

# BOREL SUMMATION OF ADIABATIC INVARIANTS

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ABSTRACT. Borel summation techniques are developed to obtain exact invariants from formal adiabatic invariants (given as divergent series in a small parameter) for a class of differential equations, under assumptions of analyticity of the coefficients; the method relies on the study of associated partial differential equations in the complex plane. The type and location of the singularities of these associated functions, important in determining exponentially small corrections to formal invariants are also briefly discussed.

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## 1. INTRODUCTION AND MAIN RESULTS

For many ordinary differential equations depending on a small parameter (say  $\epsilon$ ) it is possible to construct adiabatic invariants: these are generically divergent expansions in  $\epsilon$ , formally constant with respect to the dynamics. Under relatively mild assumptions, locally there exist actual functions which are invariant to all orders in  $\epsilon$  and, under further conditions, even within exponentially small errors (see for instance [1] and also the literature cited there).

In the present paper we show that under suitable analyticity conditions, adiabatic invariants are Borel summable and their Borel sums are *exact invariants* in the regions of regularity.

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Borel summation techniques have been substantially developed by Écalle [2, 3, 4] and used to find exact solutions starting from formal series or more general expansions in the independent variable, in the context of differential equations [5, 6, 7], difference equations [8, 9] and PDEs [10, 11].

Showing Borel summability of expansions in an external parameter presents new difficulties; in the case of ODEs, this requires the study of associated PDEs. The technique that we use applies to a wide class of differential systems but for the sake of clarity we prefer to focus on relatively simple equations and discuss later how the results and methods extend. Our prototypical equation is

$$(1.1) \quad \psi'' + (\epsilon^{-2} - V(x, \epsilon))\psi = 0 \quad (\epsilon \rightarrow 0),$$

a special case of which is the *one dimensional Schrödinger equation* in the large energy limit.

A number of different equations in which formal invariants arise can be easily brought precisely to the form (1.1).

The *parametrically perturbed pendulum*

$$(1.2) \quad \ddot{x} + \omega(t\epsilon)^2 x = 0. \quad (\epsilon \rightarrow 0)$$

after the substitution  $x(t) = \omega^{-1/2}(\epsilon t) f(\int_a^{\epsilon t} \omega(s) ds)$  becomes

$$f'' + \left( \epsilon^{-2} + \frac{3}{4} \frac{\omega'^2}{\omega^4} - \frac{\omega''}{2\omega^3} \right) f = 0.$$

*Mathieu's equation*

$$\ddot{h} - \epsilon^{-2}(\cos^2(x) + b) = h \quad (\epsilon \rightarrow 0)$$

can be transformed to

$$\epsilon^2 f'' + \left( \frac{s+b}{4s(s-1)} + \frac{3-4s+4s^2}{16s^2(s-1)^2} \epsilon^2 \right) f = 0$$

by taking  $h = \sqrt{\sin t \cos t} f(\cos^2 t)$ .

However, as it will become transparent, the equations for which the methods in this paper apply can be higher order and/or contain nonlinear terms.

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We let

$$(1.3) \quad \lambda = \epsilon^{-1}.$$

Eq. (1.1) admits the formal solutions

$$A_+(\lambda) e^{i\lambda x} \sum_{k=0}^{\infty} g_{k,+}(x) \lambda^{-k} + A_-(\lambda) e^{-i\lambda x} \sum_{k=0}^{\infty} g_{k,-}(x) \lambda^{-k}$$

with the first two terms given by

$$g_{0,\pm} = 1, g_{1\pm} = \mp \frac{1}{2} i \int V(s) ds$$

We consider (1.1) on a bounded open interval  $I \subset \mathbb{R}$  with initial condition prescribed at some  $x_0 \in I$ . Without loss of generality we may take  $I = (-1, 1)$  and  $x_0 = 0$ .

We also assume that

$$\psi(0) = f(\lambda),$$

where  $f$  is the sum of a convergent or Borel-summable series.

