# RIGOROUS WKB FOR FINITE ORDER LINEAR RECURRENCE RELATIONS WITH SMOOTH COEFFICIENTS *

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**Abstract.** We study the $\epsilon \to 0$ behavior of recurrence relations of the type $\sum_{j=0}^{l} a_j(k\epsilon, \epsilon) y_{k+j} = 0$, $k \in \mathbb{Z}$ ($l$ fixed). The $a_j$ are $C^\infty$ functions in each variable on $I \times [0, \epsilon_0]$ for a bounded interval $I$ and $\epsilon_0 > 0$. Under certain regularity assumptions we find the asymptotic behavior of the solutions of such recurrences. In typical cases there exists a fundamental set of solutions in the form $\{\exp(\epsilon^{-1} F_m(k\epsilon, \epsilon))\}_{m=1\ldots l}$ where the functions $F_m$ are $C^\infty$ in each variable on the same domain as the $a_j$, showing in particular that the formal perturbation-series solutions are asymptotic to true solutions of these recurrences. Some applications are also briefly discussed.

**Key Words.** Recurrence relations, asymptotic behavior

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**1. Introduction.** In the present paper we study the asymptotic behavior to all orders in $\epsilon \to 0$ of the solutions of one-dimensional recurrence relations of the form

$$\sum_{j=0}^{l} a_j(k\epsilon, \epsilon) y_{k+j} = 0 \tag{1}$$

which we may interpret as follows: for each fixed $k$, $y_{k+l}$ is determined from its predecessors $y_k \ldots y_{k+l-1}$ (this is assumed possible — see condition (6) below).

Under some further regularity assumptions we prove that the general solution of the recurrence can be piecewise represented as a sum

$$y_{k,\epsilon} = \sum_{m=1}^{l} C_{m,p} \exp[\epsilon^{-1} F_m(k\epsilon, \epsilon)] \tag{2}$$

where the functions $F_m$ are everywhere smooth with the exception of a small neighborhood of the points where two characteristic roots (24) cross and where the representation is different (Proposition 2.2) . In particular, a fundamental system of solutions can be chosen such that each of them has, for small–$\epsilon$ and between crossings, a WKB–like expansion:

$$y_k \sim \exp\left(\epsilon \phi(k\epsilon)\right)\left(A_0(k\epsilon) + \epsilon A_1(k\epsilon) + \ldots\right) \tag{3}$$

where $\phi$ is the root of the eikonal equation

$$\exp(\phi'(x)) = \lambda(x)$$

and $\lambda$ is one of the $l$ roots of the characteristic equation (24) and the successive amplitudes $A_i$ can be determined by perturbation–expansion. For technical reasons, we prefer the less familiar notation (2). It is essential for our arguments that a continuous branch of $\ln A_0$ can be chosen.

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One of the applications of the rigorous WKB–approach for discrete schemes is in determining the spectrum of large matrices with slowly–varying entries. Such problems appear for instance in the continuum limit of the Toda lattice. This system , described by the Hamiltonian $H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{k=1}^{n-1}\exp(x_k - x_{k-1})$ is completely integrable; this can be expressed in terms of the constancy of the spectrum of the matrix

$$\begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \dots & 0 \\ 0 & b_2 & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{4}$$

The spectral problem in case the coefficients are of the form $a_k = a(k\epsilon)$, $b_k = b(k\epsilon)$ with $a$ and $b$ smooth and satisfying some regularity conditions leads to a recurrence of the type (1) that is solved asymptotically by the methods described below [1].

In (1) the number $l$ of steps of the recurrence is fixed, $\epsilon > 0$ is a small parameter and $k \in \mathbb{Z}$ is such that $k\epsilon \in I$ where $I \subset \mathbb{R}$ is a compact interval. Some initial or boundary conditions are assumed.

The coefficients $a_0(x,\epsilon)\dots a_l(x,\epsilon) : I \times [0,\epsilon_0] \to \mathbb{C}$ are assumed $C^\infty$ in $x$ and in $\epsilon$ in some domain $I \times [0,\epsilon_0]$. We also require the existence of a uniformly asymptotic series for $a_j$: for any $t \in \mathbb{N}$,

$$\mid a_j(x,\epsilon) - \sum_{s=0}^{t} a_{j,s}(x)\epsilon^s \mid < M_{j,s}\epsilon^{t+1} \tag{5}$$

where the functions $a_{j,s}$ are $C^\infty$ in $x$ (for instance $a_j \in C^\infty(I \times [0,\epsilon_0])$.

We are also imposing the nonsingularity condition

$$\inf_{x\in I}\{|a_0(x,0)|\} > 0 \text{ and } \inf_{x\in I}\{|a_l(x,0)|\} > 0 \tag{6}$$

We begin by giving some simple examples and deriving heuristically their small-$\epsilon$ behavior. The contents of the paper will subsequently make the given solutions rigorous.

**a.** Consider the one-step recurrence relation

$$y_{k+1} = e^{f(k\epsilon)} y_k \tag{7}$$

where $y_0 = 1$, $0 \le k \le \epsilon^{-1}$ and $f$ is a $C^\infty$ function on $[0,1]$. It has the explicit solution

$$y_k = \exp((\sum_{j=0}^{k-1} f(j\epsilon)) \tag{8}$$

When $f = f_0$ is constant, $y_k = \exp(\epsilon^{-1} f_0 k\epsilon)$ and this should also be the order of magnitude of $y_k$ for a general smooth f when $\epsilon$ is very small; we then try a formal asymptotic solution

$$y_k \sim \exp(\epsilon^{-1}\Phi_0(k\epsilon) + \Phi_1(k\epsilon) + \epsilon\,\Phi_2(k\epsilon) + \dots) \tag{9}$$

in (7) . To be consistent with (7) we must have

$$\exp[\sum_{m=0}^{\infty} \epsilon^{m-1}\Phi_m(k\epsilon+\epsilon)] \sim \exp[f(k\epsilon)+\sum_{m=0}^{\infty} \epsilon^{m-1}\Phi_m(k\epsilon)],\ \epsilon\to 0 \tag{10}$$

Expanding the exponent in the LHS of (10) in a Taylor series around $k\epsilon$ and then identifying the corresponding powers of $\epsilon$ in (10) we obtain

$$\Phi_0' = f, \quad \Phi_1' = -\frac{1}{2}f' \ \dots \ \Phi_j' = C_j\ f^{(j)}, \dots \tag{11}$$

where the constants $C_j$ are seen to be $f$-independent and can thus be determined by choosing a particular $f$ for which the sum in (8) can be explicitly evaluated. Taking, e.g., $f(x) = \exp(x)$ this sum is

$$(\exp(x)-1)(\exp(\epsilon)-1)^{-1} \sim \epsilon^{-1}(\exp(x)-1)\sum_{j=0}^{\infty}\frac{B_{2j}}{(2j)!}\epsilon^{2j}, \quad \epsilon\to 0 \tag{12}$$

where $B_n$ are the Bernoulli numbers. It follows that $C_j = B_j$ $(B_{2j+1} = 0)$ and thus, for any smooth $f$

$$\sum_{j=0}^{k-1} f(j\epsilon) \sim \frac{1}{\epsilon}\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}\epsilon^{2n}\int_0^x f^{(n)}(t)dt, \quad \epsilon\to 0 \tag{13}$$

which is, of course, the Euler-Maclaurin summation formula. Proposition 2.1 below justifies our derivation of (13).

**b.** Analogously, we can obtain easily the asymptotic behavior of special functions from their generating recurrence relation. For instance, the recurrence relation

$$y_{k+1} + y_{k-1} = 2(1+k\epsilon)y_k \tag{14}$$

has, for a fundamental set of solutions the Bessel functions $J_{k+\epsilon^{-1}}(\epsilon^{-1})$ and $Y_{k+\epsilon^{-1}}(\epsilon^{-1})$.

Let $x = k\epsilon$, $\nu = k+\epsilon^{-1}$, $a = \epsilon^{-1}\nu^{-1}$. We fix $\rho < \frac{2}{3}$ and for for $x > \epsilon^{\rho}$ we take $y$ of the form (9) determining the successive terms by substituting the formal series in (14). To leading order

$$e^{\Phi_0'(x)} + e^{-\Phi_0'(x)} = 2(1+x) \tag{15}$$

and we can choose two independent solutions

$$\epsilon^{-1}\Phi_{0;\pm} = \pm\nu(\alpha-\tanh(\alpha)) \quad (\alpha \equiv (\cosh)^{-1}(1/a)) \tag{16}$$

For the next order $\Phi_{1;\pm}$ we obtain:

$$\Phi_{1;\pm}'/\Phi_{1;\pm} = -\frac{1}{2}\Phi_{0;\pm}''\coth(\Phi') \tag{17}$$

which can, again be integrated explicitly to give $\exp(\Phi_{1;\pm}) = const\,\sinh(\alpha)^{-1/2}$. That is, the asymptotic behavior is

$$(\nu\tanh(\alpha))^{-1/2}(A\exp[\nu(\alpha-\tanh(\alpha))+\dots]+B\exp[-\nu(\alpha-\tanh(\alpha))+\dots]) \tag{18}$$

and we get the familiar expressions in the theory of Bessel functions (see [2], [3]). All the successive orders are obtained easily in the same way. For $x < -\epsilon^{\rho}$ we obtain from the same equations (15), (17) similar expressions but with trigonometric instead of hyperbolic functions. For small values of $|x|$, $|x| \lesssim \epsilon^{2/3}$ the above asymptotic series becomes singular. The appropriate WKB-like series in this region is in powers of $\epsilon^{1/3}$ and the coefficients will be smooth functions of $\epsilon^{1/3}k$, but otherwise the calculation can be done explicitly in the same way; there is a region of overlap where both asymptotic series are valid, namely $\epsilon^{2/3} < |x| < \epsilon^{1/2}$ and where the series can be matched.

Proposition 2.2 can be used to make the above approach rigorous.

