Invent. math. (2001)
Digital Object Identifier (DOI) 10.1007/s002220100153

Inventiones mathematicae

On the formation of singularities of solutions of nonlinear differential systems in antistokes directions

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Oblatum 21-IX-2000 & 6-III-2001
Published online: ■ ■ ■ 2001 –

1. Introduction

In generic analytic nonlinear differential systems in the complex plane, we study the position and the type of singularities formed by solutions when an irregular singular point of the system is approached along an antistokes direction[1]. Placing the singularity of the system at infinity we look at equations of the form $\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y})$ with $\mathbf{f}$ analytic in a neighborhood of $(0, \mathbf{0})$, with genericity assumptions; $x = \infty$ is then a rank one singular point. We analyze the singularities of those solutions $\mathbf{y}(x)$ which tend to zero for $x \to \infty$ in some sectorial region, on the edges of the maximal region (also described) with this property.

After standard normalization of the differential system, it is shown that singularities occuring in antistokes directions are grouped in nearly periodical arrays of similar singularities as $x \to \infty$, the location of the array depending on the solution while the (near-) period and type of singularity are determined by the form of the differential system.

This regularity in type and position of movable singularities has been observed previously in various examples of nonlinear systems: Painlevé equations ([30], [28], [16]) third order nonlinear equations ([47], [22]) non-integrable Abel equations ([23], [34]) among others. We show these features are rather universal and find a formalism to calculate them (asymptotically).

When $\mathbf{f}$ is meromorphic and satisfies some estimates the singularities in the arrays are generically square root branch points.

[1] In the sense stemming from Stokes original papers and the one favored in exponential asymptotics literature, *Stokes* lines are those where a small exponential is purely real; on an *antistokes* line the exponential becomes purely oscillatory. In some references these definitions are interchanged.

The mechanism of singularity formation is elucidated by exponential asymptotic analysis, which also provides a general and effective calculation tool for determining the type and position of singularities. The present method generalizes that of [16]. The analysis yields two-scale asymptotic expansions of solutions, valid in a region which includes the directions along which $\mathbf{y} \to 0$ and extending, on appropriate Riemann surfaces, into regions where the solutions typically develop singularities. The expansions have the form $\mathbf{y} \sim \tilde{\mathbf{F}}(x;\xi(x)) = \sum_{m=0}^{\infty} x^{-m}\mathbf{F}_m(\xi(x))$, where $\xi(x) = C\mathrm{e}^{-\lambda x}x^{\alpha}$; λ, α and the functions $\mathbf{F}_m$ are uniquely determined, modulo trivial transformations, by $\mathbf{f}$; the constant C depends on the solution $\mathbf{y}$. $\mathbf{F}_m$ satisfy a recursive system of equations, typically simpler than the original system. In particular, for all first order equations and for Painlevé's P1 and P2 equations, the solution of the recursive system is expressible by quadratures.

The method can be interpreted as a *transasymptotic matching* technique in that the expansion $\tilde{\mathbf{F}}$ of $\mathbf{y}$ matches (and is fully determined by) its Écalle *transseries*[2] in a sector where $\mathbf{y}$ is asymptotic to a power series. The constant C in the definition of ξ is one of the constants beyond all orders in the transseries of $\mathbf{y}$. In some instances, the technique provides a connection method even in nonintegrable systems (in which case, the connection data are path-dependent). The constant C becomes thus accessible by classical asymptotics and determines the position of singularities of $\mathbf{y}$.

The expansion $\mathbf{y} \sim \tilde{\mathbf{F}}(x;\xi(x))$ satisfies Gevrey-type inequalities, and thus produces exponentially accurate estimates of $\mathbf{y}$ (see [42]).

Some examples are outlined. In the first one, a nonintegrable Abel equation, the method provides a description of the exact type of all but finitely many singularities of solutions in a sector, and of the associated the Riemann surfaces. The connection between these complicated Riemann surfaces and the numerically observed chaoticity of solutions [23] is briefly discussed.

As other examples we consider the Painlevé equations $\mathrm{P_I}$ and $\mathrm{P_{II}}$ for which we use the technique to express the asymptotic distribution of poles near the antistokes lines of the so called truncated solutions. The position of the poles only depends on an exponential asymptotics quantity, the constant beyond all orders.

2. Setting

We adopt, with few exceptions that we mention, the same conditions, notations and terminology as [13]; the results on formal solutions and their generalized Borel summability are also taken from [13].

The differential system considered has the form

$$\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n, \quad x \in \mathbb{C} \tag{1}$$

where

[2] In our context these are algebraically determinable formal combinations of series in x^{-1} and small exponentials, and generalize classical asymptotic expansions [18].

(i) $\mathbf{f}$ is *analytic* in a neighborhood $\mathcal{V}_x \times \mathcal{V}_\mathbf{y}$ of $(0, \mathbf{0})$, under the genericity conditions that:
(ii) the eigenvalues λ_j of the matrix $\hat{\Lambda} = -\left\{\frac{\partial f_i}{\partial y_j}(0, \mathbf{0})\right\}_{i,j=1,2,\dots,n}$ are linearly independent over $\mathbb{Z}$ (in particular $\lambda_j \neq 0$) and such that
(iii) $\arg \lambda_j$ are all different.

(In fact somewhat less restrictive conditions are used, namely those of [13] Sect. 1.1.2.)

By elementary changes of variables, the system (1) can be brought to the *normalized form* [13], [48]

$$\mathbf{y}' = -\hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x^{-1}, \mathbf{y}) \tag{2}$$

where $\hat{\Lambda} = \mathrm{diag}\{\lambda_j\}$, $\hat{A} = \mathrm{diag}\{\alpha_j\}$ are constant matrices, $\mathbf{g}$ is analytic at $(0, \mathbf{0})$ and $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}) + O(|\mathbf{y}|^2)$ as $x \to \infty$ and $\mathbf{y} \to 0$. Performing a further transformation of the type $\mathbf{y} \mapsto \mathbf{y} - \sum_{k=1}^{M} \mathbf{a}_k x^{-k}$ (which takes out M terms of the formal asymptotic series solutions of the equation), makes

$$\mathbf{g}(|x|^{-1}, \mathbf{y}) = O(x^{-M-1}; |\mathbf{y}|^2; |x^{-2}\mathbf{y}|) \quad (x \to \infty;\ \mathbf{y} \to 0)$$

where

$$M \geq \max_j \Re(\alpha_j)$$

and $O(a; b; c)$ means (at most) of the order of the largest among a, b, c.

Our analysis applies to solutions $\mathbf{y}(x)$ such that $\mathbf{y}(x) \to 0$ as $x \to \infty$ along some arbitrary direction $d = \{x \in \mathbb{C} : \arg(x) = \phi\}$. A movable singularity of $\mathbf{y}(x)$ is a point $x \in \mathbb{C}$ with $x^{-1} \in \mathcal{V}_x$ where $\mathbf{y}(x)$ is not analytic. The point at infinity is an irregular singular point of rank 1; it is a fixed singular point of the system since, after the substitution $x = z^{-1}$ the r.h.s of the transformed system, $\frac{dy}{dz} = -z^{-2}\mathbf{f}(z, \mathbf{y})$ has, under the given assumptions, a pole at $z = 0$.

2.1. Classical versus exponential asymptotics

In order to understand the properties of solutions of (2) for large x, one way is to find formal asymptotic solutions, then use asymptoticity relations to deduce information about the true solutions from the formal ones. It is easy to see that equation (2) admits a unique asymptotic formal power series solution [51]

$$\tilde{\mathbf{y}}_\mathbf{0} = \sum_{r=2}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{0};r}}{x^r}, \quad (\ |x| \to \infty) \tag{3}$$

The coefficients $\tilde{\mathbf{y}}_{\mathbf{0},r}$ of $\tilde{\mathbf{y}}_\mathbf{0}$ can be computed recursively by substitution in (2) and identification of the coefficients of x^{-r}; the series $\tilde{\mathbf{y}}_\mathbf{0}$ is a *formal* solution, and is usually divergent. Its Borel summability was shown, in a more general setting, by Braaksma [9].

Given an open sector of the complex x-plane, of angle less than π, there exists a true solution of (2) which is asymptotic to (3) in that sector (as $|x| \to \infty$) [51]. This solution is not unique in general.

To illustrate the way different solutions with the same asymptotic series (in a sector) can be distinguished consider the simple linear equation $f'(x) = -f(x) + x^{-1}$ with the general solution $f(x; C) = e^{-x} Ei(x) + Ce^{-x}$ where $Ei(x) = P\int_{-\infty}^{x} t^{-1}e^t dt$. Any solution $f(x; C)$ has the same (divergent) power series asymptotic expansion in the right-half plane: $f(x; C) \sim \tilde{f}_0(x) \equiv \sum_{r\geq 0} r!x^{-r-1}$ for $x \to \infty$, $\Re x > 0$. The parameter C which distinguishes different solutions multiplies the term e^{-x} which is much smaller than all the terms of the asymptotic series $\tilde{f}_0 : C$ is a constant beyond all orders.

The theory of linear equations with an irregular singular point is well developed and there are comprehensive results; we mention the works of Babbit and Varadarajan [1], Balser, Braaksma, Jurkat, Lutz, [2], [9], [4], [32], Balser, Braaksma, Ramis and Sibuya [3], Deligne [17], Jurkat [31], Katz [33], Levelt [36], Levelt and Van den Essen [37], Lutz and Schäfke [41], Manin [39], Olver [40], Malgrange [38], Ramis [42], Ramis and Martinet [43], Ramis and Sibuya [44], Sibuya [46], Turritin [49], and others – see also [50] and the references therein.

For linear equations there exist fundamental systems of solutions in terms of which one can speak of exponentially small terms. A formal analogue for nonlinear equations is represented by formal exponential series.

An n-parameter formal solution of (2) (under the assumptions mentioned) as a combination of powers and exponentials is found in the form

$$\tilde{\mathbf{y}}(x) = \sum_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \tilde{\mathbf{s}}_{\mathbf{k}}(x) \tag{4}$$

where $\tilde{\mathbf{s}}_{\mathbf{k}}$ are (usually factorially divergent) formal power series: $\tilde{\mathbf{s}}_{\mathbf{0}} = \tilde{\mathbf{y}}_{\mathbf{0}}$ (see (3)) and in general

$$\tilde{\mathbf{s}}_{\mathbf{k}}(x) = \sum_{r=0}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{k};r}}{x^r} \tag{5}$$

that can be determined by formal substitution of (4) in (2); $\mathbf{C} \in \mathbb{C}^n$ is a vector of parameters[3] (we use the notations $\mathbf{C}^{\mathbf{k}} = \prod_{j=1}^{n} C_j^{k_j}$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $|\mathbf{k}| = k_1 + \ldots + k_n$).

Note the structure of (4): an infinite sum of (generically) divergent series multiplying exponentials. They are called *formal exponential power series* [51].

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[3] In the general case when some assumptions made here do not hold, the general formal solution may also involve compositions of exponentials, logs and powers [19]. The present paper only discusses equations in the setting explained at the beginning of Sect. 2.

Formal solutions (4) of differential equations (2) were introduced by Fabry [21] and studied extensively by Cope [11].

From the point of view of correspondence of these formal solutions to actual solutions it was recognized that not all expansions (4) should be considered meaningful; also they are defined relative to a sector (or a direction).

Given a direction d in the complex x-plane the *transseries* (on d), introduced by Écalle [19], are, in our context, those exponential series (4) which are formally *asymptotic* on d, i.e. the terms $\mathbf{C}^{\mathbf{k}}\mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x}x^{\boldsymbol{\alpha}\cdot\mathbf{k}}x^{-r}$ (with $\mathbf{k}\in(\mathbb{N}\cup\{0\})^n,\ r\in\mathbb{N}\cup\{0\}$) form a well ordered set with respect to $\gg$ on d (see also [13]).[4] (For example, this is the case when the terms of the formal expansion become (much) smaller when $\mathbf{k}$ becomes larger.)

For linear systems any exponential power series solution is also a transseries: it consists of n power series multiplying exponentials since $\tilde{\mathbf{s}}_{\mathbf{k}}=0$ for $|\mathbf{k}|\geq 2$.

In the nonlinear case if a formal exponential power series (4) satisfies the condition $C_j=0$ if $\mathrm{e}^{-\lambda_j x}\not\to 0$ as $x\to\infty,\ x\in d$ then (4) is a transseries on d. In fact, it is clear that (4) is a transseries on (any direction of) the *open sector S_{trans}* defined by

$$S_{trans}=\left\{x\in\mathbb{C}\,;\ \text{if } C_j\neq 0 \text{ then } \Re(\lambda_j x)>0\ ,\ j=1,\ldots,n\right\} \tag{6}$$

This sector may be empty; it may be the whole $\mathbb{C}$ if all $C_j=0$; otherwise it lies between two antistokes lines, and has opening at most π.

If $\tilde{\mathbf{y}}_{\mathbf{0}}$ is divergent (which is generic) then the terms containing exponentials in (4) (i.e. terms with $|\mathbf{k}|\geq 1$) are much smaller than all powers of x in $\tilde{\mathbf{y}}_0$ and cannot be defined by classical asymptotic inequalities in the Poincaré sense. Hence their designation: terms beyond all orders.

Transseries and their correspondence with functions are the subject of **exponential asymptotics**, which developed substantially in the eighties with the work of M. Berry (hyperasymptotics), J. Écalle (the theory of analyzable functions), and M. Kruskal (theory of tower representations and nice functions) (see references).

From a historical point of view we must stress the importance of the fundamental work of Iwano. Generalizing earlier results of Malmquist he proved in wide generality that locally meromorphic systems of differential equations have expansions, which for the class (2) discussed in the present paper have the form

$$\mathbf{y}(x)=\boldsymbol{\phi}_{\mathbf{0}}(x)+\sum_{\mathbf{k}\in\mathbb{N}^n}\mathbf{C}^{\mathbf{k}}e^{-\boldsymbol{\lambda}\cdot\mathbf{k}x}x^{\boldsymbol{\alpha}\cdot\mathbf{k}}\boldsymbol{\phi}_{\mathbf{k}}(x) \tag{7}$$

convergent, and with $\boldsymbol{\phi}_{\mathbf{k}}$ analytic, in appropriate sectors [26], [27].

Later, Écalle introduced a very large space of expansions, relevant to differential, difference, integral and other equations. The fundamental space

[4] We note here a slight difference between our transseries and those of Écalle, in that we are allowing complex constants.

where formal solutions are sought is purely algebraic – the transseries (for example, see the expansion (4), (5) for solutions of (2)). Then a general procedure (based on Borel summation), independent of the equation where the expansions originate, is outlined to associate functions to formal expansions. As a consequence the summation procedure can be used in a broad class of problems and yields a complete isomorphism between formal expansions and a class of functions (analyzable functions). For differential equations this procedure shows that all the terms $\boldsymbol{\phi}_{\mathbf{k}}$ in (7) can be in fact obtained from $\boldsymbol{\phi}_{\mathbf{0}}$ by a form of analytic continuation (Écalle's resurgence relations). Also, Stokes phenomena can be described in detail. The paper [13] proves this procedure in the context of differential systems with a rank 1 singularity.

Braaksma has recently extended the theory to nonlinear difference equations [10].

The present paper studies the solutions in a region where the expansions (7) and those of [13] *diverge*. It is shown that in this region the *solutions* of generic systems actually do have singularities (see Theorems 2 and 3) grouped in regular arrays. Thus, a posteriori we know that expansions of the form (7) cannot converge there. Nevertheless, asymptotic representations, in terms of functions themselves singular derived in the present paper hold in this region (see (26)) and they enable finding the singularities of $\mathbf{y}$. The region where the series (26) is asymptotic to solutions and the region where (7), or (10), converge do intersect so the two expansions of the same solution can be *matched* in this region (see Theorem 1).

2.2. Further notations and results referred to

We recall that the *antistokes lines* of (2) are the $2n$ directions of the x-plane $i\overline{\lambda_j}\,\mathbb{R}_+,\ \ -i\overline{\lambda_j}\,\mathbb{R}_+,\ \ j = 1,\ldots,n$, i.e. the directions along which some exponential $e^{-\lambda_j x}$ of the general formal solution (4) is purely oscillatory.

In the context of differential systems with an irregular singular point, asymptoticity should be (generically) discussed relative to a direction towards the singular point; in fact, under the present assumptions (of nondegeneracy) asymptoticity can be defined on sectors.

A first question is to determine which are the solutions asymptotic to the power series solution (3), and to find their regularity.

Let d be a direction in the x-plane which is not an antistokes line. The solutions $\mathbf{y}(x)$ of (2) which satisfy

$$\mathbf{y}(x) \to 0 \quad (x \in d;\ |x| \to \infty) \tag{8}$$

are analytic for large x in a sector containing d, between two neighboring antistokes lines and have the same asymptotic series

$$\mathbf{y}(x) \sim \tilde{\mathbf{y}}_{\mathbf{0}} \quad (x \in d;\ |x| \to \infty) \tag{9}$$

(see Appendix 6.1 for more precise statements and details).

A sweeping correspondence between general transseries and the class of analyzable functions has been introduced in the monumental work of Écalle [18]–[20].

In the context of (2), a generalized Borel summation $\mathcal{LB}$ of transseries (4) is defined in [13]. The rest of this section states some results of [13] needed in the present paper; more details are included in Sect. 6.2 of the Appendix.

The formal solutions (4) are determined by the equation (2) that they satisfy, except for the parameters $\mathbf{C}$. Then a correspondence between actual and formal solutions of the equation is an association between solutions and constants $\mathbf{C}$. This is done using a generalized Borel summation $\mathcal{LB}$.

The operator $\mathcal{LB}$ constructed in [13] can be applied to any transseries solution (4) of (2) (valid on its open sector S_{trans}, assumed non-empty) on any direction $d \subset S_{trans}$ and yields an actual solution $\mathbf{y} = \mathcal{LB}\tilde{\mathbf{y}}$ of (2), analytic in a domain S_{an} (see (137)). Conversely, any solution $\mathbf{y}(x)$ satisfying (9) on a direction d is represented as $\mathcal{LB}\tilde{\mathbf{y}}(x)$, on d, for some unique $\tilde{\mathbf{y}}(x)$:

$$\begin{aligned}\mathbf{y}(x) &= \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{M}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x)\\ &= \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{M}\cdot\mathbf{k}} \mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}(x) \equiv \mathcal{LB}\tilde{\mathbf{y}}(x)\end{aligned} \tag{10}$$

for some constants $\mathbf{C} \in \mathbb{C}^n$, where $M_j = \lfloor \Re\alpha_j \rfloor + 1$ ($\lfloor\cdot\rfloor$ is the integer part), and

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) = \sum_{r=0}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{k};r}}{x^{-\mathbf{k}\boldsymbol{\alpha}'+r}} \qquad (\boldsymbol{\alpha}' = \boldsymbol{\alpha} - \mathbf{M}) \tag{11}$$

(for technical reasons the Borel summation procedure is applied to the series

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) \equiv x^{\mathbf{k}\boldsymbol{\alpha}'} \tilde{\mathbf{s}}_{\mathbf{k}}(x) \tag{12}$$

rather than to $\tilde{\mathbf{s}}_{\mathbf{k}}(x)$ cf. (4),(5)).

In any direction d, $\mathcal{LB}$ is a one-to-one map between the transseries solutions on d and actual solutions satisfying (9), see [13], Theorem 3.

The map $\tilde{\mathbf{y}} \mapsto \mathcal{LB}(\tilde{\mathbf{y}})$ depends on the direction d, and (typically) is discontinuous at the finitely many Stokes lines, see [13], Theorem 4.

For linear equations only the directions $\overline{\lambda_j}\,\mathbb{R}_+$, $j = 1, \ldots, n$ are Stokes lines, but for nonlinear equations there are also other Stokes lines, recognized first by Écalle (the complex conjugate directions to $p_{j;\mathbf{k}}\mathbb{R}_+$ cf. (35); see [13]). $\mathcal{LB}$ is only discontinuous because of the jump discontinuity of the vector of "constants" $\mathbf{C}$ across Stokes directions (Stokes' phenomenon); between Stokes lines $\mathcal{LB}$ does not vary with d.

The function series in (10) is uniformly *convergent* and the functions $\mathbf{y}_{\mathbf{k}}$ are analytic on domains S_{an} defined in (137) (for some $\delta > 0$, $R = R(\mathbf{y}(x), \delta) > 0$ – see Theorem 19 of Sect. 6.2).

2.3. Heuristic discussion of transasymptotic matching

There is a sharp distinction between linear and nonlinear systems with respect to the behavior beyond S_{trans}.

In the linear case there are only *finitely many* (at most n) nonzero $\mathbf{y_k}$ in (10), and (10) holds in a full (possibly ramified) neighborhood of infinity, except for jumps in the components of $\mathbf{C}$, one at each Stokes line (see [46], also [41], [13], [50] and the references therein). The map $\mathcal{LB}$ is continuous at the antistokes lines, and thus the transseries $\tilde{\mathbf{y}}$ of $\mathbf{y}$ is the same on both sides of an antistokes line. What changes at such a direction is the *classical* asymptotic expansion of $\mathbf{y}$, because classical asymptotics only retains the dominant series in $\tilde{\mathbf{y}}$, and exponentials in the transseries $\tilde{\mathbf{y}}$ exchange dominance at antistokes lines. From the point of view of exponential asymptotics, where transseries are considered rather than just the dominant series, behavior of solutions of linear equations at antistokes lines is relatively simple.

In the nonlinear case however, generically all components $\mathbf{y_k}$ are nonzero and, beyond S_{trans} (for example, in the notations of Sect. 3 below, for $\arg(x) > \pi/2$), (10) will typically blow up because of a growing exponential (e^{-x} in this example).

The divergence of (10) turns out to mark an actual change in the behavior of $\mathbf{y}(x)$, which usually develops singularities in this region. The information about the singularities is contained in (10).

