Analyzability in the context of PDEs and applications

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January 30, 2004

Abstract

We discuss the notions of resurgence, formalizability, and formation of singularities in the context of partial differential equations. The results show that Écalle's how analyzability theory extends naturally to PDEs.

1 Introduction

The study of nonlinear partial differential equations in the complex domain and especially formation of spontaneous singularities of their solutions is not a well understood subject. The theory of Écalle's analyzable functions, originally developed (mainly) for functions of one variable, provides a set of tools which are well suited to address some of these issues, but the extension to several variables is not immediate.

In the case of linear ODEs under suitable assumptions, there is a complete system of formal solutions as transseries, [11], and these are generalized Borel (multi)summable to a fundamental system of actual solutions of the system; sufficiently powerful results of a similar nature have been shown in the nonlinear case as well [11], [13], [15], [16]. This is true for difference equations as well, [6, 8, 9].

While transseries solutions can be considered and their Borel summation shown in the context of PDEs there are a number of difficulties specific to several variables. We first discuss a number of specific obstacles to extending the theory in a straightforward way, and then refer to what we expect to be a general approach to many problems and overview a number of recent results utilizing this approach.

2 Difficulties of formalizability and analyzability in PDEs

2.1 Insufficiency of formal representations

For PDEs even the notion a general formal solution appears to elude a definition that is reasonably simple and useful.
Example 1. The equation $f_t + f_x = 0$ has the general solution

$$f(t, x) = F(t - x)$$

with F any differentiable function; it is not clear to us that a worthwhile definition can be associated to the description "general formal expression in $t - x$"; on the other hand, restricting ourselves to special, well defined, combinations in $t - x$ would correspondingly limit the number of associated actual functions, precluding a complete solution of the original PDE. In this example, actual solutions outnumber by far formalizable ones.

The (apparently) opposite situation is possible as well.

2.2 Absence of general summation procedures

Let now $f_0 \in C^\infty(a,b)$. The initial value problem

$$f_t + i f_x = 0; \quad f_{|_{t=0}} = f_0(x) \tag{1}$$

always has formal solution as $t \downarrow 0$:

$$\tilde{f} = \sum_{k=0}^{\infty} t^k \frac{(-i)^k f_0^{(k)}(x)}{k!} \tag{2}$$

but it has no actual solution, if f_0 is real-valued non-analytic (cf. the proof of next proposition).

There is **no nontrivial** *summation procedure of formal Taylor series over an interval* nor a more restricted one that would associate actual solutions to (1) as the following proposition shows. (See also Remark 1).

Proposition 1 *Let S be a summation procedure defined on a differential algebra D_S of formal series of the form*

$$\tilde{g} = \sum_{k=0}^{\infty} g_k(x) t^k$$

where S is assumed to have the following (natural) properties:

- *S is linear.*
- *S commutes with differentiation.*
- *$S(\tilde{g}) \sim g_0(x)$ as $t \downarrow 0$.*
- *If $\tilde{f} \in D_S$ then $S\tilde{f} : (a,b) \times (0,\epsilon) \mapsto \mathbb{C}$ (where ϵ is allowed to depend on $\tilde{f}$).*

Assume if f_0 is real valued. Then $\tilde{f}$ in (2) is in D_S **iff** *$\tilde{f}$ is convergent (in the usual sense).*

PROOF. Assume $\tilde{f} \in D_s$. Then, by the properties of S, the function $f = S\tilde{f}$ is a differentiable function, and it is a solution of (1) in a domain $\mathcal{D} = (a, b) \times (0, \epsilon)$. If we write $f = u + iv$ we see that the pair (u, v) satisfies the Cauchy-Riemann equations in $\mathcal{D}$ and thus f is analytic in $z = t + ix$ with $(x, t) \in \mathcal{D}$. The third property of S shows that f_0 is the limit as z approaches the interval (a, b) from the upper-half plane. Since f_0 is real valued, then by the Schwarz reflection principle f extends analytically through (a, b) to a neighborhood of (a, b) in the lower half plane; in particular f is analytic on (a, b). But then (1), which is the Taylor series of f at points on (a, b) is convergent. $\square$

2.3 Obstacles to determining the formal solutions

We now contrast formal analysis of ODEs and PDEs.