**c.** Finally, let $q$ be a smooth function and consider the Cauchy problem

$$y'' = q(x)y, \quad y(0) = 0, \quad y'(0) = 1 \tag{19}$$

Assume for simplicity that $q : [0,1] \to \mathbb{R}^+$ and consider the associated Euler scheme

$$y_{k+\epsilon;\epsilon} + y_{k-\epsilon;\epsilon} = (2 + \epsilon^2 q(k\epsilon))y_{k;\epsilon}, \quad y_{0;\epsilon} = 0, \quad y_{1;\epsilon} = \epsilon \tag{20}$$

To characterize $y_{k;\epsilon}$ for small $\epsilon$ we substitute

$$y_{k;\epsilon} \sim \sum_{m=0}^{\infty} \epsilon^m \Xi_m(k\epsilon) \tag{21}$$

in (20). Note that if $\Xi_0 \neq 0$ we obtain this type of series from (9), when $\Phi_0 = 0$, by expanding the exponential.

The substitution leads to the equations:

$$\Xi_m'' = q(x)\Xi_m - 2 \sum_{1\le s; 2s\le m} \frac{\Xi_{m-2s}^{(2s)}}{(2s)!} \tag{22}$$

with the initial conditions

$$\Xi_m(0) = 0; \quad \Xi_m'(0) = -\sum_{k=1}^{m} \frac{\Xi_{m-k}^{(k+1)}(0)}{(k+1)!} \tag{23}$$

In particular, it follows that the scheme converges to the solution of the given Cauchy problem (as it should) and because, as it is easily seen, $\Xi_1 \equiv 0$ the error $|y_{k+\epsilon} - \Xi_0(k\epsilon)|$ is $O(\epsilon^2)$. Proposition 2.3 applies to this example.

**2. Main results.** In this section we consider the problem (1) and the assumptions following it and give conditions under which the exact solutions of the recurrence have asymptotic series to all orders in $\epsilon$.

PROPOSITION 2.1. *Let $\lambda_1(x), \ldots \lambda_l(x)$ be the roots of the characteristic polynomial*

$$\sum_{j=0}^{l} a_j(x,0)\lambda^j = 0 \tag{24}$$

*and assume that they are simple throughout I:*

$$\inf_{x\in I}\{|\ \lambda_m(x) - \lambda_n(x)\ |\} > 0 \quad \textit{if } m \neq n. \tag{25}$$

*(consequently, we will choose $\lambda_m$ to be $C^\infty$ functions). Suppose also that the interval $I$ is a finite union of (closed) intervals such that in each one of them the ordering of the moduli of the roots does not change, i.e.*

$$I = \bigcup_{p=1}^{P} I_p \quad \text{and } \forall p \leq P \ \exists (i_1 \ldots i_l) \text{ such that}$$

$$|\lambda_{i_1}(x)| \leq |\lambda_{i_2}(x)| \leq \ldots \leq |\lambda_{i_l}(x)| \text{ for all } x \in I_p. \tag{26}$$

*where $(i_1 \ldots i_l)$ is a permutation of $(1 \ldots l)$. Then there exists $\epsilon' \leq \epsilon_0$ and $l$ functions $F_1(x,\epsilon) \ldots F_l(x,\epsilon)$ on $I \times [0,\epsilon']$, $C^\infty$ in each variable, such that for each $\epsilon$, $\{\exp(\epsilon^{-1} F_m(k\epsilon,\epsilon))\}_{m=1\ldots l}$ form a fundamental set of solutions of the recurrence relation (1) in the sense that for any solution $y_{k,\epsilon}$ of (1) there exist constants $C_1^{(p)}, \ldots C_l^{(p)}$ such that in each $I_p$,*

$$y_{k,\epsilon} = \sum_{m=1}^{l} C_m^{(p)} \exp[\epsilon^{-1} F_m(k\epsilon,\epsilon)] \tag{27}$$

REMARK 1. *In particular, this means that for small $\epsilon$*

$$F_m(x,\epsilon) \sim \sum_{s=0}^{\infty} \Phi_{m,s}(x)\epsilon^s \tag{28}$$

*where the $\Phi_{m,s}$ are smooth; they can be computed from (1) by usual perturbation expansions in $\epsilon$. For example the first term is gotten from the eikonal equation,*

$$\exp(\Phi'_{m,0}(x)) = \lambda_m(x) \tag{29}$$

*giving the connection between the functions $F_m$ and the roots $\lambda_m$ of the characteristic polynomial. The second term is obtained from*

$$\Phi'_{n,1}(x) = -\frac{\sum_{j=0}^{l} [j^2 \Phi_{n,0}(x)''/2A_{j,0}(x) + \exp(\Phi'_{j,0}(x))A_{j,1}(x)]}{\sum_{j=0}^{l} jA_{j,0}(x)\exp(\Phi'_{j,0}(x))} \tag{30}$$

*and so on.*

Note also that if not all the functions $F_m$ have the same real part, then there exist solutions of the recurrence which are exponentially small relative to the dominant ones and which will therefore be unstable in the sense that a small "generic" perturbation of the initial condition will change completely the behavior of these solutions; a perturbation series not involving terms beyond all orders will only see the component of the solution along the dominant directions. However the relative size of the solutions could change with $x$ and then all the coefficients in the expansion (27) could be important for matching different regions.

We now address the question of the asymptotic behavior of the solutions of the recurrence when two characteristic roots cross. The previous Proposition is generalized below to the case when two of the roots of the characteristic polynomial become equal at some point in $I$ provided the roots do not coalesce too quickly (condition (31)). In this case a fundamental set of solutions has a more complicated structure. Not too close to the crossing point, a solution is still a linear combination of the form (27) but the coefficients $C_m^{(p)}$ can change at the crossing (Stokes phenomenon). Very close of the crossing, the coalescing roots bring in the asymptotic expression of the solution. series in noninteger powers of $\epsilon$. This is in some sense the discrete counterpart of the turning point-behavior of the solution of a differential equation depending on a small parameter.

Consider a subinterval $I_0$ of $I$ such that two roots of the characteristic polynomial cross once in $I_0$ (say, at $x = 0$) and except for this, the ordering of the moduli of the characteristic roots is constant in $I_0$. The crossing is assumed to be generic:

$$|\lambda_m(x) - \lambda_n(x)| > \text{const}\sqrt{|x|} \text{ ( for } m \neq n) \tag{31}$$

To avoid excessive branching of the discussion and formuli, we also assume that the coalescing roots are *complex–conjugate* for negative $x$ and *real–valued* for $x$ positive. The general case is treated in a very similar way.

Fix two constants $\frac{1}{3} < \beta < \alpha < \frac{1}{2}$. Then,

PROPOSITION 2.2. *i) For $|k| > \epsilon^{-\beta}$ the general solution of the recurrence can be written in the form*

$$y_k = \sum_{j=1}^{l} C_j \exp(\epsilon^{-1} F_j(k\epsilon, \epsilon)) \tag{32}$$

*with $F_j$ as in Lemma 3.2. The constants $C_j$ depend, in general on the sign of $k$.*

*ii) For $|k| < \epsilon^{-\alpha}$ a fundamental set of solutions can be chosen such that: $l-2$ solutions are of the form $\exp(\epsilon^{-1}F_j(k\epsilon, \epsilon))$ and two special solutions of the form*

$$\exp(F_{\pm}(k\epsilon^{1/3}, \epsilon^{1/3}))$$

*where the functions $F_j$ and $F_{\pm}$ are smooth and $\exp(F_{\pm}(x, 0)) = Ai(\Theta\, x) \pm Bi(\Theta\, x)$ where $Ai$, $Bi$ are the Airy functions (for the value of $\Theta$ see equation (89) .*

*Moreover there is a particular solution of the recurrence which has the behavior*

$$y_k \sim Ai(\Theta\, k\epsilon^{1/3})(1 + \epsilon^{1/3} A_1(\Theta\, k\epsilon^{1/3}) + ..)$$

*for large $k < \epsilon^{-\alpha}$*

*The representations in (i) and (ii) are simultaneously valid in the region $\epsilon^{-\beta} < |k| < \epsilon^{-\alpha}$ where the asymptotic series can be matched.*

**Note**. A similar result can be proven if condition (31) is replaced by

$$|\lambda_m(x) - \lambda_n(x)| > \text{const}\, x^{const'} \tag{33}$$

for some const $> 0$. Another case which is interesting for schemes that converge to differential equations for small $\epsilon$ is covered by the following proposition, in which we assume that the roots of the *complete* characteristic polynomial:

$$\sum_{j=0}^{l} a_j(x, \epsilon)\lambda^j, \qquad m = 1 \dots l \tag{34}$$

are nondegenerate in a *higher* order in $\epsilon$.

PROPOSITION 2.3. *Assume that the roots $\lambda_1(x,\epsilon)\ldots\lambda_l(x,\epsilon)$ of (34) satisfy the estimates*

$$|\lambda_m(x,\epsilon)-1| = \epsilon^q(Q_m(x)+o(\epsilon)), \quad m \le l \tag{35}$$

*for some $q \in I\!N$ where the smooth functions $Q_i$ verify*

$$\inf_{x\in I} Q_m(x) > 0, \;\inf_{x\in I} |Q_m(x)-Q_n(x)| > 0 \;\;(m \neq n)\; m,n \le l$$

*Then, the conclusion of Proposition 2.1 holds and, furthermore , for any formal series solution of the recurrence relation (1) there exists a true solution which is asymptotic to it.*

In this particular case, since, as we shall see, $F_m(x,0)=0$, it is more natural to represent the formal solutions as power series:

$$\Gamma = \sum_{i=0}^{\infty} \Psi_i(x)\epsilon^i \tag{36}$$

where the $\Psi_i$ are smooth and subject to the condition

$$\sum_{j=0}^{l} a_j(x,\epsilon)\Gamma_s(x+j\epsilon;\epsilon) = o(\epsilon^s) \;\;\forall s, \qquad x \in I. \tag{37}$$

where $\Gamma_s(x;\epsilon)=\sum_{i=0}^{s}\Psi_i(x)\epsilon^i$. The series (36) and those appearing in the previous WKB expansions are usually divergent and one could imagine that by iterating the recurrence the small error appearing in the local condition (37) could quickly reach O(1). Under the given restrictions, however, Proposition 2.3 guarantees that $\Gamma_s$ is indeed an $o(\epsilon^s)$ approximation to a genuine solution.

It can be now checked without difficulty that Propositions 2.1, 2.2 and 2.3 apply to the examples a), b) and c), respectively.

The layout of the paper is as follows: in Section 3 we prove our main results and in Section 4 we discuss some further applications of these results.

