Existence and uniqueness of solutions of nonlinear evolution systems of n-th order partial differential equations in the complex plane

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Abstract

We analyze partial differential equations of the form

$$\partial_t \mathbf{h} + (-\partial_y)^n \mathbf{h} = \mathbf{g}_2 \left(y, t, \{\partial_y^j \mathbf{h}\}_{j=0}^{n-1} \right) \partial_y^n \mathbf{h} + \mathbf{g}_1 \left(y, t, \{\partial_y^j \mathbf{h}\}_{j=0}^{n-1} \right) + \mathbf{r}(y,t); \; \mathbf{h}(y,0) = \mathbf{h}_I(y)$$

in the complex plane, for sufficiently large y in a sector, under certain analyticity and decay conditions on $\mathbf{h}$, $\mathbf{h}_I, \mathbf{g}_1, \mathbf{g}_2$ and $\mathbf{r}$ which make the nonlinearity formally small as $y \to \infty$. A similar result is shown, undeer further assumptions, for small time and any y in $\mathbb{C}$.

The results given also justify the formal asymptotic expansions of solutions and can be conversely used to reconstruct actual solutions from formal series ones by Borel summation which we prove in the process.

Due to the type of nonlinearity in the highest derivatives, the divergence of formal series solutions owed to the singularity of the system at infinity and the complex plane nature of the problem which precludes usual estimates, we use new techniques (based on Borel-Laplace duality) to control the perturbative terms. Our methods of proof and norms used are of a constructive nature.

1 Introduction

The theory of partial differential equations, when one or more of the independent variables are in the complex plane is not very developed. The classic Cauchy-Kowalevski (C-K) theorem holds for a system of first-order equations (or those equivalent to it) when the quasi-linear equations have analytic coefficients and analytic initial data is specified on an analytic but non-characteristic curve. Then, the C-K theorem guarantees the local existence and uniqueness of analytic solutions. As is well known, its proof relies on the convergence of local power series expansions and,

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without the given hypotheses, the power series may have zero radius of convergence and the C-K method does not yield solutions.

More recently, Sammartino and Caflisch ([22], [23]) proved the existence of nonlinear Prandtl boundary layer equations for analytic initial data in a half-plane. This work involved inversion of the heat operator $\partial_t - \partial_{YY}$ and using the abstract Cauchy-Kowalewski theorem for the resulting integral equation.

While this method is likely to be generalizable to certain higher-order partial differential equations, it appears unsuitable for problems where the highest derivative terms appear in a non-linear manner. These terms cannot be controlled by inversion of a linear operator and estimates of the kernel, as used by Sammartino and Caflisch.

Nonlinearity of special forms in the highest derivative was considered in a general setting, encompassing nonlinear wave equations in the real domain, in a series of profound studies by Klainerman [13], [14], Shatah [24], Klainerman and Ponce [15], Ponce [20], Ponce and Lim [21], Kenig and Staffilani [12], Klainerman and Selberg [16], Shatah and Struwe [25] and others.

The complex plane setting, as well as the type of nonlinearity allowed in our paper, do not appear to allow for an adaptation of those techniques. In fact simple examples show that existence fails outside the domain of validity of the expansions we rely upon. We use a completely different and constructive method based on recently developed generalized Borel summation techniques [10], [5], [7]. Our results are presently restricted to one spatial dimension.

Apart from the mathematical interest of understanding the question of existence and uniqueness of solutions to nonlinear PDEs in the complex plane near singular points ($y = \infty$ in our context), sectorial existence of solutions to higher order nonlinear PDEs is important in many applications.

For instance, there are nonlinear PDEs for which the initial value problem, in the absence of a regularization is relatively simple; yet ill-posed in the sense of Hadamard for any Sobolev norm on the real domain. However, the analytically continued equations into the complex spatial domain are well-posed, even without a regularization term. There have been quite a few complex domain studies involving idealized equations modeling physical phenomena (see for instance, [18], [19], [1], [2], [3] and [4]) that follow Garabedian's [11] realization that an ill-posed elliptic initial value problem in the real spatial domain may become well-posed in the complex domain. In a particular physical context, it was suggested [27] that complex domain studies would be useful in understanding small regularization for some class of initial conditions. Formal and numerical computations show the usefulness of this approach in predicting singular effects [26]. However, many of the results for small but nonzero regularization were formal and relied fundamentally on the existence and uniqueness of analytic solutions to certain higher order nonlinear partial differential equations in a sector in the complex plane, with imposed far-field matching conditions. In [7] we addressed rigorously the existence questions in some third-order nonlinear PDEs.

In a more general context, one can expect that whenever regularization appears in the form of a small coefficient multiplying the highest spatial derivative, the resulting asymptotic equation in the neighborhood of initial complex singularities will satisfy higher order (not necessarily third order) nonlinear partial differential equation with sectorial far-field matching condition in the complex plane of the type discussed here. An example of such an application is the analysis of the local behavior of solutions to the well known Kuramato-Shivashinski equation: $u_t + uu_x + u_{xx} + \nu u_{xxxx} = 0$ for small nonzero ν, near a complex-singularity. The same type of existence questions is relevant in the small time asymptotics of higher order nonlinear PDEs near initial singularities, as is illustrated further in §6. Such expansions determine relevant choice of "inner-variables" that are crucial in the complex singularity analysis of PDE [9]. It is to be noted that very little had been known about the singularity structure of PDEs in the complex plane.

We show existence and uniqueness of solutions $\mathbf{h}(y,t)$ to the initial value problem for a general class of quasilinear system of partial differential equations of the form:

$$\partial_t \mathbf{h} + (-\partial_y)^n \mathbf{h} = \mathbf{g}_2\left(y,t,\{\partial_y^j \mathbf{h}\}_{j=0}^{n-1}\right)\partial_y^n \mathbf{h} + \mathbf{g}_1\left(y,t,\{\partial_y^j \mathbf{h}\}_{j=0}^{n-1}\right) + \mathbf{r}_1(y,t) \quad \text{with} \quad \mathbf{h}(y,0) = \mathbf{h}_I(y) \tag{1}$$

where ∂_y^j denotes the j-th derivative with respect to $y \in \mathbb{C}$. The function $\mathbf{h}$ takes values in $\mathbb{C}^{m_1}$. Suitable regularity and decay conditions are imposed for large y in a sector in the complex plane (§2) and our results will hold in such a sector. These conditions make the terms in $\mathbf{h}$ on the right side of (1) formally small for large y. As will be discussed, existence cannot be expected, in general, to hold outside a specific sector.

By the transformation $y = z^{-1}$ it is seen that the problem is that of existence and uniqueness of solutions in a (sectorial) neighborhood of a point where the linear part of the principal symbol of a partial differential operator has a high order pole. Formal solutions as power series in z are expected to have zero radius of convergence, cf. also §7.2.

For problems of this type, the essence of the methodology we have introduced recently in [7] in special cases of (1) has been to use Borel-Laplace duality to regularize the problem which is recast as an integral equation in the Borel plane. This regularization, akin to Borel summation,[1] is instrumental in controlling the solution. The choice of appropriate Banach space after the Borel transform proves to be crucial, and after this choice, the contraction mapping argument itself follows from a sequence of relatively straightforward estimates in convolution Banach algebras. Borel summation methods have been also used recently in the context of the heat equation by Lutz, Miyake and Schäfke [17]. We illustrate in the Appendix, §7.2, the regularizing role of the Borel transform and discuss why it is instrumental in showing existence and uniqueness. The method we use is constructive, in the sense that it permits recovering of actual solutions from formal ones presented as classically divergent power series in inverse powers of y, see also the notes in §2. The study is done in a sector in the complex plane whose width is crucial to existence and uniqueness of solutions with prescribed decay.

Our previous results [7] were limited to a class of partial differential equations that are first order in time, t, and third order in space, y. Further, those results, motivated by a set of applications, were restricted to scalar dependent variable with no nonlinearity in its derivatives.

Among the concrete equations amenable through rather straightforward transformations to the setting [7] and thus to the present more general one are the KdV equation, the equations $H_t = H^3 H_{xxx}$ and $H_t + H_x = H^3 H_{xxx} - H^3/2$, both arising in Hele-Shaw dynamics, the equation $H_t = H^{1/3} H_{xxx}$ relevant to dendritic crystal growth, and many others. The last three of these PDEs were treated in detail in [7].

In the present paper we are generalizing the results of [7] to arbitrary order in the spatial variable. Further, the dependent function is allowed to be a vector $\mathbf{f}(y,t)$. The nonlinearity is that of a general quasi-linear equation. Indeed, it will be obvious that the result given here also generalizes easily to the case when the left side of (1) is replaced by $\partial_t \hat{\mathbf{f}} - A\partial_y^n \hat{\mathbf{f}}$, where A is a

[1] More exactly, for the technical reason of simplifying the principal symbol of the transformed operator, we are performing Borel *over*summation, meaning that the power of the factorial divided out of the coefficients of the divergent series is higher than the minimum necessary to ensure convergence.

constant matrix with positive eigenvalues, though we leave out this extra generality for the sake of relative simplicity in presentation.

2 Problem statement and main result

We study equation (1) where the inhomogeneous term $\mathbf{r}_1(y,t)$ is analytic in y the domain

$$\mathcal{D}_{\rho_0} \;=\; \Big\{(y,t)\in\mathbb{C}\times\mathbb{R}:\;\; \arg\; y \in \Big(-\frac{\pi}{2}-\frac{\pi}{2n}, \frac{\pi}{2}+\frac{\pi}{2n}\Big),\; |y|>\rho_0>0,\; 0\le t\le T\Big\} \qquad (2)$$

The restrictions on $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{r}_1$ and $\mathbf{h}_\mathbf{I}$ are better expressed once we transform the equation to a more convenient form, as is done shortly.

By taking derivatives of (1) with respect to y, i-times, i ranging from 1 to $n-1$, it is possible to consider the extended $m_1\times n$ system of equations for $\mathbf{h}$ and its first $n-1$-derivatives. This system of vector equations is of the form (see Appendix for further details)

$$\partial_t\mathbf{f}+(-\partial_y)^n\mathbf{f} = {\sum_{\mathbf{q}\succeq 0}}' \mathbf{b}_\mathbf{q}(y,t,\mathbf{f})\prod_{l=1}^{m}\prod_{j=1}^{n}\left(\partial_y^j f_l\right)^{q_{l,j}} + \mathbf{r}(y,t);\;\;\text{with}\;\; \mathbf{f}(y,0) \;=\; \mathbf{f}_I(y) \qquad (3)$$

where $\sum'$ means the sum over the multiindices $\mathbf{q}$ with

$$\sum_{l=1}^{m}\sum_{j=1}^{n} j q_{l,j}\le n \qquad (4)$$

In (3), $\mathbf{f}$ is an $m=m_1\times n$ dimensional vector, $\mathbf{q}=\{q_{l,j}\}_{j=1,l=1}^{n,m}$ is a vector of integers and the notation $\mathbf{q}\succeq 0$ means $q_{l,j}\ge 0$ for all l,j. The inequality (4) implies in particular that none of the $q_{l,j}$ can exceed n and that the summation on $\mathbf{q}$ involves *only finitely many terms.* The fact that (4) can always be ensured leads to important simplifications in the proofs. We denote

$$\langle\mathbf{q}\rangle = \sum_{(l,j)\preceq(m,n)} q_{l,j}$$

We assume that in $\mathcal{D}_{\rho_0}$ there exist constants $\alpha_r\ge 1$ (see also §5.2) and $A_r(T)$, with α_r independent of T such that

$$|y^{\alpha_r}\;\mathbf{r}(y,t)| \;<\; A_r(T) \qquad (5)$$

In this paper the absolute value $|\cdot|$ of a vector is the *max* vector norm. Additionally, we require that $\mathbf{b}_\mathbf{q}$ is analytic in $\mathbf{f}$ and in its convergent representation

$$\mathbf{b}_\mathbf{q}(y,t;\mathbf{f}) \;=\; \sum_{\mathbf{k}\succeq 0}\mathbf{b}_{\mathbf{q},\mathbf{k}}(y,t)\;\mathbf{f}^\mathbf{k}\;\;\text{where}\;\;\mathbf{k}=(k_1,k_2,...,k_m),\;\textit{and}\;\mathbf{f}^\mathbf{k}=\prod_{i=1}^{m} f_i^{k_i} \qquad (6)$$

the functions $\mathbf{b}_{\mathbf{q},\mathbf{k}}$ are analytic in y in $\mathcal{D}_{\rho_0}$. We also take, without any loss of generality, $\mathbf{b}_{\mathbf{0},\mathbf{0}}=\mathbf{0}$, altering accordingly (if needed) $\mathbf{r}(y,t)$.

We assume that in this domain, there exist positive constants β, $\alpha_{\mathbf{q}}$, and A_b, independent of $\mathbf{q}$ and $\mathbf{k}$ (with β and $\alpha_{\mathbf{q}}$ independent of T as well), such that

$$|y^{\alpha_{\mathbf{q}}+\langle\mathbf{k}\rangle\beta}\ \mathbf{b}_{\mathbf{q},\mathbf{k}}| \ < \ A_b(T) \quad \text{, where} \qquad \langle\mathbf{k}\rangle \equiv k_1 + k_2 + \cdots + k_m \tag{7}$$

The series (6) converges in the domain $\mathcal{D}_{\phi,\rho}$, defined as (cf. also (2) and (7))

$$\mathcal{D}_{\phi,\rho} = \left\{ (y,t) : \arg y \in \left(-\frac{\pi}{2} - \phi, \frac{\pi}{2} + \phi \right),\ |y| > \rho > \rho_0,\ \text{where } 0 < \phi < \frac{\pi}{2n},\ 0 \le t \le T \right\} \tag{8}$$

if

$$|\mathbf{f}| \ < \ \rho^{\beta} \tag{9}$$

Condition 1 *The solution $\mathbf{f}(\cdot,t)$ sought for (3) is required to be analytic in $\mathcal{D}_{\phi,\rho;y}$, where $\mathcal{D}_{\phi,\rho;y}$ is the projection of $\mathcal{D}_{\phi,\rho}$ on the first component, for some $\rho > 0$ (to be determined later). In the same domain, the solution and the initial condition $\mathbf{f}_I(y)$ must satisfy the conditions (cf. (5))*

$$|y|^{\alpha_r}\ |\mathbf{f}(y,t)| \ < \ A_f(T); \quad |y|^{\alpha_r}\ |\mathbf{f}_I(y,t)| \ < \ A_f(T) \tag{10}$$

for some $A_f(T) > 0$ and $(y,t) \in \mathcal{D}_{\phi,\rho}$.