1. Consider the Painlevé P1 equation

$$y'' = 6y^2 + x \tag{3}$$

a rather nontrivial example of a second order nonlinear ODE. A detailed analysis of transseries and their Borel summability in this example are discussed in [15]. We only mention a few aspects relevant to the present discussion.

Finding formal solutions of (3) is quite straightforward. Searching first for algebraic behavior, dominant balance shows that $6y^2 \sim -x$, say $y \sim 6^{-\frac{1}{2}} i x^{\frac{1}{2}}$, and then, consistent with this, $y'' = o(x)$. It follows that a formal series expansion can be gotten by taking $y_0 = 0$ and then, for $n \in \mathbb{N}$, iterating the recurrence

$$y_{n+1} = 6^{-\frac{1}{2}} i \sqrt{x - y_n''}$$

A power series solution is readily obtained in this way,

$$\tilde{y} = 6^{-\frac{1}{2}} i\sqrt{x} - \frac{1}{48x^2} + \frac{49\sqrt{6}i}{4608x^{9/2}} + \ldots \tag{4}$$

which is not classically convergent but is Borel summable to an actual solution [16]; the complete transseries can be calculated and Borel summed in a similar way, [16]. The possibility (and convenience) of the formal calculation is partly due to *asymptotic simplification*, resulting in a dominant balance equation,

$$6y^2 + x = 0$$

which can be solved exactly, from which a complete solution of the full problem follows by appropriate perturbation theory.

2. Compare this problem to the periodically forced Schrödinger equation

$$i\,\frac{\partial \psi}{\partial t} = \left[-\partial_{xx} + V(x) + \Omega(x)\,\cos\omega t\right]\,\psi \tag{5}$$

Under physically reasonable assumptions ψ is transseriable [4]:

Theorem 2 ([4]) *Assume Ω, V are compactly supported and continuous, and $\Omega > 0$ throughout the support of V. For $t > 0$ there exist $N \in \mathbb{N}$ and $\{\Gamma_k\}_{k\le N}$, $\{F_{\omega;k}(t,x)\}_{k\le N}$, $2\pi/\omega$-periodic functions of t, such that*

$$\psi(t,x) = \sum_{j\in\mathbb{Z}} e^{ij\omega t} h_j(t,x) + \sum_{k=1}^{N} e^{-\Gamma_k t} F_{\omega;k}(t,x) \tag{6}$$

with $\Re\Gamma_k > 0$ for all $k \le N$, and $h_j(t,x) = O\left(t^{-\frac{3}{2}}, |j|!^{-\frac{1}{2}}\right)$ have Borel summable power series in t,

$$h_j(t,x) = \mathcal{LB} \sum_{k \geq 3} h_{kj}(x) t^{-k/2} \tag{7}$$

The operator $\mathcal{LB}$ (Laplace-Borel) stands for generalized Borel summation [16].

Insofar as a formal analysis would be concerned, it is to be noted that there is no small parameter in (5) and largeness of t does not make any term negligibly small; a posteriori, knowledge of the transseries (6) confirms this. In this sense, (5) admits no further simplification. There is, to the knowledge of the authors, no straightforward formal way based on (5) to determine whether $\psi \to 0$, let alone its asymptotic expansion.

3 Overcoming these difficulties: the approach of asymptotic regularization

First note an implication of Écalle's analyzability techniques [5]: a wide class of problems can be regularized by suitable Borel transforms. Summability of general solutions of ODEs or difference equations, [5, 13, 14, 16] shows that, under appropriate transformations, the resulting equations admit *convergent* solutions, an indication of the regularity of the associated equation.

Transseries are obtained, by suitable inverse transforms, from these regularized solutions.

In the case of PDEs it appears that regularizing the equation is in many cases the adequate approach. The result (6) is obtained in this way.