**3. Proof of the results.** To prove Proposition 2.1 we show (Lemma 3.1) the existence of $l$ formal series solutions of the form (9) to the recurrence.

We then show (Lemma 3.2) that the the proposition is true if $P=1$ (cf.(26)). The proof is by induction on the order $l$ of the recurrence. First, we choose the particular formal solution corresponding (cf. (39)) to the root with maximum modulus —which gives the "stable" direction— and show that there is a true solution with this asymptotic behavior. Next we use this particular solution to decrease the order of the recurrence by 1.

**3.1** The functions $\Phi_{m,s}$ of the following lemma will turn out to be the functions giving the asymptotic expansions (28) and can be obtained by requiring that (1) is satisfied in all orders in $\epsilon$ by the formal solution $y_m=\exp(\epsilon^{-1}\sum_s \Phi_{m,s}(x)\epsilon^s)$.

LEMMA 3.1. *For each $m=1\ldots l$ there exists a sequence $\{\Phi_{m,s}\}_{s\in\mathbb{N}}$ of functions in $C^\infty(I)$ such that*

$$\exp(-\epsilon^{-1}\Phi_{m,0}(k\epsilon))\sum_{j=0}^{l} a_j(k\epsilon,\epsilon)\exp(\epsilon^{-1}\sum_{t=0}^{s}\epsilon^t\Phi_{m,t}((k+j)\epsilon))) = O(\epsilon^{s+1}) \tag{38}$$

*and*

$$\exp(\Phi'_{m,0}(x)) = \lambda_m(x) \tag{39}$$

*for* $\epsilon \le \epsilon_0$, $s \in I\!N$ *and* $k\epsilon \in I$

The proof of Lemma 1 is by induction on the expansion order $s$. In view of (5) and (6) we can define for each characteristic root $\lambda_m(x)$ (i.e. root of eq. (24)), a function $\Phi_{m,0} \in C^\infty(I)$ such that (39) holds.

It is then straightforward to show that $\exp(\epsilon^{-1}\Phi_{m,0})$ verifies (1) to $\mathrm{O}(\epsilon)\exp(\epsilon^{-1}\Phi_{m,0}(x))$ so that (38) holds for $s = 0$. Assuming now that $\Phi_{m,0}, \Phi_{m,1} \ldots \Phi_{m,s_0}$ are already defined so that for all $s \le s_0$ (38) is verified, one can easily check that for any $\Psi \in C^\infty(I)$,

$$\exp[-\epsilon^{-1}\Phi_{m,0}(k\epsilon)]$$

$$\sum_{j=0}^{l} \exp\{\epsilon^{-1}\sum_{t=0}^{s_0}\Phi_{m,t}((k+j)\epsilon)\epsilon^t + \epsilon^{s_0+1}\Psi((k+j)\epsilon)\}a_j(k\epsilon,\epsilon) =$$

$$\epsilon^{s_0+1}[\Psi'(k\epsilon)\sum_{j=0}^{l} j\exp\{j\Phi'_{m,o}(k\epsilon)\}a_j(k\epsilon,0) + H_{s_0}(k\epsilon,\epsilon)] + \mathrm{O}(\epsilon^{s_0+2}) \tag{40}$$

where $H_{s_0}$ is a smooth function.

Since by (6) and (25)

$$\inf_{x\in I}|\sum_{j=0}^{l} j a_j(x,0)\lambda_m^j(x)| > 0 \tag{41}$$

one can define a smooth function $\Psi(x) \equiv \Phi_{s_0+1}(x)$ such that the term in square brackets in the RHS of (40) vanishes. $\Box_{L1}$

We note at this point that the series $\sum_{s=0}^{\infty}\Phi_{m,s}(x)\epsilon^s$ is, usually, not convergent and there does not yet follow the existence of a solution asymptotic to it.

Now we address the question of existence of true solutions of the recurrence having the precribed asymptotic behavior. In what follows, we shall understand by a formal solution an expression $\tilde{Y} = \exp(\epsilon^{-1}\sum_{s=0}^{\infty}\Phi_{m,s}(x))\epsilon^s$ satisfying the conclusions of Lemma 3.1. Given $\Phi_{m,0}$, the $\Phi_{m,s}$ are uniquely determined up to integration constants.

Assume first $P = 1$ (cf. (26)). Relabelling if necessary, we assume that for $m \le n$, $|\lambda_m(x)| \le |\lambda_n(x)|$ on $I$.

Lemma 3.2. *Let* $\tilde{Y}$ *be a formal solution of (1) ($m \in \{1 \ldots l\}$ fixed). There exists a sequence* $\{Y_{m,k;\epsilon}\}_{k,\epsilon}$ *such that for any* $\epsilon \le \epsilon_0$, $Y_{m,k;\epsilon}$ *is a solution of (1) for* $k\epsilon \in I$ *having* $\Gamma_m$ *as an asymptotic series in the sense that there is a sequence of positive constants* $\{C_s\}_s$ *such that*

$$|Y_{m,k;\epsilon}\exp[-\epsilon^{-1}\sum_{t=0}^{s}\Phi_{m,t}(k\epsilon)\epsilon^t] - 1| < C_s\ \epsilon^{s+1}\ \text{for all}\ s \in I\!N \tag{42}$$

Remark 2. *The previous lemma can be restated as follows: for each m=1...l there is a function* $F_m(x,\epsilon) : I \times [0,\epsilon_0] \to \mathbb{C}$, *smooth in each variable, such that*

*$exp(\epsilon^{-1}F_m(k\epsilon,\epsilon))$ is a solution of the recurrence (1) for every $\epsilon$, $k\epsilon \in I$ and, as $\epsilon \to 0$,*

$$F_m(x,\epsilon) \sim \sum_{t=0}^{\infty} \Phi_{m,t}(x)\epsilon^t \tag{43}$$

This remark follows easily from a classical result ( see e.g. [4], page 33 and [5]) stating that for any sequence of numbers there is a smooth function having that sequence for its derivatives at the origin and from the easily proven fact that for each sequence $\{(x_n,a_n)\}_{n\in\mathbb{N}}$ , $x_n \searrow 0$ and $a_n n^k \to 0\ \forall k$ one can construct a $C^\infty$ "interpolation" function f such that $f^{(k)}(0) = 0\ \forall k$ and $f(x_n) = a_n$ for all $n$.

Some of the estimates that we need for proving Lemma 2 become to a certain extent easier by the remark that a global shift in the estimates defining an asymptotic series is unimportant:

REMARK 3.

*Let $F$ be a function such that for a fixed $s_0 \geq 0$ and any $s$ in $\mathbb{N}$*

$$\limsup_{\epsilon\to 0} \epsilon^{-s-1}|F(\epsilon) - \sum_{t=0}^{s+s_0} C_t\epsilon^t| = D_s \tag{44}$$

*Then*

$$\limsup_{\epsilon\to 0} \epsilon^{-s-1}|F(\epsilon) - \sum_{t=0}^{s} C_t\epsilon^t| \leq D_s + |C_{s+1}| \tag{45}$$

Thus, to prove (42) we need only show that for some fixed $s_0$, (we drop the subscript $\epsilon$ to ease the notation)

$$|Y_{m,k}\exp(-\epsilon^{-1}\sum_{t=0}^{s} \Phi_{m,t}(k\epsilon)\epsilon^t) - 1| < C_s\ \epsilon^{s+1-s_0} \tag{46}$$

REMARK 4. *In view of condition (25) it is easy to check that, for small $\epsilon$, the solutions $\exp(\epsilon^{-1}F_m(k\epsilon,\epsilon))$ are linearly independent, thus forming a fundamental set of solutions on $I$.*

Proof of Lemma 3.2 The proof is by induction on the order $l$ of the recurrence.

STEP 1. *For any $l$, in the same conditions as in Lemma 3.1, the estimate (42) holds for m=1 (recall that $\lambda_1(x)$ is the largest in absolute value).* First we show that if a solution is asymptotic to the formal solution corresponding to $\lambda_1$ at the left end of $I$ it remains asymptotic to it throughout $I$. We can assume without loss of generality that the left end of $I$ is at $x = 0$; at this point we choose appropriate initial conditions:

let $\eta_p$, $p = 1\ldots l$ be any functions such that as $\epsilon \to 0$

$$\eta_p(\epsilon) \sim \exp(\epsilon^{-1}\sum_{t=0}^{\infty} \Phi_{1,t}(p\epsilon)\epsilon^t) \tag{47}$$

(see Remark 2) and let

$$\tilde{Y}_{\epsilon;s}(x) = \exp(\epsilon^{-1}\sum_{t=0}^{s} \Phi_{1,t}(x)\epsilon^t) \tag{48}$$

Let also $Y_{1,k}$ be the solution of (1) satisfying the initial conditions $Y_{1,p} = \eta_p(\epsilon)$ , $p = 1 \ldots l$. It is natural to rescale the recurrence relative to its approximate solution: let

$$C_{k;s} = Y_{1,k}/\tilde{Y}_{\epsilon;s}(k\epsilon) \tag{49}$$

Then the recurrence relation for $C_{k;s}$ can be written

$$\sum_{j=0}^{l} \tilde{a}_{j,s}(k\epsilon,\epsilon) C_{k+j;s} = 0 \tag{50}$$

where $\tilde{a}_{j,s}(k\epsilon,\epsilon) = a_j(k\epsilon,\epsilon)\tilde{Y}_{\epsilon;s}((k+j)\epsilon)/\tilde{Y}_{\epsilon;s}(k\epsilon)$. It can be seen that (50) is of the same type as (1) and that, for $\epsilon$ small enough, it satisfies the corresponding assumptions (6), (25) and (26) on $I$.