It is clear that for large y such a solution $\mathbf{f}$ will indeed satisfy (9), the condition for the convergence of the infinite series in (3).

*

The general theorem proved in this paper is the following.

Theorem 2 *For any $T > 0$ and $0 < \phi < \pi/(2n)$, there exists $\tilde{\rho}$ such that the partial differential equation (3) has a unique solution $\mathbf{f}$ that is analytic in y and is $O(y^{-1})$ as $y \to \infty$ for $(y,t) \in \mathcal{D}_{\tilde{\rho},\phi}$. Furthermore, this solution satisfies $\mathbf{f} = O(y^{-\alpha_r})$ as $y \to \infty$ in $\mathcal{D}_{\tilde{\rho},\phi;y}$.*

Notes. 1. Existence necessitates the sector to be not too large; if the technique is used for reconstruction of actual solutions from formal ones as explained in 3. below, the sector must be large enough to ensure uniqueness; the width provided here ensures both.

2. The proof is made delicate by the presence of perturbation terms involving the highest derivative and the fact that, in the relevant limit $y \to \infty$ the equation is singular; the nature of this problem is sketched in §7.2.

3. As discussed earlier in [7] for special examples, this decaying behavior of the solution $\mathbf{f}$ is valid inside the specified sector and outside it one can expect infinitely many singularities with an accumulation point at infinity.

4. In §5 it is shown how our result and technique can be used to justify an asymptotic power series behavior of the solution, or to recover actual solutions from formal series expansions.

5. There is a duality between large y and small t asymptotics. In §6, it is shown how the theorem can be modified to get existence and uniqueness of solution for small t, provided a few additional conditions hold. Concrete examples are are given to show the usefulness in justifying a formal asymptotic expansion in powers of t.

3 Inverse Laplace transform and equivalent integral equation

The inverse Laplace transform (ILT) $\mathbf{G}(p,t)$ of a function $\mathbf{g}(y,t)$ analytic in y in $\mathcal{D}_{\phi,\rho;y}$ (see Condition 1) and vanishing algebraically as $|y| \to \infty$ is given by:

$$\mathbf{G}(p,t) \;=\; \left[\mathcal{L}^{-1}\{\mathbf{g}\}\right](p,t) \;\equiv\; \frac{1}{2\pi i} \int_{\mathcal{C}_D} e^{py}\; \mathbf{g}(y,t)\; dy \tag{11}$$

where $\mathcal{C}_D$ is a contour as in Fig. 1 (modulo homotopies), entirely within the domain $\mathcal{D}_{\phi,\rho;y}$ and p is restricted to the domain $\mathcal{S}_\phi$ where convergence of the integral is ensured, where

$$\mathcal{S}_\phi \;\equiv\; \{p \;:\; \arg\, p \;\in\; (-\phi,\phi), 0 \;<\; |p| \;<\; \infty\;\} \tag{12}$$

If $\mathbf{g}(y,t) \;=\; \mathbf{C}y^{-\alpha}$ for $\alpha \;>\; 0$, then $\mathbf{G}(p,t) \;=\; \mathbf{C}p^{\alpha-1}/\Gamma(\alpha)$. From the following Lemma, it is clear that the same kind of behavior for the ILT $\mathbf{G}(p,t)$ holds for small p in $\mathcal{S}_\phi$, if $\mathbf{g}$ is $O(y^{-\alpha})$ for large y.

Lemma 3 *If $\mathbf{g}(y,t)$ is analytic in y in $\mathcal{D}_{\phi,\rho;y}$, and satisfies*

$$|y^\alpha\; \mathbf{g}(y,t)| \;<\; A(T) \tag{13}$$

for $\alpha \geq \alpha_0 > 0$, then for any $\delta \in (0,\phi)$ the ILT $\mathbf{G} = \mathcal{L}^{-1}\mathbf{g}$ exists in $\mathcal{S}_{\phi-\delta}$ and satisfies

$$|\mathbf{G}(p,t)| \;<\; C\frac{A(T)}{\Gamma(\alpha)}|p|^{\alpha-1}e^{2|p|\rho} \tag{14}$$

for some $C = C(\delta,\alpha_0)$.

Proof. The proof is similar to that of Lemma 3.1 in [7]. We first consider the case when $2 \;\geq\; \alpha \;\geq \alpha_0$. Let C_{ρ_1} be the contour C_D in Fig. 1 that passes through the point $\rho_1 + |p|^{-1}$, and given by $s = \rho_1 + |p|^{-1} + ir\exp(i\phi\,\mathrm{signum}(r))$ with $r \in (-\infty,\infty)$. Choosing $2\rho > \rho_1 > (2/\sqrt{3})\rho$, we have $|s| > \rho$ along the contour and therefore, with $\arg(p) = \theta \in (-\phi+\delta, \phi-\delta)$,

$$|\mathbf{g}(s,t)| < A(T)|s|^{-\alpha} \;\;\text{and}\;\; |e^{sp}| \leq e^{\rho_1|p|+1}e^{-|r||p|\sin|\phi-\theta|}$$

Thus

$$\left|\int_{C_{\rho_1}} e^{sp}\mathbf{g}(s,t)ds\right| \leq 2A(T)e^{\rho_1|p|+1}\int_0^\infty \left|\rho_1+|p|^{-1}+ire^{i\phi}\right|^{-\alpha} e^{-|p|r\sin\delta}dr$$

$$\leq\; \tilde{K}A(T)e^{\rho_1|p|}\;\left|\rho_1+|p|^{-1}\right|^{-\alpha}\;\int_0^\infty e^{-|p|r\sin\delta}dr \leq K\delta^{-1}|p|^{\alpha-1}e^{2\rho|p|} \tag{15}$$

where $\tilde{K}$ and K are constants independent of any parameter. Thus, the Lemma follows for $2 \;\geq\; \alpha \geq \alpha_0$, if we note that $\Gamma(\alpha)$ is bounded in this range of α, with the bound only depending on α_0.

For $\alpha \;>\; 2$, there exists an integer $k \;>\; 0$ so that $\alpha - k \;\in\; (1,2]$. Taking

$$(k-1)!\mathbf{h}(y,t) = \int_\infty^y \mathbf{g}(z,t)(y-z)^{k-1}dz$$

(clearly $\mathbf{h}$ is analytic in y, in $\mathcal{D}_{\phi,\rho}$ and $\mathbf{h}^{(k)}(y,t) = \mathbf{g}(y,t)$), we get

$$\mathbf{h}(y,t) \; = \; \frac{(-y)^k}{(k-1)!}\int_1^\infty \mathbf{g}(yp,t)(p\,-\,1)^{k-1}dp \; = \; \frac{(-1)^k y^{k-\alpha}}{(k-1)!}\int_1^\infty \mathbf{A}(yp,t)p^{-\alpha}(p\,-\,1)^{k-1}dp$$

with $|\mathbf{A}(yp,t)| < A(T)$, whence

$$|\mathbf{h}(y,t)| \; < \; \frac{A(T)\Gamma(\alpha-k)}{|y|^{\alpha-k}\Gamma(\alpha)}$$

From what has been already proved, with $\alpha - k$ playing the role of α,

$$|\mathcal{L}^{-1}\{\mathbf{h}\}(p,t)| \; < \; C(\delta)\frac{A(T)}{\Gamma(\alpha)}|p|^{\alpha-k-1}e^{2|p|\rho}$$

Since $\mathbf{G}(p,t) \; = \; (-1)^k p^k \mathcal{L}^{-1}\{\mathbf{h}\}(p,t)$, by multiplying the above equation by $|p|^k$, the Lemma follows for $\alpha \; > \; 2$ as well. $\square$

Comment 1: The constant 2ρ in the exponential bound can be lowered to anything exceeding ρ, but (14) suffices for our purposes.

Comment 2: Corollary 4 below implies that for any $p \in \mathcal{S}_\phi$, the ILT exists for the functions $\mathbf{r}(y,t)$, $\mathbf{b}_{\mathbf{q},\mathbf{k}}(y,t)$, as well as for the solution $\mathbf{f}(y,t)$, whose existence is shown in the sequel.

Comment 3: Conversely, if $\mathbf{G}(p,t)$ is any integrable function satisfying the exponential bound in (14), it is clear that the Laplace Transform along a ray

$$\mathcal{L}_\theta \mathbf{G} \; \equiv \; \int_0^{\infty e^{i\theta}} \; dp \; e^{-py} \; \mathbf{G}(p,t) \tag{16}$$

exists and defines an analytic function of y in the half-plane $\Re\,[e^{i\theta}y] \; > \; 2\rho$ for $\theta \; \in \; (-\phi,\phi)$.

Comment 4: The next corollary shows that there exist bounds for $\mathbf{B}_{\mathbf{q},\mathbf{k}} \; = \; \mathcal{L}^{-1}\{\mathbf{b}_{\mathbf{q},\mathbf{k}}\}$ and $\mathbf{R} = \mathcal{L}^{-1}\{\mathbf{r}\}$ independent of arg p in $\mathcal{S}_\phi$, because of the assumed analyticity and decay properties in the region $\mathcal{D}_{\rho_0}$, which contains $\mathcal{D}_{\phi,\rho}$.

Corollary 4 *The ILT of the coefficient functions* $\mathbf{b}_{\mathbf{q},\mathbf{k}}$ *(cf. (6)) and the inhomogeneous term* $\mathbf{r}(y,t)$ *satisfy the following upper bounds for any* $p \; \in \; \mathcal{S}_\phi$

$$|\mathbf{B}_{\mathbf{q},\mathbf{k}}(p,t)| \; < \; \frac{C_1(\phi,\alpha_{\mathbf{q}})}{\Gamma(\alpha_{\mathbf{q}}+\beta|\mathbf{k}|)} \; A_b(T) \; |p|^{\beta|\mathbf{k}|+\alpha_{\mathbf{q}}-1} \; e^{2\rho_0|p|} \tag{17}$$

$$|\mathbf{R}(p,t)| \; < \; \frac{C_2(\phi)}{\Gamma(\alpha_r)} \; A_r(T) \; |p|^{\alpha_r-1} \; e^{2\rho_0|p|} \tag{18}$$

Proof. The proof is similar to that of Corollary 3.2 in[7]. From the conditions assumed we see that $\mathbf{b}_{\mathbf{q},\mathbf{k}}$ is analytic in $y \in \mathcal{D}_{\phi_1,\rho_0;y}$ for any ϕ_1 satisfying $(2n)^{-1}\pi > \phi_1 > \phi > 0$. So Lemma 30 can be applied for $\mathbf{g}(y,t) = \mathbf{b}_{\mathbf{q},\mathbf{k}}$, with $\phi_1 = \phi + ((2n)^{-1}\pi - \phi)/2$ replacing ϕ, and with δ replaced by $\phi_1 - \phi = ((2n)^{-1}\pi - \phi)/2$, and the same applies to $\mathbf{R}(p,t)$, leading to (17) and (18). In the latter case, since $\alpha_r \geq 1$, α_0 in Lemma 30 can be chosen to be 1. Thus, one can choose C_2 to be independent of α_r, as indicated in (18). □

The formal inverse Laplace transform (Borel transform) of (3) with respect to y is (see also (6))

$$\partial_t \mathbf{F} + p^n \mathbf{F} = {\sum_{\mathbf{q}\succeq 0}}' \sum_{\mathbf{k}\succeq 0} \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {\prod_{l=1}^{m}}^{*} {\prod_{j=1}^{n}}^{*} \left((-p)^j F_l\right)^{*q_{l,j}} + \mathbf{R}(p,t) \tag{19}$$

where the symbol $*$ stands for convolution

$$(f * g)(p) := \int_0^p f(s) g(p-s) ds \tag{20}$$

${\prod}^{*}$ is a convolution product (see also [5]) and $\mathbf{F} = \mathcal{L}^{-1}\mathbf{f}$. After inverting the differential operator on the left side of (19) with respect to t, we obtain the integral equation

$$\mathbf{F}(p,t) = \mathcal{N}(\mathbf{F}) \equiv$$
$$\int_0^t e^{-p^n(t-\tau)} {\sum_{\mathbf{q}\succeq 0}}' \sum_{\mathbf{k}\succeq 0} \mathbf{B}_{\mathbf{q},\mathbf{k}}(p,\tau) * \mathbf{F}^{*\mathbf{k}}(p,\tau) * {\prod_{l=1}^{m}}^{*} {\prod_{j=1}^{n}}^{*} \left((-p)^j F_l\big(p,\tau)\right)^{*q_{l,j}} d\tau + \mathbf{F}_0(p,t) \tag{21}$$

where

$$\mathbf{F}_0(p,t) = e^{-p^n t} \mathbf{F}_I(p) + \int_0^t e^{-p^n(t-\tau)} \mathbf{R}(p,\tau)\, d\tau \quad \text{and } \mathbf{F}_I = \mathcal{L}^{-1}\{\mathbf{f}_I\} \tag{22}$$

Our strategy is to reduce the problem of existence and uniqueness of a solution of (3) to the problem of existence and uniqueness of a solution of (21), under appropriate conditions.