3.0.1 Elementary illustration: regularizing the heat equation

$$f_{xx} - f_t = 0 \tag{8}$$

Since (8) is parabolic, power series solutions

$$f = \sum_{k=0}^{\infty} t^k F_k(x) = \sum_{k=0}^{\infty} \frac{F_0^{(2k)}}{k!} t^k \tag{9}$$

are divergent even if F_0 is analytic (but not entire). Nevertheless, under suitable assumptions, Borel summability results of such formal solutions have been shown by Lutz, Miyake, and Schäfke [10] and more general results of multisummability of linear PDEs have been obtained by Balser [7].

• The heat equation can be regularized by a suitable Borel summation. The divergence implied, under analyticity assumptions, by (9) is $F_k = O(k!)$ which indicates Borel summation with respect to t^{-1}. Indeed, the substitution

$$t = 1/\tau; \quad f(t,x) = t^{-1/2} g(\tau, x) \tag{10}$$

yields

$$g_{xx} + \tau^2 g_\tau + \frac{1}{2}\tau g = 0$$

which becomes after formal inverse Laplace transform (Borel transform) in τ,

$$p\hat{g}_{pp} + \frac{3}{2}\hat{g}_p - \hat{g}_{xx} = 0 \tag{11}$$

which is brought, by the substitution $\hat{g}(p,x) = p^{-\frac{1}{2}} u(x, 2p^{\frac{1}{2}});\ y = 2p^{\frac{1}{2}}$, to the wave equation, which is hyperbolic, thus *regular*

$$u_{xx} - u_{yy} = 0. \tag{12}$$

Existence and uniqueness of solutions to regular equations is guaranteed by Cauchy-Kowalevsky theory. For this simple equation the general solution is certainly available in explicit form: $u = u_-(x-y) + u_+(x+y)$ with u_-, u_+ arbitrary C^2 functions. Since the solution of (12) is related to a solution of (8) through (10), to ensure that we do get a solution it is easy to check that we need to choose $u_- = u_+ = u_0$ (up to an irrelevant additive constant which can be absorbed into u_-) which yields,

$$f(t,x) = t^{-\frac{1}{2}} \int_0^\infty y^{-\frac{1}{2}} \left[u_0\left(x + 2\,y^{\frac{1}{2}}\right) + u_0\left(x - 2\,y^{\frac{1}{2}}\right)\right] e^{-y/t} dy \tag{13}$$

which, after splitting the integral and making the substitutions $x \pm 2\,y^{\frac{1}{2}} = s$ is transformed into the usual heat kernel solution,

$$f(t,x) = t^{-\frac{1}{2}} \int_{-\infty}^\infty u_0(s) \exp\left(-\frac{(x-s)^2}{4t}\right) ds \tag{14}$$

In conclusion although there is perhaps no systematic way to formalize the general solution of the heat equation, appropriate inverse Laplace transforms allow us a complete solution of the problem (in an appropriate class of initial conditions which ensure convergence of the integrals).

Remark 1 *Proposition 1 can be also understood in the following way. Equation (1) is already regular.* Any *actual solution, if it exists with the initial condition given in the Proposition, is trivially formalizable since it is then analytic. It is thus natural that no further summable formal solutions exist.*

3.1 Nonlinear equations: regularization by Inverse Laplace Transform

In this section we briefly mention a number of our results of that substantiate regularizability.

3.1.1

Consider the third order scalar evolution PDE:

$$f_t - f_{yyy} = \sum_{j=0}^{3} b_j(y,t;f)f^{(j)} \; + \; r(y,t); \;\; f(y,0) = f_I \tag{15}$$

Formal Inverse Laplace Transform with respect to y gives[1]

$$F_t + p^3 F = \sum_{j\le 3; k<\infty} \left[B_{j,k} * (p^j F) * F^{*k}\right] + R(p,t) \tag{16}$$

where convolution is the Laplace one, $(f*g)(p) = \int_0^p f(s)g(p-s)ds$. This equation is regular in that formal power series in p converges, since the coefficients in the equation are analytic.