Throughout this paper, Borel summation is understood in the following way :

**Definition 1.** A Borel-summable series  $\tilde{y} := \sum_{k=K}^{\infty} y_k \lambda^{-k}$ ,  $K \in \mathbb{Z}$  is a formal power series with the following properties

- (i) the truncated Borel transform  $Y = \mathcal{B}\tilde{y} := \sum_{k>0} \frac{y_k}{(k-1)!} t^{k-1}$  of  $\tilde{y}$  has a nonzero radius of convergence,
- (ii)  $Y$  can be analytically continued along  $[0, +\infty)$  and
- (iii) the analytic continuation  $Y$  grows at most exponentially along  $[0, +\infty)$  and is therefore Laplace transformable along  $[0, +\infty)$ .

The Borel sum  $y$  of  $\tilde{y}$  is then given by

$$(1.4) \quad y = \mathcal{L}\mathcal{B}\tilde{y} := \sum_{k=K}^0 y_k \lambda^{-k} + \mathcal{L}Y,$$

where the sum is understood to be zero if  $K > 0$  and  $\mathcal{L}$  denotes the usual Laplace transform.

Our main results are the following :

**Theorem 2.** Assume that  $V$  is analytic in  $\epsilon$  and  $x$ :

$$(1.5) \quad V(x, \epsilon) = \sum_{k=0}^{\infty} V_k(x) \epsilon^k$$

with  $V_k$  analytic in the strip  $\mathcal{S} = \{-1 < \Re(x) < 1\}$ , real valued on the real line and, for some  $B, K, \delta > 0$ , satisfying

$$(1.6) \quad \sup_{k \in \mathbb{N}, x \in \mathcal{S}} (1 + |x|^{1+\delta}) B^{-k} |V_k(x)| = K < \infty.$$

Then given  $\lambda = \epsilon^{-1} > 0$  large enough, the general solution of (1.1) in  $(-1, 1)$  can be written in the form

$$\psi = C_1 e^{i\lambda x} \phi_+(x; \lambda) + C_2 e^{-i\lambda x} \phi_-(x; \lambda)$$

where  $\phi_+$  and  $\phi_-$  are conjugate expressions of each other and are the Borel sums of their asymptotic power series  $\sum a_k(x) \lambda^{-k}$ . The initial condition is taken to be

$$(1.7) \quad \phi(0) = f(\lambda) = 1/\lambda.$$

Furthermore,  $\phi_+$  (and  $\phi_-$ ) are uniquely determined in the following sense : if for some neighborhood  $\mathcal{V}$  of 0 and some  $\nu > 0$ ,

$$\phi(x; \lambda) = \mathcal{L}(\chi(x, \cdot)),$$

where  $\chi(\cdot, t) \in C^2(\mathcal{V})$  for  $t \in (0, \infty)$  and  $\chi(x, \cdot) \in L^1_{\nu} := L^1(\mathbb{R}^+, e^{-\nu t} dt)$  for  $x \in (-1, 1)$ , and if  $e^{i\lambda x} \phi$  solves (1.1), (1.7) on some neighborhood of 0 then

$$\phi = \phi_+.$$

**Theorem 3.** Consider a  $V$  as in Theorem 2 and take  $\psi$  to be a solution of (1.1), (1.7) in a neighborhood of 0, which can therefore be written as

$$\psi = C_1 e^{i\lambda x} \phi_+(x; \lambda) + C_2 e^{-i\lambda x} \phi_-(x; \lambda).$$

Then, in the region where the assumptions of Theorem 2 are satisfied,  $C(x; \lambda; \psi, \psi') := C_1 C_2$  is an **exact invariant** of (1.1) and is the Borel sum of an **adiabatic invariant**

$$(1.8) \quad \tilde{C} = \sum c_k(x; \psi, \psi') \lambda^{-k}, \quad i.e.$$

$$C(x; \lambda; \psi, \psi') = \mathcal{L}B\tilde{C}.$$

For example, in (1.2), in the regions where the analyticity assumptions are fulfilled there exists an *actual* invariant of the form  $C(t; \epsilon; x, \dot{x})$ .  $C$  is analytic in a sector in  $\epsilon > 0$ , *analyzable* in the sense of Écalle [2] in the given sector at  $\epsilon = 0$  (in this case, it simply means that the Taylor series of the function is Borel summable to the function itself), and it is straightforward to check that to leading order it assumes the familiar form

$$C \sim E/\omega; \quad E = \dot{x}^2 + \omega^2 x^2$$

**Remark 1.** All of our results still hold if instead of (1.6), the coefficients  $V_k$  satisfy the following weaker growth assumption at infinity:

$$|V_k(x)| \leq B^k g(|x|) \quad \text{for some } B > 0 \text{ and } g \in L^1(\mathbb{R}).$$

## 2. PROOF OF THEOREM 2

The core of the proof is contained in the case where  $V$  is independent of  $\epsilon$  and we start with this case. In the following  $\mathcal{C}$  denotes any constant the value of which is not significant to the analysis.