Also, $\max\{|\tilde{\lambda}_m(x)|; x \in I,\ m = 1\ldots l\} = 1$. Now, (46) means that for some fixed $s_0$,

$$|C_{k;s} - 1| < const_s \epsilon^{s-s_0+1} \tag{51}$$

From the definition of $\tilde{Y}_{\epsilon;s}$ it follows that

$$\sum_{j=0}^{l} \tilde{a}_{j,s}(k\epsilon,\epsilon) = O(\epsilon^s) \tag{52}$$

Rewriting the recursion relation (50) in matrix form $\mathbf{C}_{k+1} = \tilde{A}_k \mathbf{C}_k$, where

$$\tilde{A}_k = \begin{pmatrix} \frac{-\tilde{a}_{l-1}(k\epsilon,\epsilon)}{\tilde{a}_l(k\epsilon,\epsilon)} & \frac{-\tilde{a}_{l-2}(k\epsilon,\epsilon)}{\tilde{a}_l(k\epsilon,\epsilon)} & \ldots & \frac{-\tilde{a}_0(k\epsilon,\epsilon)}{\tilde{a}_l(k\epsilon,\epsilon)} \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & ..1 & 0 \end{pmatrix} \tag{53}$$

and rewriting also (52) as

$$\tilde{A}_k \mathbf{1} = \mathbf{1} + \epsilon^s \mathbf{E}_k \tag{54}$$

where $\mathbf{1}_j = 1$ and $||\mathbf{E}_k|| < const$ uniformly in $k, \epsilon$ we have

$$\mathbf{C}_k = \mathbf{1} + \epsilon^s \sum_{j=1}^{k} \tilde{A}_k \tilde{A}_{k-1} \ldots \tilde{A}_{j+1} \mathbf{E}_k \tag{55}$$

Step 1 is completed by showing (51) (and thus (46)), which follows from the stability lemma below.

LEMMA 3.3. *Let $\tilde{A}_k$ be a family of matrices of the form (53) where $\tilde{a}_j : I\times[0,\epsilon_0] \to \mathbb{C}$ (I is an interval) as in Lemma 3.2.*

*Suppose further that the roots $\tilde{\lambda}_m(k\epsilon,\epsilon)$ of the polynomial*

$$\sum_{j=0}^{l} \tilde{a}_j(k\epsilon,\epsilon)\tilde{\lambda}^j = 0, \qquad m = 1\ldots l \tag{56}$$

*satisfy*

$$\sup_{k\epsilon\in I}\{|\tilde{\lambda}_m(k\epsilon,\epsilon)|\}\le 1+const\ \epsilon,\ \ m=1..l \tag{57}$$

*and that the condition corresponding to (25) is fulfilled on I. Then there is an $\epsilon$-independent constant $C$ such that*

$$||\tilde{A}_k\tilde{A}_{k-1}\dots\tilde{A}_{j+1}||\le 1+C|k-j|\epsilon \tag{58}$$

Proof of Lemma 3. The eigenvalues of the matrix (53) are the $\tilde{\lambda}_m(k\epsilon,\epsilon)$ and the corresponding eigenvectors matrix is $(\tilde{\Lambda}_k)_{i,j} \;=\; \tilde{\lambda}_j(k\epsilon,\epsilon)^{l-i}$. We then write the product on the LHS of (58) as

$$\tilde{\Lambda}_k D_k\tilde{\Lambda}_k^{-1}\tilde{\Lambda}_{k-1}D_{k-1}\dots\tilde{\Lambda}_{j+1}D_{j+1}\tilde{\Lambda}_{j+1}^{-1} \tag{59}$$

where

$$D_p=diag(\{\tilde{\lambda}_m(p\epsilon,\epsilon)\}_{m=1\dots l}) \tag{60}$$

and the proof follows from (57) and the estimate

$$||\tilde{\Lambda}_p^{-1}\tilde{\Lambda}_{p-1}||\le 1+const\ \epsilon \tag{61}$$

which can be checked, for instance, using the following explicit formula, whose elementary proof we omit:

REMARK 5. *Let $X$ and $Y$ be two nonsingular Vandermonde-type matrices $X_{i,j}=x_j^{l-i}$, $Y_{i,j}=y_j^{l-i}$, $i,j=1\dots l$. Then,*

$$(X^{-1}Y)_{i,j}=\prod_{n\ne i}\frac{y_j-x_n}{x_i-x_n} \tag{62}$$

STEP 2. The conclusion of Lemma 3.2 for $l=1$ follows from step 1.

Now we assume that the conclusion of the lemma holds for all recurrences of order less than $l$ and prove it for order $l$, by reduction to the $l-1$ case. In view of the first step, we know that to $|\lambda_1|$ there corresponds a true solution $Y_1$ for which the asymptotic behavior is the formal solution $\tilde{Y}_1$. We shall use this solution to reduce the order of the recurrence by one. Let

$$C_k=y_k/Y_{1,k} \tag{63}$$

The recurrence relation for $C_k$ is then of the form (50) where now

$$\tilde{a}_j(k\epsilon,\epsilon)=a_j(k\epsilon,\epsilon)Y_{1,k+j}/Y_{1,k} \tag{64}$$

and obviously, the asymptotic behavior to all orders is the same as if we had made the rescaling with respect to a formal solution. The point is that now, instead of (52) we have

$$\sum_{j=0}^{l}\tilde{a}_j(k\epsilon,\epsilon)=0 \tag{65}$$

so that the $y = 1$ is an actual solution. To use this fact, let $d_k = C_{k+1} - C_k$. We get,

$$\sum_{s=0}^{l-1} b_s(k\epsilon, \epsilon) d_{k+s} = 0 \tag{66}$$

where $b_s(k\epsilon, \epsilon) = \sum_{j=s+1}^{l} \tilde{a}_j(k\epsilon, \epsilon)$.

The characteristic equation for (66) can be written as

$$\sum_{j=1}^{l} \tilde{a}_j(x, 0) \sum_{s=0}^{j-1} \tilde{\lambda}^s = 0$$

or, for $\lambda \neq 1$, as it easily follows from (65),

$$\sum_{j=0}^{l} \tilde{a}_j(x, 0) \tilde{\lambda}^j = 0 \tag{67}$$

We first check that the new recurrence satisfies the hypothesis of the Lemma. But this is easy since the new coefficients are finite combinations of $a_j$ and in particular $b_0 = \sum_{j=1}^{l} \tilde{a}_j = -a_0$ and $b_{l-1} = a_l$, and in view of (67) the same arguments as in step one apply to see that the characteristic roots have the required properties. Then we want to check that we have the required number of appropriate formal solutions. This is also straightforward because we can derive them from the formal solutions of the original equation. Indeed,

$$d_k = y_{k+1}/Y_{1,k+1} - y_k/Y_{1,k} \tag{68}$$

If we substitute for $y$ a *formal* solution $\tilde{Y}_m$ we obtain the formal expression :

$$\tilde{d}_k = \exp\{\epsilon^{-1} \sum_{t=0}^{\infty} [\Phi_{m,t}((k+1)\epsilon) - \Phi_{1,t}((k+1)\epsilon)]\epsilon^t\}$$

$$- \exp\{\epsilon^{-1} \sum_{t=0}^{\infty} [\Phi_{m,t}(k\epsilon)) - \Phi_{1,t}(k\epsilon)]\epsilon^t\} = \tag{69}$$

$$\exp\left\{\epsilon^{-1}(\Phi_m(x) - \Phi_1(x)) + \text{ series}\right\} [\lambda_m(x)/\lambda_1(x)](1 + \epsilon \times \text{series}]$$

which, since $\lambda_m(x)/\lambda_1(x)$ does not vanish can be written as an exponential of a formal series, the form required by our arguments:

$$\exp(\epsilon^{-1} \sum_{t=0}^{\infty} \Delta_{m,t}(k\epsilon)) \tag{70}$$

and then the expressions (70) for $m = 2 \ldots l$ are formal solutions for the recurrence of order $l - 1$ (66) —because, by construction, (69) are; it follows, by the induction hypothesis that there exist true solutions of (66) of the form

$$d_{m,k} = \exp(\epsilon^{-1} D_m(k\epsilon, \epsilon)) \tag{71}$$

where $D_m(\cdot,\cdot)$ are smooth functions having the asymptotic behavior given by (70).

STEP 3. To complete the proof of Lemma 2 it remains only to check that

$$\exp\left(\epsilon^{-1}F_1(k\epsilon,\epsilon)\right)\sum_{p=0}^{k}\exp(\epsilon^{-1}D_m(p\epsilon,\epsilon)) \tag{72}$$

has the asymptotic behavior needed for the original recurrence, i.e.

$$\exp(\epsilon^{-1}\sum_{t=0}^{\infty}(\Phi_{m,t}(k\epsilon)\epsilon^t)) \tag{73}$$

We let $k_1$ ($k_2$) be the left (right, respectively) end of the interval. Both $k_1$ and $k_2$ might depend on $\epsilon$. By the definition of the $d_{k;m}$ we have

$$C_{m;k} = C_{m;k_1} + \sum_{i=k_1}^{k} d_{m;i}$$

With the choice

$$C_{m;k_1} = -\sum_{i=k_1}^{k_2} d_{m;i}$$

we get the particular solution

$$C_{m;k} = \sum_{i=k}^{k_2} d_{m;i}$$

from which, referring to the definition of the $C_{m;k}$, we get a solution of the $l-1$ recurrence in the form (the choice of the sign will become clear later)

$$Y_{m;k} = -Y_{1;k}\sum_{i=k}^{k_2} d_{m;i}$$

whose the asymptotic behavior is given by the formal solution $\tilde{Y}_m$. Indeed, for any fixed large $s$ we have,

$$Y_{m;k} = -Y_{1;k}\left[\sum_{i=k}^{k_2}\exp\left(\epsilon^{-1}\sum_{t=0}^{s}\Delta_t(i\epsilon)\epsilon^t\right)\left(1+O(\epsilon^{s-1})\right)\right] =$$

$$-Y_{1;k}\left[\sum_{i=k}^{k_2}\left(\exp\left(\epsilon^{-1}\sum_{t=0}^{s}\Phi_{m;t}(i\epsilon+\epsilon)\epsilon^t - \Phi_{1;t}(i\epsilon+\epsilon)\epsilon^t\right) - \right.\right.$$

$$\exp\left(\epsilon^{-1}\sum_{t=0}^{s}\Phi_{m;t}(i\epsilon)\epsilon^t-\Phi_{1;t}(i\epsilon)\epsilon^t\right)\Bigg)\left(1+O(\epsilon^{s-1})\right)\Bigg] =$$

$$Y_{1;k}\left[\exp\left(\epsilon^{-1}\sum_{t=0}^{s}\Phi_{m;t}(k\epsilon)\epsilon^t-\Phi_{1;t}(k\epsilon)\epsilon^t\right)-\exp\left(\epsilon^{-1}\sum_{t=0}^{s}\Phi_{m;t}(k_2)-\Phi_{1;t}(k_2)\epsilon^t\right)\right]+$$

$$(k_2-k)\max_k\left|\exp\left(\epsilon^{-1}\sum_{t=0}^{s}\Delta_t(i\epsilon)\epsilon^t\right)\right|O(\epsilon^{s-1}) \tag{74}$$

Because, by assumption, $\lambda_1$ has the largest modulus, $\Re(\Phi_{m;0}(k\epsilon)-\Phi_{1;0}(k\epsilon))$ is nonincreasing in $k$ in the given region. Therefore (74) equals,

$$Y_{m;k}\left(1+\text{const}\max_k\{|\exp\left(\Phi_{m;1}(k\epsilon)-\Phi_{1;1}(k\epsilon)\right)|\}\,o(\epsilon^{s-2})\right) =$$

$$Y_{m;k}\left(1+o(\epsilon^{s-2})\right) \tag{75}$$

$\Box_{L2}$

The proof of Proposition 2.2 follows essentially the same steps but is, as expected, more involved in the regions of near–breakdown of the asymptotic series. The details are given in the next section.