The key to understanding the behavior of $\mathbf{y}(x)$ for x beyond S_{an} is to look carefully at the borderline region where (10) converges but barely so. Because of nonresonance, for $\arg(x) = \pi/2$ we have $\Re(\lambda_j x) > 0$, $j = 2, \ldots, n_1$.[5] By (37) all terms in (4) with $\mathbf{k}$ not a multiple of $\mathbf{e}_1 = (1, 0, \ldots, 0)$ are subdominant (small). Thus, for x near $i\mathbb{R}^+$ we only need to look at

$$\mathbf{y}^{[1]}(x) = \sum_{k\geq 0} C_1^k \mathrm{e}^{-kx} x^{kM_1} \mathbf{y}_{k\mathbf{e}_1}(x) \tag{13}$$

The region of convergence of (13) (thus of (10)) is then determined by the effective variable $\xi = C_1 \mathrm{e}^{-x} x^{\alpha_1}$ (since $\mathbf{y}_{k\mathbf{e}_1} \sim \tilde{\mathbf{y}}_{k\mathbf{e}_1;0}/x^{k(\alpha_1 - M_1)}$). Convergence is marginal along curves such that ξ is small enough but, as $|x| \to \infty$, is nevertheless larger than all *negative* powers of x. In this case, any term of the form $C_1^k e^{-kx} x^{kM_1} \mathbf{y}_{k\mathbf{e}_1;0}$ is much larger than the terms $C_1^l e^{-lx} x^{lM_1} \mathbf{y}_{l\mathbf{e}_1;r} x^{-r}$ if $k, l \geq 0$ and $r > 0$. Hence the leading behavior of $\mathbf{y}^{[1]}$ is expected to be

$$\mathbf{y}^{[1]}(x) \sim \sum_{k\geq 0} (C_1 \mathrm{e}^{-x} x^{\alpha_1})^k \tilde{\mathbf{s}}_{k\mathbf{e}_1;0} \equiv F_0(\xi) \tag{14}$$

(cf. (11)); moreover, taking into account all terms in $\tilde{\mathbf{s}}_{k\mathbf{e}_1}$ we get

$$\mathbf{y}^{[1]}(x) \sim \sum_{r=0}^{\infty} x^{-r} \sum_{k=0}^{\infty} \xi^k \tilde{\mathbf{y}}_{k\mathbf{e}_1;r} \equiv \sum_{j=0}^{\infty} \frac{\mathbf{F}_j(\xi)}{x^j} \tag{15}$$

Expansion (15) has a two-scale structure, with the scales ξ and x.

[5] In the notations explained below in Sect. 3 $C_j = 0$ for $j > n_1$.

It may come as a surprise that each $\mathbf{F}_j$ is a **convergent** series in ξ (though the whole expansion (15) is still divergent).

It turns out that the reshuffling (15) is meaningful and yields the correct asymptotic representation of $\mathbf{y}^{[1]}$, and therefore of $\mathbf{y}$, beyond the upper edge of S_{an}. In fact, (15) extends (9) right into the regions in $\mathbb{C}$ where $\mathbf{y}$ is singular, as near as (under mild assumptions) $O(\mathrm{e}^{-const.|x|})$ of these singularities. Once these two scales are known and once the validity of (15) is proved for our class of systems (Theorems 1 and 3 below), it is easier to calculate the $\mathbf{F}_j$ by direct substitution of (15) in (2) and identification of the powers of x (see Remark 7 and Sect. 6.9). The exact form of the second scale ξ is decisive for the domain of validity of the expansion, see Sect. 5.2.

To leading order we have $\mathbf{y} \sim \mathbf{F}_0$ (see also (14)) where $\mathbf{F}_0$ satisfies the autonomous (after a substitution $\xi = e^{\xi'}$) equation

$$\xi \mathbf{F}_0'(\xi) = \hat{\Lambda}\mathbf{F}_0(\xi) - \mathbf{g}(0, \mathbf{F}_0)$$

which can be solved in closed form for first order equations ($n = 1$) (the equation for F_0 is separable, and for $k \geq 1$ the equations are linear), as well as in other interesting cases (see e.g. Sect. 5.2, Sect. 5.3).

Assume that $\mathbf{F}_0(\xi)$ has an isolated singularity at $\xi = \xi_s$. Then $\mathbf{y}(x)$ must also be singular near x_s, if $\xi(x_s) = \xi_s$. Indeed, it is not difficult to see (see Sect. 4.6) that there must exist some $g(\xi)$ analytic at ξ_s so that that $\oint g(t)\mathbf{F}_0(t)dt = 1$ on a small circle around ξ_s. Taking x_s large we must have by (15) $\oint(1 + o(1))g(\xi(t))\mathbf{y}(t)dt = 1 + o(1)$ on a small circle around x_s. In many instances one can refine these arguments to see that the singularities of $\mathbf{y}(x)$ and $\mathbf{F}_0(\xi(x))$ must be exactly of the same type. It is clear, on the other hand, that x_s form a nearly periodic array of points as $|x_s| \to \infty$ (see Theorem 2).

In the following we will make rigorous these intuitive arguments and then proceed to explore further properties and consequences.

3. Main results

Let d be a direction in the x-plane (not an antistokes line). Consider a solution $\mathbf{y}(x)$ of (2) satisfying (8) hence (9). Let (10) be its representation as summation of a transseries $\tilde{\mathbf{y}}(x)$ (see (4)) on d. Let S_{trans} be the sector of validity of $\tilde{\mathbf{y}}(x)$ see (6).

For simplicity we *assume*, what is generically the case, that no $\overline{p_{j;\mathbf{k}}}$ (see (35)) lies on the antistokes lines bounding S_{trans}.

We *assume* that not all parameters C_j are zero, say $C_1 \neq 0$. Then S_{trans} is bounded by two antistokes lines and its opening is at most π.

Notations. It can be assumed without loss of generality (possibly after a linear transformation in x and renumbering the coordinates of $\mathbf{y}$) that

(a) $\lambda_1 = 1$, and
(b) $C_j = 0$ for $j > n_1$ (where $n_1 \in \{2, \dots, n\}$)

(c) $\arg(\lambda_1) < \arg(\lambda_2) < \ldots < \arg(\lambda_{n_1})$
(d) S_{trans} is bounded by $i\mathbb{R}_+$ and the direction $\arg(\lambda_{n_1} x) = -\pi/2$ (which are antistokes lines associated to λ_1 and λ_{n_1}), and S_{trans} is contained in the right half-plane.

The solution $\mathbf{y}(x)$ is then analytic in a region S_{an} (see (137)).

The singularities of $\mathbf{y}(x)$ that we find are related to the two antistokes directions bounding S_{trans}. We will formulate the results for the direction $i\mathbb{R}_+$ (and similar results hold for the other direction $\arg(\lambda_{n_1} x) = -\pi/2$).

The locations of singularities of $\mathbf{y}(x)$ depend on the constant C_1 (constant which may change when d crosses Stokes lines). We need its value in the sector between $i\mathbb{R}_+$ and the neighboring Stokes line in S_{trans}. Let $d' \subset S_{trans}$ be a direction in the first quadrant and consider the representation (10) of $\mathbf{y}(x)$ on d'.[6] From here on we will rename d' as d.

Fix some small, positive δ and c. Denote

$$\xi = \xi(x) = C_1 \mathrm{e}^{-x} x^{\alpha_1} \tag{16}$$

and

$$\mathcal{E} = \Big\{ x\,;\ \arg(x) \in \Big[-\frac{\pi}{2} + \delta, \frac{\pi}{2} + \delta \Big] \text{ and } \\ \Re(\lambda_j x/|x|) > c \text{ for all } j \text{ with } 2 \le j \le n_1 \Big\} \tag{17}$$

Also let

$$\mathcal{S}_{\delta_1} = \{ x \in \mathcal{E}\,;\ |\xi(x)| < \delta_1 \} \tag{18}$$

The sector $\mathcal{E}$ contains S_{trans}, except for a thin sector at the lower edge of S_{trans} (excluded by the conditions $\Re(\lambda_j x/|x|) > c$ for $2 \le j \le n_1$, or, if $n_1 = 1$, by the condition $\arg(x) \ge -\frac{\pi}{2} + \delta$), and may extend beyond $i\mathbb{R}_+$ since there is no condition on $\Re(\lambda_1 x)$ – hence $\Re(\lambda_1 x) = \Re(x)$ may change sign in $\mathcal{E}$ and $\mathcal{S}_{\delta_1}$.

Figure 1 is drawn for $n_1 = 1$; $\mathcal{E}$ contains the gray regions and extends beyond the curved boundary.

3.1. Asymptotic behavior of $\mathbf{y}(x)$ *in* S_{δ_1}

Theorem 1 *(i) There exists* $\delta_1 > 0$ *so that for* $|\xi| < \delta_1$ *the power series*

$$\mathbf{F}_m(\xi) = \sum_{k=0}^{\infty} \xi^k \tilde{\mathbf{y}}_{k\mathbf{e}_1;m}, \quad m = 0, 1, 2, \ldots \tag{19}$$

converge (for notations see (4), (5), (16) and for an estimate of δ_1 *see Proposition 4).*

[6] C_1 does not change at the possible secondary Stokes lines $\overline{d_{j,\mathbf{k}}}$, $|\mathbf{k}| \ge 1$ lying between $\mathbb{R}_+$ and $i\mathbb{R}_+$.

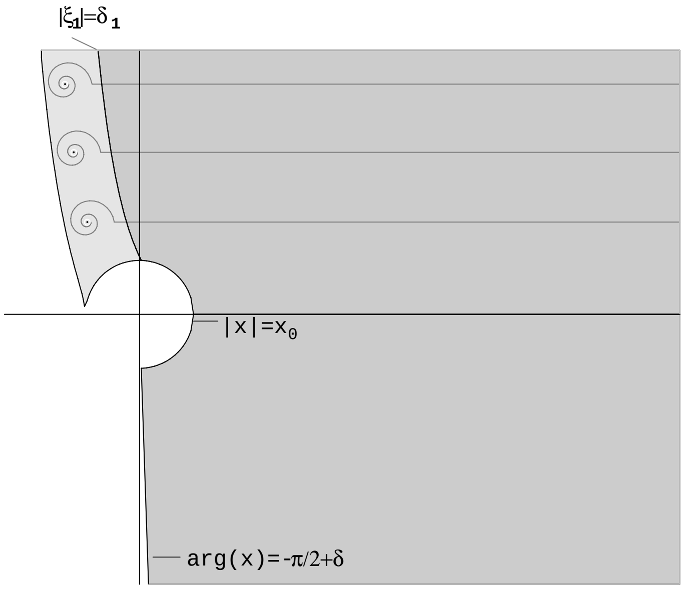

Fig. 1 Region $\mathcal{D}_x$ where (26) holds, when $n_1 = 1$. The dark gray subregion is S_{δ_1}. Curves like the spiraling gray curves surround points in X (close to singularities of $\mathbf{y}$) generate the region $\mathcal{D}_x$. The picture is drawn with $n_1 = 1, \lambda = \frac{1}{10}, \alpha = -\frac{1}{2}, \delta_1 = 3 \cdot 10^6, x_0 = 40$. In this case S_{trans} is a sector where $|\arg(x)| < \frac{\pi}{2} - 0$

Furthermore

$$\mathbf{y}(x) \sim \sum_{m=0}^{\infty} x^{-m} \mathbf{F}_m(\xi(x)) \quad (x \in \mathcal{S}_{\delta_1},\ x \to \infty) \tag{20}$$

uniformly in $\mathcal{S}_{\delta_1}$, and the asymptotic representation (20) is differentiable.

The functions $\mathbf{F}_m$ are uniquely defined by (20), the requirement of analyticity at $\xi = 0$, and $\mathbf{F}_0'(0) = \mathbf{e}_1$.

(ii) The following Gevrey-like estimates hold in $\mathcal{S}_{\delta_1}$ for some constants $K_{1,2}, B_1 > 0$:

$$|F_m(\xi(x))| \le K_1 m! B_1^m \tag{21}$$

$$\left| \mathbf{y}(x) - \sum_{j=0}^{m-1} x^{-m} \mathbf{F}_m(\xi(x)) \right| \le K_2 m! B_1^m x^{-m} \quad (m \in \mathbb{N}^+,\ x \in \mathcal{S}_{\delta_1}) \tag{22}$$

Comments. 1. It is interesting to remark that the constant beyond all orders C_1 is now classically definable in terms of the expansion (20) because this expansion is unique with its the range of validity, and with the given analyticity properties. This is in a sense a generalization of Watson's lemma in the context of transexpansions.

2. While the classical expansion (9) is valid only in any proper subsector of $\mathcal{S}_{\delta_1} \cap \{x : \arg(x) < \pi/2\}$, the representation (20) holds down to a distance going to zero as x becomes large from the (finite-plane) singularities of $\mathbf{F}_0$, near which $\mathbf{y}(x)$ also develops singularities (see Theorem 2 and Sect. 5).
3. A similar picture holds near the lower edge of S_{trans}. The constant C_j used in (16), which determines the position of singularities (see (28)) of $\mathbf{y}(x)$ related to that direction, is C_{n_1}. If $n_1 = 1$ then the value of C_1 is the one in the fourth quadrant (which may differ from the one in the first quadrant due to the Stokes phenomenon on $\mathbb{R}_+$).

3.2. Singularity analysis

We now focus on singularities of $\mathbf{y}(x)$ and their connection with singularities of $\mathbf{F}_0$.

3.2.1. Definitions (cf. Fig. 1) By (3) and (19) we have $\mathbf{F}_0(0) = 0$. Both $\mathbf{F}_0$ and $\mathbf{y}$ turn out to be analytic in S_{δ_1} (Theorems 1(i) and 2(i)); the interesting region is then $\mathcal{E} \setminus S_{\delta_1}$ (containing the light grey region in Fig. 1).

Denote by $\mathcal{P}$ a polydisk

$$\mathcal{P} = \{(x^{-1}, \mathbf{y}) : |x^{-1}| < \rho_1, |\mathbf{y}| < \rho_2\} \tag{23}$$

where $\mathbf{g}$ is analytic and continuous up to the boundary.

Let Ξ be a *finite* set (possibly empty) of points in the ξ-plane. This set will consist of singular points of $\mathbf{F}_0$ thus we assume $\mathrm{dist}(\Xi, 0) \geq \delta_1$.

Denote by $\mathcal{R}_\Xi$ the Riemann surface above $\mathbb{C} \setminus \Xi$. More precisely, we assume that $\mathcal{R}_\Xi$ is realized as equivalence classes of simple curves $\Gamma : [0, 1] \mapsto \mathbb{C}$ with $\Gamma(0) = 0$ modulo homotopies in $\mathbb{C} \setminus \Xi$.

Let $\mathcal{D} \subset \mathcal{R}_\Xi$ be *open, relatively compact, and connected*, with the following properties:

(1) $\mathbf{F}_0(\xi)$ is analytic in an $\epsilon_{\mathcal{D}}$–neighborhood of $\mathcal{D}$ with $\epsilon_{\mathcal{D}} > 0$,
(2) $\sup_{\mathcal{D}} |\mathbf{F}_0(\xi)| := \rho_3$ with $\rho_3 < \rho_2$
(3) $\mathcal{D}$ contains $\{\xi : |\xi| < \delta_1\}$.[7]

It is assumed that there is an upper bound on the length of the curves joining points in $\mathcal{D}$: $d_{\mathcal{D}} = \sup_{a,b\in\mathcal{D}} \inf_{\Gamma\subset\mathcal{D};a,b\in\Gamma} \mathrm{length}(\Gamma) < \infty$.

We also need the x-plane counterpart of this domain.

Let $R > 0$ (large) and let $X = \xi^{-1}(\Xi) \cap \{x \in \mathcal{E} : |x| > R\}$.

[7] Conditions (2), (3) can be typically satisfied since $\mathbf{F}_0(\xi) = \xi + O(\xi^2)$ and $\delta_1 < \rho_2$ (see also the examples in Sect. 5); borderline cases may be treated after choosing a smaller δ_1.

Let $\varGamma$ be a curve in $\mathcal{D}$. There is a countable family of curves γ_N in the x-plane with $\xi(\gamma_N) = \varGamma$. The curves are smooth for $|x|$ large enough and satisfy

$$\gamma_N(t) = 2N\pi i + \alpha_1 \ln(2\pi i N) - \ln \varGamma(t) + \ln C_1 + o(1) \quad (N \to \infty) \tag{24}$$

(For a proof see Appendix Sect. 6.3.)

To preserve smoothness, we will restrict to $|x| > R$ with R large enough, so that along (a smooth representative of) each $\varGamma \in \mathcal{D}$, the branches of ξ^{-1} are analytic.

If the curve $\varGamma$ is a smooth representative in $\mathcal{D}$ we then have $\xi^{-1}(\varGamma) = \cup_{N\in\mathbb{N}} \gamma_N$ where γ_N are smooth curves in $\{x : |x| > 2R\}\backslash X$.

We define $\mathcal{D}_x$ as the equivalence classes modulo homotopies in $\{x \in \mathcal{E} : |x| > R\}\backslash X$ (with ∞ fixed point) of those curves γ_N which are completely contained in $\mathcal{E} \cap \{x : |x| > 2R\}$.

Theorem 2 *(i) The functions* $\mathbf{F}_m(\xi);\ m \geq 1$, *are analytic in* $\mathcal{D}$ *(note that by construction* $\mathbf{F}_0$ *is analytic in* $\mathcal{D}$*) and for some positive* B, K *we have*

$$|F_m(\xi)| \leq Km!B^m, \quad \xi \in \mathcal{D} \tag{25}$$

(ii) For R large enough the solution $\mathbf{y}(x)$ *is analytic in* $\mathcal{D}_x$ *and has the asymptotic representation*

$$\mathbf{y}(x) \sim \sum_{m=0}^{\infty} x^{-m}\mathbf{F}_m(\xi(x)) \ \ (x \in \mathcal{D}_x,\ |x| \to \infty) \tag{26}$$

In fact, the following Gevrey-like estimates hold

$$\left|\mathbf{y}(x) - \sum_{j=0}^{m-1} x^{-j}\mathbf{F}_j(\xi(x))\right| \leq K_2 m! B_2^m |x|^{-m} \quad (m \in \mathbb{N}^+,\ x \in \mathcal{D}_x) \tag{27}$$

(iii) Assume $\mathbf{F}_0$ *has an isolated singularity at* $\xi_s \in \varXi$ *and that the projection of* $\mathcal{D}$ *on* $\mathbb{C}$ *contains a punctured neighborhood of (or an annulus of inner radius r around)* ξ_s.

Then, if $C_1 \neq 0$, $\mathbf{y}(x)$ *is singular at a distance at most* $o(1)$ $(r + o(1)$, *respectively) of* $x_n \in \xi^{-1}(\{\xi_s\}) \cap \mathcal{D}_x$, *as* $x_n \to \infty$.
The collection $\{x_n\}_{n\in\mathbb{N}}$ *forms a nearly periodic array*

$$x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1) \tag{28}$$

as $n \to \infty$.

Some of the conclusions of the theorem hold with $\mathcal{D}$ noncompact, under some natural restrictions, see Proposition 8.

Comments. 1. The singularities x_n satisfy $C_1 e^{-x_n} x_n^{\alpha_1} = \xi_s(1+o(1))$ (for $n \to \infty$). Therefore, the singularity array lies slightly to the left of the antistokes line $i\mathbb{R}_+$ if $\Re(\alpha_1) < 0$ (this case is depicted in Fig. 1) and slightly to the right of $i\mathbb{R}_+$ if $\Re(\alpha_1) > 0$.

2. In practice it is useful to normalize the system (2) so that α_1 is as small as possible (see the Comment 1. in Sect. 5.2 and Sect. 6.7).
3. By (27) a truncation of the two-scale series (26) at an m dependent on x ($m \sim |x|/B$) is seen to produce exponential accuracy $o(\mathrm{e}^{-|x/B|})$, see e.g. [42].
4. Theorem 2 can also be used to determine precisely the nature of the singularities of $\mathbf{y}(x)$. In effect, for any n, the representation (26) provides $o(\mathrm{e}^{-K|x_n|})$ estimates on $\mathbf{y}$ down to an $o(\mathrm{e}^{-K|x_n|})$ distance of an actual singularity x_n. In most instances this is more than sufficient to match to a suitable local integral equation, contractive in a tiny neighborhood of x_n, providing rigorous control of the singularity. See also Sect. 3.3 and Sect. 5.

3.3. Singularities for weakly nonlinear systems

In this section we take $\mathbf{g}$ meromorphic in a small enough neighborhood of $(0, \mathbf{0})$, but nevertheless analytic at $(0, \mathbf{0})$, and only weakly nonlinear. Such could be the case if in a sufficiently large neighborhood of zero only one component of $\mathbf{g}$ is singular, and the singular manifold of $\mathbf{g}$ is approximately a hyperplane.

Let $\mathbf{y}(x)$ be as in Theorem 1; denote $\mathbf{f}(\mathbf{y}) = -\hat{\Lambda} + \mathbf{g}(0, \mathbf{y})$. By Theorem 1 (i) we have $y_1(x) \sim \xi(x)$ and $y_j(x) = O(\xi^2) \ll y_1(x)$, $j = 2, \ldots, n$ when $1 \gg |\xi(x)| \gg |x|^{-2}$. For definiteness we assume the component g_2 to be the only one singular in some neighborhood of $(0, \mathbf{0})$. The precise *assumptions* are

$$\begin{aligned} f_1 &= -\lambda_1 y_1 + \epsilon_1(\mathbf{y}) \\ f_2(\mathbf{y}) &= \frac{-\lambda_2 y_2 - \gamma_2 y_1^2 + \epsilon_2(\mathbf{y})}{h(\mathbf{y})} \\ f_j(\mathbf{y}) &= -\lambda_j y_j - \gamma_j y_1^2 + \epsilon_j(\mathbf{y}) \quad (j \neq 1, 2) \\ h(\mathbf{y}) &= 1 - \sum_{k=1}^{n} a_j y_j + \epsilon_{n+1}(\mathbf{y}) \end{aligned} \tag{29}$$

where $a_j, \gamma_j \in \mathbb{C}$ with $\|\mathbf{a}\| < \frac{1}{2}(\rho_2)^{-1}$ (see (23)), ϵ_j are analytic and satisfy

$$|\epsilon_j(\mathbf{y})| < \epsilon \quad \text{for} \quad |\mathbf{y}| < \rho_2 \quad \text{for} \quad j = 1, \ldots, n+1 \tag{30}$$

for some small positive ϵ.