**2.1. Formal derivation of a fixed-point equation.** First, we (formally) manipulate (1.1) to obtain a fixed point problem. To leading order, the solutions of (1.1) are  $e^{i\lambda x}, e^{-i\lambda x}$  so that we look for solutions of the form

$$(2.9) \quad \psi = \psi_+ = e^{i\lambda x} \phi_+(x) \quad \text{and} \quad \psi = \psi_- = e^{-i\lambda x} \phi_-(x).$$

Then  $\phi_+$  solves

$$(2.10) \quad \phi'' + 2i\lambda\phi' - V(x)\phi = 0.$$

We then seek for solutions of (2.10) of the form  $\phi(x; \lambda) := \mathcal{L}(\chi(x, \cdot))$ , where  $\mathcal{L}$  is the usual Laplace transform. One can easily check that

$$\frac{\mathcal{L}(\chi(x, \cdot))}{\lambda} = \mathcal{L}(\Psi(x, \cdot)) \quad \text{where} \quad \Psi(x, t) = \int_0^t \chi(x, \tau) d\tau,$$

so that dividing (2.10) by  $\lambda$  leads to

$$(2.11) \quad \begin{cases} \Psi_{xx} + 2i\Psi_{xt} = V(x)\Psi \\ \Psi(x, 0) = 0 \\ \Psi(0, t) = \int_0^t d\tau = t \end{cases}$$

To control this equation, we pass to bicharacteristic coordinates, namely we let

$$(2.12) \quad s = -2ix + t$$

and obtain for  $\Phi(s, t) := \Psi(x, t)$ ,

$$(2.13) \quad \begin{cases} \Phi_{st} = \frac{1}{4}V(i(s-t)/2)\Phi \\ \Phi(s, 0) = 0 \\ \Phi(t, t) = t \end{cases}$$

Integrating (2.13) yields

$$(2.14) \quad \Phi(s, t) = J(\Phi)(s, t),$$

where

$$(2.15) \quad \begin{aligned} J(\Phi)(s, t) &= t - \frac{1}{4} \int_0^t \int_t^s V(i(s_1 - t_1)/2) \Phi(s_1, t_1) ds_1 dt_1 := \\ &t + t(s-t) \int_0^1 \int_0^1 V[i((1-\alpha)s + (\alpha-\beta)t)/2] \Phi[(1-\alpha)s + \alpha t, \beta t] d\alpha d\beta. \end{aligned}$$

## 2.2. Solving the fixed-point equation: analyticity and exponential bounds.

It is useful to note the relation between  $\Psi$  and  $\Phi$ :

$$\Psi(x, t) = \Phi(-2ix + t, t) \quad \text{or} \quad \Phi(s, t) = \Psi(i(s - t)/2, t).$$

For  $\nu > 0$  and  $\Omega$  an open set of  $\mathbb{C}^2$ , let now  $\mathfrak{B}_\nu(\Omega)$  be the Banach space of analytic functions  $F$  over  $\Omega$  with finite norm

$$(2.16) \quad \|F\|_\nu = \sup_{\Omega} |F(s, t)| e^{-\nu|t|}.$$

Let

$$P = \{(s, t) : |s| < R, |t| < R\}$$

and

$$S_1 = \{(s, t) : \Re(s), \Re(t) > 0, \max(|\Im(t)|, |\Im(s)|, |\Im(t - s)|) < 2\} \cup P.$$