4 Solution to the integral equation (21)

To establish the existence and uniqueness of solutions to the integral equation, we need to introduce an appropriate function class for both the solution and the coefficient functions.

Definition 5 *Denoting by $\overline{\mathcal{S}_\phi}$ the closure of $\mathcal{S}_\phi$ defined in (12), $\partial\mathcal{S}_\phi = \overline{\mathcal{S}_\phi} \setminus \mathcal{S}_\phi$ and $\mathcal{K} = \overline{\mathcal{S}_\phi} \times [0,T]$, we define for $\nu > 0$ (later to be taken appropriately large) the norm $\|\cdot\|_\nu$ as*

$$\|\mathbf{G}\|_\nu = M_0 \sup_{(p,t)\in\mathcal{K}} (1+|p|^2)\, e^{-\nu|p|}\, |\mathbf{G}(p,t)| \tag{23}$$

where M_0 is a constant (approximately 3.76) defined as

$$M_0 = \sup_{s\geq 0} \left\{ \frac{2(1+s^2)\left(\ln(1+s^2) + s \arctan s\right)}{s(s^2+4)} \right\} \tag{24}$$

Note*: For fixed $\mathbf{F}$, $\|\mathbf{F}\|_\nu$ is nonincreasing in ν.*

Definition 6 *We now define the following class of functions:*

$$\mathcal{A}_\phi \;=\; \left\{ \mathbf{F} : \mathbf{F}(\cdot,t) \;\; \text{analytic in } \mathcal{S}_\phi \text{ and continuous in } \overline{\mathcal{S}_\phi} \;\; \text{for } t \in [0,T] \text{ s.t. } \|\mathbf{F}\|_\nu \;<\; \infty \right\}$$

It is clear that $\mathcal{A}_\phi$ forms a Banach space.

Comment 5: Note that given $\mathbf{G} \in \mathcal{A}_\phi$, $\mathbf{g}(y,t) \;=\; \mathcal{L}_\theta\{\mathbf{G}\}$ exists for appropriately chosen θ when ρ is large enough so that $\rho \;\cos(\theta + \arg\;\, y) > \nu$, and that $\mathbf{g}(y,t)$ is analytic in y and $|y\;\mathbf{g}(y,t)|$ bounded for y on any fixed ray in $\mathcal{D}_{\phi,\rho;y}$.

Lemma 7 *For $\nu > 4\rho_0 + \alpha_r$, $\mathbf{F}_I$ in (22) satisfies*

$$\|\mathbf{F}_I\|_\nu \;<\; C(\phi) A_{f_I} (\nu/2)^{-\alpha_r+1}$$

while $\mathbf{R}$ satisfies the inequality

$$\|\mathbf{R}\|_\nu \;<\; C(\phi) A_r(T) (\nu/2)^{-\alpha_r+1}$$

and therefore

$$\|\mathbf{F}_0\|_\nu \;<\; C(\phi)(T\;A_r + A_{f_I})(\nu/2)^{-\alpha_r+1} \tag{25}$$

Proof. This proof is similar to that of Lemma 4.4 in [7]. First note the bounds on $\mathbf{R}$ in Corollary 4. We also note that $\alpha_r \;\geq\; 1$ and that for $\nu \;>\; 4\rho_0 + \alpha_r$ we have

$$\sup_p \frac{|p|^{\alpha_r \pm 1}}{\Gamma(\alpha_r)} e^{-(\nu-2\rho_0)|p|} \;\leq\; \frac{(\alpha_r \pm 1)^{\alpha_r \pm 1}}{\Gamma(\alpha_r)} e^{-\alpha_r \mp 1} (\nu - 2\rho_0)^{-\alpha_r \mp 1} \;\leq\; K \alpha_r^{1/2 \pm 1} (\nu/2)^{-\alpha_r \mp 1}$$

where K is independent of any parameter. The latter inequality follows from Stirling's formula for $\Gamma(\alpha_r)$ for large α_r.

Using the definition of the $\nu-$norm and the two equations above, the inequality for $\|\mathbf{R}\|_\nu$ follows. Since $\mathbf{f}_I(y)$ is required to satisfy the same bounds as $\mathbf{r}(y,t)$, a similar inequality holds for $\|\mathbf{F}_I\|_\nu$. Now, from the relation (22),

$$|\mathbf{F}_0(p,t)| < |\mathbf{F}_I(p)| + T \sup_{0 \leq t \leq T} |R(p,t)|$$

Therefore, (25) follows. $\square$

Comment 6: Not all Laplace-transformable analytic functions in $\mathcal{D}_{\phi,\rho;y}$ belong to $\mathcal{A}_\phi$. In our assumptions, the coefficients need not be bounded near $p = 0$ and hence do not belong in $\mathcal{A}_\phi$. It is then useful to introduce the following function class:

Definition 8

$$\mathcal{H} \;\equiv\; \left\{ \mathbf{H} : \mathbf{H}(p,t) \text{ analytic in } \mathcal{S}_\phi, \;\, |\mathbf{H}(p,t)| < C|p|^{\alpha-1} e^{\rho|p|} \right\}$$

for some positive constants C and α and ρ which may depend on $\mathbf{H}$.

Lemma 9 *If $\mathbf{H} \in \mathcal{H}$ and $\mathbf{F} \in \mathcal{A}_\phi$, then for $\nu > \rho + 1$, for any j, $\mathbf{H} * F_j$ belongs to $\mathcal{A}_\phi$, and satisfies the following inequality*[2]*:*

$$\|\mathbf{H} * F_j\| \le \left\| |\mathbf{H}| * |F_j| \right\|_\nu \ \le \ C\ \Gamma(\alpha)(\nu - \rho)^{-\alpha}\ \|\mathbf{F}\|_\nu \tag{26}$$

The proof is a vector adaptation of that of Lemma 4.6 in [7].
PROOF. ¿From the elementary properties of convolution, it is clear that $\mathbf{H} * F_j$ is analytic in $\mathcal{S}_\phi$ and is continuous on $\overline{\mathcal{S}_\phi}$. Let $\theta = \arg\ p$. We have

$$|\mathbf{H} * F_j(p)| \ \le \ ||\mathbf{H}| * |F_j|(p)| \le \int_0^{|p|} |\mathbf{H}(se^{i\theta})||F_j(p - se^{i\theta})|ds$$

But

$$|\mathbf{H}(se^{i\theta})| \ \le \ Cs^{\alpha-1}e^{|s|\rho}$$

and

$$\int_0^{|p|} s^{\alpha-1}e^{|s|\rho}|F_j(p - se^{i\theta})|ds \le \|F_j\|_\nu e^{\nu|p|}|p|^\alpha \int_0^1 \frac{s^{\alpha-1}e^{-(\nu-\rho)|p|s}}{M_0(1 + |p|^2(1-s)^2)}ds \tag{27}$$

If $|p|$ is large, noting that $\nu - \rho \ \ge \ 1$, we obtain from Watson's lemma,

$$\int_0^{|p|} s^{\alpha-1}e^{|s|\rho}|F_j(p - se^{i\theta})|ds \le K\Gamma(\alpha)\|F_j\|_\nu \frac{e^{\nu|p|}}{M_0(1 + |p|^2)}|\nu - \rho|^{-\alpha} \tag{28}$$

Now, for any other $|p|$, we obtain from (27),

$$\int_0^{|p|} s^{\alpha-1}e^{|s|\rho}|F_j(p - se^{i\theta})|ds \le K|\nu - \rho|^{-\alpha}\|F_j\|_\nu \frac{e^{\nu|p|}\Gamma(\alpha)}{M_0}$$

Thus (28) must hold in general as it subsumes the above relation when $|p|$ is not large. From this relation, (26) follows by applying the definition of $\| \cdot \|_\nu$. □

Corollary 10 *For $\mathbf{F} \in \mathcal{A}_\phi$, and $\nu > 4\rho_0 + 1$, we have $\mathbf{B}_{\mathbf{q},\mathbf{k}} * F_l \in \mathcal{A}_\phi$ and*

$$\|\mathbf{B}_{\mathbf{q},\mathbf{k}} * F_l\|_\nu \le \left\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{F}| \right\|_\nu \ \le \ K\ C_1(\phi, \alpha_{\mathbf{q}})\ (\nu/2)^{-\langle \mathbf{k} \rangle \beta - \alpha_{\mathbf{q}}} A_b(T)\ \|\mathbf{F}\|_\nu$$

PROOF. The proof follows simply by using Lemma 9, with $\mathbf{H}$ replaced by $\mathbf{B}_{\mathbf{q},\mathbf{k}}$ and using the relations in Corollary 4. □

Lemma 11 *For $\mathbf{F} \in \mathcal{A}_\phi$, with $\nu > 4\rho_0 + 1$, for any j, l,*

$$|\mathbf{B}_{\mathbf{q},\mathbf{k}} * (p^j F_l)| \ \le \ \frac{KC_1|p|^j e^{\nu|p|}A_b(T)}{M_0(1 + |p|^2)}\|\mathbf{F}\|_\nu \ \left(\frac{\nu}{2}\right)^{-\beta\langle \mathbf{k} \rangle - \alpha_{\mathbf{q}}}$$

[2] In the following equation, $\| \cdot \|_\nu$ is extended naturally to functions which are only continuous in $\mathcal{K}$.

Proof.
From the definition (20), it readily follows that

$$|\mathbf{B}_{\mathbf{q},\mathbf{k}} * (p^j F_l)| \;\leq\; |p|^j \; |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |F_l|$$

The rest follows from Corollary (10), and definition of $\|\cdot\|_\nu$. $\square$

Lemma 12 *For* $\mathbf{F},\ \mathbf{G} \;\in\; \mathcal{A}_\phi$ *and* $j \;\geq\; 0$

$$|(p^j F_{l_1}) * G_{l_2}| \;\leq\; |p|^j \;\big|\, |\mathbf{F}| * |\mathbf{G}| \,\big| \tag{29}$$

Proof. Let $p \;=\; |p|\; e^{i\theta}$. Then,

$$|(p^j F_{l_1}) * G_{l_2}| \;\;=\;\; \left| \int_0^p \; \tilde{s}^j \; F_{l_1}(\tilde{s}) G_{l_2}(p - \tilde{s}) \; d\tilde{s} \right| \leq |p|^j \;\; \int_0^{|p|} ds |\mathbf{F}(s\, e^{i\theta})| \; |\mathbf{G}(p - s e^{i\theta})| \tag{30}$$

from which the lemma follows. $\square$

Corollary 13 *If* $\mathbf{F} \;\in\; \mathcal{A}_\phi$*, then*

$$\left| {\prod_{l=1}^{m}}^{\!*} \; {\prod_{j=1}^{n}}^{\!*} \left(p^j F_l\right)^{*q_{l,j}} \right| \;\leq\; |p|^{\sum j q_{l,j}} \left| {\prod_{l=1}^{m}}^{\!*} \; {\prod_{j=1}^{n}}^{\!*} |\mathbf{F}|^{*q_{l,j}} \right| \tag{31}$$

where $\sum j q_{l,j}$ *extends over all* $(l,j) \preceq (m,n)$.

Proof. This follows simply from repeated application of Lemma 12. $\square$

Lemma 14 *For* $\mathbf{F},\ \mathbf{G} \;\in\; \mathcal{A}_\phi$*,*

$$\Big| |\mathbf{F}| * |\mathbf{G}| \Big| \;\leq\; \frac{e^{\nu |p|}}{M_0 (1 + |p|^2)} \|\mathbf{F}\|_\nu \|\mathbf{G}\|_\nu$$

Proof.

$$\Big| |\mathbf{F}| * |\mathbf{G}| \Big| \;=\; \left| \int_0^p |\mathbf{F}(\tilde{s})| \; |\mathbf{G}(p - \tilde{s})| \; d\tilde{s} \right| \leq \int_0^{|p|} ds |\mathbf{F}(s\, e^{i\theta})| \; |\mathbf{G}(p - s\, e^{i\theta})| \tag{32}$$

Using the definition of $\|\cdot\|_\nu$, the above expression is bounded by

$$\frac{e^{\nu|p|}}{M_0^2} \|\mathbf{F}\|_\nu \; \|\mathbf{G}\|_\nu \;\; \int_0^{|p|} \frac{ds}{(1+s^2)[1 \;+\; (|p| - s)^2]} \;\leq\; \frac{|p|^j \;\, e^{\nu |p|}}{M_0 (1 \;+\; |p|^2)} \; \|\mathbf{F}\|_\nu \; \|\mathbf{G}\|_\nu$$

The last inequality follows from the definition (24) of M_0 since

$$\int_0^{|p|} \frac{1}{(1+s^2)[1 \;+\; (|p| - s)^2]} \;=\; 2 \; \frac{\ln \; (|p|^2 + 1) \;+\; |p| \;\tan^{-1} \; |p|}{|p| \; (|p|^2 \;+\; 4)}$$

$\square$

Corollary 15 *For* $\mathbf{F}$, $\mathbf{G} \in \mathcal{A}_\phi$, *then*

$$\big\| |\mathbf{F}| * |\mathbf{G}| \big\|_\nu \;\leq\; \|\mathbf{F}\|_\nu \|\mathbf{G}\|_\nu$$

PROOF. The proof follows readily from Lemma 14 and definition of $\|\cdot\|_\nu$. $\square$

Lemma 16 *For* $\nu > 4\rho_0 + 1$,

$$\left| \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*k} * \prod_{l=1}^{m}{}^{*} \prod_{j=1}^{n}{}^{*} \left(p^j F_l\right)^{*q_{l,j}} \right| \leq \frac{e^{\nu|p|}|p|^{\sum jq_{l,j}}}{M_0(1+|p|^2)} \|\mathbf{F}\|_\nu^{\langle \mathbf{q}\rangle + \langle \mathbf{k}\rangle - 1} \big\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{F}| \big\|_\nu \tag{33}$$

if $(\mathbf{q}, \mathbf{k}) \neq (\mathbf{0}, \mathbf{0})$ *and is zero if* $(\mathbf{q}, \mathbf{k}) = (\mathbf{0}, \mathbf{0})$.