Choose $x_0 > 0$ large enough so that the function $\mathbf{g}(x^{-1}, \mathbf{y})$ is analytic in $\mathcal{P}$ if $\rho_1 = x_0^{-1}$ (see (23)) and

$$\|\mathbf{g}\|_{\mathcal{P}} < \epsilon \tag{31}$$

Theorem 3 *For almost all values of the parameters γ_j, a_j with $j = 1, \dots, n$ the following holds. If ϵ is small, $\mathbf{F}_0(\xi)$ and $\mathbf{y}(x)$ are not entire. $\mathbf{F}_0$ has isolated square root branch points on its circle of analyticity and correspondingly $\mathbf{y}$ has arrays of branch points for large x.*

More precisely, there exists ξ_s so that $\mathbf{F}_0$ is analytic for $|\xi| < |\xi_s|$ and $\mathbf{y}$ is analytic if $|\xi(x)| < |\xi_s| - \delta(x)$ where $\delta(x) \to 0$ as $x \to \infty$. Furthermore, $\mathbf{F}_0$ is analytic on the Riemann surface of the square root at ξ_s and there is an array $\tilde{x}_n \in \mathcal{S}_{\delta_1}$ with $\xi(\tilde{x}_n) = \xi_s + o(1)$ as $|\tilde{x}_n| \to \infty$ so that $\mathbf{y}(x)$ is analytic on the Riemann surface generated by curves that encircle at most one of the $\tilde{x}_n$. Near ξ_s and $\tilde{x}_n$ we have

$$\mathbf{F}_0(\xi) = \mathbf{F}_A\big((\xi - \xi_s)^{1/2}\big) \qquad \mathbf{y}(x) = \mathbf{y}_A\big((x - \tilde{x}_n)^{1/2}\big) \tag{32}$$

respectively, where $\mathbf{y}_A$ and $\mathbf{F}_A$ are analytic at zero.

4. Proofs and further results

4.1. Further results needed

A possible proof of Theorem 1 using only classical asymptotics concepts is sketched in Sect. 6.10. The proof given in this section uses some results in exponential asymptotics. We need the following facts proved in [13].

Denote by $\overline{d}$ the complex conjugate of d. We have

$$\mathbf{y}_{\mathbf{k}}(x) = \int_{\overline{d}} \mathbf{Y}_{\mathbf{k}}(p) \mathrm{e}^{-px} dp \tag{33}$$

where the functions $\mathbf{Y}_{\mathbf{k}}$ have the form

$$\mathbf{Y}_{\mathbf{k}}(p) = p^{-\mathbf{k}\boldsymbol{\alpha}'-1} \mathbf{A}_{\mathbf{k}}(p) \tag{34}$$

(Lemma 20 in [13]), with $\mathbf{A}_{\mathbf{k}}$ analytic near zero, and along curves towards infinity avoiding the points $p_{j;\mathbf{k}}$ defined as

$$p_{j;\mathbf{k}} = \lambda_j - \mathbf{k} \cdot \boldsymbol{\lambda}\,, \quad j = 1, \dots, n_1\,, \quad \mathbf{k} \in \mathbb{Z}_+^{n_1} \tag{35}$$

Because in S_{trans} we have $\Re(\lambda_j x) > 0$ for $j \le n_1$ and since $\mathbf{k} \ge 0$ it follows that (1) the $p_{j;\mathbf{k}}$ have no accumulation point and (2) only finitely many of them are in $\overline{S}_{trans} = \{\overline{d} : d \subset S_{trans}\}$ (see [13] for more details). In particular, if $p_{j;\mathbf{k}} \notin \overline{d}$ then there exists $a_1 > 0$ and an a_1-neighborhood $\overline{d}_{a_1}$ of $\overline{d}$ (i.e. $\overline{d}_{a_1} = \{x; \operatorname{dist}(x, \overline{d}) < a_1\}$) where all $\mathbf{A_k}$, $\mathbf{k} \geq 0$ are analytic.

There exist positive constants K_1, ν_0 such that for all $\mathbf{k} \geq 0$ we have ([13], Prop. 22(ii) for $\mathbf{W_k} = \mathbf{Y_k}$ in $\mathcal{D}'_{m,\nu}$ and $\mathcal{T}_{\mathbf{k}\cdot\boldsymbol{\beta}'-1}$)

$$\sup_{p\in\overline{d}_{a_1}} |\mathbf{Y_k}(p)\mathrm{e}^{-|\nu_0 p|}| \leq K_1^{|\mathbf{k}|} \tag{36}$$

Also, $\tilde{\mathbf{y}}_\mathbf{k}$ is the classical asymptotic expansion of $\mathbf{y_k}$ ([13], Theorem 3):

$$\mathbf{y_k}(x) \sim \tilde{\mathbf{y}}_\mathbf{k}(x) \quad \left(x \in d\mathrm{e}^{i\theta},\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right);\ x \to \infty\right) \tag{37}$$

In the following we need a better estimate of $\mathbf{A_k}$ (see (34) and (33)), than given in [13].

Proposition 4 *For some $a_1, \delta_2 > 0$ and all $\mathbf{k}$ and l we have*

$$|\mathbf{A}_\mathbf{k}^{(l)}(p)| \leq \left|\Gamma(-\mathbf{k}\cdot\boldsymbol{\alpha}')\right|^{-1} l!\, a_1^{-l}\, \mathrm{e}^{\nu_0|p|+a_1}\, \delta_2^{-|\mathbf{k}|} \tag{38}$$

uniformly in $\overline{d}_{a_1}$.

The proof amounts to minor modifications in a proof in [13]. We detail these modifications in Sect. 6.6.

4.2. Proof of Theorem 1

(a) *Analyticity at* 0. From (33), (34), and Watson's lemma [5] we get as $x \to \infty$ along d

$$\begin{aligned}\mathbf{y_k}(x) &\sim \sum_{m\geq 0} \frac{\mathbf{A}_\mathbf{k}^{(m)}(0)}{m!} \int_{\overline{d}} p^{m-\mathbf{k}\cdot\boldsymbol{\alpha}'-1}\mathrm{e}^{-px}dp \\ &= \sum_{m\geq 0} \frac{\mathbf{A}_\mathbf{k}^{(m)}(0)}{m!x^{m-\mathbf{k}\cdot\boldsymbol{\alpha}'}} \int_0^\infty s^{m-\mathbf{k}\cdot\boldsymbol{\alpha}'-1}\mathrm{e}^{-s}dp = \sum_{m\geq 0} \frac{\mathbf{A}_\mathbf{k}^{(m)}(0)\Gamma(m-\mathbf{k}\cdot\boldsymbol{\alpha}')}{m!x^{m-\mathbf{k}\cdot\boldsymbol{\alpha}'}} \\ &=: \sum_{m\geq 0} \mathbf{y}_{\mathbf{k};m} \frac{1}{x^{m-\mathbf{k}\cdot\boldsymbol{\alpha}'}}\end{aligned} \tag{39}$$

so that from Proposition 4 it follows that $\mathbf{F}_m(\xi) = \sum_{k=0}^\infty \mathbf{y}_{k\mathbf{e}_1;m}\xi^k$ converges if $|\xi| < \delta_1$, where $\delta_1 < \delta_2$, in which ball we also have

$$|\mathbf{F}_m(\xi)| \leq K_1 B_1^m m! \tag{40}$$

for some B_1.

(b) *Asymptoticity.* We first note that using again Watson's lemma, by (38) we have

$$
\begin{aligned}
&\left|\mathbf{y}_{\mathbf{k}} - x^{\mathbf{k}\cdot\boldsymbol{\alpha}'} \sum_{l=0}^{M} \frac{\tilde{\mathbf{y}}_{\mathbf{k};l}}{x^l}\right| \\
&\quad = \left| \int_{\overline{d}} p^{-\mathbf{k}\cdot\boldsymbol{\alpha}'-1} \left(\mathbf{A}_{\mathbf{k}}(p) - \sum_{l=0}^{M} l!^{-1} \mathbf{A}_{\mathbf{k}}^{(l)}(0) p^l \right) \mathrm{e}^{-px} dp \right| \\
&\quad\quad \le \mathrm{e}^{a_1} \left| \Gamma(-\mathbf{k}\cdot\boldsymbol{\alpha}')^{-1} \right| \delta_2^{-|\mathbf{k}|} {a_1}^{-M-1} (M+1)! \int_{\overline{d}} \times \\
&\quad\quad\quad\quad |p^{-\mathbf{k}\cdot\boldsymbol{\alpha}'-1} p^{M+1} \mathrm{e}^{\nu_0|p|} \mathrm{e}^{-px} |d|p| \\
&\quad\quad\quad \le \left| \Gamma(-\mathbf{k}\cdot\boldsymbol{\alpha}')^{-1} \delta_2^{-|\mathbf{k}|} x^{\mathbf{k}\cdot\boldsymbol{\alpha}'+M+1} {a_1}^{-M-1} \right| \\
&\quad\quad\quad\quad\quad \le \delta_2^{-|\mathbf{k}|} (M+1)! {a_1}^{-M-1} \left| x^{\mathbf{k}\cdot\boldsymbol{\alpha}'+M+1} \right|
\end{aligned} \tag{41}
$$

It is then convenient to write, for $x \in \mathcal{S}_{\delta_1}$,

$$
\mathbf{y}(x) = \sum_{k=0}^{\infty} C_1^k \mathrm{e}^{-kx} x^{k\alpha_1} \mathbf{y}_{k\mathbf{e}_1}(x) + \sum_{\mathbf{k}\succ 0;\mathbf{k}\neq k\mathbf{e}_1}^{\infty} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{\mathbf{k}\boldsymbol{\alpha}} \mathbf{y}_{\mathbf{k}}(x) \tag{42}
$$

and (cf. (18), (17))

$$
= \sum_{k=0}^{\infty} C_1^k \mathrm{e}^{-kx} x^{k\alpha_1} \mathbf{y}_{k\mathbf{e}_1}(x) + O(\mathrm{e}^{-c|x|}) := \mathbf{y}^{[1]}(x) + O(\mathrm{e}^{-c|x|})
$$

Now,

$$
\begin{aligned}
\sum_{k=0}^{\infty} C_1^k \mathrm{e}^{-kx} x^{k\alpha_1} \mathbf{y}_{k\mathbf{e}_1}(x) &= \sum_{k=0}^{\infty} C_1^k \mathrm{e}^{-kx} x^{k\alpha_1} \left(\sum_{m=0}^{M} \tilde{\mathbf{y}}_{k\mathbf{e}_1;m} x^{-m} \right) \\
&\quad + \sum_{k=0}^{\infty} C_1^k \mathrm{e}^{-kx} x^{k\alpha_1} \left(\mathbf{y}_{k\mathbf{e}_1}(x) - \sum_{m=0}^{M} \tilde{\mathbf{y}}_{k\mathbf{e}_1;m} x^{-m} \right)
\end{aligned} \tag{43}
$$

and Theorem 1 follows from (41). Differentiability simply follows from the fact that (20) holds in a nontrivial sector.

4.3. Special Gevrey estimates

In the proofs of the main theorems we need optimal estimates of high order derivatives of functions of the form $\boldsymbol{\varphi}(z, \sum_{k=1}^{m} \mathbf{a}_k z^k)$ in terms of estimates of $\mathbf{a}_k$, when $\sum_{k=0}^{\infty} \mathbf{a}_k z^k$ are Gevrey type series. Because of the truncations involved, the estimates do not follow from Gevrey theory.

Let $\boldsymbol{\varphi}$ be an analytic function in the polydisk $\mathcal{P} = \{|z| < \rho_1, |\mathbf{y}| < \rho_2\} \subset \mathbb{C} \times \mathbb{C}^n$ and continuous up to the boundary.

Assume first that the series

$$\mathbf{a}(z) = \sum_{k=1}^{\infty} \mathbf{a}_k z^k$$

converges and denote by $\mathbf{a}^{[m]}$ the truncation

$$\mathbf{a}^{[m]}(z) = \sum_{k=0}^{m} \mathbf{a}_k z^k$$

Then $\frac{d^m}{dz^m}\boldsymbol{\varphi}(z; \mathbf{a}(z))|_{z=0}$ is a polynomial in $\mathbf{a}_1, \dots, \mathbf{a}_m$, of degree 1 in $\mathbf{a}_m$:

$$\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\boldsymbol{\varphi}(z; \mathbf{a}(z))|_{z=0} = \partial_{\mathbf{y}}\boldsymbol{\varphi}(0; \mathbf{a}_0)\mathbf{a}_m + \frac{1}{m!}\frac{d^m}{dz^m}\boldsymbol{\varphi}\big(z; \mathbf{a}^{[m-1]}(z)\big)\Big|_{z=0} \quad (44)$$

Relation (44) is meaningful even when $\mathbf{a}(z)$ is only a formal sum (with no convergence conditions)–in the sense that the LHS is the coefficient of z^m in the formal series expansion of $\boldsymbol{\varphi}(z; \mathbf{a}(z))$ at $z = 0$. We are primarily interested in the Gevrey-1 character of $\mathbf{a}(z)$ (meaning that for some $c_1, c_2 > 0$ we have $|\mathbf{a}_k| < c_1 c_2^k k!$; see also [42]). Proposition 5 below is formulated in a way that permits an inductive proof of Gevrey type inequalities, when $\mathbf{a}_k$ are defined recursively.

Proposition 5 *Assume $\rho_2 - |\mathbf{a}_0| > 0$. There exists a positive C so that: for any $B, K > 0$ and any $\{\mathbf{a}_k\}_{k=1\dots m}$ such that $|\mathbf{a}_k| < KB^k k!$, $k = 1, 2, \dots, m$ we have*

$$\frac{1}{m!}\left|\frac{d^m}{dz^m}\boldsymbol{\varphi}(z; \mathbf{a}^{[m-1]}(z))\right|_{z=0} \le K_2 B^{m-1}(m-1)!(1 + Cm^{-1}\log^2 m) \quad (45)$$

for some K_2 (see (55) for an estimate of K_2).

For the proof we need the following result.

Lemma 6 *Let a, b satisfy $1 < a < b$. There exists $C = C(a; b)$ such that if $Bmz := Z \in (a, b)$ then*

$$\left|\frac{\sum_{k=1}^{m} k! m^{-k} Z^k}{m! m^{-m} Z^m [1 - Z^{-1}]^{-1} + m^{-1} Z} - 1\right| \le Cm^{-1}(\ln m)^2 \quad (m \in \mathbb{N}) \quad (46)$$

Proof of Lemma 6. In this proof (and in the proof of Proposition 5) we write $O(f(m))$ for terms that go to zero not slower than than $f(m)$ uniformly in B, Z (and K).

Let $k_m = \lfloor m/Z \rfloor$. For $k \le k_m$ the terms $k! m^{-k} Z^k$ are decreasing in k, and increasing for $k \ge k_m$. Thus

$$\sum_{k=1}^{k_m} B^k z^k k! \le \frac{Z}{m} + \frac{2Z^2}{m^2} + m\frac{6Z^3}{m^3} = Bz(1 + O(m^{-1})) \quad (m \to \infty) \quad (47)$$

Denote $p_m = \lfloor 2\ln m/\ln Z \rfloor$. For m large enough we have

$$m \geq k_m + p_m,\ 1 - p_m/m > 1/2,\ \ p_m > k_m$$

and for $p \leq p_m$

$$1 \geq \prod_{j=0}^{p}\left(1 - \frac{j}{m}\right) \geq \mathrm{e}^{-2\frac{p(p+1)}{2m}} \geq \mathrm{e}^{-\frac{p_m(p_m+1)}{m}} \tag{48}$$

Denote $\sigma_{N_1}^{N_2} = \sum_{k=N_1}^{N_2} \frac{k!(Bz)^k}{m!(Bz)^m}$; we have

$$\sigma_{m-p_m}^{m} = \sum_{k=0}^{p_m} Z^{-k} \prod_{j=0}^{k}\left(1 - \frac{j}{m}\right)^{-1} \geq \frac{1 + O(m^{-2})}{1 - Z^{-1}} \tag{49}$$

while using (4.3) it follows that

$$\sigma_{m-p_m}^{m} \leq \mathrm{e}^{\frac{p_m(p_m+1)}{m}} \sum_{k=0}^{\infty} Z^{-k} = \frac{1 + O(m^{-1}(\ln m)^2)}{(1 - (mBz)^{-1})} \tag{50}$$

For $k \in (k_m, m - p_m)$, because the terms in the sum are increasing we get

$$\sigma_{k_m}^{m-p_m} \leq m \frac{\mathrm{e}^{p_m(p_m+1)/m}}{Z^{2\ln m/\ln Z}} = m^{-1} + O(m^{-2}(\ln m)^2) \tag{51}$$

Combining (49) (50), (51), and (47), Lemma 6 follows. □

Proof of Proposition 5. We keep the requirement of uniformity with respect to B, K in the notation $O(\cdot)$, as in the Proof of Lemma 6.

Let $\rho = \min\{\rho_1, \rho_2 - |\mathbf{a}_0|\}$ (cf. the beginning of Sect. 4.3). For small s and $\mathbf{y}$ we have

$$\begin{aligned} &\left|\boldsymbol{\varphi}(s,\mathbf{y}) - \boldsymbol{\varphi}(0,\mathbf{0}) - \partial_s\boldsymbol{\varphi}(0,\mathbf{0})s - \partial_{\mathbf{y}}\boldsymbol{\varphi}(0,\mathbf{0})\cdot\mathbf{y}\right| \\ \leq\ &\frac{2(n+1)^2\|\boldsymbol{\varphi}\|}{\rho^2}\left(|s|^2 + \|\mathbf{y}\|^2\right) = \nu_1\left(|s|^2 + \|\mathbf{y}\|^2\right) \end{aligned} \tag{52}$$

We choose a circle of radius r_m, where

$$(m-1)!(Br_m)^{m-1} = Br_m \tag{53}$$

For large m we have $Bmr_m = e + O(m^{-1})$, and we also see that the assumptions of Lemma 6 are satisfied. In particular we have for $|s| = r_m$ that

$$\left|\sum_{k=1}^{m-1} \mathbf{a}_k s^k\right| \leq KBr_m(1 + (1 - 1/e)^{-1})(1 + O(m^{-1}(\ln m)^2)) \tag{54}$$

Noting that $\oint_{r_m} s^{m+1}\mathbf{a}^{[m-1]}(s)ds = 0$ we have, using (53) and (52),

$$\begin{aligned}\left|\frac{1}{m!}\frac{d^m}{dz^m}\boldsymbol{\varphi}(z;\mathbf{a}^{[m-1]})\right|_{z=0} &= \left|\frac{1}{2\pi i}\oint_{r_m}\mathrm{d}s\frac{\boldsymbol{\varphi}(s;\mathbf{a}^{[m]}(s))}{s^{m+1}}\right| \\ &\leq \nu_1(1+K^2B^2)|r_m|^{2-m}(1+O(m^{-1}(\ln m)^2)) \\ &= \nu_1(1+K^2B^2)B^{-1}(B^{m-1}(m-1)!)(1+O(m^{-1}(\ln m)^2)) \quad (55)\end{aligned}$$

□

Remark 7 A direct calculation shows that the expansion in (26) is a formal solution of (2) for large x iff the functions $\mathbf{F}_m$ are solutions of the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_0 = \xi^{-1}\big(\hat{\Lambda}\mathbf{F}_0 - \mathbf{g}(0,\mathbf{F}_0)\big) \quad (56)$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_m + \hat{N}\mathbf{F}_m = \alpha_1\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_{m-1} + \mathbf{R}_{m-1} \quad \text{for } m \geq 1 \quad (57)$$

where $\hat{N}$ is the matrix

$$\xi^{-1}(\partial_{\mathbf{y}}\mathbf{g}(0,\mathbf{F}_0) - \hat{\Lambda}) \quad (58)$$

and the function $\mathbf{R}_{m-1}(\xi)$ depends only on the $\mathbf{F}_k$ with $k < m$:

$$\xi\mathbf{R}_{m-1} = -\Big[(m-1)I + \hat{A}\Big]\mathbf{F}_{m-1} - \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathbf{g}\left(z;\sum_{j=0}^{m-1}z^j\mathbf{F}_j\right)\Bigg|_{z=0} \quad (59)$$

(see also (44)).

4.4. Proof of Theorem 2 (i)

By Theorem 1, $\mathbf{F}_m,\ m \geq 0$ are analytic for $|\xi| < \delta_1$. Furthermore, $\mathbf{F}_m$ are analytic in the $\epsilon_{\mathcal{D}}$–neighborhood of $\mathcal{D}$ since by assumption, $\mathbf{F}_0$ is analytic there and equations (57) are linear for $m \geq 1$.

We set as initial conditions for (57) in $\mathcal{D}$ the values of $\mathbf{F}_m(\xi_0)$ provided by Theorem 1 at a point $\xi_0 \in \mathcal{D}$ with $|\xi_0| \in (\frac{1}{2}\delta_1, \delta_1)$.

For $\xi \in \mathcal{D}$ and $m \geq 1$, (57) can be integrated, yielding the recursive system

$$\begin{aligned}\mathbf{F}_m(\xi) &= \hat{M}(\xi;\xi_0)\mathbf{F}_m(\xi_0) + \alpha_1\big(\mathbf{F}_{m-1}(\xi) - \hat{M}(\xi;\xi_0)\mathbf{F}_{m-1}(\xi_0)\big) \\ &\quad - \alpha_1\int_{\xi_0}^{\xi}\hat{M}(\xi;s)\hat{N}(s)\mathbf{F}_{m-1}(s)ds + \int_{\xi_0}^{\xi}\hat{M}(\xi;s)\mathbf{R}_{m-1}(s)ds \quad (60)\end{aligned}$$

where $\hat{M}(\xi;\zeta)$ is the fundamental solution of

$$\frac{\mathrm{d}\hat{M}}{\mathrm{d}\xi} + \hat{N}\hat{M} = 0 \quad \text{with} \quad \hat{M}(\xi;\zeta)|_{\xi=\zeta} = I \tag{61}$$

Direct estimates in (60) and (59) using (40) give

$$\begin{aligned} \|\mathbf{F}_m\|_{\mathcal{D}} \le MK_1 m! B_1^m + (|\alpha_1| + M)\|\mathbf{F}_{m-1}\|_{\mathcal{D}} \\ + 2|\alpha_1| d_{\mathcal{D}} M\delta_2^{-1}(\|\mathbf{g}\| + \|\hat{\Lambda}\|)\|\mathbf{F}_{m-1}\|_{\mathcal{D}} \\ + 2\delta_2^{-1} M\left[(m+A)\,\|F_{m-1}\| + d_{\mathcal{D}}\left\|\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathbf{g}(z;\mathbf{F}(z)^{[m-1]})|_{z=0}\right\|_{\mathcal{D}}\right] \end{aligned} \tag{62}$$

where

$$\begin{aligned} M = \sup_{\xi,\zeta\in\mathcal{D}} \|\hat{M}(\xi;\zeta)\|;\ \ \|\mathbf{F}\|_{\mathcal{D}} = \sup_{\xi\in\mathcal{D}} |\mathbf{F}(\xi)|;\ \ \|\mathbf{g}\| = \sup_{|z|<\rho_1,|\mathbf{y}|<\rho_2} |\mathbf{g}(z,\mathbf{y})| \\ A = \max_{\xi\in\mathcal{D}} \|\hat{N}(\xi)\| \le (\delta_2/2)^{-1}\,(\|g\| + \|\Lambda\|) \end{aligned} \tag{63}$$

Choosing K, B large enough, the proof of (25) is immediate induction from (62) and Proposition 5.