**Lemma 4.** *Assume  $V(x)$  is analytic in  $S = \{|x| < R\} \cup \{x : |\Re(x)| < 1\}$  and  $|V(x)| < K(1 + |x|)^{-1-\delta}$  in  $S$  for some  $\delta > 0$ . Then*

$$J : \mathfrak{B}_\nu(S_1) \rightarrow \mathfrak{B}_\nu(S_1)$$

*is a contraction for  $\nu > 0$  large enough.*

*Proof.* First we check that  $J$  is well-defined on  $\mathfrak{B}_\nu(S_1)$ . Clearly,

$$(s, t) \in P \Rightarrow i((1 - \alpha)s + (\alpha - \beta)t)/2 \in \{x : |x| < R\}$$

for  $\alpha, \beta \in [0, 1]$ . Also

$$|\Re(i((1 - \alpha)s + (\alpha - \beta)t)/2)| = |(1 - \alpha)\Re(s) + (\alpha - \beta)\Re(t)|/2 < 1,$$

provided this inequality holds for  $(\alpha, \beta) = (0, 0), (0, 1), (1, 0), (1, 1)$  since the above expression is linear in  $(\alpha, \beta)$ . The condition is thus

$$\max(|\Im(t)|, |\Im(s)|, |\Im(t - s)|) < 2, \quad \text{i.e. } (s, t) \in S_1.$$

It follows easily that  $J$  maps analytic-over- $S_1$  functions to themselves. Also,  $\|t\|_\nu \leq \nu^{-1}$ . We are left with proving that  $J$  is contractive (in a nonlinear case, one would restrict  $J$  to a ball of radius  $\mathcal{C}\nu^{-1}$ ) and this follows from the following estimate

$$(2.17) \quad \left| \int_s^t V(i(s_1 - t_1)/2) F(s_1, t_1) ds_1 \right| \leq \mathcal{C} \|F\|_\nu e^{\nu|t_1|} \int_{-\infty}^{\infty} (1 + |u|)^{-1-\delta} du = \mathcal{C} \|F\|_\nu e^{\nu|t_1|},$$

whence by integration

$$\|J(F)\|_\nu \leq \mathcal{C}\nu^{-1} \|F\|_\nu.$$

□

Applying Lemma 4 with  $R = 1$ , by the fixed-point theorem, we obtain a solution  $\Phi \in \mathfrak{B}_\nu(S_1)$  of (2.14).

**2.3. Solving the original equation.** This part of the proof is standard but we include it for convenience. To go back to the original equation, we just need to reverse and justify the transformations that lead us from (1.1) to (2.14) : since  $\Phi \in \mathfrak{B}_\nu(S_1)$ ,  $\Psi$  is analytic in a neighborhood of  $(-1, 1) \times (0, \infty)$  and

$$|\Psi(x, t)| \leq \mathcal{C}e^{\nu|t|}.$$

Clearly,  $\Psi$  satisfies (2.11). We claim that given any  $x_0 \in (-1, 1)$  there exists  $\epsilon > 0$  such that each partial derivative of  $\Psi$  is exponentially bounded on  $(x_0 - \epsilon, x_0 + \epsilon) \times (0, \infty)$  so that  $\phi = \lambda \mathcal{L}(\Psi(x, \cdot))$  is well-defined and will solve (2.10) with the initial condition  $\phi(0; \lambda) = 1/\lambda$ . Indeed,

$$\begin{aligned} \Psi_x &= -2i\Phi_s, & \Psi_{xx} &= -4\Phi_{ss} \\ \Psi_t &= \Phi_s + \Phi_t, & \Psi_{xt} &= -2i(\Phi_{ss} + \Phi_{st}) \end{aligned}$$

But since  $\Phi$  is analytic in  $U = P \cup S_1$ , letting  $s_0 = -2ix + t_0$  for  $x \in (x_0 - \epsilon, x_0 + \epsilon)$  and  $t_0 \in \mathbb{R}^+$ , we have that  $(s_0, t_0) \in U$  and by Cauchy's formula,

$$|\Phi_s(s_0, t_0)| = (2\pi)^{-1} \left| \int_{\mathcal{S}(s_0, \epsilon)} \Phi(s, t_0)/(s - s_0)^2 ds \right| \leq \mathcal{C}_\epsilon e^{\nu|t_0|},$$

where  $\mathcal{S}(s_0, \epsilon)$  denotes the circle centered at  $s_0$  of radius  $\epsilon = (1 - |x_0|)/2$ . The bounds on the other derivatives can be obtained similarly and we therefore take Laplace transforms in (2.11) to obtain a solution of (1.1),(1.7) on  $(-1, 1)$ , of the form

$$\psi(x; \lambda) = \psi_+ = e^{i\lambda x} \phi_+ = \lambda e^{i\lambda x} \mathcal{L}(\Psi(x, \cdot)) \quad \text{where} \quad |\Psi(x, t)| \leq \mathcal{C}e^{\nu|t|}.$$

Working with  $\phi_- = \overline{\phi_+}$  (as defined in (2.9)), we obtain similarly a solution of the form  $\psi(x; \lambda) = e^{-i\lambda x} \phi_-$ .