PROOF. For $(\mathbf{k}, \mathbf{q}) = (\mathbf{0}, \mathbf{0})$ we have $\mathbf{B}_{\mathbf{q},\mathbf{k}} = 0$ (see comments after eq. (6)). If $\mathbf{k} \neq \mathbf{0}$, we can use Corollary 13 to argue that the left hand side of (33) is bounded by

$$|p|^{\sum jq_{l,j}} \left| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{F}| * |\mathbf{F}|^{*(\langle \mathbf{k}\rangle - 1)} * \prod_{l=1}^{m}{}^{*} \prod_{j=1}^{n}{}^{*} |\mathbf{F}|^{*q_{l,j}} \right|$$

Using Corollaries 10 and 15, the proof if $\mathbf{k} \neq 0$ follows. Similar steps work for the case $\mathbf{k} = \mathbf{0}$ and $\mathbf{q} \neq \mathbf{0}$, except that $\mathbf{B}_{\mathbf{q},\mathbf{k}}$ is convolved with $p^{j_1} F_{l_1}$ for some (j_1, l_1), for which the corresponding $q_{l_1,j_1} \neq 0$, and we now use Lemma 12 and the definition of $\|\cdot\|_\nu$. $\square$

Corollary 17 *For* $\nu > 4\rho_0 + 1$,

$$\left| \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*k} * \prod_{l=1}^{m}{}^{*} \prod_{j=1}^{n}{}^{*} \left(p^j F_l\right)^{*q_{l,j}} \right| \leq \frac{KC_1 A_b(T) e^{\nu|p|}|p|^{\sum jq_{l,j}}}{M_0(1+|p|^2)} \left(\frac{\nu}{2}\right)^{-\langle \mathbf{k}\rangle\beta - \alpha_{\mathbf{q}}} \|\mathbf{F}\|_\nu^{\langle \mathbf{q}\rangle + \langle \mathbf{k}\rangle} \tag{34}$$

PROOF. The proof follows immediately from Corollary 10 and Lemma 16. $\square$

Lemma 18 *For* $\nu > 4\rho_0 + 1$ *we have*

$$\left| \int_0^t e^{-p^n(t-\tau)} \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*k} * \prod_{l=1}^{m}{}^{*} \prod_{j=1}^{n}{}^{*} \left(p^j F_l\right)^{*q_{l,j}} \, d\tau \right|$$
$$\leq \frac{CA_b(T)e^{\nu|p|}}{M_0(1+|p|^2)} \left(\frac{\nu}{2}\right)^{-\langle \mathbf{k}\rangle\beta - \alpha_{\mathbf{q}}} \|\mathbf{F}\|_\nu^{\langle \mathbf{q}\rangle + \langle \mathbf{k}\rangle} \left(\frac{\nu}{2}\right)^{-\beta\langle \mathbf{k}\rangle - \alpha_{\mathbf{q}}} T^{(n-\sum jq_{l,j})/n} \tag{35}$$

where the constant C *is independent of* T, *but depends on* ϕ.

PROOF. The proof follows from Lemmas 11 and 16 and the fact that for $0 \le l \le n$,

$$|p|^l \int_0^t e^{-|p|^n \cos(n\theta)(t-\tau)} d\tau \;\le\; \frac{T^{(n-l)/n}}{\cos^{l/n}(n\phi)} \sup_{\gamma} \frac{1-e^{-\gamma^n}}{\gamma^{n-l}} \tag{36}$$

□

Definition 19 *For* $\mathbf{F}$ *and* $\mathbf{h}$ *in* $\mathcal{A}_\phi$*, and* $\mathbf{B}_{\mathbf{q},\mathbf{k}} \in \mathcal{H}$*, as above, define* $\mathbf{h}_0 = \mathbf{0}$ *and for* $k \ge 1$*,*

$$\mathbf{h}_{\mathbf{k}} \;\equiv\; \mathbf{B}_{\mathbf{q},\mathbf{k}} * [(\mathbf{F}+\mathbf{h})^{*\mathbf{k}} - \mathbf{F}^{*\mathbf{k}}]. \tag{37}$$

Lemma 20 *For* $\nu > 4\rho_0 + 1$*, and for* $\mathbf{k} \;\ne\; 0$*,*

$$\|\mathbf{h}_{\mathbf{k}}\|_\nu \;\le\; \langle\mathbf{k}\rangle \Big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\Big)^{\langle\mathbf{k}\rangle - 1} \Big\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\Big\|_\nu \tag{38}$$

and is zero for $\mathbf{k} = 0$*.*

PROOF. The case of $\mathbf{k} = 0$ follows from definition of $\mathbf{h}_0$. The general expression above for $\mathbf{k} \ne 0$ is proved by induction. The case of $\langle\mathbf{k}\rangle \;=\; 1$ is obvious from (37). Assume formula (38) holds for all $\langle\mathbf{k}\rangle \;\le\; l$. Then all multiindices of length $l+1$ can be expressed as $\mathbf{k} + \mathbf{e}_i$, where $\mathbf{e}_i$ is an the m dimensional unit vector in the i-th direction for some i, and $\mathbf{k}$ has length l.

$$\|\mathbf{h}_{\mathbf{k}+\mathbf{e}_i}\|_\nu \;=\; \|\mathbf{B}_{\mathbf{q},\mathbf{k}} * (F_i + h_i) * (\mathbf{F}+\mathbf{h})^{*\mathbf{k}} - \mathbf{B}_{\mathbf{q},\mathbf{k}} * F_i * \mathbf{F}^{*\mathbf{k}}\|_\nu \;=\; \|\mathbf{B}_{\mathbf{q},\mathbf{k}} * h_i * (\mathbf{F}+\mathbf{h})^{*\mathbf{k}} + F_i * \mathbf{h}_{\mathbf{k}}\|_\nu$$

Using (38) for $\langle\mathbf{k}\rangle = l$, we get

$$\le \||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\|_\nu \left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^l + l\|\mathbf{F}\|_\nu \Big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\Big)^{l-1} \Big\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\Big\|_\nu$$

$$\le (l+1)\Big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\Big)^l \Big\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\Big\|_\nu$$

Thus (38) holds for $\langle\mathbf{k}\rangle = l+1$. □

Definition 21 *For* $\mathbf{F} \in \mathcal{A}_\phi$ *and* $\mathbf{h} \in \mathcal{A}_\phi$*, and* $\mathbf{B}_{\mathbf{q},\mathbf{k}}$ *as above define* $\mathbf{g}_{\mathbf{0}} = \mathbf{0}$*, and for* $\langle\mathbf{q}\rangle \ge 1$*,*

$$\mathbf{g}_{\mathbf{q}} \;\equiv\; \mathbf{B}_{\mathbf{q},\mathbf{k}} * {}^*\prod_{l=1}^{m} {}^*\prod_{j=1}^{n} \left(p^j [F_l + h_l]\right)^{*q_{l,j}} - \;\mathbf{B}_{\mathbf{q},\mathbf{k}} * {}^*\prod_{l=1}^{m} {}^*\prod_{j=1}^{n} \left(p^j F_l\right)^{*q_{l,j}} \tag{39}$$

Lemma 22 *For* $\nu > 4\rho_0 + 1$*,* $\mathbf{g}_{\mathbf{0}} = 0$ *and for* $\langle\mathbf{q}\rangle \ge 1$

$$|\mathbf{g}_{\mathbf{q}}| \;\le\; \frac{e^{\nu|p|}|p|^{\sum j q_{l,j}} \langle\mathbf{q}\rangle}{M_0(1+|p|^2)} \Big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\Big)^{\langle\mathbf{q}\rangle - 1} \Big\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\Big\|_\nu \tag{40}$$

and is zero for $\mathbf{q} = 0$*.*

Proof. The case for $\mathbf{q} = 0$ follows by definition of $\mathbf{g_0}$. We prove the other cases by induction. The case $\langle \mathbf{q} \rangle = 1$ is clear from (39), since only linear terms in $\mathbf{F}$ are involved. Assume the inequality (40) holds for a particular $\mathbf{q}$. We now show that it holds when $\mathbf{q}$ is replaced by $\mathbf{q}+\mathbf{e}$, where $\mathbf{e}$ is a $m \times n$ dimensional unit vector, say in the (l_1, j_1) direction. So,

$$
\begin{aligned}
&|\mathbf{g}_{\mathbf{q}+\mathbf{e}}| \\
&= \left| \mathbf{B}_{\mathbf{q},\mathbf{k}} * \left[p^{j_1} (F_{l_1} + h_{l_1}) \right] * {}^{*}\prod_{l=1}^{m} {}^{*}\prod_{j=1}^{n} \left[p^j (\mathbf{F} + \mathbf{h}) \right]^{*q_{l,j}} - \mathbf{B}_{\mathbf{q},\mathbf{k}} * \left[p^{j_1} F_{l_1} \right] * {}^{*}\prod_{l=1}^{m} {}^{*}\prod_{j=1}^{n} \left[p^j \mathbf{F} \right]^{*q_{l,j}} \right| \\
&\le \left| \mathbf{B}_{\mathbf{q},\mathbf{k}} * \left(p^{j_1} h_{l_1} \right) \right| * \left| {}^{*}\prod_{l=1}^{m} {}^{*}\prod_{j=1}^{n} \left[p^j (\mathbf{F} + \mathbf{h}) \right]^{*q_{l,j}} \right| + \left| (p^{j_1} F_{l_1}) * \mathbf{g_q} \right| \qquad (41)
\end{aligned}
$$

Using Lemma 16 and equation (40), we get the following upper bound

$$
\begin{aligned}
|\mathbf{g}_{\mathbf{q}+\mathbf{e}}| \le{}& \frac{|p|^{j_1 + \sum j q_{l,j}} e^{\nu |p|}}{M_0 (1 + |p|^2)} \left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu \right)^{\sum q_{l,j}} \left\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \right\|_\nu \\
&+ \frac{|p|^{j_1 + \sum j q_{l,j}} \langle \mathbf{q} \rangle e^{\nu |p|}}{M_0 (1 + |p|^2)} \left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu \right)^{\langle \mathbf{q} \rangle - 1} \|\mathbf{F}\|_\nu \left\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \right\|_\nu \\
\le{}& \frac{|p|^{\sum j (q_{l,j} + e_{l,j})} (\langle \mathbf{q} + \mathbf{e} \rangle) e^{\nu |p|}}{M_0 (1 + |p|^2)} \left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu \right)^{\langle \mathbf{q} \rangle} \left\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \right\|_\nu
\end{aligned}
$$

Therefore (40) holds when $\mathbf{q}$ is replaced by $\mathbf{q}+\mathbf{e}$ and the induction step is proved. □

Lemma 23 *For $\mathbf{F}$ and $\mathbf{h}$ in $\mathcal{A}_\phi$, $\nu > 4\rho_0 + 1$,*

$$
\begin{aligned}
&\left| \mathbf{B}_{\mathbf{q},\mathbf{k}} * \left(\mathbf{F} + \mathbf{h} \right)^{*\mathbf{k}} * {}^{*}\prod_{l=1}^{m} {}^{*}\prod_{j=1}^{n} \left(p^j (F_l + h_l) \right)^{*q_{l,j}} - \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {}^{*}\prod_{l=1}^{m} {}^{*}\prod_{j=1}^{n} \left(p^j F_l \right)^{*q_{l,j}} \right| \\
&\qquad \le \frac{|p|^{\sum j q_{l,j}} (\langle \mathbf{q} \rangle + \langle \mathbf{k} \rangle) e^{\nu |p|}}{M_0 (1 + |p|^2)} \left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu \right)^{\langle \mathbf{k} \rangle + \langle \mathbf{q} \rangle - 1} \| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \|_\nu \qquad (42)
\end{aligned}
$$

if $(\mathbf{q}, \mathbf{k}) \ne (\mathbf{0}, \mathbf{0})$ and is zero otherwise.