The proof of Proposition 2.3 is very easy, using an estimate of the form (58) for the matrices corresponding to the original recurrence, estimate which is straightforward to obtain from Remark 5 and the hypothesis of the proposition.

**4. Proof of Proposition 2.2.** We assume at first that the crossing occurs between the two largest characteristic roots and explain at the end of the proof how the general case is reduced to this one.

The layout is as follows. We first study the small region around the crossing point (the interior region) where $\exp(\epsilon^{-1}\sum\Phi_{1,2;t}(x)\epsilon^t)$ fail to be formal solutions (and the series occuring at the exponent cease to be asymptotic series). The new formal solutions are to leading order combinations of Airy functions. Their formal properties (domain of asymptoticity, growth in $x$) are examined. Next we show that there exist true solutions of the recurrence that are asymptotic to them. It is also shown that there exists a particular true solution which is asymptotic, to leading order, to the function $\mathrm{Ai}(x\epsilon^{-2/3})$ and which is important for the matching problem (it gives the exponentially decaying formal solution).

We then show that the formal solutions coming from the exterior region continue to represent correctly the solutions of the recurrence far enough into the interior region (down to $|x|\sim\epsilon^{2/3}$) to allow for matching with the interior ones, which are valid up to $|x|\sim\epsilon^{1/2}$.

**A. The interior region of the crossing interval**

This is the region $|x|\ll\epsilon^{\frac{1}{2}}$; for definiteness we fix $\alpha\in(\frac{1}{2},\frac{2}{3})$ and take it to be

$$D_\alpha=\{k:|k|<\epsilon^{-\alpha}\}$$

or, in terms of $\xi := k\epsilon^{1/3} := k\delta$ which is, for reasons that will become clear later, the natural variable in this region,

$$D_\alpha = \{\xi : |\xi| < \delta^{1-3\alpha}\} \tag{76}$$

The basic steps of the proof of existence of solutions with given asymptotics are the same as for Lemma 3.2. We will first obtain a solution corresponding to (one of the two) largest eigenvalues and with it reduce the problem to a lower order, nondegenerate recurrence.

Because in the variable $\xi$ $D_\alpha$ is unbounded, a slight extension of Lemma 3.2 is needed for the interior region. We now allow the interval $I$ in Lemma 3.2 to be of the form (76) but strengthen the other hypothesis. In order to make the correspondence with Lemma 3.2, note that $\xi$ plays the role of $x$ and $\delta$ is the counterpart of $\epsilon$. We require the same conditions as in Lemma 3.2 and in addition,

**a)** $a_j(\xi,\delta)$ are assumed to have asymptotic series $\sum_t a_{j,t}(\xi)\delta^t$ valid throughout the region $D_\alpha$ which are smooth in the sense that the all their formal derivatives with respect to $\xi$, $\sum_t a_{j,t}^{(m)}(\xi)\delta^t$, exist and $|a_{j,t}^{(m)}(\xi)| < \text{const}\,\delta^{const'\,t}$

**b)** The roots of the characteristic polynomial

$$\sum_{j=0}^{l} a_j(\xi,0)\lambda(\xi)^j = 0 \tag{77}$$

are nondegenerate:

$$\inf_{D_\alpha} |\lambda_m(\xi) - \lambda_n(\xi)| > \text{const} > 0 \;\; (m \neq n) \tag{78}$$

and the polynomial itself is nondegenerate in the sense

$$\inf_{D_\alpha} \left\{ |a_0(k\delta)|, |a_l(k\delta)|, \frac{1}{|a_j(k\delta)|} \right\} > \text{const} > 0 \tag{79}$$

**c)**

$$|\lambda_m((k+1)\delta,\delta) - \lambda_m(k\delta,\delta)| < \frac{\text{const}}{|k|+1}$$

in $D_\alpha$ where $\lambda_m((k+1)\delta,\delta)$ are the roots of the complete polynomial $\sum_{j=0}^{l} a_j(\xi,\delta)\lambda^j$

Lemma 4.1. *Under these assumptions for any formal solution of (1) of the form*

$$S := \exp(\delta^{-1}\Phi_0(\xi) + \sum_{m=0}^{\infty} \Phi_m(\xi)\,\delta^m)$$

*where the exponent is assumed to be a smooth asymptotic series in the sense defined in a), there exists a true solution of the recurrence which is asymptotic to it in $D_a$.*

**Proof**. The proof follows closely the proof of Lemma 3.2. We only emphasize the differences: For the recurrence (50) we have also to verify condition c) :

$$\left|\tilde\lambda_m((k+1)\delta,\delta) - \tilde\lambda_m(k\delta,\delta)\right| = \left|\frac{\lambda_m((k+1)\delta,\delta)}{\lambda_1((k+1)\delta,\delta)} - \frac{\lambda_m(k\delta,\delta)}{\lambda_1(k\delta,\delta)}\right| <$$

$$\text{const } (|k|+1)^{-1}$$

The equivalent of Lemma 3.3 states now that there is an $\delta$-independent constant C such that

$$\|A_k A_{k-1}\dots A_{j+1}\| \le (1 + C|k-j|^{\text{const}}) \tag{80}$$

Indeed, by the Remark 5 a diagonal term of the matrix $T := \Lambda_p^{-1}\Lambda_{p-1}$ is of the form

$$T_{m,m} = \prod_{n\neq m} \frac{\lambda_m((p-1)\delta,\delta) - \lambda_n(p\delta,\delta)}{\lambda_m(p\delta,\delta) - \lambda_n(p\delta,\delta)} =$$

$$= \prod_{n\neq m}\left(1 + \frac{\lambda_m((p-1)\delta,\delta) - \lambda_m(p\delta,\delta)}{\lambda_m(p\delta,\delta) - \lambda_n(p\delta,\delta)}\right) = 1 + O((|k|+1)^{-1}) \tag{81}$$

by c). Similarly, the moduli of the nondiagonal terms are seen to be less than $\text{const}/(|k|+1)$. Therefore $T = \mathbf{I} + R$ where the $\|R\|$ is $O((|k|+1)^{-1})$, hence the inequality (80) follows.

The last part of the proof of Lemma 3.3 applies here without any significant change. $\Box_{L4.1}$

The reduction to the nondegenerate case.

We consider the initial recurrence in the neighborhood of a crossing point, say x=0 where $\lambda_1(0) = \lambda_2(0) = 1$ (the value at zero can be chosen through a trivial global rescaling of the recurrence). It is convenient to consider rescaled variables $\delta = \epsilon^{1/3}$ and $\xi = k\delta$. In these variables, the coefficients $a_j(x,\epsilon) = a_j(\xi\delta^2,\delta^3)$ have smooth asymptotic series in $\delta$ in $D_\alpha$ which are in fact obtained through series expansion in $x$ from (5):

$$a_j(\xi\delta^2,\delta^3) \sim \sum_{s\ge 0} P_{j,s}(\xi)\delta^s \tag{82}$$

where

$$P_{j,0} = a_j(0,0); \quad P_{j,1}(\xi) = 0; \quad P_{j,2}(\xi) = (D_x a_j)(0,0)\xi;$$

and in general $P_{j,s}(\xi)$ are polynomials in $\xi$ of degree at most $s/2$ for $s$ even and $(s-3)/2$ if $s$ is odd. To avoid complicating the notation we write $a_j(\xi,\delta) \equiv a_j(\xi\delta^2,\delta^3)$. We have first to find formal solutions for this new recurrence.

Lemma 4.2. *There exist $l$ linearly independent formal solutions in $D_\alpha$, of the form*

$$\exp\left(\delta^{-1}\sum_{t=0}^{\infty}\Psi_{m,t}(\xi)\delta^t\right) \tag{83}$$

*where $\Psi_{m,t}(\xi)$ are smooth in $\xi$ and satisfy the estimates*

$$|\Psi_{m,t}(\xi)| < const_{m,t} + const'_{m,t}|\xi|^{\frac{t}{2}+1}$$

This means in particular that the domain of formal validity of the power series is then $\xi^{1/2}\delta \ll 1$ i.e., $x \ll 1$. The domain in which it is actually asymptotic to the solution is however much smaller ($x \ll \sqrt{\epsilon}$) as we shall see.

Proof of Lemma 4.2.

The formal solutions corresponding to the nondegenerate roots give rise automatically to acceptable formal solutions in the new variables $\xi, \delta$. Indeed,

$$\exp\left(\epsilon^{-1}\sum_{t=0}^{\infty}\Phi_{m,t}(x)\epsilon^t\right) = \tag{84}$$

$$\exp\left(\delta^{-3}\sum_{t=0}^{\infty}\Phi_{m,t}(\xi\,\delta^2)\delta^{3\,t}\right) =$$

$$\exp\left(\delta^{-3}\Phi_{m,t}(0) + \delta^{-1}\sum_{s,t\geq 0}^{\infty}\Phi_{m,t}^{(s)}(0)\xi^s\delta^{2\,s+3\,t-2}\right)$$

The term in $\delta^{-3}$ is merely a multiplicative constant so it can be dropped and we are left with a formal solution of the form

$$\exp\left(\delta^{-1}\sum_{t=0}^{\infty}\chi_{m,t}(\xi)\delta^t\right) \tag{85}$$

where the $\chi_{m,t}(\xi)$ are in fact polynomials in $\xi$ of degree $\leq t/2+1$.