In the following we will rely on the fact that Borel summable series are closed under algebraic operations.

**Lemma 5.** *Borel summable series form a field.*

This result follows from the general theory of Borel summation (see [12], [13], [14], [15]); for convenience of the reader we nevertheless provide a short self-contained proof. The fact that Borel summable series form an *algebra* is straightforward. Given a nonzero series  $\tilde{y}$ , we want to construct its multiplicative inverse  $(\tilde{y})^{-1}$ . Up to factoring out a monomial  $y_K \lambda^{K-1}$  in the expansion of  $\tilde{y}$ , we may always assume that

$$\tilde{y} = \lambda(1 + \tilde{f}) \quad \text{for some Borel summable } \tilde{f} = o(1).$$

The inverse  $g$  of  $\tilde{y}$  must then satisfy

$$(2.18) \quad (1 + \tilde{f})\tilde{g} = 1/\lambda.$$

If  $F = \mathcal{B}\tilde{f}$  is the Borel transform of  $\tilde{f}$ , defined on a fixed (star-shaped-about-the-origin) neighborhood  $\Omega$  of  $[0, \infty)$  and if  $G$  denotes the Borel transform of the inverse  $g$  we are looking for, we must have

$$(2.19) \quad G = 1 - F * G,$$

where

$$(2.20) \quad (F * G)(t) = \int_0^t F(q)G(t-q)dq := t \int_0^1 F(\alpha t)G((1-\alpha)t)d\alpha.$$

Given  $A > 0$  and  $\nu > 0$ , consider the norm

$$(2.21) \quad \|F\| := \sup_{t \in \Omega} (A + |t|)^2 |F(t)| e^{-\nu t},$$

defined for  $F \in \mathcal{B}_{\nu, A}(\Omega)$ , the space of (exponentially bounded) analytic functions over  $\Omega$ , equipped with the above norm. (2.19) will have a solution  $G$  provided  $G \rightarrow F * G$  is a contraction in that space. Now,

$$(2.22) \quad \begin{aligned} |F * G| &\leq \|F\| \|G\| e^{\nu|t|} \int_0^{|t|} (A+u)^{-2} (A+|t|-u)^{-2} du \\ &= \|F\| \|G\| e^{\nu|t|} \left( \int_0^{|t|/2} + \int_{|t|/2}^{|t|} \right) (A+u)^{-2} (A+|t|-u)^{-2} du \end{aligned}$$

and

$$\int_0^{|t|/2} (A+u)^{-2} (A+|t|-u)^{-2} du \leq (A+|t|/2)^{-2} \int_0^\infty (A+u)^{-2} du \leq \frac{C}{A} (A+|t|)^{-2}.$$

Working similarly with the last integral in (2.22), we obtain that

$$(2.23) \quad \|F * G\| \leq \frac{C}{A} \|F\| \|G\|.$$

Hence, fixing  $A > 0$  large enough it follows that  $G \rightarrow F * G$  is contractive.

Taking Laplace transforms and applying Watson's lemma, we obtain the (unique) solution  $\tilde{g}$  of (2.18) defined by

$$\mathcal{L}G \sim \tilde{g}.$$

□

**2.4. Uniqueness of  $\phi_+$ .** Take  $\phi = \mathcal{L}(\chi(x, \cdot))$  and  $\mathcal{V}$  a neighborhood of the origin as in the statement of Theorem 2. As in Step 2.1, using the same notations to go from  $\phi$  to  $\Phi$ , it follows that  $\Phi$  solves (2.14), so that we only need to prove that  $J$  is a contraction for the norm

$$\|F\|_Y = \sup_x \int_0^\infty |F(x, t)| e^{-\nu t} dt$$

and observe that since  $L_\nu^1 \subset L_{\nu'}^1$  when  $\nu < \nu'$ , we may choose  $\nu > 0$  as large as we please. Rewriting (2.15) in terms of  $\chi \in Y$ , we have