4.5. Proof of Theorem 2 (ii)

We will prove (26) at each point $x = x_a \in \mathcal{D}_x$ (with uniform estimates on $\mathcal{D}_x$). x_a is the endpoint of a curve γ_N in $\mathcal{D}_x$ with $\xi(\gamma_N) = \Gamma$ curve in $\mathcal{D}$ and satisfying (24).

Denote $a = \xi(x_a)$. If $|a| < \delta_1$ then (26) follows from Theorem 1 so we assume $|a| \ge \delta_1$. Then we can choose Γ to go from 0 along a direction up to the circle $|\xi| = \delta_1$, not re-entering the circle.

Let t_0 such that $\xi_0 = \Gamma(t_0) \in (\frac{1}{2}\delta_1, \delta_1)$ and denote $\Gamma^0 = \Gamma|_{[t_0,1]}$, $\gamma^0 = \gamma_N|_{[t_0,1]}$; then γ^0 lies in a bounded region.

We prove (26) in a small, connected, simply connected neighborhood $\mathcal{N}_{\gamma^0}$ of γ^0.

Denote $\boldsymbol{\delta}(x) = \mathbf{y}(x) - \mathbf{F}^{[m]}(x)$.

To estimate $\boldsymbol{\delta}(x)$ we use a contraction argument if m is not too large (*Case I*) and a direct argument for m large (*Case II*).

Let c_0 be positive and small and let m_a be the maximal integer such that $m!(2B/|a|)^m \le c_0$.

Case I: $m \le m_a$

First the differential equation satisfied by $\mathbf{F}^{[m]}(x)$ will be written, then the equation of $\boldsymbol{\delta}(x)$, and finally the independent variable will be changed to ξ, yielding (65).

With the notations introduced in Sect. 4.3 and denoting by $\mathbf{F}(x)$ the formal series $\sum_{k=0}^{\infty} x^{-k}\mathbf{F}_m(\xi(x))$ we get from Remark 7

$$\frac{d}{dx}\mathbf{F}^{[m]}(x) = \left(-\hat{\Lambda} + \frac{1}{x}\hat{A}\right)\mathbf{F}^{[m]}(x) + \mathbf{g}(x^{-1}, \mathbf{F}^{[m]}(x))^{[m]} + \frac{1}{x^{m+1}}\hat{A}\mathbf{F}_m(\xi(x)) + \frac{\alpha_1}{x^{m+1}}\xi\frac{d\mathbf{F}_m}{d\xi}(\xi(x)) - \frac{m}{x^{m+1}}\mathbf{F}_m(\xi(x)) \quad (64)$$

Using (64) and (2), a direct calculation yields the equation for $\boldsymbol{\delta}(x)$. The map $\xi(x)$ is a biholomorphism of $\mathcal{N}_{\gamma^0}$ onto a neighborhood $\mathcal{N}_{\Gamma^0}$ of Γ^0; changing the independent variable from x to ξ we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\boldsymbol{\delta} + \hat{N}\boldsymbol{\delta} = \mathbf{T}_0\left(\frac{1}{x(\xi)}\right)\boldsymbol{\delta} + \mathbf{T}_1\left(\frac{1}{x(\xi)}, \boldsymbol{\delta}\right) + \mathbf{T}_2\left(\frac{1}{x(\xi)}\right) \quad (65)$$

where

$$\mathbf{T}_0\left(\frac{1}{x}\right) = \frac{1}{x\xi}\frac{1}{\alpha/x - 1}\left[-\alpha\hat{\Lambda} + \hat{A} + \alpha\partial_y\mathbf{g}(0, \mathbf{F}_0)\right]$$

$$\mathbf{T}_1\left(\frac{1}{x}, \boldsymbol{\delta}\right) = \frac{1}{\xi}\frac{1}{\alpha/x - 1}\left[\mathbf{g}\left(\frac{1}{x}, \mathbf{F}^{[m]} + \boldsymbol{\delta}\right) - \mathbf{g}\left(\frac{1}{x}, \mathbf{F}^{[m]}\right) - \partial_{\mathbf{y}}\mathbf{g}(0, \mathbf{F}_0)\boldsymbol{\delta}\right]$$

$$\mathbf{T}_2\left(\frac{1}{x}\right) = \frac{1}{\xi}\frac{1}{\alpha/x - 1}\left[\mathbf{g}\left(\frac{1}{x}, \mathbf{F}^{[m]}\right) - \mathbf{g}\left(\frac{1}{x}, \mathbf{F}^{[m]}\right)^{[m]} - \frac{\alpha_1}{x^{m+1}}\xi\frac{d}{d\xi}\mathbf{F}_m + \frac{1}{x^{m+1}}(m\hat{I} + \hat{A})\mathbf{F}_m\right]$$

where $\mathbf{T}_{0,1,2}$ are clearly well defined for small enough ξ and $\boldsymbol{\delta}$. Furthermore, they are well defined for $\xi \in \mathcal{N}_{\Gamma^0}$ if $|\boldsymbol{\delta}(\xi)| < (\rho_2 - \rho_3)/2$ and for R large enough (see Appendix Sect. 6.5).

As in (60) we obtain for $\boldsymbol{\delta}$ the integral equation

$$\boldsymbol{\delta} = \mathcal{J}(\boldsymbol{\delta}) \qquad \text{where} \qquad \mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1$$

$$\text{with} \quad \mathcal{J}_0(\xi) = \hat{M}(\xi;\xi_0)\boldsymbol{\delta}(\xi_0) + \int_{\xi_0}^{\xi}\hat{M}(\xi;s)\mathbf{T}_2\left(\frac{1}{x(s)}\right)ds$$

$$\mathcal{J}_1(\boldsymbol{\delta})(\xi) = \int_{\xi_0}^{\xi}\hat{M}(\xi;s)\mathbf{T}_1\left(\frac{1}{x(s)}, \boldsymbol{\delta}(s)\right)ds + \int_{\xi_0}^{\xi}\hat{M}(\xi;s)\mathbf{T}_0\left(\frac{1}{x(s)}\right)\boldsymbol{\delta}(s)ds \quad (66)$$

Let $\mathcal{B}$ be the Banach space of $\mathbb{C}^n$-valued analytic functions $\boldsymbol{\delta}$ on $\mathcal{N}_{\Gamma^0}$, continuous up to the boundary, and satisfying $\boldsymbol{\delta}(\xi_0) = 0$, with the norm $\|\boldsymbol{\delta}\| = \sup_{\xi\in\mathcal{N}_{\Gamma^0}}|\boldsymbol{\delta}(\xi)|$.

The integral operator $\mathcal{J}$ of (66) is defined on the ball of radius $(\rho_2-\rho_3)/2$ in $\mathcal{B}$. We will show that it invariates a ball in $\mathcal{B}$ and that it is a contraction there. As a consequence, the integral equation (66) has a solution $\boldsymbol{\delta}$ which is analytic on $\mathcal{N}_{\Gamma^0}$, therefore $\mathbf{y}(x)$ is analytic on $\mathcal{N}_{\gamma_N^0}$; we will also obtain estimates for $\boldsymbol{\delta}$, which will prove (26) in *Case I*.

We will denote by *const* a constant independent of a, N, m, B, c_0, R. It will be assumed that $B, R > 1$, $c_0 < 1$.

Note first that the assumption of *Case I* implies

$$m/|a| < const \tag{67}$$

To estimate $\mathcal{J}(\boldsymbol{\delta})$ note first that

$$\|\mathbf{T}_0\| < \frac{const}{|a|} \tag{68}$$

By (21) when a is large we have $|\boldsymbol{\delta}(\xi_0)| \le K_2(m+1)!B_1^{m+1}a^{-m-1}$. By Theorem 2 (i), since ξ varies in a compact set independent of a, and then $|\hat{M}| \le M$ as in (63). Also estimating derivatives with the Cauchy formulas on circles $|x-x'| < \rho_1/2$; $|\mathbf{y}-\mathbf{y}'| < \epsilon_{\mathcal{D}}/2$ and taking a large so that $|a| < 2|a-d_{\mathcal{D}}|$ we get

$$\|\mathbf{T}_1\| \le \frac{2\|\boldsymbol{\varphi}\|\|\boldsymbol{\delta}\|}{\epsilon_{\mathcal{D}}}\left(\frac{4}{\rho_1|a|}+\frac{2\|\boldsymbol{\delta}\|}{\epsilon_{\mathcal{D}}}\right) < const\left(\frac{1}{|a|}+\|\boldsymbol{\delta}\|\right)\|\boldsymbol{\delta}\| \tag{69}$$

and

$$\|\partial_{\boldsymbol{\delta}}\mathbf{T}_1\| \le \frac{8}{|a|\rho_1\epsilon_{\mathcal{D}}}+\frac{4\|\boldsymbol{\delta}\|}{\epsilon_{\mathcal{D}}^2} \tag{70}$$

Also

$$\left\|\int_{\xi_0}^{\xi}\hat{M}(\xi;s)\mathbf{T}_2\left(\frac{1}{x(s)}\right)ds\right\| \le \frac{2MKd_{\mathcal{D}}(2B)^m m!}{|a|^m}\left(\frac{2|\alpha_1|d_{\mathcal{D}}}{|a|\epsilon_{\mathcal{D}}}+\frac{m}{|a|}\right) \tag{71}$$

and using (67)

$$\le const\,\frac{(2B)^m m!}{|a|^m} \tag{72}$$

so that,

$$\|\mathcal{J}_0(\boldsymbol{\delta})\| \le \left\|\hat{M}\boldsymbol{\delta}(\xi_0)+\int_{\xi_0}^{\xi}\hat{M}(\xi;s)\mathbf{T}_2\left(\frac{1}{x(s)}\right)ds\right\|$$
$$\le\ const\,\frac{(2B)^m m!}{|a|^m} \le\ const\,c_0 \tag{73}$$

From (68), (69) we get

$$\|\mathcal{J}_1(\boldsymbol{\delta})\| \le const\,\|\boldsymbol{\delta}\| \left(\frac{1}{|a|} + \|\boldsymbol{\delta}\|\right) \tag{74}$$

Also, from (68), (69), (70)

$$\|\mathcal{J}(\boldsymbol{\delta}_1) - \mathcal{J}(\boldsymbol{\delta}_2)\| \le const\,\left(\frac{1}{|a|} + \|\boldsymbol{\delta}\|\right)\|\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2\| \tag{75}$$

It is easy to see that for positive constants R, K_0 large enough, and c_0 small enough the following holds: if $\|\boldsymbol{\delta}\| < K_0 c_0$ then : $\|\boldsymbol{\delta}\| < \frac{\rho_2-\rho_3}{2}$, $\|\mathcal{J}(\boldsymbol{\delta})\| < K_0 c_0$, also

$$\|\mathcal{J}_1(\boldsymbol{\delta})\| < \frac{1}{4}\|\boldsymbol{\delta}\| \tag{76}$$

and $\|\mathcal{J}(\boldsymbol{\delta}_1) - \mathcal{J}(\boldsymbol{\delta}_2)\| \le \lambda\|\boldsymbol{\delta}_1 - \boldsymbol{\delta})\|$ with $\lambda < 1$. This shows the existence and analyticity of $\boldsymbol{\delta}$.

Finally, to obtain the needed estimate (26) note that using (73), (76), we get

$$\|\boldsymbol{\delta}\| = \|\mathcal{J}(\boldsymbol{\delta})\| \le \|\mathcal{J}_0\| + \|\mathcal{J}_1(\boldsymbol{\delta})\| \le const\,\frac{(2B)^m}{|a|^m}m! + \frac{1}{4}\|\boldsymbol{\delta}\|$$

so that using (67) and Lemma 20

$$\|\boldsymbol{\delta}\| < const\,\frac{(2B)^m}{|a|^m}m! < const\,\frac{(2B)^{m+1}}{|a|^{m+1}}(m+1)!$$

$$< const\,\frac{(4B)^{m+1}}{|x|^{m+1}}(m+1)!$$

which concludes the proof in *Case I*.

Case II: $m > m_a$

In this case

$$m!(2B/|a|)^m > c_0 \tag{77}$$

Since

$$\|\boldsymbol{\delta}\| < \left\|\mathbf{y}(x) - \mathbf{F}_0 - \frac{1}{x}\mathbf{F}_1\right\| + \sum_{k=2}^{m}\frac{1}{|x|^k}\|\mathbf{F}_k\|$$

using the result of *Case I* to estimate the first term

$$\le const\,\frac{(5B)^2 2!}{|x|^2} + m\,\max\left\{\frac{K2!B^2}{|x|^2},\,\frac{Km!B^m}{|x|^m}\right\}$$

and since $(2B/|a|)^2 2! < c_0 < (2B/|a|)^m m!$

$$= const\, \frac{8B^2 2!}{|x|^2} + m\, K\, m!\, \frac{(2B)^m}{|a|^m} < const\,(m+1)!\, \left(\frac{2B}{|a|}\right)^{m+1} \left(\frac{2B}{|a|}\right)^{-1}$$

From (77) $|a|/(2B) < const\, m\, c_0^{-1/m} < const\, c_0^{-m}$; using this, (77) and (140) we finally get

$$< K(c_0)\,(m+1)!\,(4B/c_0)^{m+1}\,|x|^{-m-1}$$

where $K(c_0)$ is a constant dependent of c_0. □

We should stress that while the estimates in this proof clearly show the Gevrey character of the expansion, they are very far from optimal. In fact the substantial increase in B in the arguments was artificially introduced to make the calculations less cumbersome.

The following is an extension, in some respects, of Theorem 2 (ii).

Proposition 8 *Assume $\mathcal{D}$ is not necessarily compact, Γ is a curve of possibly infinite length in $\mathcal{D}$ with the following properties:*
(a) For some $\epsilon > 0$, $\mathbf{T}_{1,2}(z, \boldsymbol{\delta})$ and $\hat{N}(z)$ are analytic for z in an ϵ neighborhood of Γ and for $|\boldsymbol{\delta}| < \epsilon$ and in addition $\mathbf{T}_{1,2}(z, \boldsymbol{\delta}) = O(z\boldsymbol{\delta}, \boldsymbol{\delta}^2)$
(b) $\hat{M}(\xi, \xi_{1,0})$ is bounded in an ϵ neighborhood of Γ and for some K and all $\xi \in \Gamma$ we have $\int_{\xi_{1,0}}^{\xi} \left|\hat{M}(\xi, \xi_{1,0})\right| \mathrm{d}|s| < K$ (where $|\hat{M}|$ is some Euclidian norm of the matrix $\hat{M}(\xi, \xi_{1,0})$).

Then the conclusions of Theorem 2 (ii) hold in the x domain $\mathcal{D}_x$ corresponding to $\mathcal{D}$.

Noting that $\left|\hat{M}(\xi, \xi_{1,0})\right| \mathrm{d}|s|$ is a finite measure along Γ, the proof is virtually identical to the proof of Theorem 2.

4.6. Proof of Theorem 2 (iii)

We need the following result which is in some sense a converse of Morera's theorem.

Lemma 9 *Let $B_r = \{\xi : |\xi| < r\}$ and assume that $f(\xi)$ is analytic on the universal covering of $B_r\backslash\{0\}$. Assume further that for any circle around zero $\mathcal{C} \subset B_r\backslash\{0\}$ and any $g(\xi)$ analytic in B_r we have $\oint_{\mathcal{C}} f(\xi)g(\xi)d\xi = 0$. Then f is in fact analytic in B_r.*

Proof. Let $a \in B_r\backslash\{0\}$. It follows that $\int_a^{\xi} f(s)ds$ is single-valued in $B_r\backslash\{0\}$. Thus f is single-valued and, by Morera's theorem, analytic in $B_r\backslash\{0\}$. Since by assumption $\oint_{\mathcal{C}} f(\xi)\xi^n d\xi = 0$ for all $n \geq 0$, there are no negative powers of ξ in the Laurent series of $f(\xi)$ about zero: f extends as an analytic function at zero. $\square$

To show Theorem 2 (iii), assume ξ_s is an isolated singularity of $\mathbf{F}_0$ (thus $\xi_s \neq 0$) and $X = \{x : \xi(x) = \xi_s\}$. By Lemma 9 there is a circle $\mathcal{C}$ around ξ_s and a function $g(\xi)$ analytic in $B_r(\xi - \xi_s)$ such that $\oint_{\mathcal{C}} \mathbf{F}_0(\xi)g(\xi)d\xi = 1$. In a neighborhood of $x_n \in X$ the function $f(x) = \mathrm{e}^{-x}x^{\alpha_1}$ is a biholomorphism and for large x_n

$$\oint_{f^{-1}(\mathcal{C})} \mathbf{y}(x)\frac{g(f(x))}{f(x)}dx$$
$$= -\oint_{\mathcal{C}} \left(1 + O(x_n^{-1})\right)\left(\mathbf{F}_0(\xi) + O(x_n^{-1})\right)g(\xi)d\xi = 1 + O\left(x_n^{-1}\right) \neq 0 \quad (78)$$

It follows from Lemma 9 that for large enough x_n $\mathbf{y}(x)$ is not analytic inside $\mathcal{C}$ either. Since the radius of $\mathcal{C}$ can be taken $o(1)$ Theorem 2 (iii) follows.

Note. In many cases the singularity of $\mathbf{y}$ is of the *same type* as the singularity of $\mathbf{F}_0$. See Sect. 5 for further comments.

4.7. Proof of Theorem 3

As in Sect. 4.5 we can reduce to the study of (2) in $\mathcal{N}_{\Gamma^0}$, where the function $\xi(x)$ is biholomorphic and we can change variables to ξ. In this variable both (2) and (56) assume the form (where $x = x(\xi)$ and $\mathbf{F}$ is $\mathbf{F}_0$ or $\mathbf{y}$)

$$\xi\frac{\mathrm{d}F_1}{\mathrm{d}\xi} = F_1 + \epsilon_1^{[1]}(x^{-1}, \mathbf{F})$$
$$\xi\frac{\mathrm{d}F_2}{\mathrm{d}\xi} = \frac{\lambda_2 F_2 - \gamma_2 F_1^2 + \epsilon_2^{[1]}(x^{-1}, \mathbf{F})}{h(\xi, \mathbf{F})}$$
$$\xi\frac{\mathrm{d}F_j}{\mathrm{d}\xi} = \lambda_j F_j - \gamma_j F_1^2 + \epsilon_j^{[1]}(x^{-1}, \mathbf{F}) \quad (j \neq 1, 2) \quad (79)$$

where

$$h(\xi, \mathbf{F}) = 1 - \sum_{j=1}^{n} a_k F_k + \epsilon_{n+1}^{[1]}(x^{-1}, \mathbf{F}),$$

with $\epsilon_j^{[1]}$ analytic in $\mathcal{P}$ and $\|\epsilon_j^{[1]}\|_{\mathcal{P}} < \epsilon$, $i = j, \ldots, n+1$ (for $\mathbf{F}_0$, we have $\epsilon_j^{[1]}(x^{-1}, \mathbf{F}_0) = \epsilon_j(\mathbf{F}_0)$).

Generically a_2 is nonzero. Then the analytic change of variables to $F_1, h, F_3, \dots, F_n$ leads to a system of the form

$$\xi \frac{dF_1}{d\xi} = F_1 + \epsilon_1^{[1]} \tag{80}$$

$$\xi h \frac{dh}{d\xi} = h\Big[\lambda_2 - \sum_{j\neq 2} a_j\lambda_j F_j - F_1^2 \sum_{j\geq 3} a_j\gamma_j\Big] + \Big[-\lambda_2 + \lambda_2 \sum_{j\neq 2} a_j F_j - a_2\gamma_2 F_1^2\Big] - \epsilon_2^{[2]} \tag{81}$$

$$\xi \frac{dF_j}{d\xi} = \lambda_j F_j - \gamma_j F_1^2 + \epsilon_j^{[1]} \quad (j > 2) \tag{82}$$

The substitution $F_1 = \xi + f_1$, $F_j = b_j\xi^2 + f_j$ $(j > 2)$, with $b_j = (\lambda_j - 2)^{-1}\gamma_j$, in (80)–(82) yields

$$\xi \frac{df_1}{d\xi} = f_1 + \epsilon_1^{[3]}$$

$$\xi h \frac{dh}{d\xi} = \lambda_2 h + a_1(\lambda_2 - h)(\xi + f_1) + \sum_{j\geq 3}^{n} a_j(\lambda_2 - \lambda_j h)\big(b_m\xi^2 + f_j\big) + (f_1 + \xi)^2 \sum_{j=3}^{n} a_j h_j - \lambda_2 - \epsilon_2^{[3]}$$

$$\xi \frac{df_j}{d\xi} = \lambda_j f_j + 2\xi\gamma_j f_1 + \gamma_j f_1^2 - \epsilon_j^{[3]} \qquad (j > 2) \tag{83}$$

According to the hypothesis of Theorem 3 it is useful to analyze first the equation (describing the leading order behavior of h)

$$hh' = \big(\lambda_2\xi^{-1} + d_1 + d_2\xi\big)h + \big(-\lambda_2\xi^{-1} + d_3 + d_4\xi\big); \quad h(0) = 1 \tag{84}$$

(this Abel type equation cannot be solved in closed form, in general). In integral form,

$$h(\xi)^2 = 1 + d_1^{[2]}\xi^2 + d_2^{[2]}\xi + \int_0^\xi (d_3^{[2]}s + d_4^{[2]})h(s)ds + 2\lambda_2 \int_0^\xi (h(s) - 1)s^{-1}ds \tag{85}$$

Lemma 10 *(i) Equation (84) has a unique solution h_0 analytic at $\xi = 0$, with $h_0(0) = 1$.*

(ii) For a generic set of $d_1, \dots, d_4$ the solution h_0 is not entire and, on the boundary of the disk of analyticity, h_0 has square root branch points.