Proof. It is clear from (37) that the left side of (42) is simply

$$
\left| \mathbf{h_k} * {}^{*}\prod_{l=1}^{m} {}^{*}\prod_{j=1}^{n} \left(p^j (F_l + h_l) \right)^{*q_{l,j}} + \mathbf{F}^{*\mathbf{k}} * \mathbf{g_q} \right|
$$

However, from Corollary 13, Lemmas 14 and 20,

$$
\left| \mathbf{h_k} * {}^{*}\prod_{l=1}^{m} {}^{*}\prod_{j=1}^{n} \left(p^j (F_l + h_l) \right)^{*q_{l,j}} \right| \le \frac{|p|^{\sum j q_{l,j}} \langle \mathbf{k} \rangle e^{\nu |p|}}{M_0 (1 + |p|^2)} \left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu \right)^{\langle \mathbf{k} \rangle + \langle \mathbf{q} \rangle - 1} \left\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \right\|_\nu
$$

and from Corollary 13, Lemmas 14 and 22,

$$\left|\mathbf{F}^{*\mathbf{k}} * \mathbf{g}_{\mathbf{q}}\right| \;\le\; \frac{|p|^{\sum j q_{l,j}} \langle \mathbf{q} \rangle e^{\nu|p|}}{M_0(1+|p|^2)} \left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{\langle \mathbf{k}\rangle + \langle \mathbf{q} \rangle - 1} \left\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \right\|_\nu$$

Combining these two inequalities, the proof of the lemma follows. □

Lemma 24 *For $\nu > 4\rho_0 + 1$ we have*

$$\Bigg\| \int_0^t e^{-p^n(t-\tau)} \Bigg[\mathbf{B}_{\mathbf{q},\mathbf{k}} * \Big(\mathbf{F} + \mathbf{h}\Big)^{*\mathbf{k}} * {\prod_{l=1}^{m}}^{*} {\prod_{j=1}^{n}}^{*} \left(p^j (F_l + h_l)\right)^{*q_{l,j}} - \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {\prod_{l=1}^{m}}^{*} {\prod_{j=1}^{n}}^{*} \left(p^j F_l\right)^{*q_{l,j}} \Bigg] d\tau \Bigg\|_\nu$$
$$\le\; A_b(T) C(\phi) (\langle \mathbf{q}\rangle + \langle \mathbf{k}\rangle) \left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{\langle \mathbf{k}\rangle + \langle \mathbf{q}\rangle - 1} T^{(n - \sum j q_{l,j})/n} \left(\frac{\nu}{2}\right)^{-\beta\langle \mathbf{k}\rangle - \alpha_{\mathbf{q}}} \|\mathbf{h}\|_\nu \quad (43)$$

Proof. The proof follows from Corollary 10 and Lemma 23 and the definition of $\|\cdot\|_\nu$ along with the bound (36). □

Lemma 25 *For $\mathbf{F} \in \mathcal{A}_\phi$, and $\nu > 4\rho_0 + \alpha_r$ large enough so that $\left(\frac{\nu}{2}\right)^{-\beta} (\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu) < 1$ (see Note after Definition (5)), $\mathcal{N}(\mathbf{F})$ defined in (21) satisfies the following bounds*

$$\|\mathcal{N}(\mathbf{F})\|_\nu \;\le\; \|\mathbf{F}_0\|_\nu \;+\; C(\phi) A_b(T) {\sum_{\mathbf{q}\succeq 0}}' \sum_{\mathbf{k}\succeq 0} T^{(n-\sum j q_{l,j})/n} \left(\frac{2^\beta \|\mathbf{F}\|_\nu}{\nu^\beta}\right)^{\langle \mathbf{k}\rangle} \left(\frac{\nu}{2}\right)^{-\alpha_{\mathbf{q}}} \|\mathbf{F}\|_\nu^{\langle \mathbf{q}\rangle} \quad (44)$$

$$\|\mathcal{N}(\mathbf{F}+\mathbf{h}) - \mathcal{N}(\mathbf{F})\|_\nu$$
$$\le C(\phi) A_b(T) \|\mathbf{h}\|_\nu {\sum_{\mathbf{q}\succeq 0}}' \sum_{\mathbf{k}\succeq 0} T^{(n-\sum j q_{l,j})/n} \left(\frac{\nu}{2}\right)^{-\beta\langle \mathbf{k}\rangle - \alpha_{\mathbf{q}}} (\langle \mathbf{q}\rangle + \langle \mathbf{k}\rangle) \left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{\langle \mathbf{q}\rangle + \langle \mathbf{k}\rangle - 1} \quad (45)$$

Proof. The proofs are immediate from the expression (21) of $\mathcal{N}(\mathbf{F})$ and Lemmas 18, 20 and 24. The condition $\left(\frac{\nu}{2}\right)^{-\beta} (\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu) < 1$ guarantees the convergence of the infinite series involving summation in $\mathbf{k}$. Note that with respect to the multi-index $\mathbf{q}$ we only have a finite sum, due to (4). Note also that the summation over $\mathbf{k}$ can also be bounded by a more explicit function, if so desired. □

Comment 7: Lemma 25 is the key to showing the existence and uniqueness of a solution in $\mathcal{A}_\phi$ to (21), since it provides the conditions for the nonlinear operator $\mathcal{N}$ to map a ball into itself as well the necessary contractivity condition.

Lemma 26 *If there exists some $b > 1$ so that*

$$(\nu/2)^{-\beta} b \|\mathbf{F}_0\|_\nu < 1 \tag{46}$$

and

$$C(\phi) A_b(T) {\sum_{\mathbf{q} \succeq 0}}' \sum_{\mathbf{k} \succeq 0} T^{(n - \sum j q_{l,j})/n} \left(\frac{\nu}{2}\right)^{-\langle \mathbf{k} \rangle \beta - \alpha_{\mathbf{q}}} \|b\mathbf{F}_0\|_\nu^{\langle \mathbf{k} \rangle + \langle \mathbf{q} \rangle} \; < \; 1 - \frac{1}{b} \tag{47}$$

then the nonlinear mapping $\mathcal{N}$, as defined in (21), maps a ball of radius $b\|\mathbf{F}_0\|_\nu$ into itself. Further, if

$$C(\phi) A_b(T) {\sum_{\mathbf{q} \succeq 0}}' \sum_{\mathbf{k} \succeq 0} T^{(n - \sum j q_{l,j})/n} (\langle \mathbf{q} \rangle + \langle \mathbf{k} \rangle) \left(\frac{\nu}{2}\right)^{-\langle \mathbf{k} \rangle \beta - \alpha_{\mathbf{q}}} (3b)^{\langle \mathbf{k} \rangle + \langle \mathbf{q} \rangle - 1} \|\mathbf{F}_0\|_\nu^{\langle \mathbf{k} \rangle + \langle \mathbf{q} \rangle - 1} \; < \; 1 \tag{48}$$

then $\mathcal{N}$ is a contraction there.

Proof. This is a simple application of Lemma 25, if we note that $\|\mathbf{F}\|_\nu^k \; < \; b^k \|\mathbf{F}_0\|_\nu^k$ and the fact that for both $\mathbf{F}$ and $\mathbf{F} + \mathbf{h}$ in the ball of radius $b\|\mathbf{F}\|_0$, $\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu \; \leq \; 3b\|\mathbf{F}_0\|_\nu$. $\square$

Lemma 27 *For any given $T \; > \; 0$ and ϕ in the interval $(0, \; (2n)^{-1}\pi)$, for all sufficiently large ν, there exists a unique $\mathbf{F} \; \in \; \mathcal{A}_\phi$ that satisfies the integral equation (21).*

Proof. We choose $b = 2$ for definiteness. It is clear from the bounds on $\|F_0\|_\nu$ in Lemma 7 that for given T, since $\alpha_r \geq 1$, we have $b(\nu/2)^{-\beta}\|\mathbf{F}_0\|_\nu \; < \; 1$ for all sufficiently large ν. Further, it is clear by inspection that all conditions (46), (47) and (48) are satisfied for all sufficiently large ν. The lemma now follows from the contractive mapping theorem. $\square$

4.1 Behavior of ${}^s\mathbf{F}$ near $p = 0$

Proposition 28 *For some $K_1 \; > \; 0$ and small p we have $|{}^s\mathbf{F}| \; < \; K_1 |p|^{\alpha_r - 1}$ and thus $|{}^s\mathbf{f}| \; < \; K_2 |y|^{-\alpha_r}$ for some $K_2 > 0$ in $\mathcal{D}_{\phi,\rho}$ as $|y| \to \infty$.*

Proof. The idea of the proof here is to think of the solution ${}^s\mathbf{F}$ to (21) as a solution to a linear equation of the form

$${}^s\mathbf{F} = \mathcal{G}\,({}^s\mathbf{F}) + \mathbf{F}_0 \quad \text{or} \quad {}^s\mathbf{F} = (1 - \mathcal{G})^{-1}\mathbf{F}_0 \tag{49}$$

Here, the suitably chosen operator $\mathcal{G}$, while depending on ${}^s\mathbf{F}$, is thought of a known quantity (in effect, ${}^s\mathbf{F}$ *is* now known). The expression of a suitable $\mathcal{G}$ is, however, somewhat involved, and this is given in the following.

Convergence in $\| \cdot \|_\nu$ implies uniform convergence on compact subsets of $\mathcal{K}$ and we can interchange summation and integration in (21). With ${}^s\mathbf{F}$ the unique solution of (21) we define

$$\mathbf{G}_i = {\sum}^{\dagger} \mathbf{B}_{\mathbf{q},\mathbf{k}} * {}^s\mathbf{F}^{*\mathbf{k}_{ic}} * {}^*\prod_{l=1}^{m} {}^*\prod_{j=1}^{n} \left[(-p)^j \; ({}^sF_l)\right]^{*q_{l,j}}$$

where $\sum^{\dagger}$ stands for the sum over $\mathbf{k} \neq 0, k_i \neq 0$ and $k_{i'} = 0$ for $i' < i$, the notation ${}^s\mathbf{F}^{*\mathbf{k}_{ic}}$ stands for $({}^sF_1)^{*k_1} * ({}^sF_2)^{*k_2} * \cdots * ({}^sF_m)^{*k_m}$, except that the term ${}^sF_i^{*k_i}$ is replaced by ${}^sF_i^{*(k_i-1)}$ in this convolution.

$$\hat{\mathbf{G}}_{j',l'} = \sum\nolimits^{\ddagger} \mathbf{B}_{\mathbf{q},\mathbf{0}} * {}^*\prod_{l=1}^{m}{}^{\ddagger}\, {}^*\prod_{j=1}^{n}{}^{\ddagger} \left[(-p)^j({}^sF_l)\right]^{*q_{l,j}}$$

where $\sum^{\ddagger}$ stands for the sum over $\mathbf{q} \succ 0,\ q_{l',j'} \neq 0,$ and $q_{l,j} = 0$ for $l+j < l'+j'$ and for $l+j = l'+j',$ when $l < l'$. Also, $\prod^{\ddagger}$ indicates that the exponent of the term $l = l',\ j = j'$ is changed from $q_{l',j'}$ to $q_{l',j'} - 1$. Now we define a linear operator $\mathcal{G}$ by

$$\mathcal{G}\mathbf{Q} = \int_0^t e^{-p^n(t-\tau)} \left[\sum_{i=1}^{m} \mathbf{G}_i * Q_i + \sum_{j'=1}^{n}\sum_{l'=1}^{m} \hat{\mathbf{G}}_{j',l'} * \left((-p)^j Q_{l'}\right)\right]$$

Then, by carefully comparing with (21), one finds that ${}^s\mathbf{F}$ satisfies (49).

For $a > 0$ small enough define $\overline{\mathcal{S}}_a = \overline{\mathcal{S}} \cap \{p : |p| \leq a\}$. Since ${}^s\mathbf{F}$ is continuous in $\overline{\mathcal{S}}$ we have $\lim_{a\downarrow 0} \|\mathcal{G}\| = 0$, where the norm is taken over $C(\overline{\mathcal{S}}_a)$.

By (5), (10), (22) and Lemma 30, we see that $\|\mathbf{F}_0\|_\infty \leq K_3|a|^{\alpha_r-1}$ in $\overline{\mathcal{S}}_a$ for some $K_3 > 0$ independent of a. Then, as $a \downarrow 0$, we have

$$\max_{\overline{\mathcal{S}}_a} |{}^s\mathbf{F}(p,t)| = \|{}^s\mathbf{F}\| \leq (1 - \|\mathcal{G}\|)^{-1} \max_{\overline{\mathcal{S}}_a} \|\mathbf{F}_0\| \leq 2K_3|a|^{\alpha_r-1}$$

and thus for small p we have $|\mathbf{F}(p,t)| \leq 2K_3|p|^{\alpha_r-1}$ and the proposition follows. $\square$

4.2 Completion of proof of Theorem 2.

Lemma 30 shows that if $\mathbf{f}$ is any solution of (3) satisfying Condition 1, then $\mathcal{L}^{-1}\{\mathbf{f}\} \in \mathcal{A}_{\phi-\delta}$ for $0 < \delta < \phi$ for ν sufficiently large. For large y, the series (6) converges uniformly and thus $\mathbf{F} = \mathcal{L}^{-1}\{\mathbf{f}\}$ satisfies (21), which by Lemma 27 has a unique solution in $\mathcal{A}_\phi$ for any $\phi \in (0,\ (2n)^{-1}\pi)$. Conversely, if ${}^s\mathbf{F} \in \mathcal{A}_{\tilde{\phi}}$ is the solution of (21) for $\nu > \nu_1$, then from Comment 5, ${}^s\mathbf{f} = \mathcal{L}\,{}^s\mathbf{F}$ is analytic in y in $\mathcal{D}_{\phi,\rho}$ for $0 < \phi < \tilde{\phi} < (2n)^{-1}\pi$, for sufficiently large ρ, where in addition from Proposition 33, ${}^s\mathbf{f} = O(y^{-\alpha_r})$. This implies that the series in (3) converges uniformly and by the properties of Laplace transforms, ${}^s\mathbf{f}$ solves (1) and satisfies condition (1).

5 Formal expansions and their rigorous justification

5.1 In a heuristic calculation of a formal solution to (3) relying on the smallness of $\mathbf{f}$ and our assumptions on the nonlinearity, the most important terms for large y (giving the "dominant balance") are $\mathbf{f}_t$ on the left side of (3) and $\mathbf{r}(y,t)$ on the right side. This suggests that, to leading order, $\mathbf{f}(y,t) \sim \mathbf{f}_I(y) + \int_0^t \mathbf{r}(y,t)dt$ (since the functions $\mathbf{f}_I(y)$ and $\mathbf{r}(y,t)$ decay at a rate $y^{-\alpha_r}$, which for $\alpha_r > 1$ is much less than y^{-1}, other terms in the differential equation (3) should not contribute). We then decompose $\mathbf{f}(y,t) = A_1(t)y^{-\alpha_r} + \tilde{\mathbf{f}}$ and substitute into (3); the equation for $\tilde{\mathbf{f}}$ will generally satisfy an equation of the same form as $\mathbf{f}$ in (3), but with α_r replaced by a larger number; this procedure generates in principle a formal asymptotic series for the solution, as is the case in the examples in [7].