$$(2.24) \quad \begin{aligned} K(\Phi)(s, t) &:= -4(J(\Phi)(s, t) - t) \\ &= t(s-t) \int_0^1 \int_0^1 \int_0^{\beta t} V(x_{\alpha\beta}) \chi(x_{\alpha\beta}, \tau) d\tau d\alpha d\beta, \end{aligned}$$

where

$$x_{\alpha\beta} = \frac{i}{2} ((1-\alpha)s + (\alpha-\beta)t).$$

So

$$(2.25) \quad |K(\Phi)| \leq \|\chi\|_Y |t(s-t)| \int_0^1 \int_0^1 |V(x_{\alpha\beta})| e^{\nu\beta t} d\alpha d\beta \leq C\nu^{-1} \|\chi\|_Y,$$

where we used (2.17) in the last inequality.

**2.5.  $\psi_+$  and  $\psi_-$  are linearly independent.** Observe that  $\psi_+$  and  $\psi_-$  defined in (2.9) are conjugate expressions of each other and arguing by contradiction, suppose that for some  $A = A(\lambda) \in S^1$

$$\psi_+ = A\psi_-$$

Let  $B(x; \lambda) := \phi_+/\phi_-$ . Then

$$(2.26) \quad B = Ae^{-2i\lambda x}.$$

By the Lemma 5,  $B$  defined by (2.26) is asymptotic to a Borel summable power series and since  $\phi_+(0; \lambda) = \phi_-(0; \lambda) = 1/\lambda$ , we have for  $x = 0$ ,

$$(2.27) \quad A(\lambda) = B(0; \lambda) = 1 + o(1) \quad \text{as } \lambda \rightarrow \infty,$$

whereas for any other fixed value of  $x \neq 0$ , there exist  $K = K(x) \in \mathbb{Z}$  and  $b = b(x) \in \mathbb{C}$  such that

$$A(\lambda)e^{-2i\lambda x} = B = \lambda^K(b + o(1)).$$

Combining this equation with (2.27), we obtain

$$(1 + o(1))e^{-2i\lambda x} = \lambda^K(b + o(1)),$$

which is clearly impossible.

**2.6. The case (1.5).** We explain here how to adapt the argument for a potential  $V = V(x, \epsilon)$  depending on  $\epsilon = 1/\lambda$ . According to (1.5), if  $\phi(x) = \mathcal{L}(\chi(x, \cdot))$ , the inverse Laplace transform of  $V\phi$  is given by:

$$(2.28) \quad \mathcal{L}^{-1}V\phi = V_0(x)\chi + \sum_{k=1}^{\infty} \frac{V_k(x)}{(k-1)!} t^{k-1} * \chi,$$

where  $*$  is the convolution product defined in (2.20). Hence, taking as before  $\Psi(x, t) = \int_0^t \chi(x, \tau) d\tau$ , we obtain

$$\frac{V\phi}{\lambda} = \mathcal{L} \left( V_0(x)\Psi + \sum_{k=1}^{\infty} \frac{V_k(x)}{(k-1)!} t^{k-1} * \Psi \right).$$

Following the analysis of the  $\epsilon$ -independent case, we end up with the following operator :

$$\tilde{J}(\Phi)(s, t) = J(\Phi)(s, t) - \frac{1}{4} \sum_{k>0} \int_0^t \int_t^s \frac{V_k(i(s_1 - t_1)/2)}{(k-1)!} [t^{k-1} * \Phi](s_1, t_1) ds_1 dt_1,$$

where  $J$  is defined by (2.15). Again, we must prove that  $\tilde{J}$  is contractive. To do so, instead of (2.16), we use the norm  $\|\cdot\|$  defined in (2.21). For  $A$  large enough, it follows from (2.23) that

$$(2.29) \quad \|F * G\| \leq \|F\| \|G\| \text{ holds for all } F, G \text{ analytic in } \Omega.$$

In order to control  $\|\tilde{J}\|$ , we perform a few side computations. We first estimate  $\|t^{k-1}\|$  :

$$\|t^{k-1}\| \leq \mathcal{C} \sup_{t \in \mathbb{R}^+} (t^{k+1}e^{-\nu t} + t^{k-1}e^{-\nu t}) = \mathcal{C} \left[ \left( \frac{k+1}{\nu} \right)^{k+1} e^{-(k+1)} + \left( \frac{k-1}{\nu} \right)^{k-1} e^{-(k-1)} \right].$$