Proof. (i) It is straightforward to check that, since $\lambda_2 \notin \mathbb{N}$ (see Sect. 2) then (85) has a (unique) formal solution of the form $\tilde{h} = 1 + \sum_{k=1}^{\infty} \tilde{h}_k \xi^k$ (where $h_1 = (d_2^{[2]} + d_4^{[2]})(2 - d_5^{[2]})^{-1}$). To show $\tilde{h}$ converges we take $h = 1 + \sum_{k=1}^{M-1} \tilde{h}_k \xi^k + \xi^M h_M(\xi)$ in (85):

$$\begin{aligned} &2\xi^M h_M(\xi) \\ &= Q(\xi)\xi^{2M} h_M(\xi)^2 + \xi^M R(\xi) - d_5^{[2]} \int_0^{\xi} \Big(t^M + \sum_{k=0}^{2M} b_k t^{M+k}\Big) h_M(t)dt \end{aligned} \tag{86}$$

with $Q(\xi)$, $R(\xi)$ analytic, or

$$\begin{aligned} 2h_M(\xi) &= Q(\xi)\xi^M h_M(\xi)^2 + R(\xi) - \\ &\quad d_5^{[2]} \int_0^1 \Big(s^{M-1} + \sum_{k=0}^{2M} b_k s^{M+k} \xi^{k+1}\Big) h_M(\xi s)ds = 2\mathcal{J}(h_M) \end{aligned}$$

$\mathcal{J}$ is manifestly contractive in the sup norm for small ξ, if $M > \left|d_5^{[2]}\right|$.

(ii) The proof, elementary but delicate, is given in Sect. 6.8. □

Lemma 11 *Let ξ_0 be a branch point singularity on the boundary of the disk of analyticity of h_0 (see Lemma 10 (ii)). Assume $\boldsymbol{\epsilon}^{[3]}$ in (83) is small enough and analytic in a (large enough) neighborhood in ξ, $\mathbf{F}$ of $(\xi_0, \mathbf{F}_0(\xi_0))$. Then*

(i) For some $0 < \delta_1 < \delta_2$, $\mathbf{F}(\xi)$ and $h(\xi)$ are analytic in the cut annulus $\{\xi : |\xi - \xi_0| \in (\delta_1, \delta_2),\ \arg(\xi - \xi_0) \neq 0\}$.

(ii) h and $\mathbf{F}$ have a square root branch point at some ξ_s with $\xi_s - \xi_0 = O(\|\boldsymbol{\epsilon}^{[3]}\|)$.

Proof. We substitute $h = h_0 + f_2$ cf. Lemma 10; $\mathbf{f}$ satisfies a system of the form

$$\xi \mathbf{f}' = \hat{N}(\xi)\mathbf{f} + \boldsymbol{\epsilon}(\xi, \mathbf{f}^2)$$

or

$$\mathbf{f} = \mathbf{f}_1 + \hat{M}(\xi) \int_{\xi}^{\xi} \hat{M}^{-1}(s)\boldsymbol{\epsilon}(s, \mathbf{f}^2(s))ds$$

where the matrices $\hat{N}$, $\hat{M}$, $\hat{M}^{-1}$ and $\boldsymbol{\epsilon}$ are analytic in $\mathcal{N}_{\Gamma^0}$. Part (i) follows now in the same way as Theorem 2 (ii), and $\|y - \mathbf{F}_0\|_{\mathcal{N}_{\Gamma^0}} = O(\|\boldsymbol{\epsilon}^{[3]}\|)$. (ii) In a small neighborhood of ξ_0 by part (i) and Proposition 10 (generically) $\frac{d}{d\xi}h \neq 0$ and we may change variables in (83) so that h is the independent variable (and $\xi = \xi(h)$. We note that

$$\frac{d\xi}{dh} = \frac{\xi h}{A(f,\xi)h + B(f,\xi) + \epsilon(f,h,\xi)}$$

and h, ϵ are small while generically $B(f(\xi_0), \xi_0)$ is not small. Then (83) in the variable h, with initial condition $\xi(h_0 + O(\|\boldsymbol{\epsilon}^{[3]}\|)) = \xi_0 + O(\|\boldsymbol{\epsilon}^{[3]}\|)$, has a solution which is analytic near $h = 0$. Furthermore it is easy to see that (under the same genericity assumptions) we have $\partial_h \xi_{h=0} = 0$ but $\det \partial_{hh}(f_1, \xi, \ldots, f_n)_{h=0} \neq 0$ and then $F_j(\xi) = F_j^1((\xi - \xi_0)^{1/2})$ with F_j^1 locally analytic.

5. Examples

5.1. Example 1

We first illustrate how singularities of solutions are found (using transasymptotic matching) on a first order Abel equation[8]:

$$u' = u^3 - z \tag{87}$$

the first example on which nonintegrability was shown using Kruskal's poly-Painlevé analysis [34].

The study of (87) is done in the following steps. Classical asymptotics of differential equations [51] shows (and it also follows from the analysis below) that for $z \to \infty$ with $\arg z \in \left(\frac{3}{10}\pi, \frac{9}{10}\pi\right)$ there is a one parameter family of solutions $u = u(z; C)$ such that $u(z; C) = z^{1/3}(1 + o(1))$. Then $u \sim \tilde{u} = z^{1/3} \sum_{k=0}^{\infty} \frac{\tilde{u}_k}{z^{5k/3}}$. The parameter C may be chosen to be the constant beyond all orders, see Sect. 5.1.2.

After proper normalization of (87) (see Sect. 5.1.1) Theorems 1 and 2 are applicable and provide a global asymptotic description of $u(z; C)$ in a region where the solution is analytic and surrounds its singularities for large z (Proposition 13). These are algebraic branch points of order $-1/2$ (see (102)) and their location, dependent on C, is determined asymptotically. Conversely, the C of a particular solution can be determined from the asymptotic location of one singularity.

5.1.1. Normalization Formal solutions provide a good guide in finding the normalization transformations. A transformation bringing the equation to its normal form also brings its transseries solutions to the form (10). It is simpler to look for substitutions with this latter property, and then the first step is to find the transseries solutions of (87).

Power series solutions. Since at this stage we are merely looking for useful transformation hints, rigor is naturally not required. Substituting of $u \sim Az^p$ in (87) and looking for maximal balance [5] give $p = 1/3$, $A^3 = 1$. Then $u \sim Az^{1/3} + Bz^q$ with $q < 1/3$ determines $B = \frac{1}{9}A^2$, $q = -4/3$. Inductively, one obtains a power series formal solution $\tilde{u}_0 = Az^{1/3}(1 + \sum_{k=1}^{\infty} \tilde{u}_{0,k} z^{-5k/3})$.

[8] The authors are grateful to A. Fokas for pointing out to this example.

General transseries solutions of (87). In order to determine the form of the exponentials in the transseries of u, the method is to look for transcendentally small corrections beyond $\tilde{u}_0$, by linear perturbation theory. Substituting $u = \tilde{u}_0 + \delta$ in (87) yields to leading order in δ, the equation

$$\delta' = \left(3A^2 z^{2/3} + \frac{2}{3z}\right)\delta \tag{88}$$

whence $\delta \propto z^{2/3} \exp\left(\frac{9}{5}A^2 z^{5/3}\right)$. In (4) the exponentials have *linear* exponent, with negative real part. The independent variable should thus be $x = -(9/5)A^2 z^{5/3}$ and $\Re(x) > 0$. Then $\tilde{u}_0 = x^{1/5}\sum_{k=0}^{\infty} u_{0;k} x^{-k}$, which compared to (8) suggests the change of dependent variable $u(z) = Kx^{1/5}h(x)$. Choosing for convenience $K = A^{3/5}(-135)^{1/5}$ yields

$$h' + \frac{1}{5x}h + 3h^3 - \frac{1}{9} = 0 \tag{89}$$

The next step is to achieve leading behavior $O(x^{-2})$. This is easily done by subtracting out the leading behavior of h (which can be found by maximal balance, as above). With $h = y + 1/3 - x^{-1}/15$ we get the normal form

$$y' = -y + \frac{1}{5x}\,y + g(x^{-1}, y) \tag{90}$$

where

$$g(x^{-1}, y) = -3(y^2 + y^3) + \frac{3y^2}{5x} - \frac{1}{15x^2} - \frac{y}{25x^2} + \frac{1}{3^2 5^3 x^3} \tag{91}$$

We see that

$$\lambda = 1, \quad \alpha = 1/5, \text{ and thus } \xi = Cx^{1/5}\mathrm{e}^{-x} \tag{92}$$

5.1.2. Definition of C for a given solution $u(z)$ After normalization (90) the results in [14] apply, and the constant C is uniquely associated to a $u(z)$ on a direction $\arg(z) = \phi$ as the limit

$$C = \lim_{\substack{z\to\infty \\ \arg(z)=\phi}} \xi(z)^{-1}\left(u(z) - \sum_{k\le|x(z)|}\frac{\tilde{u}_k}{z^{(5k-1)/3}}\right) \tag{93}$$

This limit exists for all $\phi \in \left(\frac{3}{10}\pi, \frac{9}{10}\pi\right)$ and is piecewise constant, with one jump discontinuity at the midpoint of this interval. The value of C relevant to the singularities of u is the one nearest to the edge of S_{trans} where these singularities are calculated, as follows from Theorem 1.

5.1.3. Finding the two-scale expansion (26) Having the second scale given by (92) and all the conditions of Theorem 1 satisfied, the simplest way to calculate the functions F_k in $\tilde{y} = \sum_{k=0}^{\infty} x^{-k} F_k(\xi)$ is by substituting $y = \tilde{y}$ in (90) and solving the differential equations, as in the proof of Theorem 2 (i); the equation for $F_0(\xi)$ is, cf. (56),

$$\xi F_0' = F_0(1 + 3F_0 + 3F_0^2); \qquad F_0'(0) = 1 \tag{94}$$

and, cf. (57),

$$\xi F_k' = (3F_0 + 1)^2 F_k + R_k(F_0, \dots, F_{k-1}) \qquad \text{(for } k \geq 1 \text{ and where } R_1 = \frac{3}{5} F_0^3) \tag{95}$$

The first term F_0 of the expansion of u is then given by

$$\xi = \xi_0 F_0(\xi)(F_0(\xi) + \omega_0)^{-\theta} (F_0(\xi) + \overline{\omega_0})^{-\overline{\theta}} \tag{96}$$

with $\xi_0 = 3^{-1/2} \exp(-\frac{1}{6}\pi\sqrt{3})$, $\omega_0 = \frac{1}{2} + \frac{i\sqrt{3}}{6}$ and $\theta = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The functions F_k, $k \geq 1$ can also be obtained in closed form, order by order.

By Theorem 1, the relation $y \sim \tilde{y}$ holds in the sector

$$S_{\delta_1} = \left\{ x \in \mathbb{C} : \arg(x) \geq -\frac{\pi}{2} + \delta,\ |Cx^{1/5} e^{-x}| < \delta_1 \right\}$$

for some $\delta_1 > 0$ and any small $\delta > 0$.

Theorem 3 insures that $y \sim \tilde{y}$ holds in fact on a larger region, surrounding singularities of F_0 (and thus of y). To apply this result we need the surface of analyticity of F_0 and an estimate for the location of its singularities.

Lemma 12 *(i) The function F_0 is analytic on the universal covering $\mathcal{R}_{\Xi}$ of $\mathbb{C} \setminus \Xi$ where*

$$\Xi = \left\{ \xi_p = (-1)^{p_1} \xi_0 \exp(p_2 \pi \sqrt{3}) : p_{1,2} \in \mathbb{Z} \right\} \tag{97}$$

and its singularities are algebraic of order $-1/2$, located at points lying above Ξ.

(ii) (The first Riemann sheet) The function F_0 is analytic in $\mathbb{C} \setminus \big((-\infty, \xi_0] \cup [\xi_1, \infty)\big)$.

(iii) The Riemann surface associated to F_0 is represented in Fig. 2.

Proof. Singularities of F_0. The RHS of (94) is analytic except at $F_0 = \infty$, thus F_0 is analytic except at points where $F_0 \to \infty$. From (96) it follows that $\lim_{F_0 \to \infty} \xi \in \Xi$ and (i) follows straightforwardly; in particular, as $\xi \to \xi_p \in \Xi$ we have $(\xi - \xi_p)^{1/2} F_0(\xi) \to \sqrt{-\xi_p/6}$.

(ii) We now examine on which sheets in $\mathcal{R}_{\Xi}$ these singularities are located, and start with a study of the first Riemann sheet (where $F_0(\xi) = \xi + O(\xi^2)$ for small ξ). Finding which of the points ξ_p are singularities of F_0

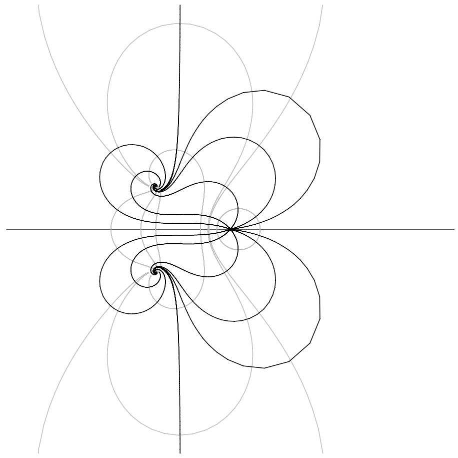

Fig. 2 The dark lines represent the phase portrait of (98), as well as the lines of steepest variation of $|\xi(u)|$. The light gray lines correspond to the orthogonal field, and to the lines $|\xi(u)| = const$.

on the first sheet can be rephrased in the following way. On which constant phase (equivalently, steepest ascent/descent) paths of $\xi(F_0)$, which extend to $|F_0| = \infty$ in the plane F_0, is $\xi(F_0)$ uniformly bounded?

Constant phase paths are governed by the equation $\Im(\mathrm{d}\ln\xi) = 0$. Thus, denoting $F_0 = X + iY$, since $\xi'/\xi = \left(F_0 + 3F_0^2 + 3F_0^3\right)^{-1}$ one is led to the *real* differential equation $\Im(\xi'/\xi)\mathrm{d}X + \Re(\xi'/\xi)\mathrm{d}Y = 0$, or

$$Y(1 + 6X + 9X^2 - 3Y^2)\mathrm{d}X \\ - (X + 3X^2 - 3Y^2 + 3X^3 - 9XY^2)\mathrm{d}Y = 0 \quad (98)$$

We are interested in the field lines of (98) which extend to infinity. Noting that the singularities of the field are $(0, 0)$ (unstable node, in a natural parameterization) and $P_\pm = (-1/2, \pm\sqrt{3}/6)$ (stable foci, corresponding to $-\overline{\omega_0}$ and $-\omega_0$), the phase portrait is easy to draw (see Fig. 2) and there are only two curves starting at $(0, 0)$ so that $|F_0| \to \infty,\ \xi$ bounded, namely $\pm\mathbb{R}^+$, along which $\xi \to \xi_0$ and $\xi \to \xi_1$, respectively.

(iii) Thus Fig. 2 encodes the structure of singularities of F_0 on $\mathcal{R}_\Xi$ in the following way. A given class $\gamma \in \mathcal{R}_\Xi$ can be represented by a curve composed of rays and arcs of circle. In Fig. 2, in the F_0-plane, this corresponds to a curve γ' composed of constant phase (dark gray) lines or constant modulus (light gray) lines. Curves in $\mathcal{R}_\Xi$ terminating at singularities of F_0 correspond in Fig. 2 to curves so that $|F_0| \to \infty$ (the four dark gray separatrices $S_1, \dots, S_4$). Thus to calculate where, on a particular Riemann sheet of $\mathcal{R}_\Xi$, is F_0 singular, one needs to find the limit of ξ in (96), as $F_0 \to \infty$ along along γ' followed by S_i. This is straightforward, since the branch of the complex powers $\theta, \overline{\theta}$, is calculated easily from the index of γ' with respect to $P_\pm$. $\square$

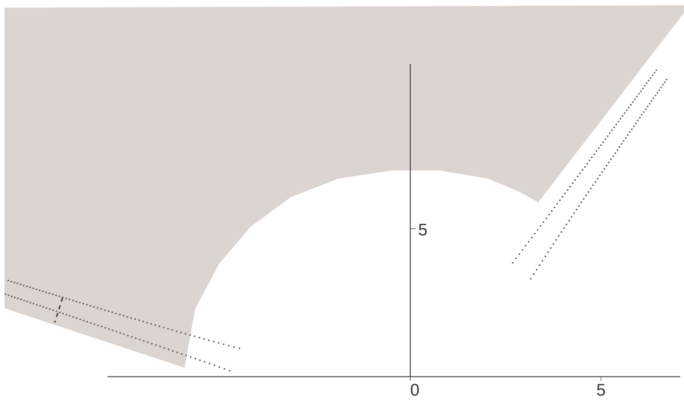

Fig. 3 Singularities on the boundary of S_{trans} for (87). The gray region lies in the projection on $\mathbb{C}$ of the Riemann surface where (99) holds. The short dotted line is a generic cut delimiting a first Riemann sheet.

Theorem 2 can now be applied on relatively compact subdomains of $\mathcal{R}_{\Xi}$ and used to determine a uniform asymptotic representation $y \sim \tilde{y}$ in domains surrounding singularities of $y(x)$, and to obtain their asymptotic location. Going back to the original variables, similar information on $u(z)$ follows. For example, using Theorem 2 for the first Riemann sheet (cf. Lemma 12 (ii))

$$\mathcal{D} = \{|\xi| < K \mid \xi \notin (-\infty, \xi_1) \cup (\xi_0, +\infty)\ ,\ |\xi - \xi_0| > \epsilon, |\xi - \xi_1| > \epsilon, \}$$

(for any small $\epsilon > 0$ and large positive K) the corresponding domain in the z-plane is shown in Fig. 3.

In general, we fix $\epsilon > 0$ small, and some $K > 0$ and define $\mathcal{A}_K = \{z : \arg z \in \left(\frac{3}{10}\pi - 0, \frac{9}{10}\pi + 0\right),\ |\xi(z)| < K\}$ and let $\mathcal{R}_{K,\Xi}$ be the universal covering of $\Xi \cap \mathcal{A}_K$ and $\mathcal{R}_{z;K,\epsilon}$ the corresponding Riemann surface in the z plane, with ϵ– neighborhoods of the points projecting on $z(x(\Xi))$ deleted.

Proposition 13 *(i) The solutions $u = u(z; C)$ described in the beginning of Sect. 5 have the asymptotic expansion*

$$u(z) \sim z^{1/3}\left(1 + \frac{1}{9}z^{-5/3} + \sum_{k=0}^{\infty} \frac{F_k\left(C\xi(z)\right)}{z^{5k/3}}\right) \qquad (as\ z \to \infty;\quad z \in \mathcal{R}_{z;K,\epsilon}) \quad (99)$$

where

$$\xi(z) = x(z)^{1/5}\mathrm{e}^{-x(z)},\ \textit{and}\ x(z) = -\frac{9}{5}z^{5/3} \qquad (100)$$

(ii) In the "steep ascent" strips $\arg(\xi) \in (a_1, a_2),\ |a_2 - a_1| < \pi$ *starting in $\mathcal{A}_K$ and crossing the boundary of $\mathcal{A}_K$, the function u has at most one*

singularity, when $\xi(z)=\xi_0$ *or* ξ_1, *and* $u(z)=z^{1/3}\mathrm{e}^{\pm 2\pi i/3}(1+o(1))$ *as* $z\to\infty$ *(the sign is determined by* $\arg(\xi)$*).*

(iii) The singularities of $u(z;C)$, *for* $C\neq 0$, *are located within* $O(\epsilon)$ *of the punctures of* $\mathcal{R}_{z;K,0}$.

Applying Theorem 2 to (90) it follows that for $n\to\infty$, a given solution y is singular at points $\tilde{x}_{p,n}$ such that $\xi(\tilde{x}_{p,n})/\xi_p=1+o(1)$ ($|\tilde{x}_{p,n}|$ large).

Now, y can only be singular if $|y|\to\infty$ (otherwise the r.h.s. of (90) is analytic). If $\tilde{x}_{p,n}$ is a point where y is unbounded, with $\delta=x-\tilde{x}_{p,n}$ and $v=1/y$ we have

$$\frac{\mathrm{d}\delta}{\mathrm{d}v}=vF_s(v,\delta) \tag{101}$$

where F_s is analytic near $(0,0)$. It is easy to see that this differential equation has a unique solution with $\delta(0)=0$ and that $\delta'(0)=0$ as well.

The result is then that the singularities of u are also algebraic of order $-1/2$.

Proposition 14 *If* z_0 *is a singularity of* $u(z;C)$ *then in a neighborhood of* z_0 *we have*

$$u=\pm\sqrt{-1/2}(z-z_0)^{-1/2}A_0\big((z-z_0)^{1/2}\big) \tag{102}$$

where A_0 *is analytic at zero and* $A_0(0)=1$.

Notes. 1. The local behavior near a singularity could have been guessed by local Painlevé analysis and the method of dominant balance, with the standard ansatz near a singularity, $u\sim Const.(z-z_0)^p$. Our results however are **global**: Proposition 13 gives the behavior of *a fixed* solution at infinitely many singularities, and gives the **position** of these singularities as soon as C_1 (or the position of only one of these singularities) is known (and in addition show that the power behavior ansatz is correct in this case).

2. Eq. (90) can be brought to a form similar to that in Theorem 3 by the substitution $y=v/(1+v)$ in (90). The result has the form

$$v'=-v-27\,\frac{v^3}{1+v}-10\,v^2+\frac{1}{5t}v+g^{[1]}(t^{-1},v) \tag{103}$$

where $g^{[1]}$ is a now an $O(t^{-2},v^{-2})$ polynomial of total degree 5. The singularities of v are at the points where $v(t)=-1$, and are square root branch points, as in Theorem 3, whose technique of proof would have also applied, if the more explicit formula (96) was unavailable.

5.2. Example 2: The Painlevé equation P_{I}

The Painlevé functions were studied asymptotically in terms of doubly periodic functions by Boutroux (see, for example, [24]). Solutions of the $\mathrm{P_I}$ equation turn out to have arrays of poles and they can be asymptotically represented by elliptic functions whose parameters change with the direction in the complex plane. Joshi and Kruskal carried out this type of expansions for generic solutions, which have poles throughout a neighborhood of infinity, to sufficiently many orders to determine how the parameters of the elliptic functions vary [29], [30], and applied this method to solve the connection problem. However, there exist special, one-parameter, families of solutions of $\mathrm{P_I}$ (the truncated solutions, important in applications) that are free of poles in some sectors. These solutions have the same classical asymptotic expansion in the pole free sector to all orders and cannot be distinguished by classical asymptotics there. They differ by a constant C *beyond all orders* which can be determined by exponential asymptotic methods. The results of the present paper apply to this special family of solutions and give an asymptotic representation uniformly valid at the ultimate array of poles (neighboring the pole free sector) and make the link between the position of these poles position and the value of C.