Multiplying by the integrating factor of the l.h.s. and integrating yields

$$F(p,t) = \mathcal{N}F(p,t)$$
$$= F_0(p,t) + \sum_{j\le 3; k<\infty} \int_0^t (-1)^j e^{-p^3(t-\tau)} \left[(p^j F) * B_{j,k} * F^{*k}\right](p,\tau)d\tau$$

The regularity of this equation plays a crucial role in the proofs in [1] where we find the actual solutions of equation (15).

3.1.2

Similar methods were later extended [3] to equations of the form

$$\mathbf{u}_t + \mathcal{P}(\partial_{\mathbf{x}}^{\mathbf{j}})\mathbf{u} + \mathbf{g}\left(\mathbf{x}, t, \{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\}\right) = 0; \;\; \mathbf{u}(\mathbf{x},0) = \mathbf{u}_I(\mathbf{x})$$

with $\mathbf{u} \in \mathbb{C}^r$, for $t \in (0,T)$ and large $|\mathbf{x}|$ in a poly-sector S in $\mathbb{C}^d$ ($\partial_{\mathbf{x}}^{\mathbf{j}} \equiv \partial_{x_1}^{j_1}\partial_{x_2}^{j_2}...\partial_{x_d}^{j_d}$ and $j_1 + ... + j_d \le n$). The principal part of the

[1] For technical convenience, in [1] we used oversummation. The paper [3] shows that in fact Borel summability holds in the correct variable, in the more general setting decribed in the next section.

constant coefficient n-th order differential operator $\mathcal{P}$ is subject to a cone condition. The nonlinearity $\mathbf{g}$ and the functions $\mathbf{u}_I$ and $\mathbf{u}$ satisfy analyticity and decay assumptions in S.

The paper [3] shows existence and uniqueness of the solution of this problem and finds its asymptotic behavior for large $|\mathbf{x}|$.

Under further regularity conditions on $\mathbf{g}$ and $\mathbf{u}_I$ which ensure the existence of a formal asymptotic series solution for large $|\mathbf{x}|$ to the problem, we prove its Borel summability to the actual solution $\mathbf{u}$.

In special cases motivated by applications we show how the method can be adapted to obtain short-time existence, uniqueness and asymptotic behavior for small t, of sectorially analytic solutions, without size restriction on the space variable.

3.2 Nonlinear Stokes phenomena and movable singularities

In the context of ODEs it was shown [15], under fairly general assumptions, that the information contained in the regularized problem (equivalently, in the transseries) can be used to determine more global behavior of solutions of nonlinear equations, in particular the fact that they form spontaneous singularity close to anti-Stokes lines. The method, transasymptotic matching, was extended to difference equations [8, 9, 6].

In nonlinear partial differential equations, formation of singularities is a very important phenomenon but no general methods to address this issue existed.

The method of regularization that we described provides such a method. We briefly discuss the main points of [2].

At present our methods apply to nonlinear evolution PDEs with *one space variable*; even for these, substantial new difficulties arise with respect to [15].

Consider the modified Harry Dym equation (arising in Hele-

Shaw dynamics)

$$H_t - H^3 H_{xxx} + H_x - \frac{1}{2} H^3 = 0; \quad H(x,0) = \frac{1}{\sqrt{x}}$$

in an appropriate sector.

Small time behavior. From [1] it follows that there exists a unique solution to above problem, and it has Borel summable series for small t and small $y = x - t$:

$$H(x,t) = y^{-1/2} - t\left(\frac{15}{8y^5} + \frac{1}{2y^{3/2}}\right) + t^2\left(\frac{25875}{128y^{19/2}} + \frac{195}{32y^6} + \frac{3}{8y^{5/2}}\right) + ...$$

Singularity manifolds near anti-Stokes lines. To apply the method of transasymptotic matching, we look on a scale where the asymptotic expansion becomes formally invalid: $y = x - t = O(t^{2/9})$. The transition variable is thus

$$\eta = \frac{x-t}{t^{2/9}}, \quad \tau = t^{7/9}, \quad H(x,t) = t^{-1/9}\ G(\eta,\tau)$$