By Stirling's formula, it follows that

$$(2.30) \quad \|t^{k-1}\| \leq \mathcal{C}(k-1)! k^{3/2} \nu^{-k+1}.$$



Next, we estimate for  $t > 0$  the following quantity :

$$e^{-\nu t}(A+t)^2 \int_0^t e^{\nu t_1}(A+t_1)^{-2} dt_1 = \left( \int_0^{t/2} + \int_{t/2}^t \right) e^{\nu(t_1-t)} \left( \frac{A+t}{A+t_1} \right)^2 dt_1.$$

On the one hand

$$\int_0^{t/2} e^{\nu(t_1-t)} \left( \frac{A+t}{A+t_1} \right)^2 dt_1 \leq e^{-\nu t/2} \int_0^{t/2} \left( \frac{A+t}{A+t_1} \right)^2 dt_1 \leq e^{-\nu t/2} \frac{(A+t)^2}{A} \leq \frac{\mathcal{C}}{\nu^2},$$

while on the other hand

$$\int_{t/2}^t e^{\nu(t_1-t)} \left( \frac{A+t}{A+t_1} \right)^2 dt_1 \leq \mathcal{C} \int_{t/2}^t e^{\nu(t_1-t)} dt_1 \leq \mathcal{C}/\nu.$$

Hence,

$$(2.31) \quad e^{-\nu t}(A+t)^2 \int_0^t e^{\nu t_1}(A+t_1)^{-2} dt_1 \leq \mathcal{C}/\nu.$$

We can now estimate  $\|\tilde{J}(\Phi)\|$ . Instead of (2.17), we have

$$\begin{aligned} \left| \int_s^t V_k(i(s_1 - t_1)/2)[t^{k-1} * \Phi](s_1, t_1) ds_1 \right| &\leq \mathcal{C} \|t^{k-1} * \Phi\| e^{\nu|t_1|} (A+|t_1|)^{-2} B^k \int_{-\infty}^{\infty} (1+|u|)^{-1-\delta} du \\ &\leq \mathcal{C} \nu \left( \frac{B}{\nu} \right)^k (k-1)! k^{3/2} e^{\nu|t_1|} (A+|t_1|)^{-2} \|\Phi\|, \end{aligned}$$

where we used (2.29) and (2.30) in the last inequality. Whence by integration and by (2.31), we finally obtain

$$\|\tilde{J}(\Phi)\| \leq \mathcal{C} \left( \nu^{-1} + \sum_{k>0} k^{3/2} \left( \frac{B}{\nu} \right)^k \right) \|\Phi\| \leq \frac{\mathcal{C}}{\nu} \|\Phi\|.$$

**2.7. Further generalizations.** It is not difficult to modify the proof given to allow for higher order equations, or for nonlinear dependence on  $\psi$  which does not affect the highest derivative. In the nonlinear case, the initial condition must obviously be left in the form of a general sum of a convergent or Borel summable series. The strategies of dealing with nonlinearities are described in [7] and [11].

### 3. PROOF OF THEOREM 3

This follows easily from the explicit expression of  $C$ , Theorem 2 and Lemma 5. Indeed, differentiating  $\psi$  (with respect to  $x$ ), we obtain

$$\begin{aligned} \psi &= C_1 e^{i\lambda x} \phi_+ + C_2 e^{-i\lambda x} \phi_- \\ \psi' &= C_1 e^{i\lambda x} A + C_2 e^{-i\lambda x} B \end{aligned}$$

where

$$A = \phi'_+ + i\lambda\phi_+ \quad \text{and} \quad B = \phi'_- - i\lambda\phi_-$$

So that

$$C := C_1 C_2 = \frac{\begin{vmatrix} \psi & \phi_- \\ \psi' & B \end{vmatrix} \cdot \begin{vmatrix} \phi_+ & \psi \\ A & \psi' \end{vmatrix}}{\begin{vmatrix} \phi_+ & \phi_- \\ A & B \end{vmatrix}^2}$$

By construction,  $\phi^+ \sim \sum a_k(x)\lambda^{-k}$  and is uniquely determined so that using Lemma 5, we may conclude that

$$C \sim \sum c_k(x, \psi, \psi')\lambda^{-k}$$

where all considered power series are Borel summable.  $\square$

#### 4. DISCUSSION OF SINGULARITIES OF $\Psi$

So far we have assumed that  $V$  was analytic. If this is not so, singularities of  $\Psi$  can originate in the singular points of  $V$ . We restrict the analysis to the relatively common situation where  $V$  has a branch-point at some point  $x_0$  (we may assume without loss of generality  $x_0 = 0$ ) of the form

$$V(x) = x^{-\beta}V_1(x),$$

with  $V_1$  analytic at 0. We assume  $0 < \Re\beta < 1$  (for different  $\Re\beta$ , the analysis can be done similarly).