We note that the behavior of the triply truncated[9] solutions, which in a sector have $C = 0$ and, consequently are pole-free in larger sectors, does not follow immediately from our analysis. But this case can be treated by a similar methodology since after continuation across a Stokes line the value of C becomes equal to a Stokes multiplier, generically nonzero.

This example extends the asymptotic expansions of [16] to larger regions of the complex plane, and also to all orders.

We consider solutions of the Painlevé $\mathrm{P_I}$ equation (in the form of [25], which by rescaling gives the form in [24])

$$\frac{d^2y}{dz^2} = 6y^2 + z \tag{104}$$

in a region centered on a Stokes line, say $d = \{z : \arg z = \pi\}$.

To bring (104) to a normal form the transformations are suggested by the general methodology explained in Sect. 5.1.1. There is a one parameter family of solutions for each of the behaviors $y \sim \pm\sqrt{\frac{-z}{6}}$ for large z along d. We will study the family with $y \sim +\sqrt{\frac{-z}{6}}$, since the other can be treated similarly. Its transseries can be obtained as in the previous example, namely determining first the asymptotic series $\tilde{y}_0$, then by linear perturbation theory around it one finds the form of the small exponential, and notices the exponential is determined up to one multiplicative parameter. We get the

[9] They are also known as "*doublement tronquées*".

transseries solution

$$\tilde{y} = \sqrt{\frac{-z}{6}} \sum_{k=0}^{\infty} \xi^k \tilde{y}_k \tag{105}$$

where

$$\xi = \xi(z) = Cx^{-1/2}\mathrm{e}^{-x}; \quad \text{with } \ x = x(z) = \frac{(-24z)^{5/4}}{30} \tag{106}$$

and $\tilde{y}_k$ are power series, in particular

$$\tilde{y}_0 = 1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} - \frac{7^2}{2^8 \cdot 3}\frac{1}{z^5} - \ldots - \frac{\tilde{y}_{0;k}}{(-z)^{5k/2}} - \ldots$$

We note that in the sector $|\arg(z) - \pi| < \frac{2}{5}\pi$ the constant C of a particular solution y (see (109)) changes only once, on the Stokes line $\arg(z) = \pi$ [13].

As in Example 1, the form of the transseries solution (105), (106) suggests the transformation

$$x = \frac{(-24z)^{5/4}}{30}; \ y(z) = \sqrt{\frac{-z}{6}}\, Y(x)$$

which, in fact, coincides with Boutroux's (cf. [24]); $\mathrm{P_I}$ becomes

$$Y''(x) - \frac{1}{2}Y^2(x) + \frac{1}{2} = -\frac{1}{x}Y'(x) + \frac{4}{25}\frac{1}{x^2}Y(x) \tag{107}$$

For the present techniques to apply equation (107) needs to be fully normalized and to this end we subtract the $O(1)$ and $O(x^{-1})$ terms of the asymptotic behavior of $Y(x)$ for large x. It is convenient to subtract also the $O(x^{-2})$ term (since the resulting equation becomes simpler). Then the substitution

$$Y(x) = 1 - \frac{4}{25x^2} + h(x)$$

transforms $\mathrm{P_I}$ to

$$h'' + \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625x^4} = 0 \tag{108}$$

Written as a system, with $\mathbf{y} = (h, h')$ this equation satisfies the assumptions in Sect. 2, with $\lambda_{1,2} = \pm 1$, $\alpha_{1,2} = -1/2$, and then $\xi(x) = C\mathrm{e}^{-x}x^{-1/2}$. The results of the present paper apply to the normal form (108) of $\mathrm{P_I}$ and we will prove Proposition 15 below which shows in (i) how the constant C beyond all orders is associated to a truncated solution $y(z)$ of $\mathrm{P_I}$ for $\arg(z) = \pi$ (formula (109)) and gives the position of one array of poles z_n of the solution associated to C (formula (110)), and in (ii) provides uniform asymptotic expansion to all orders of this solution in a sector centered on $\arg(z) = \pi$ and one array of poles (except for small neighborhoods of these poles) in formula (112).

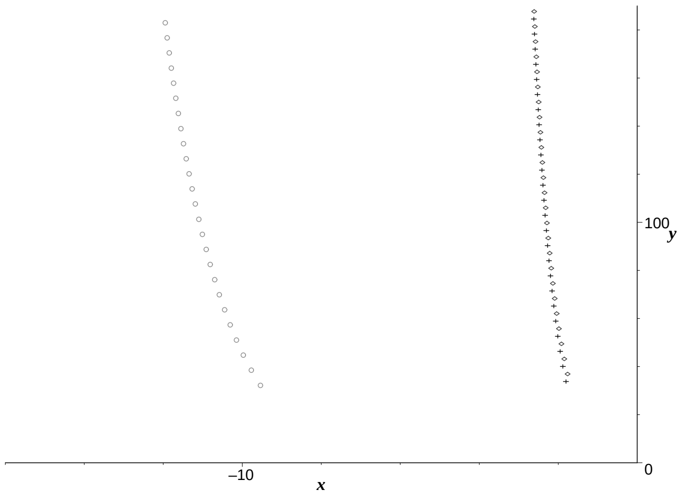

Fig. 4 Poles of (108) for $C = -12$ ($\diamond$) and $C = 12$ ($+$), calculated via (116). The light circles are on the second line of poles for to $C = -12$.

Proposition 15 *(i) Let y be a solution of (104) such that $y(z) \sim \sqrt{-z/6}$ for large z with $\arg(z) = \pi$. For any $\phi \in (\pi, \pi + \frac{2}{5}\pi)$ the following limit determines the constant C (which does not depend on ϕ in this range) in the transseries $\tilde{y}$ of y:*

$$\lim_{\substack{|z|\to\infty \\ \arg(z)=\phi}} \xi(z)^{-1} \left(\sqrt{\frac{6}{-z}}y(z) - \sum_{k\le|x(z)|} \frac{\tilde{y}_{0;k}}{z^{5k/2}} \right) = C \tag{109}$$

(Note that the constants $\tilde{y}_{0;k}$ do not *depend on C). With this definition, if $C \neq 0$, the function y has poles near the antistokes line $\arg(z) = \pi + \frac{2}{5}\pi$ at all points z_n, where, for large n*

$$z_n = -\frac{(60\pi i)^{4/5}}{24} \left(n^{\frac{4}{5}} + iL_n n^{-\frac{1}{5}} + \left(\frac{L_n^2}{8} - \frac{L_n}{4\pi} + \frac{109}{600\pi^2} \right) n^{-\frac{6}{5}} \right) + O\left(\frac{(\ln n)^3}{n^{\frac{11}{5}}} \right) \tag{110}$$

with $L_n = \frac{1}{5\pi} \ln \left(\frac{\pi i C^2}{72} n \right)$, or, more compactly,

$$\xi(z_n) = 12 + \frac{327}{(-24z_n)^{5/4}} + O(z_n^{-5/2}) \ \ (z_n \to \infty) \tag{111}$$

(ii) Let $\epsilon \in \mathbb{R}^+$ and define

$$\mathcal{Z} = \left\{ z : \arg(z) > \frac{3}{5}\pi;\ |\xi(z)| < 1/\epsilon;\ \ |\xi(z) - 12| > \epsilon \right\}$$

(the region starts at the antistokes line $\arg(z) = \frac{3}{5}\pi$ *and extends slightly beyond the next antistokes line,* $\arg(z) = \frac{7}{5}\pi$*). If* $y \sim \sqrt{-z/6}$ *as* $|z| \to \infty$, $\arg(z) = \pi$*, then for* $z \in \mathcal{Z}$ *we have*

$$y \sim \sqrt{\frac{-z}{6}} \left(1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} + \sum_{k=0}^{\infty} \frac{30^k H_k(\xi)}{(-24z)^{5k/4}} \right) \qquad (|z| \to \infty,\ z \in \mathcal{Z}) \quad (112)$$

The functions H_k *are rational, and* $H_0(\xi) = \xi(\xi/12 - 1)^{-2}$*. The expansion (112) holds uniformly in the sector* $\pi^{-1}\arg(z) \in (3/5, 7/5)$ *and also on one of its sides, where* H_0 *becomes dominant, down to an* $o(1)$ *distance of the actual poles of* y *if* z *is large.*

Proof. We prove the corresponding statements for the normal form (108). To go back to the variables of (104) mere substitutions are needed, which we omit.

Most of Proposition 15 is a direct consequence of Theorems 1 and 2. For the one-parameter family of solutions which are small in the right half plane we then have

$$h \sim \sum_{k=0}^{\infty} x^{-k} H_k(\xi(x)) \qquad (113)$$

As in the first example we find H_k by substituting (113) in (108).

The equation of H_0 is

$$\xi^2 H_0'' + \xi H_0' = H_0 + \frac{1}{2} H_0^2$$

The general solution of this equation are the Weierstrass elliptic functions of $\ln\xi$, as expected from the general knowledge of the asymptotic behavior of the Painlevé solutions (see [24]). For our special initial condition, H_0 analytic at zero and $H_0(\xi) = \xi(1 + o(1))$, the solution is a degenerate elliptic function, namely,

$$H_0(\xi) = \frac{\xi}{(\xi/12 - 1)^2}$$

Next, one of the two free constants in the general solution H_1 is determined by the condition of analyticity at zero of H_1 (this constant multiplies terms in $\ln\xi$). It is interesting to note that the remaining constant is only determined in the *next* step, when solving the equation for H_2! This pattern is typical (see Sect. 6.9). Continuing this procedure we obtain successively:

$$H_1 = \left(216\,\xi + 210\,\xi^2 + 3\,\xi^3 - \frac{1}{60}\,\xi^4 \right) (\xi - 12)^{-3} \qquad (114)$$

$$H_2 = \left(1458\xi + 5238\xi^2 - \frac{99}{8}\xi^3 - \frac{211}{30}\xi^4 + \frac{13}{288}\xi^5 + \frac{\xi^6}{21600} \right) (\xi - 12)^{-4} \qquad (115)$$

We omit the straightforward but quite lengthy inductive proof that all H_k are rational functions of ξ. The reason the calculation is tedious is that this property holds for (108) but *not* for its generic perturbations, and the last potential obstruction to rationality, successfully overcome by (108), is at $k = 6$. On the positive side, these calculations are algorithmic and are very easy to carry out with the aid of a symbolic language program.

In the same way as in Example 1 one can show that the corresponding singularities of h are double poles: all the terms of the corresponding asymptotic expansion of $1/h$ are *analytic* near the singularity of h! All this is again straightforward, and lengthy because of the potential obstruction at $k = 6$. We prefer to rely on an existing direct proof, see [16].

Let ξ_s correspond to a zero of $1/h$. To leading order, $\xi_s = 12$, by Theorem 2 (iii). To find the next order in the expansion of ξ_s one substitutes $\xi_s = 12 + A/x + O(x^{-2})$, to obtain

$$1/h(\xi_s) = \frac{(A - 109/10)^2}{12^3 x^2} + O(1/x^3)$$

whence $A = 109/10$ (because $1/h$ is analytic at ξ_s) and we have

$$\xi_s = 12 + \frac{109}{10x} + O(x^{-2}) \tag{116}$$

Given a solution h, its constant C in ξ for which (113) holds can be calculated from asymptotic information in any direction above the real line by near least term truncation, namely

$$C = \lim_{\substack{x\to\infty \\ \arg(x)=\phi}} \exp(x) x^{1/2} \left(h(x) - \sum_{k\le|x|} \frac{\tilde{h}_{0,k}}{x^k} \right) \tag{117}$$

(this is a particular case of much more general formulas [14]) where $\sum_{k>0} \tilde{h}_{0,k} x^{-k}$ is the common asymptotic series of all solutions of (108) which are small in the right half plane. □

General comments. 1. The expansion scales, x and $x^{-1/2}e^{-x}$ are crucial. Only for this choice one obtains an expansion which is valid both in S_{trans} and near poles of (108). For instance, the more general second scale $x^a e^{-x}$ introduces logarithmic singularities in H_j, except when $a \in -\frac{1}{2} + \mathbb{Z}$. With these logarithmic terms, the two scale expansion would only be valid in an $O(1)$ region in x, what is sometimes called a “patch at infinity”, instead of more than a sector. Also, $a \in -\frac{1}{2} - \mathbb{N}$ introduces obligatory singularities at $\xi = 0$ precluding the validity of the expansion in S_{trans}. The case $a \in -\frac{1}{2} + \mathbb{N}$ produces instead an expansion valid in S_{trans} but not near poles. Indeed, the substitution $h(x) = g(x)/x^n,\ n \in \mathbb{N}$ has the effect of changing α to $\alpha + n$ in the normal form. This in turn amounts to restricting the analysis to a region far away from the poles, and then all H_j will be entire. In general it is useful thus to make (by substitutions in (2)) $a = \alpha$ minimal compatible with the assumptions (a1) and (a2), as this ensures the widest region of analysis.

2. The pole structure can be explored beyond the first array, in much of the same way: For large ξ induction shows that $H_n \sim Const_n.\xi^n$, suggesting a reexpansion for large ξ in the form

$$h \sim \sum_{k=0}^{\infty} \frac{H_k^{[1]}(\xi_2)}{x^k};\ \xi_2 = C^{[1]}\xi x^{-1} = C\,C^{[1]}x^{-3/2}\mathrm{e}^{-x} \tag{118}$$

By the same techniques it can be shown that (118) holds and, by matching with (113) at $\xi_2 \sim x^{-2/3}$, we get $H_0^{[1]} = H_0$ with $C^{[1]} = -1/60$. Hence, if x_s belongs to the first line of poles, i.e. $\xi(x_s) = \xi_s$ cf. (116), the second line of poles is given by the condition

$$x_1^{-3/2}\mathrm{e}^{-x_1} = -60 \cdot 12C$$

i.e., it is situated at a logarithmic distance of the first one:

$$x_1 - x_s = -\ln x_s + (2n+1)\pi i - \ln(60) + o(1)$$

(see Fig. 4). Similarly, on finds $x_{s,3}$ and in general $x_{s,n}$. The second scale for the $n-$th array is $x^{-n-1/2}\mathrm{e}^{-x}$.

The expansion (113) can be however matched directly to an *adiabatic invariant*-like expansion valid throughout the sector where h has poles, similar to the one in [30]. In this language, the successive expansions of the form (118) pertain to the separatrix crossing region. We will not pursue this issue here.

5.3. Example 3: The Painlevé equation P2

This equation reads:

$$y'' = 2y^3 + xy + \alpha \tag{119}$$

(Incidentally, this example also shows that for a given equation distinct solution manifolds associated to distinct asymptotic behaviors may lead to different normalizations.) After the change of variables

$$x = (3t/2)^{2/3}; \quad y(x) = x^{-1}(t\,h(t) - \alpha)$$

one obtains the normal form equation

$$h'' + \frac{h'}{t} - \left(1 + \frac{24\alpha^2+1}{9t^2}\right)h - \frac{8}{9}h^3 + \frac{8\alpha}{3t}h^2 + \frac{8(\alpha^3-\alpha)}{9t^3} = 0 \tag{120}$$

and

$$\lambda_1 = 1,\ \ \alpha_1 = -1/2;\ \xi = \frac{e^{-t}}{\sqrt{t}};\ \ \xi^2 F_0'' + \xi F_0' = F_0 + \frac{8}{9}F_0^3$$

The initial condition is (always): F_0 analytic at 0 and $F_0'(0) = 1$. This implies

$$F_0(\xi) = \frac{\xi}{1 - \xi^2/9}$$

Distinct normalizations (and sets of solutions) are provided by

$$x = (At)^{2/3}; \quad y(x) = (At)^{1/3}\left(w(t) - B + \frac{\alpha}{2At}\right)$$

if $A^2 = -9/8$, $B^2 = -1/2$. In this case,

$$w'' + \frac{w'}{t} + w\left(1 + \frac{3B\alpha}{tA} - \frac{1-6\alpha^2}{9t^2}\right)w$$
$$-\left(3B - \frac{3\alpha}{2tA}\right)w^2 + w^3 + \frac{1}{9t^2}\left(B(1+6\alpha^2) - t^{-1}\alpha(\alpha^2-4)\right) \quad (121)$$

so that

$$\lambda_1 = 1, \alpha_1 = -\frac{1}{2} - \frac{3}{2}\frac{B\alpha}{A}$$

implying

$$\xi^2 F_0'' + \xi F_0' - F_0 = 3BF_0^2 - F_0^3$$

and, with the same initial condition as above, we now have

$$F_0 = \frac{2\xi(1+B\xi)}{\xi^2+2}$$

The first normalization applies for the manifold of solutions such that $y \sim -\frac{\alpha}{x}$ (for $\alpha = 0$ y is exponentially small and behaves like an Airy function) while the second one corresponds to $y \sim -B - \frac{\alpha}{2}x^{-3/2}$.

6. Appendix

6.1. Some results in classical asymptotics

The notations and assumptions are those of Sect. 2.

Theorem 16 *Let $\mathbf{y}(x)$ be a solution of (2) satisfying (8) on a direction d which is not an antistokes line. Let S be the open sector bounded by two consecutive antistokes lines which contains d.*

Then

(i) for any $d' \subset S$ the solution $\mathbf{y}(x)$ is analytic on d' for x large enough, and tends to 0 along d'. Also

(ii) (9) holds on d'.

These facts follow from the proof of Theorem 12.1 of [51] (for more general contexts see also the proofs of [26], [27]) and from the proof (of a similar theorem) presented in [50]. Unfortunately, (i) and (ii) of Theorem 16 were not formulated in these references as results in their own right. They also follow from the more general results of [13], but their essence is of a classical asymptotics nature and the ideas of exponential asymptotics are not really needed. (To compare the results obtained using classical versus exponential asymptotics approaches see Theorem 19 and the Remark following it, Sect. 6.2.) We therefore include here a self-contained proof of Theorem 16. The iteration argument of [51] is set up as iterations of contractive operators on appropriate Banach spaces of analytic functions.

Proof of (i). Setting. Fix $\eta > 0$ and let $S_\eta \subset S$ be the open subsector whose bounding directions form an angle η with the boundary of S. We assume η is small enough, so that $d \subset S_\eta$.

Let $x_0 \in d$ and let $D = x_0 + S_\eta$. It will be shown that if $|x_0|$ is large enough, then the solution $\mathbf{y}(x)$ satisfying (8) is analytic in D, and tends to 0 as $|x| \to \infty$, $x \in D$.

Denote $y_j(x_0) = y_j^0$.

Note that for each index j, $\Re(\lambda_j x)$ has the same sign for all $x \in D$ (there are no antistokes lines in S_η). Divide the coordinates of $\mathbf{y}$ into the two sets I_+, I_-

$$I_\pm = \{j = 1, \ldots, n\,;\ \Re(\lambda_j x) \in \mathbb{R}_\pm\ ,\ x \in D\} \tag{122}$$

Integral equations. Equation (2) can be written in the integral form

$$\begin{aligned} y_j(x) &= x^{\alpha_j} e^{-\lambda_j x} a_j + x^{\alpha_j} e^{-\lambda_j x} \int_{\Pi_j(x)} x_1^{-\alpha_j} e^{\lambda_j x_1} g_j\left(x_1{}^{-1}, \mathbf{y}(x_1)\right) dx_1 \\ &\equiv \psi_j(x) + \mathcal{J}_j(\mathbf{y})(x) \quad , \quad j = 1, \ldots, n \end{aligned} \tag{123}$$

where the paths of integration $\Pi_j(x) \subset D$ are: the segment $[x_0, x]$ if $j \in I_+$ and the half-line from ∞ to x, along the direction of $x - x_0$ for $j \in I_-$.

Since the solution $\mathbf{y}(x)$ goes to 0 along d we see that in (123) its constants of integration a_j are

$$a_j = 0 \ \text{ for } \ j \in I_- \quad , \quad a_j = y_j(x_0) x_0^{-\alpha_j} e^{\lambda_j x_0} \ \text{ for } \ j \in I_+ \tag{124}$$

By assumption $\mathbf{g}(x^{-1}, \mathbf{y})$ is analytic at $(0, \mathbf{0})$, say for $|x|^{-1} \le r$ and $|\mathbf{y}| \le \rho_2$, and satisfies $|\mathbf{g}(x^{-1}, \mathbf{y})| < const\ \left(|x|^{-2} + |\mathbf{y}|^2\right)$ (see Sect. 2).

Let $\mathcal{B}_0$ be the Banach space of functions $\mathbf{y}(x)$ analytic on D and continuous on $\overline{D}$ (with the sup norm). Let $\mathcal{F}$ be the closed subset of functions $\mathbf{y} \in \mathcal{B}_0$ with $\|\mathbf{y}\| \le \rho$ (where $\rho > 0$ will be chosen small enough) and satisfying $y_j(x_0) = y_j^0$ for $j \in I_+$ and $y_j(x_0) = 0$ for $j \in I_-$.

Relations (123), (124) can be viewed as an equation $\mathbf{y} = \boldsymbol{\psi} + \mathcal{J}(\mathbf{y})$ on $\mathcal{F}$ (if $|x_0| > r^{-1}$ and $\rho < \rho_2$). For ρ small we show that if $\mathbf{y} \in \mathcal{F}$ then $\boldsymbol{\psi} + \mathcal{J}(\mathbf{y}) \in \mathcal{F}$ and the fact that $\mathcal{J}$ is a contraction on $\mathcal{F}$. It will follow that the integral equation has a unique solution thus proving (i).