5.2 When the formal procedure in the preceding section gives the leading order behavior $\mathbf{f}(y,t) \sim A_1(t)y^{-\alpha_r}$, the validity of this asymptotic relation is rigorously shown in $\mathcal{D}_{\phi,\tilde{\rho}}$ as follows. We write as before $\mathbf{f}(y,t) = A_1(t)y^{-\alpha_r} + \tilde{\mathbf{f}}$ in (3). With α_r in the equation for $\tilde{\mathbf{f}}$ larger than the α_r of $\mathbf{f}$, our theorem guarantees that $\tilde{\mathbf{f}}$ is indeed $o(y^{-\alpha_r})$. We can recursively use this procedure on $\tilde{\mathbf{f}}$, and so on, to justify the asymptotic expansion of $\mathbf{f}$ for large y in $\mathcal{D}_{\phi,\rho;y}$.

These arguments also show that the assumption $\alpha_r \geq 1$ is not crucial, since the terms in the asymptotic series for $\mathbf{f}$ for which the exponent does not satisfy this condition can be subtracted out explicitly, as there are generally finitely many of them.

5.3 Conversely, if the inverse Laplace transform ${}^s\mathbf{F}$ of the solution ${}^s\mathbf{f}$ has a convergent Puiseux series about $p = 0$, which is for instance the case in the examples treated in [7], then by Watson's Lemma ${}^s\mathbf{f}$ will have for large y an asymptotic series in inverse powers of y, and the representation ${}^s\mathbf{f} = \mathcal{L}\{{}^s\mathbf{F}\}$ makes ${}^s\mathbf{f}$ (by definition) the Borel sum of its asymptotic series. (The large width of the sector is then needed to guarantee the uniqueness).

6 Solution and asymptotics for small t

We consider solution small time and its asymptotics when variable $yt^{-1/n}$ is large. In this case, it is convenient to use a scaled variable

$$\zeta = yt^{-1/n}, \hat{\mathbf{f}}(\zeta,t) = \mathbf{f}(t^{1/n}\zeta,t) \tag{50}$$

We seek to determine asymptotics of $\hat{f}$ for $|\zeta|$ large. Accordingly, it convenient to introduce change of variable in the Borel-plane as well:

$$s = pt^{1/n}, \hat{\mathbf{F}}(s,\tau;t) = \mathbf{F}(t^{-1/n}s, t\tau) \tag{51}$$

It is to be noted that

$$\hat{\mathbf{f}}(\zeta,t) = t^{-1/n}\int_0^\infty e^{-s\zeta}\hat{\mathbf{F}}(s,1;t)ds \tag{52}$$

The integral equation (21) becomes

$$\hat{\mathbf{F}}(s,1;t) = \hat{\mathbf{F}}_{\mathbf{0}}(s,1;t) \ + \\ \int_0^1 e^{-s^n(1-\tau)}{\sum_{\mathbf{q}\succeq 0}}' \sum_{\mathbf{k}\succeq 0} t^{1-\sum_{j=1}^n \sum_{l=1}^m \frac{j}{n} q_{l,j}} \left\{ \hat{\mathbf{B}}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {\prod_{l=1}^m}^{*} {\prod_{j=1}^n}^{*} \left((-s)^j \hat{F}_l\right)^{*q_{l,j}} \right\} (s,\tau;t)d\tau \tag{53}$$

From the definition of $\hat{\mathbf{F}}(s,\tau;t)$ in terms of $\mathbf{F}$, it follows that $\hat{\mathbf{F}}(s,\tau;t) = \hat{\mathbf{F}}(s\tau^{1/n}, 1; t\tau)$. Hence (53) implies that

$$\hat{\mathbf{F}}(s,\tau;t) = \hat{\mathcal{N}}(\hat{\mathbf{F}})(s,\tau;t) \equiv \hat{\mathbf{F}}_{\mathbf{0}}(s,\tau;t) \ +$$
$$\int_0^1 e^{-s^n\tau(1-\tau')} {\sum_{\mathbf{q}\succeq 0}}' \sum_{\mathbf{k}\succeq 0} \tau t^{1-\sum_{j=1}^n \sum_{l=1}^m \frac{j}{n} q_{l,j}} \left\{ \hat{\mathbf{B}}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {}^*\prod_{l=1}^m {}^*\prod_{j=1}^n \Big((-s)^j \hat{F}_l \Big)^{*q_{l,j}} \right\} (s, \tau\tau', t) d\tau' \tag{54}$$

It is convenient to introduce the domain

$$\hat{\mathcal{S}}_\phi \ \equiv \ \left\{ s \ : \ \arg \ s \in (-\phi, \ \phi) \ , \ \ 0 < |s| < \infty \ , \ \ 0 \ < \ \phi < \frac{\pi}{2n} \right\} \tag{55}$$

The corresponding domain in the ζ-plane will be denoted by $\hat{\mathcal{D}}_{\phi,\rho}$, *i.e.*

$$\hat{\mathcal{D}}_{\phi,\rho} \equiv \left\{ \zeta : \ arg \ \zeta \in \left(-\frac{\pi}{2} - \frac{\pi}{2n}, \frac{\pi}{2} + \frac{\pi}{2n} \right) \ , \ |\zeta| > \rho \right\}$$

We introduce the following norm in $\hat{\mathcal{S}}_\phi$:

$$\|\hat{\mathbf{F}}(\cdot,\cdot;t)\|_\nu = \sup_{0\le\tau\le 1, s\in\hat{\mathcal{S}}_\phi} (1+|s|^2) e^{-\nu|s|} |\hat{\mathbf{F}}(q,\tau;t)| \tag{56}$$

With this norm, it is clear that the class $\hat{\mathcal{A}}_\phi$ of analytic functions in $\hat{\mathcal{S}}_\phi$, which is continuous up to the boundary and have finite $\|.\|_\nu$, forms a Banach space. For the purposes of this section, it is also convenient to define $c_{\mathbf{q},\mathbf{k}}(\nu,t)$ to be upper bounds of $\|\hat{\mathbf{B}}_{\mathbf{q},\mathbf{k}} * \hat{\mathbf{F}}\|_\nu / \|\hat{\mathbf{F}}\|_\nu$. We also define $C(\phi)$ as:

$$C(\phi) \equiv \sup_{0\le l\le n} \frac{1}{\cos^{l/n}(n\phi)} \sup_\gamma \frac{1-e^{-\gamma^n}}{\gamma^{n-l}}$$

Lemma 29 *If the following conditions hold for some $b \ > \ 1$ and $\nu > 1$,*

$$C(\phi) {\sum_{\mathbf{q}\succeq 0}}' \sum_{\mathbf{k}\succeq 0} t^{1-\sum_{j=1}^n \sum_{l=1}^m \frac{j}{n} q_{l,j}} \ c_{\mathbf{q},\mathbf{k}}(\nu,t) \|b\hat{\mathbf{F}}_{\mathbf{0}}\|^{\langle\mathbf{k}\rangle+\langle\mathbf{q}\rangle} \ \le \ (b-1)\|\hat{\mathbf{F}}\|_\nu$$

and

$$C(\phi) {\sum_{\mathbf{q}\succeq 0}}' \sum_{\mathbf{k}\succeq 0} t^{1-\sum_{j=1}^n \sum_{l=1}^m \frac{j}{n} q_{l,j}} \ c_{\mathbf{q},\mathbf{k}}(\nu,t) (\langle\mathbf{q}\rangle + \langle\mathbf{k}\rangle) \|3b\hat{\mathbf{F}}_{\mathbf{0}}\|^{\langle\mathbf{k}\rangle+\langle\mathbf{q}\rangle-1} \le 1$$

then the mapping $\hat{\mathcal{N}}$ defined in (54) is a contraction mapping and therefore there will be a unique solution $\hat{\mathbf{F}}$ to the integral equation (54).

Proof. First, note that using Lemma 16 (with $\rho_0 = 0$), with variable s replacing p, and using definition of $c_{\mathbf{q},\mathbf{k}}$, it follows that

$$\left| \left\{ \hat{\mathbf{B}}_{\mathbf{q},\mathbf{k}} * \hat{\mathbf{F}}^{*k} * {}^*\prod_{l=1}^m {}^*\prod_{j=1}^n \left(s^j \hat{F}_l \right)^{*q_{l,j}} \right\} (s, \tau\tau'; t) \right| \le \ \frac{e^{\nu|s|}|s|^{\sum jq_{l,j}}}{M_0(1+|s|^2)} c_{\mathbf{q},\mathbf{k}}(\nu,t) \|\hat{\mathbf{F}}\|_\nu^{\langle\mathbf{q}\rangle+\langle\mathbf{k}\rangle}$$

Hence using integration similar to the proof of Lemma 18, we obtain

$$\left\| \int_0^1 \tau e^{-s^3\tau(1-\tau')} \hat{\mathbf{B}}_{\mathbf{q},\mathbf{k}} * \hat{\mathbf{F}}^{*k} * {}^*\prod_{l=1}^{m} {}^*\prod_{j=1}^{n} \left(s^j \hat{F}_l \right)^{*q_{l,j}} (s, \tau\tau'; t) d\tau' \right\|_\nu$$

$$\leq C(\phi) c_{\mathbf{q},\mathbf{k}}(\nu, t) \|\hat{\mathbf{F}}\Big\|_\nu^{\langle \mathbf{q} \rangle + \langle \mathbf{k} \rangle}$$

From the expression for $\hat{\mathcal{N}}$, it follows from the above that it maps a ball of radius $b\|\hat{\mathbf{F}}_{\mathbf{0}}\|_0$ back to itself when when the first condition of the Lemma is satisfied. Again, using Lemma 23 (with variable s replacing p) and using integration as in Lemma 24, it follows that

$$\Bigg\| \int_0^1 \tau \hat{\mathbf{B}}_{\mathbf{q},\mathbf{k}} * \Bigg\{ (\hat{\mathbf{F}} + \hat{\mathbf{h}})^{*\mathbf{k}} * {}^*\prod_{l=1}^{m} {}^*\prod_{j=1}^{n} \left(s^j [\hat{F}_l + \hat{h}_l] \right)^{*q_{l,j}} - \hat{\mathbf{F}}^{*\mathbf{k}} * {}^*\prod_{l=1}^{m} {}^*\prod_{j=1}^{n} \left(s^j \hat{F}_l \right)^{*q_{l,j}} \Bigg\} (s, \tau\tau'; t)$$

$$e^{-s^3\tau(1-\tau')} d\tau' \Bigg\|_\nu \leq C(\phi) (\langle \mathbf{q} \rangle + \langle \mathbf{k} \rangle) c_{\mathbf{q},\mathbf{k}}(\nu, t) \left(\|\hat{\mathbf{h}}\|_\nu + \|\hat{\mathbf{F}}\|_\nu \right)^{\langle \mathbf{q} \rangle + \langle \mathbf{k} \rangle - 1} \|\hat{\mathbf{h}}\|_\nu$$

For $\hat{\mathbf{F}}$ and $\hat{\mathbf{h}} + \hat{\mathbf{F}}$ both inside the ball of radius $b\|\hat{\mathbf{F}}_{\mathbf{0}}\|_\nu$, it follows that from expression for $\hat{\mathcal{N}}$ and the the second condition of the Lemma that $\hat{\mathcal{N}}$ is contractive.

□

6.1 Example 1: Modified Harry-Dym equation

Consider the initial value problem for the scalar function $H(x, t)$ given by

$$H_t + H_x = -\frac{H^3}{2} + H^3 H_{xxx} \ , \quad H(x, 0) = x^{-1/2} \tag{57}$$

In a formal procedure involving involving substitution of a power-series in t into (57), one can determine all terms of the asymptotic expansion in the form

$$H(x, t) = \sum_{n=0}^{\infty} t^n H_n(x - t) \tag{58}$$

where $H_0(x) = x^{-1/2}$. The recurrence relation for $n \geq 0$ is given by

$$(n + 1) H_n = -\frac{1}{2} \sum_{n_j \geq 0, \sum_{j=1}^{3} n_j = n} H_{n_1} H_{n_2} H_{n_3} + \sum_{n_j \geq 0, \sum_{j=1}^{4} n_j = n} H_{n_1} H_{n_2} H_{n_3} H_{n_4}''' \tag{59}$$

¿From the recurrence relation above, it can be proved by induction that $H_n(x) = x^{-1/2} P_n(x^{-1}, x^{-9/2})$, where where P_n is a homogeneous polynomial of degree n, *i.e.*, each term of $P_n(a, b)$ is of the form $a^j b^{n-j}$ for $0 \leq j \leq n$. Thus, the formal asymptotic expansion is of the form

$$H(x, t) \ \sim \ (x - t)^{-1/2} \sum_{n=0}^{\infty} P_n \left(\frac{t}{x - t}, \frac{t}{(x - t)^{9/2}} \right) \tag{60}$$

We will justify asymptotic expansion (60) for $x-t >> t^{2/9}$, $arg\ x \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$. It is convenient to introduce scaled variables

$$y = \frac{2}{3}(x-t)^{3/2} \ \ ; \ G(y,t) = H(x(y,t),t) \tag{61}$$

The equation for G is given by

$$\mathbf{N}[G](y,t) := G_t + \frac{1}{2}G^3 - \frac{3y}{2}G^3 G_{yyy} - \frac{3}{2}G^3 G_{yy} + \frac{1}{6y}G^3 G_y = 0 \tag{62}$$

To determine the residual when operator $\mathbf{N}$ acts on the first N terms of the asymptotic expansion (60), it is convenient to introduce another change of variable:

$$\zeta_1 = ty^{-2/3} \ ; \ \zeta_2 = ty^{-3} \ ; \ G(y,t) = y^{-1/3}M(\zeta_1,\zeta_2)$$