We let  $P = \{(s, t) : |s| < \frac{1}{2\epsilon}, |t| < \epsilon\}$  and consider the region

$$S_1 = \{(s, t) : \Re(s) > \Re(t) > 0, \max(|\Im(t)|, |\Im(s)|, |\Im(t-s)|) < 2\} \cap P$$

Assume that  $(s, t) \rightarrow V_1(i(s-t)/2)$  is analytic in  $S_1$  and that  $V$  satisfies the following estimate (similar to that in Lemma 4) : for some  $\delta > 0, K > 0$ ,

$$|V(x)| \leq K(1 + |x|)^{-1-\delta} \quad \text{if } |x| > 1.$$

**Proposition 6.** *There exists a solution  $\Phi \in C^2(S_1)$  of (2.15). Furthermore, letting  $v_0 = V_1(0)$ ,  $\Phi$  satisfies*

$$(4.32) \quad \Phi(s, t) = t + \frac{v_0}{4} \left\{ s \frac{s^{2-\beta} - (s-t)^{2-\beta}}{(1-\beta)(2-\beta)(3-\beta)} + \frac{(s-t)^{2-\beta}}{(1-\beta)(3-\beta)} t + \frac{t^{3-\beta}}{(1-\beta)(2-\beta)(3-\beta)} \right\} + O(t^3),$$

as  $t \rightarrow 0$ .

#### Remarks

(1) In particular, it follows easily that for  $s \neq 0$ ,  $\Phi$  does not extend analytically at  $t = 0$ . Indeed, assuming the contrary, we would have  $Ct^{3-\beta} = A(t) + O(t^3)$  with  $A$  analytic; as  $t \rightarrow 0$  this forces  $A(t) \sim Ct^{3-\beta}$ . But  $A$  is analytic and we must have for some  $n \in \mathbb{N}$  and  $C' \in \mathbb{R}$ ,  $A(t) \sim C't^n$ , which is a contradiction.

(2) Proposition 6 provides information about the analytic continuation (still denoted  $\Phi$ ) of the solution to (2.15). Indeed, we can work as in Lemma 4 to show analyticity in a region where  $t > \epsilon > 0$ . Lemma 4 and Proposition 6 provide uniqueness in the space of exponentially bounded analytic functions over the corresponding region, so that by obvious imbeddings the analytic continuation  $\Phi$  coincides with the fixed point of (2.15). Changing variables, i.e. going back to  $\Psi$ , we conclude that  $\Psi$  is nonanalytic at  $t = 0$ . This implies in turn a Stokes transition on  $\phi$  (and thus the adiabatic constant constructed in Theorem 3).

(3) Furthermore, the fixed point procedure that leads to relation (4.32) can easily (and rigorously) provide more detailed information about the singularity manifold; it suffices to construct the space of functions in which the fixed point equation

(4.33) below is considered in such a way that the norms entail information about the singularity type to be proved: the details are quite straightforward but at the same time quite long and we will not elaborate on them in the present paper. Similar constructions can be found in [7].

*Proof.* From (2.15) we have

$$(4.33) \quad \Phi(s, t) = t - \frac{1}{4} \int_0^t \int_t^s \frac{V_1(i(s_1 - t_1))}{(s_1 - t_1)^\beta} \Phi(s_1, t_1) ds_1 dt_1$$

The fact that  $J$  is defined and contractive with norm  $O(t)$  on the functions defined in  $S_1$  with the sup norm, follow as in Lemma 4.

We have  $(1 - J)\Phi = t$  and thus for small  $t$ ,  $\Phi = t + Jt + O(J^2t) = t + Jt + O(t^3)$  and, with  $v_0 = V_1(0)$  the result follows.  $\square$

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