Lemma. *Let*

$$|x_0| \geq \max_{j=1,\dots,n} |\Re\alpha_j|(|\lambda_j|\,\sin\eta)^{-1} \max\left\{1+\sqrt{2}\,,\ \left(\sqrt{2}\sin\eta\right)^{-1}\right\}$$

If: (i) $j \in I_+$ and $x(t) = x_0 + t(x - x_0)$, $t \in [0,1]$, or
(ii) $j \in I_-$ and $x(t) = x + t(x - x_0)$, $t \geq 0$
then

$$\left|\frac{x}{x(t)}\right|^{\Re\alpha_j} e^{\frac{1}{2}\Re[\lambda_j(x(t)-x)]} \leq 1 \qquad (125)$$

The proof of this lemma is straightforward (the left side of (125) is increasing in t in case (i) and decreasing in case (ii)). The following estimates can be used: $|\cos\arg[\lambda_j(x - x_0)]| \geq \sin\eta > 0$, $\cos[\arg(x - x_0) - \arg x_0] \geq -\cos(2\eta) > -1$, and for (i) $\Re[\lambda_j(x(t) - x)] = -(1 - t)|\lambda_j(x-x_0)|\cos\arg[\lambda_j(x-x_0)] < -(1-t)|\lambda_j(x-x_0)|\sin\eta$, while for (ii) $\Re[\lambda_j(x(t)-x)] = t|\lambda_j(x-x_0)|\cos\arg[\lambda_j(x-x_0)] < -t|\lambda_j(x-x_0)|\sin\eta$.

The set $\mathcal{F}$ is invariant under iterations

Let $\mathbf{y} \in \mathcal{F}$. For $j \in I_+$

$$|\psi_j(x) + \mathcal{J}_j(\mathbf{y})(x)| \leq const\ |y_j^0| \left|\frac{x}{x_0}\right|^{\Re\alpha_j} e^{\Re[\lambda_j(x_0-x)]}$$
$$+ const\ |x - x_0| \int_0^1 \left|\frac{x}{x(t)}\right|^{\Re\alpha_j} e^{\Re[\lambda_j(x(t)-x)]} \left(|x(t)|^{-2} + \|\mathbf{y}\|^2\right) dt$$

and using (125) and that $|x(t)| \geq |x_0|(1-\cos^2(2\eta))$ the last term is bounded by

$$const\ |\mathbf{y}^0| + const\ |x - x_0| \left(|x_0|^{-2} + \|\mathbf{y}\|^2\right) \int_0^1 e^{-\frac{1}{2}(1-t)|\lambda_j(x-x_0)|\sin\eta}\, dt$$
$$< const\ |y_j^0| + const\ \left(|x_0|^{-2} + \|y\|^2\right)$$

Similar estimates hold for $j \in I_-$, hence (for some $K > 0$)

$$\|\boldsymbol{\psi} + \mathcal{J}(\mathbf{y})\| \leq K\left(|\mathbf{y}^0| + |x_0|^{-2} + \|\mathbf{y}\|^2\right).$$

Let ρ be small, such that $\rho < (3K)^{-1}$. Then if $|\mathbf{y}^0| < \rho(3K)^{-1}$, and $|x_0|^{-2} < \rho(3K)^{-1}$ we have $\boldsymbol{\psi} + \mathcal{J}(\mathbf{y}) \in \mathcal{F}$.

Contraction. Let $\mathbf{y}$ and $\mathbf{y}'$ be in $\mathcal{F}$. Writing $g_j(x^{-1},\mathbf{y}) = g_{j,0}(x^{-1}) + \sum_{k=1,\dots,n} g_{j,k}(x^{-1},\mathbf{y})y_k$ with $g_{j,k}$ analytic for $|x^{-1}| \leq r$ and $|\mathbf{y}| \leq \rho_2$ and $g_{j,k} = O(x^{-2}) + O(|\mathbf{y}|)$ for $k \geq 1$, and $g_{j,0} = O(x^{-2})$ then $|g_j(x^{-1},\mathbf{y}) - g_j(x^{-1},\mathbf{y}')| \leq const(|x|^{-2} + \rho)|\mathbf{y} - \mathbf{y}'|$ so that, with estimates similar to the above, we get $\|\mathcal{J}(\mathbf{y}) - \mathcal{J}(\mathbf{y}')\| < const(|x_0|^{-2} + \rho)|\mathbf{y} - \mathbf{y}'|$. For small ρ and large x_0 the operator $\mathcal{J}$ is a contraction on $\mathcal{F}$, and part (i) of Theorem 16 is proved.

Remark. In the estimates above the smaller η (i.e. the closer x to an antistokes line) the larger x_0 must be. This is closely related to the fact (which is the object of the present paper) that solutions (which are analytic in a "sector"–more precisely, in a region described in Theorem 16) develop (generically) singularities on the edges of this "sector".

Proof of (ii). Let $\boldsymbol{\phi}(x)$ be a solution of (2) satisfying (9)–which is known to exist [51]. Let $\mathbf{y}(x)$ be a solution satisfying (8). Let $\mathbf{u}(x) = \mathbf{y}(x) - \boldsymbol{\phi}(x)$. It is enough to show that $\mathbf{u}(x) = O(|x|^{-r})$ for all $r \in \mathbb{N} \cup \{0\}$.

The function $\mathbf{u}(x)$ has limit 0 along d and satisfies

$$\mathbf{u}' = -\hat{\Lambda}\mathbf{u} + \frac{1}{x}\hat{A}\mathbf{u} + \mathbf{h}\left(x^{-1}, \mathbf{u}\right) \tag{126}$$

where

$$\mathbf{h}\left(x^{-1}, \mathbf{u}\right) = \mathbf{g}\left(x^{-1}, \mathbf{u} + \boldsymbol{\phi}(x)\right) - \mathbf{g}\left(x^{-1}, \boldsymbol{\phi}(x)\right) \tag{127}$$

As in the proof of (i) we write (126) in integral form (similar to (123))

$$\begin{aligned} u_j(x) &= x^{\alpha_j} e^{-\lambda_j x} a_j + x^{\alpha_j} e^{-\lambda_j x} \int_{\Pi_j(x)} x_1^{-\alpha_j} e^{\lambda_j x_1} p_j\left({x_1}^{-1}, \mathbf{u}(x_1)\right) dx_1 \\ &\equiv \psi_j(x) + \mathcal{J}_j(\mathbf{u})(x) \quad , \quad j = 1, \ldots, n \end{aligned} \tag{128}$$

where the paths of integration are those of (123) and the constants a_j satisfy the analogue of (124)

$$a_j = 0 \;\text{ for }\; j \in I_- \quad , \quad a_j = u_j(x_0) x_0^{-\alpha_j} e^{\lambda_j x_0} \;\text{ for }\; j \in I_+ \tag{129}$$

where $\mathbf{u}_0 = \mathbf{y}(x_0) - \boldsymbol{\phi}(x_0)$.

Let D be as in (i), where $\boldsymbol{\phi}$ is analytic. Consider the Banach space $\mathcal{B}_r$ of functions $\mathbf{u}(x)$ analytic on D, continuous on $\overline{D}$, with the norm $\|\mathbf{u}\| = \sup_{x \in D} |x^r \mathbf{u}(x)|$ (see also [50]).

Let $\mathcal{F}$ be the closed subset of functions $\mathbf{u} \in \mathcal{B}_r$ with $\|\mathbf{u}\| \le \rho$ (where $\rho > 0$ will be chosen small enough) and satisfying $\mathbf{u}(x_0) = \mathbf{u}_0$.

Note that $\boldsymbol{\phi}(x) = O(x^{-2})$ (cf. (3)) hence $|\boldsymbol{\phi}(x)| < M|x|^{-2}$ (for $x \in D$ and x_0 large enough). Then since $\mathbf{g}(x^{-1}, \mathbf{y})$ was assumed $O(x^{-2}) + O(|\mathbf{y}|^2)$ it follows that for $x \in D$ (cf. (127))

$$\left|\mathbf{h}\left(x^{-1}, \mathbf{u}\right)\right| < const \left(|x|^{-2}|\mathbf{u}| + |\mathbf{u}|^2\right)$$

(for x_0 large enough, so that $|\boldsymbol{\phi}(x)| < \rho/2$ and for $|\mathbf{u}| < \rho/2$).

The same estimates as in the proof of (i) (the only difference being that the α_j of (i) should be replaced here by $r + \alpha_j$) show that equation $\mathbf{u} = \boldsymbol{\psi} + \mathcal{J}(\mathbf{u})$ has a unique solution in $\mathcal{F}$ if $|x_0|$ is large enough (depending on r, as expected). Hence $\|\mathbf{u}\|_r < \infty$ which concludes the proof of (ii). $\square$

6.2. Summary of some results in [13]

This subsection contains details on results of [13] cited, referred to, or relevant for the present paper. A simple consequence of a Lemma in [13] (needed for the present paper) is formulated and proved at the end of this section (Theorem 19).

Since the Theorems, Lemmas and some formulas cited in this section are from [13], to avoid repetition we will follow by a * sign any result cited from [13].

The setting is the same as in the present paper: the equation studied is

$$\mathbf{y}' = -\hat{\Lambda}\mathbf{y} - \frac{1}{x}\hat{B}\mathbf{y} + \mathbf{g}(x^{-1}, \mathbf{y}) \tag{130}$$

(same as (2) with $\hat{B} = -\hat{A}$) having transseries solutions

$$\tilde{\mathbf{y}}(x) = \sum_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{-\boldsymbol{\beta}\cdot\mathbf{k}} \tilde{\mathbf{s}}_{\mathbf{k}}(x) \equiv \sum_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{-\mathbf{M}\cdot\mathbf{k}} \tilde{\mathbf{y}}_{\mathbf{k}}(x) \tag{131}$$

(same as (4), (5), (11) for $\beta_j = -\alpha_j$, $M_j = -\lfloor \Re\beta_j \rfloor + 1$, $j = 1, \ldots, n$).

Since the association between actual and formal solutions depends on directions (Stokes phenomena) a sector in the complex x-plane is chosen as follows. Fix some non-empty open sector $S' \subset \mathbb{C}$ and consider those transseries (131) valid in S' (as explained in Sect. 2.1). Some constants $C_1, \ldots, C_n$ may be required to be zero in S', say $C_j = 0$ for $j = n_1 + 1, \ldots, n$ (with $n_1 \geq 0$). Let S_{trans} be the (non-empty, open) maximal sector of validity of any transseries (131) with $C_j = 0$ for $j = n_1 + 1, \ldots, n$ (see (6)).

To simplify the notations it can be assumed (after trivial changes of coordinates) that $\lambda_1 = 1$ (see also Sect. 2.2).

1. The construction of actual solutions associated to transseries solutions valid in S_{trans} is done in [13] using a generalized Borel summation as follows.

Denote by $\mathbf{Y}(p)$ the *formal* inverse Laplace transform of $\mathbf{y}(x)$ (i.e. $\mathbf{Y}(p) = (2\pi i)^{-1} \int_{a-i\infty}^{a+i\infty} e^{px}\mathbf{y}(x)\, dx$, its convergence following from subsequent analysis). Using the usual properties of the inverse Laplace transform (e.g. the transform of $y'(x)$ is $-pY(p)$, multiplication is transformed into convolution, etc.) the differential equation (130) is transformed into a convolution equation (eq. (1.13)*).

The Stokes lines in the Borel plane (p-plane) are defined as the complex conjugates of the Stokes lines in the direct (x) space. For linear equations the Stokes lines in the p-plane are $d_{j,\mathbf{0}} = \lambda_j\mathbb{R}_+$, $j = 1, \ldots, n$. For nonlinear equations, there also are other Stokes lines (which play a role only in the higher order terms of the transseries, with $|\mathbf{k}| \geq 2$, hence they are "transparent" in the linear case) namely $d_{j,\mathbf{k}} = (\lambda_j - \mathbf{k}\cdot\lambda)\mathbb{R}_+$ with $j = 1,\dots,n$, $\mathbf{k} \in (\mathbb{N}\cup\{0\})^n$ (note that $p_{j,\mathbf{k}} \in d_{j,\mathbf{k}}$ cf.(35)).

Once the sector S_{trans} is fixed, there are only finitely many lines $\overline{d_{j,\mathbf{k}}}$ in this sector.

In any proper subsector of any of the n sectors formed by the Stokes lines $d_{j,\mathbf{0}}$ the convolution equation has a unique solution $\mathbf{Y_0}(p)$ which is analytic at $p = 0$ (Lemma 16*). $\mathbf{Y_0}(p)$ is in fact the Borel transform of the asymptotic series $\tilde{\mathbf{y}}_\mathbf{0}(x)$ (see (3)) (i.e. $\mathbf{Y_0}(p) = \sum_r \tilde{\mathbf{y}}_{\mathbf{0},r} p^r/r!$). But $\mathbf{Y_0}$ has singularities on the Stokes lines at $p \in \lambda_j\mathbb{Z}_+,\ j = 1,\dots,n$ (hence the classical Laplace transform cannot be taken on $\mathbb{R}_+$ and the classical Borel sum of $\tilde{\mathbf{y}}_\mathbf{0}$ does not exist).

Denote by $\mathbf{Y}_\mathbf{0}^+$ the analytic continuation of $\mathbf{Y_0}$ on directions above $d_{1,\mathbf{0}} = \mathbb{R}_+$ (but below the neighboring Stokes line), respectively by $\mathbf{Y}_\mathbf{0}^-$ for the continuation below $\mathbb{R}_+$; they exist see Lemma 16*. It is shown that as p approaches $\mathbb{R}_+$ from above (or below) $\mathbf{Y}_\mathbf{0}^+(p)$ (respectively, $\mathbf{Y}_\mathbf{0}^-(p)$) tends to a distribution on $\mathbb{R}_+$ in an adequate space of distributions–the staircase distributions, introduced in [13] (Lemma 16*). In general, the two distributions are different. (Of course, a similar picture holds at any other Stokes line.)

Higher order functions $\mathbf{Y_k}$, $|\mathbf{k}| \geq 1$ are then constructed (Lemma 20*) by solving the convolution equation on the Stokes lines.

Consider for example the line $\mathbb{R}_+ = d_{1,\mathbf{0}}$. Fix a solution $\mathbf{Y_0}$ of the convolution equation in the space of staircase distributions on $\mathbb{R}_+$. The construction of the higher order $\mathbf{Y_k}$'s with $k_j = 0$ if $j \geq n_1 + 1$[10] is done in the proof of Lemma 20* as follows. In view of the sought-for expansion (10), after introducing it in (130) and identifying the coefficients one obtains (a recursive system of) differential equations for $\mathbf{y_k}$; formal inverse Laplace transform yields (a recursive system of) convolution equations for $\mathbf{Y_k}$. Once a staircase distribution solution $\mathbf{Y_0}$ on $\mathbb{R}_+$ is chosen the general solution of this system with regularity (34) depends on n_1 free constants $C_1,\dots,C_{n_1}$, in the form $\mathbf{C^kY_k}$. Outside the Stokes line $\mathbb{R}_+$ the solutions $\mathbf{Y_k}$ are, in fact, analytic up to the nearest direction (of positive argument ψ_+, respectively negative argument ψ_-) which is either a Stokes line $d_{j,\mathbf{k}}$ which lies in the right half-plane (i.e. the half-plane orthogonal to $d_{1,\mathbf{0}} = \mathbb{R}_+$) – where some of the constants C_j may change, or is an antistokes line associated to $\lambda_1 = 1$, i.e. $i\mathbb{R}_+$ or $i\mathbb{R}_-$ – where the transseries is no longer defined[11] – (Lemma 20* (i)–(iv)).

It is interesting to note that the $\mathbf{Y_k}$ multiplied by Stokes constants are generated as differences between different branches of $\mathbf{Y_0}$ (Theorem 4*, Proposition 23*).

[10] Other $\mathbf{Y_k}$'s are not needed since the corresponding $\tilde{\mathbf{y}}_k$ cannot be present in a transseries on the fixed S_{trans}.

[11] $\mathbf{Y_k}$ may be analytic beyond the antistokes lines $\pm i\mathbb{R}_+$ but the analysis in [13] stops there.

Then (a generalized) Laplace transform is applied to $\mathbf{Y_k}$ (in the space of staircase distributions on the Stokes line $\mathbb{R}_+$ under consideration) yielding (analytic) functions $\mathbf{y_k} = \mathcal{L}\mathbf{Y_k}$.

The last step in the summation of transseries is showing that the sum (10) converges (Lemma 20* (v); more details are found in the proof of Lemma 17 of this section).

The reconstruction of a solution from a transseries is concluded showing that the function $\mathbf{y}(x)$ obtained as the sum of (10) is a solution of the differential equation (130)–which follows easily because of appropriate convergence and since all functions have been constructed from formal objects satisfying the equation (Lemma 20* (v)).

The correspondence between transseries (i.e. the constants $\mathbf{C}$) and actual solutions given by the summation of Lemma 20* is not unique. This is due to the non-uniqueness of staircase distribution solutions $\mathbf{Y_0}$ on $\mathbb{R}_+$ (on which the higher order $\mathbf{Y_k}$ depend): there is a one parameter family of solutions $\mathbf{Y}_0^\alpha$ (among which $\mathbf{Y}_0^\pm$ for $\alpha = 0, 1$) – they are special averages of analytic continuations of the germ of analytic function $\mathbf{Y_0}$ at $p = 0$.

2. Conversely, any solution of (130) satisfying (9) on a direction d in the right half-plane is the sum of a transseries, which is unique once α is fixed (Theorem 3*(iii)).

The proof is done in the following steps (Propositions 24* and 25*). There exists a solution $\mathbf{y_0}^\alpha(x)$ asymptotic to the transseries $\tilde{\mathbf{y}}_0$ (i.e. $\tilde{\mathbf{y}}$ with all $C_j = 0$) as constructed at part **1.** above. It is also shown that any two solutions $\mathbf{y}^{1,2}(x)$ with the same asymptotic expansion $\mathbf{y}^{1,2} \sim \tilde{\mathbf{y}}_0$ on d differ by exponentially small terms: $\mathbf{y}^1(x) - \mathbf{y}^2(x) = \sum_{j=1,\dots,n} C_j e^{-\lambda_j x} x^{-\beta_j}(\mathbf{e}_j + o(1))$ on d. Thus the difference $\mathbf{y}(x) - \mathbf{y}_0^\alpha(x)$ fixes the constants $C_j = C_j(\alpha, d)$, which are then used to construct a solution $\mathbf{y}_t(x)$ from the transseries $\tilde{\mathbf{y}}$ with these constants (using part **1.** above). The last step is showing that $\mathbf{y}(x) \equiv \mathbf{y}_t(x)$.

3. It has been thus established that (given a direction d, and a parameter α) there is a one-to-one correspondence between transseries solutions and actual solutions of (130). The correspondence is built using a (family of) generalized Borel summation(s) $\mathcal{LB}_\alpha$ on d. It is shown that the operator $\mathcal{LB}_\alpha$ is compatible with all algebraic operations (performed on transseries, respectively functions on d).

The Stokes phenomenon is analyzed in Theorems 4* and 5*.

4. We now state the maximal domain of analyticity[12] of $\mathbf{y}(x)$ implied by formula (2.41)* of Lemma 20*.

Lemma 17 *Let* $\mathbf{y}(x)$ *be a solution of (130) satisfying (9) on a direction d above (but close enough to)* $\mathbb{R}_+$.

---

[12] This domain is improperly stated in [13].

Let C_j^-, $j = 1, \ldots, n$ be the constants such that $\mathbf{y}(x)$ is represented on d as

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_{\mathbf{0}}^- + \sum_{|\mathbf{k}|>0} (\mathbf{C}^-)^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda} x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}{}^- \tag{132}$$

(see Theorem 3(iii)).*

Then for any $\epsilon, \delta > 0$ there is $x_1 > 0$ such that $\mathbf{y}(x)$ is analytic on the domain

$$D_{an}^- = \Big\{ x \; ; \; |x| > x_1 \, , \, arg(x) \in \Big[-\psi^- - \frac{\pi}{2} + \epsilon, -\psi^- + \frac{\pi}{2} - \epsilon \Big], \\ \Big| C_j^- x^{M_j} e^{-\lambda_j x} \Big| < \delta^{-1} \, , \; j = 1, \ldots, n \Big\} \tag{133}$$

The constants ϵ, δ are the same for all solutions of (130) with transseries valid in the same sector S_{trans} as $\tilde{\mathbf{y}}$.

The *Proof* follows immediately from Lemma 20*(v), but we provide the details.

All functions $\mathbf{Y}_{\mathbf{k}}{}^-(\cdot e^{i\phi})$ are Laplace transformable in the space of staircase distributions on $\mathbb{R}_+$ with exponential weight $e^{-\nu p}$ (for $\nu > 0$ large enough) if $\phi \in (\psi_-, \psi_+)$ (see Lemma 20*(v)).

For $\phi \in (\psi_-, 0)$ the $\mathbf{Y}_{\mathbf{k}}{}^-(pe^{i\phi})$, $(p \geq 0)$ are analytic (Lemma 20*(iii)), so the Laplace transform in the space of staircase distributions coincides with the classical Laplace transform (see Lemma 6* and the classical properties of the Laplace transform and of the spaces L^1_ν) where the Laplace transform is defined as

$$\mathcal{L}F(x) = \int_d e^{-px} F(p)\, dp \quad , \quad \text{for } x \in \overline{d} \tag{134}$$

(Note that in (134) $p \in d$ and $x \in \overline{d}$ so that $\Re(px) > 0$.)

Remark 18 If F is analytic on a direction $d = e^{-i\eta}\mathbb{R}_+$ ($\eta \in \mathbb{R}$) and

$$\|F\|_\nu \equiv \int_d e^{-\nu p} \, |F(p)| \; dp \; < \infty$$

then its Laplace transform (134) is analytic for $x \in \overline{d} \equiv e^{i\eta}\mathbb{R}_+$, $|x| > \nu$.

Furthermore, for any $\epsilon > 0$ $\mathcal{L}F$ has analytic continuation on the domain $\arg x \in [\eta - \frac{\pi}{2} + \epsilon, \eta + \frac{\pi}{2} - \epsilon] \equiv I$ and $|x| > \nu(\sin\epsilon)^{-1}$ and $\mathcal{L}F$ satisfies

$$|\mathcal{L}F(x)| < \|F\|_\nu \tag{135}$$

The proof is immediate, noting that the path of integration of (134) can be rotated to any other direction in the interval I – since I is so that $\Re(xp) > 0$.