Substituting into (17), we obtain the following equivalent equation

$$-\frac{G}{9} - \frac{2}{9}\eta G_\eta + \frac{7}{9}\tau G_\tau + \frac{\tau}{2} G^3 - G^3 G_{\eta\eta\eta} = 0$$

The natural formal expansion solution in this regime is

$$G(\eta,\tau) = \sum_{k=0}^{\infty} \tau^k G_k(\eta) \tag{17}$$

with matching conditions at large η, to ensure the solution agrees with the one obtained in [1]:

$$G_0(\eta) \ \sim\ \eta^{-1/2}; \quad G_k(\eta) \ \sim\ \frac{A_k}{\eta^{k+1/2}}$$

We show that the series (17) is actually *convergent* and equals $H(x,t)$ in the Borel summed region (the radius of convergence shrinks however with η). The convergence problem is subtle and required a rather delicate construction of suitable invariant domains. Having shown that, it is intuitively clear (and not difficult to prove) that if G_0 is singular, then H is singular. The leading order solution G_0 satisfies

$$G_0 + 2\,\eta\, G_0' + 9\, G_0^3\, G_0''' = 0$$

while for $k \geq 1$,

$$G_0^3 \mathcal{L}_k G_k = R_k$$

where

$$\mathcal{L}_k u = u''' + \frac{2}{9G_0^3}\eta u' - \left(\frac{7k-1}{9G_0^3} + \frac{3G_0'''}{G_0}\right) u$$

and the right hand side R_k is given by

$$R_k(\eta) = \frac{1}{2} \sum_{k_1+k_2+k_3=k-1} G_{k_1}G_{k_2}G_{k_3} + \sum_{k_j<k, \sum k_j = k} G_{k_1}G_{k_2}G_{k_3}G_{k_4}'''$$

The nonlinear ODE of G_0 with asymptotic condition has been studied in [17] and computational evidence suggested clusters of singularities η_s, where $G_0(\eta) \sim e^{\pi i/3} \left(\frac{\eta_s}{3}\right)^{1/3} (\eta - \eta_s)^{2/3}$.

$$G_0 + 2\,\eta\, G_0' + 9\, G_0^3\, G_0''' = 0 \tag{18}$$

For a rigorous singularity analysis of (18) we now used transasymptotic matching as developed for ODEs [15]. The asymptotic behavior of G_0 is of the form

$$G_0(\eta) \;\sim\; \eta^{-1/2} U(\zeta)$$

where

$$\zeta = -\ln C + \frac{9}{8}\ln\eta + \frac{i4\sqrt{2}}{27}\eta^{9/4} + (2n-1)\pi i$$

(where C is the Stokes constant) and $U(\zeta)$ satisfies algebraic equation

$$\frac{1}{4} e^{\zeta+2} = e^{-2\sqrt{U}} \left(\frac{\sqrt{U}+1}{\sqrt{U}-1}\right)$$

Singularities of $U(\zeta)$ occur at $\zeta_s = \ln 4 - 2 - i\pi$, corresponding for $n \in \mathbb{N}$ large to

$$\frac{i4\sqrt{2}}{27}\eta_s^{9/4} + \frac{9}{8}\ln\ \eta_s = -2 + \ln 4 - (2n-1)i\pi + \ln C$$

The **Theorem** that we prove in [2] is that *For a singularity $\hat{\eta}_s$ of U* [2] *there exists a domain $\mathcal{D}$ around the singularity* [3] *such that the expansion is convergent for small τ.*

In particular, for small τ, the singularity of $G(\eta,\tau) = H(x,t)$ approaches the singularity of U and is, to the leading order, of the same type, $(\eta - \eta_s)^{2/3}$.

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[2]with $|\hat{\eta}_s|$ large enough and with arg $\hat{\eta}_s$ close to the anti-Stokes line arg $\eta = -\frac{4\pi}{9}$.

[3]that extends to ∞ with arg $\eta \in \left(-\frac{2\pi}{9} + \delta, \frac{2\pi}{9} - \delta\right)$ for some $\frac{\pi}{9} > \delta > 0$, and includes a region $\mathcal{S}$ around the the singularity $\hat{\eta}_s$ but excludes an open neighborhood.

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