For $m = 1, 2$ it is more convenient to write first the possible formal series solutions for the equation, in the form:

$$\sum_{t=0}^{\infty}\chi_t(\xi)\delta^t \tag{86}$$

and then show that we can write them in the form (85).

Substituting (86) in the recurrence we get

$$\sum_{j=0}^{l}a_{j,\xi,\delta}\sum_{t=0}^{\infty}\chi_t(\xi+j\delta)\delta^t$$

The term of order $s$ in (86) is gotten by differentiating the auxilliary equation

$$\sum_{j=0}^{l}a_j(\xi,\delta)\chi(\xi+j\delta,\delta) = 0$$

$s$ times with respect to $\delta$. $s$ times with respect to $\delta$. We get (see (82) )

$$\sum_{j=0}^{l}\sum_{t=0}^{s} P_{s-t}(\xi)\,(D_\xi + D_\delta)^t\,\chi\Big|_{delta=0} = 0$$

which after expansion, change of order of summation and use of (82) gives,

$$\sum_{j=0}^{l}\left[\frac{1}{\binom{s}{s-2}}\sum_{\sigma=0}^{s-3}\sum_{t=\sigma}^{s}\left(\binom{s}{t}\binom{t}{\sigma}P_{s-t}(\xi)j^{t-\sigma}D_\xi^{t-\sigma}\chi_\sigma(\xi)\right)+\right.$$

$$\left. D_x a_j(0,0)\xi\,\chi_{s-2} + a_j(0,0)j^2\chi_{s-2}\right] = 0 \tag{87}$$

It follows that $\chi_0$ is obtained as a solution of the homogeneous Airy equation

$$\chi''(\xi) = \Theta^3\xi\,\chi(\xi) \tag{88}$$

where

$$\Theta^3 = \frac{\sum_{j=0}^{l} D_x a_j(0,0)}{\sum_{j=0}^{l} a_j(0,0)\,j^2} \tag{89}$$

and that, given $\chi_0 ... \chi_{s-3}$, we get $\chi_{s-2}$ as a solution of an inhomogeneous Airy equation of the form

$$\chi''(\xi) = \Theta^3\xi\,\chi(\xi) + R(\xi)$$

where $R(\xi)$ is a linear combination of higher derivatives of $\chi_0..\chi_{s-3}$. To avoid cumbersome notations, we shall assume in the following that $\Theta$ is one. We can check that the assumption of genericity ($|\lambda_1(x) - \lambda_2(x)| >$ const $\sqrt{x}$) implies $\sum_{j=0}^{l} D_x a_j(0,0) \neq 0$. We shall assume for definiteness that it is negative.

It follows by an obvious induction that the $\chi_s$ are smooth. Now we show that they satisfy the inequalities stated in the Lemma 4.2.

Remark Consider the inhomogeneous Airy equation $f''(\xi) = \xi\, f(\xi) + R(\xi)$ and assume $R(\xi) \sim \xi^\rho \exp(2A/3\xi^{3/2})$ with $A = \pm 1$ for $x \to \infty$ and $A = \pm i$ at $-\infty$ . Then $f(\xi) \sim \xi^{\rho-1}\exp(2A/3\xi^{3/2})$. This estimate follows immediately from the explicit form of the solution:

$$f(\xi) = \mathrm{Ai}(\xi)\int^\xi R(t)Bi(t)\,dt - Bi(\xi)\int^\xi R(t)Ai(t)\,dt$$

At this point we can show by induction that the solution $\chi_n(\xi)$ grows at most like $\exp(2A/3\xi^{3/2})\xi^{n/2}$. So we assume that this holds for $s \leq n$ and we show that it is true for $n+1$. Using the remark and (87) the induction step is: with $p_n = n/2$

$$\max_{0\leq\sigma\leq n;\sigma\leq t\leq n+3}\left\{\left[\frac{1}{2}(n+1-t)\right] + \frac{(-1)^n+1}{2} + \frac{1}{2}(t-\sigma) + p_n\right\} \leq p_{n+1} + 1$$

which is straightforward.

Finally, we argue that there are two linear independent formal solutions of this type that can be written in the form (85) which is convenient for our approach. For this we have to choose $\chi_0$, which is a solution of the homogeneous Airy equation such that it does not vanish in $D_\alpha$. Since the Wronskian of the couple $\mathrm{Ai}(\xi)$, $\mathrm{Bi}(\xi)$ is a nonzero constant, any combination with real nonzero constants of the form $C_1\mathrm{Ai}(\xi)+iC_2\mathrm{Bi}(\xi)$ is an everywhere nonzero solution (and the derivative is also nonzero). We can choose two linear independent solutions in this way, say the ones for which $\chi_0 = \mathrm{Ai}(\xi)\pm i\mathrm{Bi}(\xi)$. That they are formally linearly independent with respect to the solutions gotten from (84) follows easily, for instance from the fact that they correspond to different roots of the characteristic polynomial.

To show that there is an actual solution for each formal solution in this region we first single out a true solution corresponding to the dominant characteristic root and then use it to reduce the problem to a regular one. Then we show that they give the expected asymptotic behavior for the solutions of the original equation.

The ideas are similar to those used in the regular case with the exception that extra care is needed along the degenerate directions.

Choose

$$\chi_0(\xi) = \mathrm{Ai}(\xi) + i\,Bi(\xi)$$

and consider the nonzero formal solution that, has the leading order $\chi_0$. We proceed as in Step 1, Section 3 to construct a rescaled recurrence with respect to the truncation of our formal solution. Exactly the same argument as there shows that the new coefficients $\tilde{a}$ have smooth asymptotic series.

What is new here is that we must provide for the estimate of the type c) to which end we examine the complete characteristic equation $P(\xi,\delta;\tilde{\lambda}) = 0$. P is a polynomial in $\tilde{\lambda}$ (actually it is, to leading order, a polynomial with constant coefficients) $C^\infty$ in $\xi$ and $\delta$. P(0,0,$\tilde{\lambda}$) has a double root $\tilde{\lambda} = 1$ but by assumption the second derivative does not vanish so that we can obtain the roots of the polynomial perturbatively. After series expansion, we obtain:

$$\sum_{j=0}^{l} \left\{ \left[a_j(0,0)\left(1+\delta\, C_j + \delta^2 E_j + O(\delta^3)\right) + \xi D_x a_j\, \delta^2 + O(\delta^3)\right]\right.$$

$$\left.\left(1 + j\chi_1(\xi)\delta + \left(j\chi_2(\xi)\delta^2 + j(j-1)/2\chi_1^2\right) + O(\delta^3)\right)\right\} = 0 \tag{90}$$

where we have taken $\tilde{\lambda} \sim 1 + \chi_1(\xi)\delta + \chi_2(\xi)\delta^2 + O(\delta^3)$ and

$$C_j = \frac{1}{\chi_0}\left(j\chi_0' + \chi_1\right)$$

$$E_j = \frac{1}{\chi_0^2}\left(j^2\chi_0''\chi_0 + j\chi_1'\chi_0 - j\chi_1\chi_0' - \chi_1^2\right)$$

Using the relations (82) we get two solutions for $\chi_1$: $\chi_1 = 0$ —actually, as expected, we get a root $\tilde{\lambda}_1 = 1 + O(\delta^s)$— and $\chi_1 = 1 - 2\chi_0'\chi_0^{-1}\delta + O(\delta^2)$. We see that in the first order in $\delta$ there is no root— crossing, which is not a surprize since a generic perturbation tends separate coalescing roots. The asymptotic series are uniformly valid in our domain $D_\alpha$. Now we show that

$$\left|\frac{\lambda_2((k+1)\epsilon)-\lambda_2(k\epsilon)}{\lambda_1(k\epsilon)-\lambda_2(k\epsilon)}\right| < \frac{\text{const}}{|k|+1} \tag{91}$$

(this explains the condition c) at the begining of this section).

We have

1)

$$\left|(\log\,\chi_0(\xi))'\right| > \text{const}\sqrt{\xi}+\text{const}'$$

This is obvious since it holds asymptotically and the function does not vanish. Hence

$$\left|\tilde{\lambda}_2(\delta\,k)) - \tilde{\lambda}_1(\delta\,k)\right| > \text{const}\sqrt{\delta|k|+1}$$

2)

$$\left|(\log\,h_0(\xi))''\right| < \text{const}/\sqrt{\xi+1}+\text{const}'$$

Using the asymptotic series for the $\tilde{\lambda}_{1,2}$ and the estimates 1.) and 2.) we see that

$$\left|\frac{\tilde{\lambda}_2(\delta\,(k+1)) - \tilde{\lambda}_2(\delta\,k)}{\tilde{\lambda}_2(\delta\,k) - \tilde{\lambda}_1(\delta\,k)}\right| < \frac{(\sqrt{\xi+1}+\text{const}')\delta^2}{(\text{const}\sqrt{\xi}+\text{const}')\delta} < \frac{\text{const}}{|k|+1}$$

The similar estimates for the other roots are better but this of course does not improve the overall rate of convergence. $\tilde{\lambda}_1$ =1+O($\delta^s$) and can be obviously made less than $1+\text{const}\,\delta^2$ in $D_\alpha$ so that also $|\tilde{\lambda}_1((k+1)\delta) - \tilde{\lambda}_1(k\delta)| < 1+\text{const}'\,\delta^2$ which is enough for our purposes. For $m \neq 1,2$ we can for instance use the fact that the derivative of the polynomial at these points does not vanish and settle for a crude bound $|\tilde{\lambda}_m(\delta\,(k+1)) - \tilde{\lambda}_m(\delta\,k)| < \text{const}\,\delta^2$ which can be obtained immediately from (90).

Now, to see that there is a true solution of the recurrence which is asymptotic to our formal series starting with Ai+iBi, we only have to repeat the same arguments as in the regular case.

