text{for } 1 \leq k \leq n-1 \quad , \quad \text{where } \alpha = 2A_2 + 0.06\, A_2^2 \tag{4.44}$$

For $1 \leq k \leq n-1$ we have, using (4.44), (4.29)

$$0 < \frac{(7n-7k-6)(7n-7k-4)}{2} b_{n-k-1} d_{n-k-1} T_k \tag{4.45}$$

$$\leq A_2 \frac{(7n-7k-6)(7n-7k-4)}{2} d_{n-k-1} T_k$$

$$= 2A_2 d_{n-k} T_k \leq 2\alpha A_2 d_k d_{n-k} \tag{4.46}$$

and using (4.41) (see (4.32))

$$0 < Q_n^+ \leq \frac{\alpha A_2}{d_n}\sum_{k=1}^{n-1} d_k d_{n-k} = \alpha A_2 D_n \leq \frac{\alpha A_2 E(N_0)}{n^2} < \frac{0.24\,\alpha A_2}{n^2} \tag{4.47}$$

for $n \geq 5$. The term Q_n^- (see (4.33)) is estimated similarly, using (4.43):

$$0 < Q_n^- \leq \frac{A_2^2 E(N_0)}{2n^2} < \frac{0.12\, A_2^2}{n^2} \text{ for } n \geq 5 \tag{4.48}$$

Then (4.47), (4.48) implies (4.36) which proves Lemma 5.

4.2. **Proof of Lemma 6.** A direct calculation yields:

$$b_1 = 1\ ,\ b_2 = \frac{169}{160} = 1.061\cdots\ ,\ b_3 = \frac{743}{680} = 1.092\cdots\ ,\ b_4 = \frac{426573}{382976} = 1.113\cdots$$
$$b_5 = \frac{71300607}{63289600} = 1.126\cdots\ ,\ b_6 = \frac{1406520669011}{1239463526400} = 1.134\cdots$$
$$b_7 = \frac{135335882622883}{118668949344000} = 1.140\cdots\ ,\ b_8 = \frac{6575066918153233021}{5744440195153920000} = 1.144\cdots$$

Then (4.38) holds, and also (4.39) is true for $n = 8$. Estimate (4.39) is shown by induction.

Let $n \geq 9$. Assuming (4.39) for all k with $8 \leq k \leq n-1$, then in particular $A_1 \leq b_k \leq A_2$ for all k with $0 \leq k \leq n-1$ hence (4.36) holds. Using (4.31) in (4.39) for $k = n-1$ we get

$$A_1 + \frac{B}{n-1} + Q_n \leq b_n \leq A_2 - \frac{B}{n-1} + Q_n$$

which in view of (4.36) implies (4.39) for $k = n$. Lemma 6 is proved.

Note 7. *Substantially sharper estimates of b can be obtained using more terms in the expansion of* $\mathbf{Y}_0^{(r)}(0)$ *that can be easily obtained from* [7].

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