Then we find

$$\begin{aligned}\mathbf{N}[y^{-\frac{1}{3}}M] = \frac{\zeta_2^{\frac{2}{7}}}{\zeta_1^{\frac{9}{7}}}\Big\{ & \zeta_1 M_{\zeta_1} + \zeta_2 M_{\zeta_2} + \frac{1}{2}\,\zeta_1 M^3 + \frac{5}{6}\zeta_2 M^4 + 5\,\zeta_1\zeta_2 M^3 M_{\zeta_1} + \frac{185}{2}\,{\zeta_2}^2 M^3 M_{\zeta_2} \\ & + \frac{10}{3}\,{\zeta_1}^2\zeta_2 M^3 M_{\zeta_1\zeta_1} + 51\,\zeta_1{\zeta_2}^2 M^3 M_{\zeta_1\zeta_2} + 162\,{\zeta_2}^3 M^3 M_{\zeta_2\zeta_2} + \frac{4}{9}\,{\zeta_1}^3\zeta_2 M^3 M_{\zeta_1\zeta_1\zeta_1} \\ & + \zeta_1\zeta_2^2\left(6\zeta_1 + 27\zeta_2^2\right) M^3 M_{\zeta_1\zeta_1\zeta_2} + \frac{81}{2}\,{\zeta_2}^4 M^3 M_{\zeta_2\zeta_2\zeta_2}\Big\}\end{aligned} \tag{63}$$

From this representation, it is clear that if $Q_N(\zeta_1,\zeta_2)$ is some N-th order polynomial, then

$$\mathbf{N}\left[y^{-\frac{1}{3}}Q_N(ty^{-2/3},ty^{-3})\right](y,t) = t^{-1}y^{-\frac{1}{3}}Q_{4N+1}(ty^{-2/3},ty^{-3}) \tag{64}$$

where Q_{4N+1} is some polynomial of degree $4N+1$. Again to avoid proliferation of symbols, we denote a generic polynomial of order n by Q_n. Note that the symbol Q_n will generally denote different polynomials at different stages; but since we are only worried about the general form of expansion, this should not cause confusion. When we choose

$$g(y,t) = y^{-\frac{1}{3}}Q_N(ty^{-2/3},ty^{-3}) = y^{-\frac{1}{3}}\sum_{j=0}^{N} P_j(ty^{-2/3},ty^{-3}) = y^{-\frac{1}{3}}\sum_{j=0}^{N} t^n P_j(y^{-2/3},y^{-3}) \tag{65}$$

with coefficients of P_j chosen in accordance to the terms of the asymptotic expansion (60), then the coefficients in the resulting $Q_{4N+1}(ty^{-2/3},ty^{-3})$ in (64) upto order N are zero. Then is clear that

$$Q_{4N+1}(a,b) = P_{N+1}(a,b)\left[1+O(a,b)\right]$$

for some homogeneous $N+1$-st degree polynomial. We introduce the following transformation into (57):

$$H(x(y,t),t) = g(y,t) + y^{-2}f(y,t) \tag{66}$$

where $g(y,t)$ is the truncated asymptotic terms in (65) upto terms of $O(t^N)$. In the proof below, we will choose, without any loss of generality $N \geq 3$. In order to justify the asymptotic expansion

(60), we will prove that in the sector $arg\ y \in \left(-\frac{2}{3}\pi + \phi, \frac{2}{3}\pi - \phi\right)$, with $0 < \phi < \frac{\pi}{6}$, for $y >> t^{\frac{1}{3}}$, $|f(y,t)| << |y^{5/3} P_N \left(ty^{-2/3}, ty^{-3}\right)|$ for any $N \geq 3$.

With change of variable, the equation for $f(y,t)$ is in the form

$$f_t - f_{yyy} = \sum_{j=0}^{3} \sum_{k=0}^{3} b_{j,k}(y,t) f^k f^{(j)} + r(y,t) \tag{67}$$

where

$$r(y,t) = t^{-1} y^{5/3} Q_{4N+1}(ty^{-2/3}, ty^{-3}) \tag{68}$$

where $Q_{4N+1}(ty^{-2/3},\ ty^{-3})$ is a polynomial of degree $4N+1$ in the form

$$Q_{4N+1}(ty^{-2/3}, ty^{-3}) = (N+1) P_{N+1}\left(ty^{-2/3}, ty^{-3}\right)\ \left[1 + O(ty^{-2/3},\ ty^{-3})\right]$$

Equation (67) is clearly a special case ($n=3$ and $m=1$) more general vector equation being discussed in this paper. The precise form of $b_{j,k}$ is unimportant, except to note that they are of the form:

$$b_{0,0} = y^{-2/3} Q_{2N}(ty^{-2/3}, ty^{-3}) + y^{-3} Q_{3N}(ty^{-2/3}, ty^{-3})$$

$$b_{0,1} = \frac{Q_N}{y^{7/3}} + \frac{Q_{2N}}{y^{14/3}}\ ,\ b_{0,2} = \frac{Q_0}{y^4} + \frac{Q_N}{y^{19/3}}\ ,\ b_{0,3} = \frac{Q_0}{y^8}$$

$$b_{1,0} = \frac{Q_N}{y^2}\ ,\ b_{1,1} = \frac{Q_{2N}}{y^{11/3}}\ ,\ b_{1,2} = \frac{Q_N}{y^{16/3}}\ ,\ b_{1,3} = \frac{Q_0}{y^7}$$

$$b_{2,0} = \frac{Q_{3N}}{y}\ ,\ b_{2,1} = \frac{Q_{2N}}{y^{8/3}}\ ,\ b_{2,2} = \frac{Q_N}{y^{13/3}}\ ,\ b_{2,3} = \frac{Q_0}{y^6}$$

$$b_{3,0} = Q_{3N}\ \text{ with constant term } 0\ , b_{3,1} = \frac{Q_{2N}}{y^{5/3}}\ ,\ b_{3,2} = \frac{Q_N}{y^{10/3}}\ ,\ b_{3,3} = \frac{Q_0}{y^5}$$

Note that

$$F_0(p,t)\ =\ \int_0^t e^{-p^3(t-\tau)}\ R(p,\tau)\ d\tau \tag{69}$$

where

$$R(p,t) = t^{-1} p^{-8/3} Q_{4N+1}(tp^{2/3}, tp^3)\ ,\ \text{ where } \tilde{Q}_{4N+1}(a,b) = \tilde{P}_{N+1}(a,b)[1 + O(a,b)]$$

where $\tilde{P}_{N+1}$ is some polynomial of degree $N+1$ containing only terms of the form $a^j b^{N+1-j}$, $0 \leq j \leq (N+1)$. $B_{j,k}$ appearing in (70) have the forms:

$$B_{0,0} = p^{-1/3} Q_{2N}(tp^{2/3}, tp^3) + p^2 Q_{3N}(tp^{2/3}, tp^3)$$

$$B_{0,1} = p^{4/3} Q_N + p^{11/3} Q_{2N}\ ,\ B_{0,2} = p^3 Q_0 + p^{16/3} Q_N\ ,\ B_{0,3} = p^7 Q_0$$

$$B_{1,0} = p Q_N\ ,\ B_{1,1} = p^{8/3} Q_{2N}\ ,\ B_{1,2} = p^{13/3} Q_N\ ,\ B_{1,3} = p^6 Q_0$$

$$B_{2,0} = Q_{3N}\ ,\ B_{2,1} = p^{5/3} Q_{2N}\ ,\ B_{2,2} = p^{10/3} Q_N\ ,\ B_{2,3} = p^5 Q_0$$

$$B_{3,0} = p^{-1} Q_{3N}\ \text{ with constant term } 0\ , B_{3,1} = p^{2/3} Q_{2N}\ ,\ B_{3,2} = p^{7/3} Q_N\ ,\ B_{3,3} = p^4 Q_0$$

where symbol Q_m appearing in each $B_{j,k}$ is a generic symbol for a polynomial of degree m in the variables $tp^{2/3}$ and tp^3. ¿From expression for $R(p,t)$, it follows that

$$\hat{R}(s,\tau;t) = t^{-1/9} s^{-8/3} Q_{4N+1}\left(t^{7/9} \tau s^{2/3}, \tau s^3\right)$$

Lemma 30 *For $0 < t \leq 1$ and $\nu > 1$ sufficiently large,*

$$\|\hat{F}_0(.,.;t)\|_\nu \leq t^{8/9}\nu^{8/3}\tilde{P}_{N+1}(t^{7/9}\nu^{-2/3}, \nu^{-3}) \equiv t^{8/9}\delta$$

for some homogenous polynomial $\tilde{P}_{N+1}(a,b)$ of degree $N+1$ with positive coefficients, where each term is of the form $a^j b^{N+1-j}$.

Proof. We note that for $s \in \mathcal{S}_\phi$,

$$|\hat{F}_0(s,\tau;t)| = (t\tau)^{8/9}|\int_0^1 \frac{d\tau'}{\tau'} e^{-s^3\tau(1-\tau)}\tilde{Q}_{4N+1}\left(t^{7/9}\tau\tau' s^{2/3}, \tau\tau' s^3\right)| < t^{8/9}|s|^{-8/3}\hat{Q}_{4N+1}\left(t^{7/9}|s|^{2/3}, |s|^3\right)$$

for some $4N+1$ degree polynomial $\hat{Q}_{4N+1}(a,b)$, whose lowest order term is of the form $a^j b^{N+1-j}$. The Lemma follows from definition of the norm $\|.\|_\nu$ and using the fact that both $t^{7/9}\nu^{-2/3}$ and ν^{-3} are sufficiently small to ignore terms to be able to bound polynomial terms higher than $N+1$. $\square$

Lemma 31 *For any $\hat{F} \in \mathcal{A}_\phi$, for ν large enough and $t \in [0,1]$,*

$$\||B_{j,k}| * |\hat{F}|\|_\nu \leq c_{j,k}(\nu,t)\|\hat{F}\|_\nu$$

where

$$c_{0,0} = C\left(\frac{1}{t^{2/3}\nu^3} + \frac{t^{1/9}}{\nu^{2/3}}\right) \; ; \; c_{0,1} = C\left(\frac{1}{t^{4/9}\nu^{7/3}} + \frac{1}{\nu^{14/3}t^{11/9}}\right)$$

$$c_{0,2}(\nu,t) = C\left(\frac{1}{t\nu^4} + \frac{1}{t^{16/9}\nu^{19/3}}\right)\|\hat{F}\|_\nu \; ; \; c_{0,3} = \frac{C}{t^{7/3}\nu^8}\|\hat{F}\|_\nu$$

$$c_{1,0}(\nu,t) = \frac{C}{t^{1/3}\nu^2} \; ; \; c_{1,1} = \frac{C}{t^{8/9}\nu^{11/3}} \; ; \; c_{1,2} = \frac{C}{t^{13/9}\nu^{16/3}} \; ; \; c_{1,3} = \frac{C}{t^2\nu^7}$$

$$c_{2,0} = \frac{C}{\nu} \; ; \; c_{2,1} = \frac{C}{\nu^{8/3}t^{5/9}} \; ; \; c_{2,2} = \frac{C}{\nu^{13/3}t^{10/9}} \; ; \; c_{2,3} = \frac{C}{\nu^6 t^{5/3}}$$

$$c_{3,0} = C\left(\frac{t^{10/9}}{\nu^{2/3}} + \frac{t^{1/3}}{\nu^3}\right) \; ; \; c_{3,1} = \frac{C}{\nu^{5/3}t^{2/9}} \; ; \; c_{3,2} = \frac{C}{\nu^{10/3}t^{7/9}} \; ; \; c_{3,3} = \frac{C}{\nu^5 t^{4/3}}\|\hat{F}\|_\nu$$

where constant C is independent of ν and t.

Proof. ¿From Lemma 9, with $\rho = 0$ and s replacing p, we note that $\alpha > 0$ and $\nu > 1$,

$$\||s|^{\alpha-1} * |\hat{F}|\|_\nu \leq \frac{C}{\nu^\alpha}\|\hat{F}\|_\nu$$

The Lemma follows from bounds on $|\hat{B}_{j,k}(s,\tau;t)|$ that follow from bounds on $B_{j,k}$ and noticing that any polynomial of the form $Q_m(t^{7/9}\nu^{-2/3}, \nu^{-3}$ has bounds independent of ν and t. $\square$

Lemma 32 *For any $t \in (0,1)$, and $\nu > 1$ large enough, the integral equation obtained by Borel-transforming (67) and introducing $s = pt^{1/3}$ variable:*

$$\hat{F}(s,\tau;t) = \int_0^1 \tau \sum_{j=0}^3 t^{1-j/3} \sum_{k=0}^3 (-1)^j e^{-s^3\tau(1-\tau')}\left[(s^j\hat{F}) * \hat{B}_{j,k} * \hat{F}^{*k}\right](s,\tau\tau';t)d\tau' + \hat{F}_0(s,\tau;t) \tag{70}$$

has a unique solution in $\hat{\mathcal{A}}_\phi$.

Proof. Examining the bounds on $c_{j,k}$ from Lemma 31, it follows from Lemma 29 that $\mathcal{N}$ is indeed a contraction map for any $t \in (0, 1]$ and ν sufficiently large but, importantly, independent of t.

$\square$

6.1.1 Behavior of solution $\hat{F}_s$ near $p = 0$

Proposition 33 *For some $K_1 > 0$ and small s we have $|\hat{F}_s| < K_1 t^{8/9}|s|^{-8/3} P_{N+1}(t^{7/9}|s|^{2/3}, |s|^3)$, thus*

$$|f_s(\zeta, t)| < K_2 t^{5/9}|\zeta|^{5/3}|P_{N+1}(t^{7/9}|\zeta|^{-2/3}, |\zeta|^{-3})$$

for some $K_2 > 0$ as $|\zeta| \to \infty$ in $\hat{\mathcal{D}}_{\rho,\phi}$.