The next step is to use this particular solution to lower the order of the recurrence. We mimic the construction done in the proof of Lemma 3.2 to get a lower order recurrence in the variable $d_k$

$$\sum_{s=0}^{l-1} b_s(k\delta,\delta)d_{k+s} = 0$$

(see (66)) and want to check that this new recurrence satisfies the hypothesis of of Lemma 4.1. To leading order, the characteristic polynomial of the above equation has no double roots (it now has only one root equal to 1).

As in the regular case, $b_0(\xi,0) = -\tilde{a}(\xi,0) = -a_0(0,0)$; $b_{l-1}(\xi,0) = -\tilde{a}_l(\xi,0) = -a_l(0,0)$.

Noting that the coefficients of the recurrence (66) have asymptotic series valid throughout the domain (as finite sums of terms of the series of $\alpha_j(\xi,\delta)Y_{1,k+j}/Y_{1,k}$), the boundedness of the coefficients is also trivial.

As in the proof of Lemma 3.2, the polynomial in $b_j$ has the same roots as the polynomial in $\tilde{a}_j$ (except for the eliminated one) for which we have already obtained the estimates of type c).

Now we have a regular problem for which we know that to each formal solution there is a genuine solution asymptotic to it.

It remains to check that we can recover the asymptotic behavior of the solutions of the original recurrence from those of the reduced one. For the solutions corresponding to the characteristic roots that are less than one, exactly the same proof as in Step 3 of Lemma 3.2 works. For any formal solution of our original equation that corresponds to the largest eigenvalue and which does not vanish, the proof is the one given in Lemma 4.1.

In the crossing region however there might be a special interest in finding a particular solution which is not of exponential type and which is small for large $\xi$ ( the Airy–like solution). To this end, a slightly different argument is necessary. We can obtain a formal solution of the reduced equation which is, to leading order,

$$\frac{\mathrm{Bi}(\xi+\delta)\mathrm{Ai}(\xi)-\mathrm{Bi}(\xi)\mathrm{Ai}(\xi+\delta)}{(\mathrm{Ai}(\xi)-i\mathrm{Bi}(\xi))(\mathrm{Ai}(\xi+\delta)-i\mathrm{Bi}(\xi+\delta))} \tag{92}$$

(suggested by computing (68) for two formal solutions of the original equation, corresponding to Ai±iBi)). The asymptotic behavior of (92) for large $\xi$ is

$$\frac{\delta}{(\mathrm{Ai}(\xi)-i\mathrm{Bi}(\xi))^2}(1+\text{power series}) \quad (1 \ll |\xi| \ll \delta^{-1/2}) \tag{93}$$

Writing the asymptotic representation of Ai±iBi in the form

$$\sim \xi^{-1/4}\exp(2/3\xi^{3/2})(1+series)$$

is sufficient to see this. It is important to note that there are no powers of $\xi$ multiplying the asymptotics (93); its leading order does not vanish and (93) can be written as an exponential of a formal series, the form required by our arguments.

We now apply the construction in Step 3, Section 3 to recover the solutions of the initial recurrence.

Using the asymptotic behavior of the Airy functions for large argument, we get for the reconstructed solution the representation

$$Y \sim \xi^{(-1/4)}\exp(\frac{2}{3}\xi^{3/2})\sum_{k=j}^{\delta^{-\alpha}}(k\delta)^{1/2}\exp(-\frac{4}{3}(k\delta)^{\frac{3}{2}})(1+power\ series)$$

for which the Euler–Maclaurin summation formula gives the asymptotic representation,

$$\xi^{-1/4}\exp\left(-\frac{2}{3}\xi^{3/2}\right)$$

for large positive $\xi$

In conclusion there is a true solution of the recurrence which behaves like the Airy function for positive $k$ (also for negative $k$ when this is properly interpreted) and our argument shows what initial conditions have to be chosen in order to obtain it. For negative $k$ we see that in fact all the solutions corresponding to the largest two eigenvalues are comparable.

**B.** The exterior region.

Now we want to show that the solutions coming from the exterior region remain asymptotic to the true solutions as long as $|x| \gg \epsilon^{2/3}$

The problem that arises here is that the characteristic polynomial has a virtually degenerate root for small $x$ and this leads to a lesser smoothness of the asymptotic series and ultimately to its collapse at $|x| \sim \epsilon^{2/3}$. Let $\beta$ be as in Proposition 2.2 and define the exterior region by

$$E_\beta = \{x : \ |x| > \epsilon^\beta\}$$

In what follows we make the following conventions. We write

$$g(\epsilon) = O(\epsilon^\infty)$$

if $\lim_{\epsilon\to 0}\epsilon^{-k}\, g(\epsilon) = 0$ for all $k$; also $\mathcal{F}(x^\gamma)$ will denote a generic function such that it together with its derivatives of arbitrary order $s$ satisfy the estimates

$$\left|\mathcal{F}(x^\gamma)^{(s)}(x)\right| < A_s + B_s x^{\gamma-s} \tag{94}$$

with the convention that $|\mathcal{F}(x^0)| \equiv |\mathcal{F}(\ln(x))| < A + B|\ln|x||$.

The first step is to study the asymptotic properties of the formal solutions i.e of the (possibly divergent) expressions for which

$$\sum_{j=0}^{l} a_j(x;\epsilon)\exp\left(\sum_{t=0}^{\infty}\Phi_t(x+j\epsilon)\epsilon^t\right) \equiv 0 \tag{95}$$

We show by induction that $\Phi_t$ and their derivatives behave like $x^{\frac{3}{2}(1-t)}$ and its derivatives. We place ourselves in the assumption of genericity of the crossing which means in particular that $\frac{\partial P(\lambda;x)}{\partial\lambda} > const\sqrt{|x|}$.

LEMMA 4.3. *i) If in (95) the asymptotic series for the $a_j$ are of the form $\sum_{k=0}^{\infty}\mathcal{F}_k(x^{1-\frac{3}{2}k})\epsilon^k$ then, in the formal solution (95) we have $\Phi_t = \mathcal{F}(x^{\frac{3}{2}(1-t)})$.*

*ii) The same conclusion is true if $\frac{\partial P(\lambda;x)}{\partial\lambda} > const > 0$ uniformly in $x$ and $a_j \sim \sum_{k=0}^{\infty}\mathcal{F}_k(x^{\frac{1}{2}-\frac{3}{2}k})\epsilon^k$*

Note that the coefficients $a_j$ of our initial recurrence are smoother then it is assumed in i) but this smoothness does not withstand a rescaling as done for (65).

The proof of the Lemma is by induction on $t$.

i) It is easy to see from the eikonal equation that $\Phi_0=\mathcal{F}(x^{3/2})$. Assume that the conclusions of the lemma hold for all $t' < t$. After a formal series expansion of the exponent of (95) one gets

$$\sum_{j=0}^{l} a_j(x;\epsilon)\exp\left[\epsilon^t\left(j\Phi_t^{(s)}+\sum_{\substack{s\geq 1\\ t'+s=t+1}}\frac{j^s}{s!}\Phi_{t'}^{(s)}\right)+\sum_{k=0}^{t}\epsilon^{k-1}\sum_{\substack{s\geq 1\\ t'+s\leq t+1}}\frac{j^s}{s!}\Phi_{t'}^{(s)}+O(\epsilon^{t+1})\right]=0 \tag{96}$$

which, using the induction hypothesis, can be rewritten as

$$\sum_{j=0}^{l}\left(\sum\mathcal{F}_{k;j}(x^{1-\frac{3}{2}k})\epsilon^k\right)\exp\left(j\Phi_0'+\epsilon^t\left(j\Phi_t'+\mathcal{F}(x^{1-\frac{3}{2}t})\right)+\right.$$

$$\left.\sum_{k=1}^{t-1}\epsilon^k\mathcal{F}_k(x^{\frac{1-3k}{2}})+O(\epsilon^{t+1})\right)=0 \tag{97}$$

After expanding in powers of $\epsilon$ and collecting the term in $\epsilon^t$ we obtain the equation for $\Phi_t$ in the form:

$$\sum_{j=0}^{l}\left(ja_{j;0}(x)\epsilon^{\Phi_0'(x)}\,\Phi_t'(x)+\mathcal{F}(x^{1-\frac{3t}{2}})\right)=0 \tag{98}$$

or

$$\Phi_t'(x)=\frac{\mathcal{F}(x^{1-\frac{3t}{2}})}{\frac{\partial P(\lambda;x)}{\partial\lambda}} \tag{99}$$

where the derivative of the polynomial is evaluated at $\epsilon=0$, $\lambda=\exp(\Phi_0')$ thus proving (i).

For (ii) the same proof works, replacing everywhere $\mathcal{F}(x^{1-\frac{3t}{2}})$ with $\mathcal{F}(x^{\frac{1}{2}-\frac{3t}{2}})$.

**B1**. Rescaling first the recurrence with respect to the approximate solution we show that there is a genuine solution corresponding to the maximal eigenvalue. Let $\Psi_m$ be any function such that $\Psi_m(x;\epsilon)\sim\sum_0^\infty\Phi_{m;t}(x)\epsilon^t$ and take

$$\tilde{Y}_m:=\exp(\epsilon^{-1}\,\Psi_m(x;\epsilon)) \tag{100}$$

The existence of a solution corresponding to the asymptotics $\exp(\epsilon^{-1}\,\Psi_1(x;\epsilon))$ is again equivalent to a solution which is 1 to all orders in $\epsilon$ of the recurrence

$$\sum_{j=0}^{l}\tilde{a}_{j,s}(k\epsilon,\epsilon)C_{k+j}=0 \tag{101}$$

where $\tilde{a}_{j,s}(k\epsilon,\epsilon)=a_j(k\epsilon,\epsilon)\tilde{Y}_1((k+j)\epsilon)/\tilde{Y}_1(k\epsilon)$. The formal solutions of the equation (101) are $\tilde{Y}_m/\tilde{Y}_1$. We need the roots of the new characteristic polynomial

$$\tilde{P}(\tilde{\lambda}):=\sum_{j=0}^{l}\tilde{a}_{j,s}(k\epsilon,\epsilon)\tilde{\lambda}^j=0 \tag{102}$$

It is easy to see that the polynomial (102) has a root which is 1 to all orders in $\epsilon$. Let now $G(x;\epsilon)$ be one of the differences $G_m(x;\epsilon) = \Psi_m(x;\epsilon) - \Psi_1(x;\epsilon)$. We have,

$$\sum_{j=0}^{l} \tilde{a}_{j,s}(k\epsilon,\epsilon)\exp(\epsilon^{-1}G(x+j\epsilon;\epsilon)) = o(\epsilon^t)$$

for all $t$ and so, after series expansion

$$\sum_{j=0}^{l} \tilde{a}_{j,s}(k\epsilon,\epsilon)\lambda_0^j(1+\epsilon H_j(x;\epsilon)) = o(\epsilon^\infty) \tag{103}$$

where $H_j(x;\epsilon)$ are some smooth functions of $x,\epsilon$ and $\lambda_0 := \exp(G_x(x;\epsilon))$. Using Lemma 4.3 and the genericity assumptions it is not difficult to see that

$$|H_j(x;\epsilon)| < \text{const}\,|x|^{-1/2} \tag{104}$$

If we look for solutions of the characteristic polynomial (102) in the form $\lambda_0+\gamma$ we get

$$0 = \tilde{P}(\tilde{\lambda}) = \sum_{j=0}^{l} P^{(j)}(\lambda_0)\gamma^j$$

(where the derivatives are taken with respect to $\lambda$). Using (103) we obtain $\gamma$ as the unique small solution of the equation

$$\gamma = \frac{\epsilon\sum_{j=0}^{l}\tilde{a}_j(x;\epsilon)\lambda_0^j H_j(x;\epsilon)}{\sum_{j=1}^{l}P^{(j)}(\lambda_0)\gamma^{j-1}} \tag{105}$$

which is a contraction for small enough $\epsilon$ (and small $\gamma$ ) in the region $|x|>\epsilon^\beta$ as it is easy to check. We then obtain from (104) and (105)

$$|\gamma| < \text{const}\frac{\epsilon}{|x|} \tag{106}$$

again valid for $|x|>\epsilon^\beta$.

We shall also need estimates for $\gamma(x+\epsilon)-\gamma(x)$. Differentiating (105) with respect to $x$ and using Lemma 4.3

$$|\gamma(x+\epsilon)-\gamma(x)| < \text{const}\,\epsilon\,|x|^{-2} \tag{107}$$

Now we proceed as in the regular case in rewriting the recurrence in matrix form and evaluating the terms in the product (59). In the matrices $T := \tilde{\Lambda}_k^{-1}\tilde{\Lambda}_{k-1}$ the off–diagonal elements are estimated by $T_{mn} < \text{const}\,\epsilon|x|^{-1}$ and for the diagonal elements we have $T_{mm} = 1 + O(\epsilon/x)$. Indeed,

$$T_{mm} = \prod_{n \neq m} \left( 1 + \frac{\lambda_m((p-1)\epsilon, \epsilon) - \lambda_n(p\epsilon, \epsilon)}{\lambda_m(p\epsilon, \epsilon) - \lambda_n(p\epsilon, \epsilon)} \right)$$

Since by the assumption of genericity the roots of the polynomial are separated by at least const$\sqrt{(|x|)}$ each term in the product above can be estimated by

$$1 + \text{const}\,\epsilon(\lambda_0'(x) + \gamma'(x))|x|^{-1/2} + O\left(\epsilon(\lambda_0'(x) + \gamma'(x))|x|^{-1/2}\right)$$

$$< 1 + \text{const}\left(\epsilon|x|^{-1} + \epsilon^2|x|^{-5/2}\right) < 1 + \text{const}\,\epsilon|x|^{-1}$$

in our region $E_\beta$. The nondiagonal terms are estimated in a very similar way.

We derive the estimate $\|T(x) - \mathbf{I}\| < K\epsilon/x$ for some constant $K$. Assume for definiteness that we are on the left of the crossing point. We get,

$$\left\| \prod_{k;-k\epsilon > \epsilon^\beta} T(k\epsilon) \right\| < \text{const}\,\epsilon^{-K/3}$$

Finally we have to control the product of the norms of the diagonal matrices $D_k$. Since they all have one eigenvalue equal to 1 to all orders in $\epsilon$ and for $i > 2$ $|\tilde{\lambda}_i(x;\epsilon)| < 1 - \text{const}$ the only nontrivial contribution comes from $\tilde{\lambda}_2$ and this only if $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ have the same modulus to leading order in $\epsilon$. Referring to the decomposition $\tilde{\lambda} = \lambda_0 + \gamma$ we have in this case, using Euler-Maclaurin summation formula,

$$\prod_{k;-k\epsilon > \epsilon^\beta} |\lambda_0(k\epsilon)| = \exp\left( \epsilon^{-1} \sum_k \Re((\Psi_2)_x(k\epsilon;\epsilon) - (\Psi_1)_x(k\epsilon;\epsilon)) \right) \sim$$

$$\exp\left\{ \Re\left( \Phi_{2;1}(-\epsilon^{2/3}) - \Phi_{1;1}(-\epsilon^{2/3}) \right) \right\} < \epsilon^{-const}$$

Also,

$$\prod_{k;-k\epsilon > \epsilon^\beta} \left| 1 + \frac{\gamma(k\epsilon)}{\lambda_0(k\epsilon)} \right| < \prod_{k;-k\epsilon > \epsilon^\beta} \left| 1 + \frac{\text{const}}{|k|} \right| < \epsilon^{-const}$$

so that also $\prod \tilde{\lambda}$ is less than $\epsilon^{-const}$. At this point the same arguments as in the regular case show that there is a true solution behaving asymptotically as the formal solution corresponding to the largest eigenvalue.

**B2**

Let $Y_1$ be a solution of the recurrence relation such that $Y_1 \sim \exp\left(\epsilon^{-1} \sum \Phi_{1;t}\epsilon^t\right)$ in $E_\beta$. We now follow the same steps that led to equations (65) and (66).

It is a matter of straightforward induction to derive from Lemma 4.3 that the coefficients $\tilde{a}_j$ have the behavior

$$\tilde{a}_j(x;\epsilon) \sim a_{j;0}(x)e^{j\ \Phi_0'(x)} + \sum_{k \geq 1} \mathcal{F}_{j;k}(x^{\frac{1}{2} - \frac{3}{2}k})\epsilon^k$$

and then clearly

$$b_j(x;\epsilon) \sim \sum_{k\ge 0} \mathcal{F}_{j;k}(x^{\frac{1}{2}-\frac{3}{2}k})\epsilon^k \tag{108}$$

It is also easy to check that the recurrence (66) is now nondegenerate in the sense that:

$$\inf_{E_\beta} \left\{ |b_0(k\epsilon)|, |b_{l-1}(k\epsilon)|, \frac{1}{|b_j(k\epsilon)|} \right\} > \text{const} > 0 \tag{109}$$

and the characteristic polynomial of the new recurrence does not have coalescing roots (the root $\lambda = 1$ of (65) has been eliminated in the reduction):

$$\inf_{E_\beta}\{ |\, |\lambda_m(x)| - |\lambda_n(x)|\, | \} > \text{const} > 0 \quad (\text{for } m \ne n) \tag{110}$$

We are now left with a problem of the following type. Taking a recurrence of the form

$$\sum_{j=0}^{l} a_j(x;\epsilon) y_{k+j} = 0 \tag{111}$$

under the following conditions:

$$\tilde{a}_j(x;\epsilon) \sim \sum_{k\ge 0} \mathcal{F}_{j;k}(x^{\frac{1}{2}-\frac{3}{2}k})\epsilon^k \tag{112}$$

$$\inf_{E_\beta} \left\{ |a_0(k\epsilon)|, |a_{l-1}(k\epsilon)|, \frac{1}{|a_j(k\epsilon)|} \right\} > \text{const} > 0 \tag{113}$$

$$\inf_{E_\beta}\{ |\, |\lambda_m(x)| - |\lambda_n(x)|\, | > \text{const} > 0 \ (m \ne n)\} \tag{114}$$

where $\lambda_m(x)$ are the roots of the polynomial

$$P(\lambda) := \sum_{j=0}^{l} a_{j,0}(k\epsilon)\lambda^j = 0 \tag{115}$$

we want to show that

Lemma 4.4. *Given a formal solution to (111):*

$$\exp\left( \epsilon^{-1} \sum_{t=0}^{\infty} \epsilon^t \Phi_t(x) \right)$$

*where $\Phi_t(x) = \mathcal{F}_t(x^{\frac{3}{2}(1-t)})$, there is a solution asymptotic to it for $0 < x \in E_\beta$ (and correspondingly one when $x$ negative).*

Proof: induction on $l$.

a) We show that we can find a solution corresponding to the root the has the largest modulus (this will simultaneously prove the lemma for $l = 1$). All the arguments in B1 above apply here. Actually, now we could get some better estimates since we do not have small denominators in (106), (107) and in the estimates of the matrices but this would not affect the final result.

b) We assume that the conclusion is true for all recurrences of order less than $l-1$ and show it holds for recurrences of order $l$. By the arguments above, there is a true solution asymptotic to the formal solution defined by the maximum eigenvalue. Using it to reduce the order of the recurrence we obtain an order–$l-1$ scheme, which satisfies the conditions of Lemma 4.4 as it is easy to check and for which we thus know the asymptotic behavior of the solutions. It remains to verify that they can be used to produce solutions of the higher–order recurrence with the stated asymptotic behavior. For definiteness we study the subregion $x < -\epsilon^{\beta}$. All the arguments in Step 3, Section 3 apply if we take $k_1$ to be $-\epsilon^{\beta}$. The only change is that in (75) $\Phi_{m;1}(x)$ are not uniformly bounded. Instead, using Lemma 4.3 we get $|\Phi_{j;1}| = \mathcal{F}(\ln(|x|)) < K|\ln\epsilon|$ for some fixed constant $K$ so the RHS of (75) changes to $Y_{m;k}\left(1 + O(\epsilon^{s-2-K})\right)$

The conclusion is that after the first reduction we end up with a recurrence that is nondegenerate in the sense of Lemma 4.4 and for which we can control the small–$\epsilon$ behavior of the solutions. Now, the reconstruction of the solutions of the original recurrence from the solutions of the reduced one amounts to merely repeating without any significant change the construction and estimates in part b) above. At this point in the proof it is clear that if the crossing roots are not the largest, on can reduce the order of the recurrence to the actual level at which the roots cross, and then apply the arguments above.

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