Proof. Convergence in $\|\cdot\|_\nu$ implies uniform convergence on compact subsets of $\mathcal{K}$ and we can interchange summation and integration in (70). With $\hat{F}_s$ the unique solution of (70) we let

$$\hat{G}_j = \sum_{k=0}^{3} (-1)^j \hat{B}_{j,k} * \hat{F}_s^{*k}$$

and define the linear operator $\mathcal{G}$ by

$$\mathcal{G}\hat{Q} = \int_0^1 \tau e^{-s^3\tau(1-\tau')} \sum_{j=0}^{3} t^{1-j/3}(s^j \hat{Q}) * G_j \, d\tau$$

Clearly $\hat{F}_s$ also satisfies the linear equation

$$\hat{F}_s = \mathcal{G}\hat{F}_s + \hat{F}_0 \quad \text{or} \quad \hat{F}_s = (1 - \mathcal{G})^{-1}\hat{F}_0$$

For $a > 0$ small enough define $\overline{\mathcal{S}}_a = \overline{\mathcal{S}} \cap \{s : |s| \le a\}$. Since $\hat{F}_s$ is continuous in $\overline{\mathcal{S}}$ we have $\lim_{a\downarrow 0} \|\mathcal{G}\| = 0$, where the norm is taken over $C(\overline{\mathcal{S}}_a)$.

We know see that

$$\|\hat{F}_0\|_\infty < t^{8/9}|a|^{-8/3} P_{N+1}(t^{7/9}|a|^{2/3}, |a|^3)$$

in $\overline{\mathcal{S}}_a$ for $N + 1$st degree polynomial with positive constants independent of t and a. Then, as $a \downarrow 0$, we have

$$\max_{\overline{\mathcal{S}}_a} |\hat{F}(s, t)| = \|F\|_\infty \le (1 - \|\mathcal{G}\|)^{-1} \max_{\overline{\mathcal{S}}_a} \|F_0\| \le 2t^{8/9}|a|^{-8/3} P_{N+1}(t^{7/9}|a|^{2/3}, |a|^3)$$

and thus for small s we have

$$|\hat{F}_s(s, t)| \le 2t^{8/9}|s|^{-8/3} P_{N+1}(t^{7/9}|s|^{2/3}, |s|^3)$$

the proposition for follows from applying Watson's Lemma in (52).

$\square$

Theorem 34 *For $t << 1$ and $x - t >> t^{2/9}$, with arg $(x - t) \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$, the asymptotic expansion of $H(x, t)$, the solution to the nonlinear PDE, is of the form (60).*

PROOF. We note that $x - t >> t^{2/9}$ corresponds to $y \;>>\; t^{1/3}$ and therefore $\zeta >> 1$. Further $arg\ (x-t) \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$ corresponds to $arg\ \zeta \in \left(-\frac{2}{3}\pi, \frac{2}{3}\pi\right)$. Lemma 32 and Proposition 33 implies that for any $\phi \in (0, \frac{\pi}{6})$ for large $y \in \mathcal{D}_\phi$ with $\zeta = y/t^{1/3} >> 1$,

$$|f(y,t)| = O\left(|y|^{5/3} P_{N+1}(t|y|^{-2/3}, t|y|^{-3}\right) = O\left(|y|^{5/3} t^{N+1} P_{N+1}(|y|^{-2/3}, |y|^{-3}\right)$$

With the change of variable, this implies that

$$(y(x,t))^{-2} f(y(x,t),t) = O\left(t^{N+1}|x-t|^{-1/2} P_{N+1}(|x-t|^{-1}, |x-t|^{-9/2}\right) = o(g(y(x,t),t)$$

This completes the proof. $\square$

7 Appendix

7.1 Derivation of equation (3) from (1)

We start from (1), where $\mathbf{h}$ is an m_1 dimensional vector field depending on y and t. We define the $m = m_1 \times n$-dimensional vector field as $\mathbf{f} = \left(\mathbf{h}, \partial_y \mathbf{h}, {\partial_y}^2 \mathbf{h}, ..., {\partial_y}^{(n-1)} \mathbf{h}\right)$. Then it is clear from (71) that $\mathbf{g}_1$ and $\mathbf{g}_2$ only depend on $\mathbf{f}$. So, for showing that (1) implies (3) it is enough to show that for $1 \le n' \le n$,

$$\partial^{n'-1} \left[\mathbf{g}_1(y,t,\mathbf{f}) + \mathbf{g}_2(y,t,\mathbf{f})\partial_y \mathbf{f}\right]$$

is of the form on the right hand side of (3). We do so in two steps.

Lemma 35 *For any $n' \ge 1$,*

$$\partial_y^{n'-1} \mathbf{g_1}(y,t,\mathbf{f}(y,t)) = \sum_{\mathbf{q}\succeq 0}{}^{\ddagger} \mathbf{b_q}(y,t,\mathbf{f}) \prod_{l=1}^{m} \prod_{j=1}^{n'-1} \left(\partial_y^j f_l\right)^{q_{l,j}} \tag{71}$$

for some $\mathbf{b_{q,k}}$, depending on n', g_1, and its first $n'-1$ derivatives with respect to its arguments, and where $\sum^{\ddagger}$ means the sum with the further restriction

$$\sum_{l,j} j q_{l,j} \le n' - 1$$

PROOF. The proof is by induction. We have, with obvious notation,

$$\partial_y \mathbf{g}_1(y,t,\mathbf{f}(y,t)) = \mathbf{g}_{1,y} + \mathbf{g}_{1,\mathbf{f}} \cdot \partial_y \mathbf{f}$$

which is of the form (71). Assume (71) holds for $n' = k \ge 1$, i.e.

$$\partial_y^{k-1} \mathbf{g_1}(y,t,\mathbf{f}) = \sum_{\mathbf{q}\succeq 0}{}^{\ddagger} \mathbf{b_q}(y,t,\mathbf{f}) \prod_{l=1}^{m} \prod_{j=1}^{k-1} \left(\partial_y^j f_l\right)^{q_{l,j}}$$

Taking a y derivative, we get

$$\partial_y^k \mathbf{g_1}(y,t,\mathbf{f}) = \sum_{\mathbf{q}\succeq 0} \left(\sum_{i=1}^m \frac{\partial}{\partial f_i} \mathbf{b_q}(y,t,\mathbf{f}) \partial_y f_i + \partial_1 \mathbf{b_q}(y,t,\mathbf{f}) \right) \prod_{l=1}^m \prod_{j=1}^{k-1} \left(\partial_y^j f_l\right)^{q_{l,j}}$$

$$+ \sum_{\mathbf{q}\succeq 0} \mathbf{b_q}(y,t,\mathbf{f}) \sum_{l'=1}^m \sum_{j'=1}^{k-1} q_{l',j'} \left(\partial_y^{j'} f_{l'}\right)^{q_{l',j'}-1} \left(\partial_y^{j'+1} f_{l'}\right) \prod_{l=1}^{m\backslash l'} \prod_{j=1}^{k-1\backslash j'} \left(\partial_y^j f_l\right)^{q_{l,j}}$$

where the notation $\partial_1 \mathbf{b_q}$ denotes partial with respect to the first argument of $\mathbf{b}$ and $\prod_{l=1}^{m\backslash l'}$ stands for the product with $l = l'$ term excluded. It is easily seen from the above expressions that the product of the number of derivatives times the power, when added up at most equals

$$j' + 1 + j'(q_{l',j'} - 1) + \sum_{j\neq j'} \sum_{l \neq l'} j q_{l,j} = \sum_{l,j} j q_{l,j} + 1 \leq \ k - 1 + 1 = k$$

Thus, (71) holds for $n' = k + 1$, with a different $\mathbf{b}$. The induction step is proved. $\square$

Lemma 36 *For any n' between 1 and n,*

$$\partial^{n'-1} \left[\mathbf{g_2}(y,t,\mathbf{f})\partial_y \mathbf{f}\right] = \sum_{\mathbf{q}\succeq 0}^{\ddagger} \mathbf{b_q}(y,t,\mathbf{f}) \prod_{l=1}^m \prod_{j=1}^{n'} \left(\partial_y^j f_l\right)^{q_{l,j}} \tag{72}$$

for some $\mathbf{b_{q,k}}$, depending on n', $\mathbf{g}_2$ and its first $n'-1$ derivatives with respect to each argument, and with further restriction

$$\sum_{l,j} j q_{l,j} \ \leq \ n'$$

Proof. It is clear that the $n'-1$ derivative of $\mathbf{g}_2 \partial_y \mathbf{f}$ is a linear combination of

$$\partial_y^{n''} \left[\mathbf{g}_2\right] \partial_y^{n'-n''} \mathbf{f}$$

for n'' ranging from 0 to $n'-1$. Since the derivatives of $\mathbf{g}_2$ are of the form (71), just like $\mathbf{g}_1$, it follows that in the above expression the sum of the product of the number of derivatives and the power to which they are raised will be

$$\sum_{j=1}^{n''} \sum_{l=1}^m j q_{l,j} + n' - n'' \ \leq \ n'' + n' - n'' = n'$$

Hence the lemma follows. $\square$

7.2 Simple illustrations of regularization by Borel summation

(i) A vast literature has emerged recently in the field of Borel summation starting with the fundamental contributions of Écalle, see e.g. [10] and it is impossible to give a quick account of the breadth of this field. See for example [6] for more references. Yet, in the context of relatively general PDEs, very little is known. In this section we discuss informally and using admittedly rather trivial examples, the regularizing power of Borel summation techniques.

(ii) Singular perturbations give rise to nonanalytic behavior and divergent series in both ODEs and PDEs. We start by looking at a simple ODE. Infinity is an irregular singular point of the equation $f' - f = 1/x$ as can be easily seen after the transformation $x = 1/z$ in which variable the coefficients of the first order equation have, after normalization, a double pole at $x = 0$. This is manifested in the factorial divergence of the formal power series solution $\tilde{f} = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{x^{k+1}}$

The *Borel transform* is by definition the formal (term-by-term) inverse Laplace transform and gives $\mathcal{B}(\tilde{f})(p) = \sum_{k=0}^{\infty}(-1)^k p^k$ which is a convergent series. In effect, after formal inverse Laplace transform, the equation becomes $(p+1)F(p) + 1 = 0$ ($F(p) := \mathcal{L}^{-1} f$) which is regular at $p = 0$. It is easily seen that F is Laplace transformable and $\mathcal{L}F = f$ is a solution of the original equation.

(iii) In the context of PDEs, consider the heat equation

$$h_t - h_{yy} = 0 \tag{73}$$

first for *small* t. Power series solutions $\tilde{h} = \sum_{k=0}^{\infty} H_k(y) t^k$, even with H_0 real-analytic, generally have zero radius of convergence. Indeed, the recurrence relation for the H_k is $kH_k = H''_{k-1}$ meaning that for H_0 analytic but not entire, H_k will roughly behave like $k!\, const^k$ for large k. The factorial divergence suggests Borel summation, conventionally performed with respect to a large variable; substituting $T = 1/t$ in (73) yields

$$\frac{\partial^2 h}{\partial y^2} + T^2 \frac{\partial h}{\partial T} + \frac{1}{2} h = 0 \tag{74}$$

Inverse Laplace transform of (74) with respect to T yields

$$\hat{h}_{yy} - p\hat{h}_{pp} - \frac{3}{2}\hat{h}_p = 0 \tag{75}$$

followed by $\hat{h}(p, y) = p^{-1/2} u(2\sqrt{p}, y)$, $z = 2\sqrt{p}$ transforms (75) into the wave equation

$$u_{yy} - u_{zz} = 0$$

which is now regular (in fact the explicit solution of the new problem with the induced initial conditions immediately yields, by Laplace transform, the heat kernel solution of (73) [8]).

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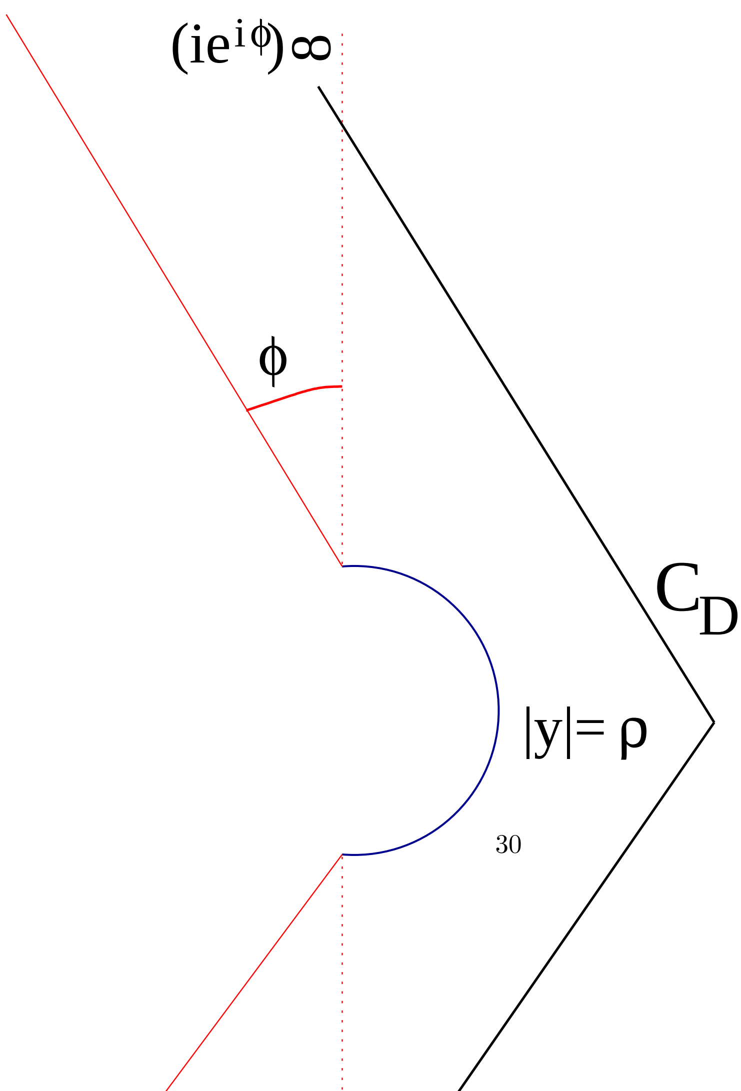

(ie^{iφ})∞
φ
C_D
|y|= ρ