To prove Lemma 17 let ϵ be small, positive and let $\phi \in [\psi_- + \epsilon, -\epsilon]$. For any $\delta > 0$ there is $\nu > 0$ so that the functions $\mathbf{Y}_{\mathbf{k}}{}^-(pe^{i\phi})$, $(p \geq 0)$ satisfy $\|\mathbf{Y}_{\mathbf{k}}{}^-(\cdot e^{i\phi})\|_\nu < \delta^{|\mathbf{k}|}$ for all multi-indices $\mathbf{k}$ considered (i.e. $|\mathbf{k}| \geq 0$, $k_j = 0$ for $j > n_1$) see Proposition 22*(ii). Using Remark 18 and (132) the result is immediate. □

Remark. Similarly to Lemma 17 there is $x_1 > 0$ such that $\mathbf{y}(x)$ is analytic on $D_{an}^+ = D_{an}^+(\epsilon, \delta, \mathbf{C}^+)$ where

$$D_{an}^+ = \left\{ x \ ; \ |x| > x_1 \, , \ \arg(x) \in \left[-\psi^+ - \frac{\pi}{2} + \epsilon, -\psi^+ + \frac{\pi}{2} - \epsilon \right] \text{and} \right.$$
$$\left. \left| C_j^+ x^{M_j} e^{-\lambda_j x} \right| < \delta^{-1} \ j = 1, \dots, n \right\} \quad (136)$$

so that the domain of analyticity of a solution $\mathbf{y}(x)$ includes domains of the form $D_{an}^- \cup D_{an}^+$.

Theorem 19 *Let $\mathbf{y}(x)$ be a solution of (130) satisfying (9) on $d = \mathbb{R}_+$. Let $\epsilon > 0$ be small.*

There exists δ, $R > 0$ such that $\mathbf{y}(x)$ is analytic (at least) on

$$S_{an} = S_{an}\,(\mathbf{y}(x); \epsilon) = S_\epsilon^+ \cup S_\epsilon^- \quad (137)$$

where

$$S_\epsilon^\pm = \left\{ x \, ; \ |x| > R \, , \ arg(x) \in \left[-\frac{\pi}{2} \mp \epsilon, \frac{\pi}{2} \mp \epsilon \right] and \right.$$
$$\left. \left| C_j^- e^{-\lambda_j x} x^{-\beta_j} \right| < \delta^{-1} \ for \ j = 1, \dots, n \right\} \quad (138)$$

The constant δ is the same for all solutions of (130) with transseries valid in the same sector S_{trans} as $\tilde{\mathbf{y}}$. (However, R does depend on $\mathbf{C}$.)

Proof. From Lemma 17 using the expansion of $\mathbf{Y}$ in terms of $\mathbf{Y_k}^-$, respectively $\mathbf{Y_k}^+$ it follows that $\mathbf{y}(x)$ is analytic for $|x| > x_1$, $\arg x \in [-\psi^+ - \frac{\pi}{2} + \epsilon, -\psi^- + \frac{\pi}{2} - \epsilon] \equiv I$ and $\left| C_j^\pm e^{-\lambda_j x} x^{-M_j} \right| < \delta^{-1}$, $j = 1, \dots, n$. Since $M_j = -\lfloor \Re \beta_j \rfloor + 1$ and the sector I is larger than the sector where all exponentials in the transseries of $\mathbf{y}(x)$ are bounded, the result follows. $\square$

Remark. Fix a direction d (not an antistokes line) and consider solutions satisfying (8). It is interesting to compare the result on the domain of analyticity of these solutions as given by Theorem 19 (obtained using results of exponential asymptotics) to the result of Theorem 16(i) (obtained in a classical setting). There are special families of solutions for which the sector of analyticity given by Theorem 19 is, in fact, larger than the sector between two consecutive antistokes lines (i.e. solutions having the corresponding C_j zero). For Painlevé P1 equation these special solutions are called *triply truncated*.

Convention. The Borel summation used in the present paper is $\mathcal{LB} \equiv \mathcal{LB}_{\frac{1}{2}}$.

6.3. Proof of (24)

Proof. This follows from the definition $\xi = C_1 x^{\alpha_1} e^{-x}$ and from the asymptotic behavior of the functional inverse W of se^s (see e.g. [12]). For large $t > 0$ the branch of W which is real has the expansion (convergent, as it is not difficult to show)

$$W(t) = \ln t - \ln\ln t + \frac{\ln\ln t}{\ln t} + \frac{\frac{1}{2}(\ln\ln t)^2 - \ln\ln t}{(\ln t)^2} + \cdots$$

□

6.4. Points on γ_N^0 have the same magnitude

Let R be large, so that

$$\rho_3 + \frac{3KB}{R} < \rho_2 \tag{139}$$

Lemma 20 *There is a small enough neighborhood $\mathcal{N}_{\gamma_N^0}$ of γ_N^0 so that any $x', x'' \in \mathcal{N}_{\gamma_N^0}$ satisfy*

$$\frac{1}{2} < \frac{|x'|}{|x''|} < 2 \tag{140}$$

Proof. Using (24) for $t \in [t_0, 1]$ we get the uniform estimate

$$|\gamma_N(t_0)| - |\gamma_N(t)| = \Im\left(\ln(\Gamma(t)) - \ln(\Gamma(t_0))\right) + o(1) \quad (N \to \infty)$$

Therefore $\lim_{N\to\infty} \gamma_N(t')/\gamma_N(t'') = 1$ uniformly for $t, t' \in [t_0, 1]$. So for N large $|x/x'| < 3/2$ for all x, x' on γ_N between x_0 and a, so in a small enough neighborhood $|x/x'| < 1/2$ which proves the Lemma. □

6.5. Special estimates

We show that the second argument of $\mathbf{g}$ in (65) has absolute value less than ρ_2 (cf. (23)).

We need the following lemma.

Lemma 21 *Let $A, c_0 > 0$. There exist $A_0, \kappa > 0$ such that*

$$\sum_{k=1}^{m} \frac{k!}{A^k} \le \kappa \left(\frac{1}{A} + c_0\right) \tag{141}$$

if $A \ge A_0$ and $m!A^{-m} \le c_0$.

Before giving the proof of the lemma, we show how (141) is used.

Let R be large, and c_0 small, so that

$$K\kappa\left(\frac{2B}{R}+c_0\right)<\frac{\rho_2-\rho_3}{2} \tag{142}$$

In view of Theorem 2(i), for $x\in\mathcal{N}_{\gamma^0}$

$$\left|\sum_{k=1}^{m}\frac{1}{x^k}\mathbf{F}_k\left(\xi(x)\right)\right|\le\sum_{k=1}^{m}\frac{1}{|x|^k}k!KB^k$$

and in view of Lemma 20 the last term is bounded by

$$\le\sum_{k=1}^{m}\left(\frac{2B}{|a|}\right)^k k!K \tag{143}$$

Using (143), Lemma 21 and the bound ρ_3 on $\mathbf{F}_0$ and (142) we have

$$|\mathbf{F}^{[m]}(x)|\le|\mathbf{F}_0\left(\xi(x)\right)|+K\kappa\left(\frac{2B}{|a|}+c_0\right)<\frac{\rho_2+\rho_3}{2}<\rho_2$$

Finally, if $|\boldsymbol{\delta}(x)|<\frac{\rho_2-\rho_3}{2}$ on $\mathcal{N}_{\gamma^0}$ then

$$|\boldsymbol{\delta}+\mathbf{F}^{[m]}|<\frac{\rho_2-\rho_3}{2}+\frac{\rho_2+\rho_3}{2}<\rho_2=\rho_2$$

Proof of Lemma 21. Estimates like (141) are common in proofs using least term truncation of factorially divergent series (see e.g. [14]). The proof of (141) is included here for completeness.

The series

$$\sum_{k\ge1}\frac{k!}{A^k} \tag{144}$$

is divergent. Its terms decrease for $k\le A$ and increase for $k>A$; the term with $k=\lfloor A\rfloor$ (or $k=\lfloor A\rfloor+1$) is called the least term (see e.g. [5]).

Case I: $m\le A$

In this case the terms in the l.h.s. of (141) are decreasing, hence

$$\sum_{k=1}^{m}\frac{k!}{A^k}\le\frac{1}{A}+\frac{2}{A^2}(m-1)<\frac{3}{A}$$

Case II: $A<m\le eA/2$

The terms in the l.h.s. of (141) are increasing for $A<k<m$, not exceeding the second term: $m!/A^m<2!/A^2$.

Indeed, this is a simple estimate using Stirling's formula and the fact that the function $F(A) = A^{3/2}\alpha^{\alpha e A}$ is decreasing for $A > A_0$ (if A_0 is large enough).

Then as in Case I

$$\sum_{k=1}^{m} \frac{k!}{A^k} \leq \frac{1}{A} + \frac{2}{A^2}(m-1) < \frac{e+1}{A}$$

Case III: $eA/2 < m$

Denote $p = \lfloor A \rfloor$ and $q = \lfloor \frac{1+A}{2} p - \frac{1}{2} \rfloor$.

Write

$$\sum_{k=1}^{m} \frac{k!}{A^k} = S_{III} + S_{II} + S_I$$

where

$$S_{III} = \sum_{k=1}^{p} \frac{k!}{A^k} \ , \ \ S_{II} = \sum_{k=p+1}^{q} \frac{k!}{A^k} \ , \ \ S_I = \sum_{k=q+1}^{m} \frac{k!}{A^k}$$

and estimate each sum separately.

To estimate S_I

$$S_I \leq \sum_{k=q+1}^{m} \frac{k!}{p^k} \leq \frac{m!}{p^m} \left[\sum_{k=q+1}^{m-1} \frac{p^{m-k}}{(q+2)^{m-k}} \right]$$

and since $p/(q+2) < 2/(A+1) < 1$ this is less than

$$\frac{m!}{p^m} \frac{A+1}{A-1} < \frac{m!}{p^m} \frac{e+2}{e-2}$$

To estimate S_{II} note that

$$S_{II} < (q-p)\frac{q!}{A^q} < \frac{m!}{A^m}(q-p)\left(\frac{A}{q+1}\right)^{m-q}$$

and since $\frac{A}{q+1} < \rho_0 < 1$ if $A > A_0$ (for A_0 large enough)

$$< \frac{m!}{A^m}(q-p)\rho_0^{m-q} < \frac{m!}{p^m}\rho_1$$

Finally

$$S_{III} < \frac{1}{A} + \frac{2}{A^2}(p-1) < \frac{3}{A}$$

The result of Lemma 21 follows. $\square$

6.6. Proof of Proposition 4

A consolidation of one of the norms in Example **(3a)** in [13] is first needed. For convenience we repeat that part. The notations are those in [13].

(3a) For $\Re(\beta) > 0$ and $\phi_1 \neq \phi_2$, let $\mathcal{T}_\beta(\mathcal{E} \cup \overline{\mathcal{V}}) = \{f : f(p) = p^\beta F(p)\}$, where F is analytic in the interior of $\mathcal{E} \cup \mathcal{V}$ and continuous in its closure. We use the family of (equivalent) norms

$$\|f\|_{\nu,\beta} = \left|\Gamma(\beta+1)\right| K \sup_{s\in\mathcal{E}\cup\overline{\mathcal{V}}} \left|e^{-\nu p} f(p)\right| \tag{145}$$

It is clear that convergence of f in $\|\|_{\nu,\beta}$ implies uniform convergence of F on compact sets in $\mathcal{E} \cup \mathcal{V}$ (for p near zero, this follows from Cauchy's formula). $\mathcal{T}_\beta$ are thus Banach spaces and focusing spaces in $\|\|_{\nu,\beta}$ by (145). The spaces $\{\mathcal{T}_\beta\}_\beta$ are isomorphic to each-other. Convolution is defined as

$$p^{-\beta_1-\beta_2-1}(f_1 * f_2)(p) = \int_0^1 t^{\beta_1} F_1(pt)(1-t)^{\beta_2} F_2(p(1-t))\mathrm{d}t = F(p) \tag{146}$$

where F is manifestly analytic, and the application

$$(\cdot * \cdot) : \mathcal{T}_{\beta_1} \times \mathcal{T}_{\beta_2} \mapsto \mathcal{T}_{\beta_1+\beta_2+1} \tag{147}$$

is continuous:

$$\begin{aligned}
&\|f_1 * f_2\|_{\nu,\beta_1+\beta_2+1}\\
&\quad= \left|\Gamma(\beta_1+\beta_2+1)\right| K \sup_p \left| e^{-\nu p}\int_0^p s^{\beta_1} F_1(s)(p-s)^{\beta_2} F_2(p-s)\mathrm{d}s\right|\\
&\qquad\qquad\le \frac{\Gamma(\beta_1+\beta_2+2)}{K}\sup_p\\
&\qquad\int_0^p \left|\frac{KF_1(s)e^{-\nu s}s^{\beta_1}}{\Gamma(\beta_1+1)}\frac{KF_2(p-s)e^{-\nu(p-s)}(p-s)^{\beta_2}}{\Gamma(\beta_2+1)}\right| \mathrm{d}|s|\\
&\qquad\qquad\qquad\le \|f_1\|_{\nu,\beta_1}\|f_2\|_{\nu,\beta_2}
\end{aligned} \tag{148}$$

Estimating the norm of $\mathbf{Y_k}$ exactly as in [13] but using this inequality instead of (2.8) of [13] we get that

$$|\mathbf{Y_k}(p)| \le \left|\Gamma(-\mathbf{k}\cdot\boldsymbol{\alpha}')^{-1}\right| \delta_2^{-|\mathbf{k}|} e^{\nu_0|p|}$$

and thus, using straightforward Cauchy estimates in $\overline{d}_{a_1}$ of derivatives, (38) is proved. □

6.7. Note on normalization of the α_i

The reference [13] uses a transformation that makes $\beta_j < 0$ ($\alpha_j > 0$ in the present notation). To determine the singularities of $\mathbf{y}$ it is now important to make α_j as *small* as possible, as explained in Comment 1, Sect. 5.2. In some cases we must then allow for $\mathbf{m} < 0$ in [13], Eq. (2.43). This does not affect the estimates (2.44) through (2.46) in the space $\mathcal{T}_{\mathbf{k}\boldsymbol{\beta}'-1}$, the only one that relevant to the present paper. Minor modifications of the proof following Lemma 20 in [13] are needed. For completeness we redo redo here the whole proof.

For $|\mathbf{k}| > 1$ with $\mathbf{W_k} := \mathbf{Y_k}$ and $\mathbf{R_k} := \mathbf{T_k}$, the functions $\mathbf{W_k}$ satisfy the equations

$$(1 + J_{\mathbf{k}})\mathbf{W_k} = \hat{Q}_{\mathbf{k}}^{-1}\mathbf{R_k} \tag{149}$$

with $\hat{Q}_{\mathbf{k}} := (-\hat{\Lambda} + p + \mathbf{k} \cdot \boldsymbol{\lambda})$ (notice that for $|\mathbf{k}| > 1$ and $p \in \mathcal{S}_0'$ we have $\det \hat{Q}_{\mathbf{k}}(p) \neq 0$).

$$(J_{\mathbf{k}}\mathbf{W})(p) := \hat{Q}_{\mathbf{k}}^{-1}\Bigg((\hat{B} + \mathbf{m}\cdot\mathbf{k})\int_0^p \mathbf{W}(s)\mathrm{d}s - \sum_{j=1}^{n}\int_0^p W_j(s)\mathbf{D}_j(p-s)\mathrm{d}s\Bigg) \tag{150}$$

Proposition 22 *(i) For large ν and constants K_1 and $K_2(\nu)$ independent of $\mathbf{k}$, with $K_2(\nu) = O(\nu^{-1})$ we have $\|Q_{\mathbf{k}}^{-1}\| \le \frac{K_1}{|\mathbf{k}|}$ and*

$$\|J_{\mathbf{k}}\| \le K_2(\nu) \tag{151}$$

(ii) For large ν, the operators $(1 + J_{\mathbf{k}})$ defined in $\mathcal{D}'_{m,\nu}$, and also in $\mathcal{T}_{\mathbf{k}\boldsymbol{\beta}'-1}$ for $|\mathbf{k}| > 1$ and in $\mathcal{T}_1$ for $|\mathbf{k}| = 1$ are simultaneously invertible. Given $\mathbf{Y}_0$ and $\mathbf{C}$, the $\mathbf{W_k}$, $|\mathbf{k}| \ge 1$ are uniquely determined. For any $\delta > 0$ there is a large enough ν, so that

$$\|\mathbf{W_k}\| \le \delta^{|\mathbf{k}|},\ k = 0, 1, \ldots \tag{152}$$

(in the $\mathcal{D}'_{m,\nu}$ topology, (152) hold uniformly in $\phi \in [\psi_- + \epsilon, 0]$ and $\phi \in [0, \psi_+ - \epsilon]$ for any small $\epsilon > 0$). □

Proof. (i) follows immediately from (148).

(ii) From (149) and (i) we get, for some K and $j \ge 1$ $\|\mathbf{W_k}\| \le K\|\mathbf{R_k}\|$. We first show inductively that the $\mathbf{W_k}$ are bounded. Choosing a suitably large $\nu(\epsilon)$ we can make $\max_{|\mathbf{k}|\le 1}\|\mathbf{W_k}\|_\nu \le \epsilon$ for any positive ϵ (uniformly in ϕ). We show by induction that $\|\mathbf{W_k}\|_\nu \le \epsilon$ for all k. In the same way as in [13] we get

$$\|\mathbf{W_k}\|_\nu \le K\|\mathbf{R_k}\|_\nu \le \sum_{\mathbf{l}\le\mathbf{k}} \kappa_1^{|\mathbf{l}|}\epsilon^{|\mathbf{k}|}\sum_{(\mathbf{i}_{mp})} 1$$
$$\le \epsilon^{|\mathbf{k}|}\sum_{s=0}^{|\mathbf{k}|}\kappa_1^s 2^{n_1(|\mathbf{k}|+s)}2^{s+n_1} \le (C_1\epsilon)^{|\mathbf{k}|} \tag{153}$$

where C_1 does not depend on ϵ, $\mathbf{k}$. Choosing ϵ so that $\epsilon < C_1^{-2}$ we have, for $|\mathbf{k}| \ge 2$ $(C_1\epsilon)^{|\mathbf{k}|} < \epsilon$ completing the induction step. But as we now know that $\|\mathbf{W_k}\|_\nu \le \epsilon$, the same inequalities (153) show that in fact $\|\mathbf{W_k}\|_\nu \le (C_1\epsilon)^{|\mathbf{k}|}$. Choosing ϵ small enough, the first part of Proposition 22, (ii) follows. □

6.8. Proof of Lemma 10 (ii)

1. *Generically h_0 is not entire.* Assume h_0 is analytic in a neighborhood of the disk $B_R = \{|\xi| \le R\}$ and let $M_R = \sup_{|z|=R} h_0(z)$. We have

$$\sup_{|z|\le R}\left\{z^{-1}(h_0(z)-1)\right\} = \sup_{|z|=R}\left\{z^{-1}(h_0(z)-1)\right\} \le R^{-1}(M+1)$$

and thus, for some constants C_i we have

$$M^2 \le C_1R^2 + C_2 + (C_3R^2 + C_4)M \tag{154}$$

whence

$$M \le C_5R^2 + C_6 \tag{155}$$

If h_0 is entire it then follows that h_0 is a quadratic polynomial in ξ. But it is straightforward to check that (84) does not, generically, admit quadratic solutions. Thus the radius of analyticity of h_0 is finite, say R_0, and[13]

$$\sup_{|z|<R_0}\left\{z^{-1}(h_0(z)-1)\right\} \le R_0^{-1}(M_{R_0}+1) \text{ and } M_{R_0} \le C_5R_0^2 + C_6 \tag{156}$$

2. *h_0^2 is uniformly continuous on $\overline{B}_{R_0}$*. Indeed, if $\xi, \xi' \in \overline{B}_{R_0}$ we have by (85) and (156) that h_0^2 is in fact Lipschitz:

$$\left|h_0^2(\xi) - h_0^2(\xi')\right| \le Const.|\xi - \xi'| \tag{157}$$

3. If $\xi_0 \in \partial B_{R_0}$ and $h_0(\xi_0) \ne 0$ then ξ_0 is a regular point of eq. (84) and thus h_0 is analytic at ξ_0.

[13] An *upper* bound for R_0 can be found by comparing $h_0'''(0)$ with its estimate from Cauchy's formula and (154)

4. *If $\xi_s \in \partial B_{R_0}$ is a singular point of h_0 and $-\lambda_2\xi_s^{-1} + d_3 + d_4\xi_s \neq 0$ then ξ_s is a square root branch point of h_0, i.e. $h_0(\xi) = h_1((\xi - \xi_s)^{1/2})$ where h_1 is analytic at zero.* From parts **2** and **3** above, $h_0(\xi_s) = 0$. It is convenient to look at the equation for $\xi(h_0)$ derived from (84):

$$\frac{\mathrm{d}\xi}{\mathrm{d}h_0} = \frac{2h_0}{(\lambda_2\xi^{-1} + d_1 + d_2\xi)h_0 + (-\lambda_2\xi^{-1} + d_3 + d_4\xi)}; \quad \xi(0) = \xi_s \quad (158)$$

whose unique solution is analytic near zero. The claim now follows by noting that $\xi'(0) = 0$ and $\xi''(0) = 2(-\lambda_2\xi_s^{-1} + d_3 + d_4\xi_s)$.
5. We now restrict the analysis to a smaller but generic set of coefficients. We denote by K_s the following subset of parameters (see (79) and (84))

$$K_s = \Big\{(d,\gamma) = (d_j,\gamma_j)_{j=1,\dots,n} \in \mathbb{C}^{2n} : h_0 \text{ not entire and} \\ P(\xi) = -2\lambda_2 + \lambda_2 d_1\xi + d_3\xi^2 \text{ has distinct roots}\Big\} \quad (159)$$

We show in parts 6 through 8 that if $\xi_s \in \partial B_{R_0}$ is a singular point of h_0 and $P(\xi_s) = 0$, then a generic small variation of (a,γ) in K_s makes $P(\xi_s) \neq 0$, and by part 4, ξ_s becomes a square root branch point of h_0.

We thus assume that $P(\xi_s) = 0$. The substitution $\xi - \xi_s = t$ in (84) gives

$$2h_0h_0' = B_1h_0 + tB_2 \quad (160)$$

where

$$B_1 = 2\lambda_2(t+\xi_s)^{-1} - d_1 + (t+\xi_s)d_2; \;\; B_2 = \frac{t+\xi_s-\xi}{t+\xi_s}d_3$$

and the roots of $P(\xi)$ are $\xi_1 \neq \xi_s$.

We now study h_0 in the following smaller region. Let t_0 be small on the segment $[0,-\xi_s]$. Choose $\Delta = \{t : |t - t_0| < |t_0|\} \subset B_{R_0} - \xi_s$ to be a disk tangent at $t = 0$ to $B_{R_0} - \xi_s$ which does not contain the points $\xi_s - \xi$ and $-\xi_s$. Then h_0 is analytic in Δ and continuous in $\overline{\Delta}\backslash\{0\}$ (while h_0^2 is continuous in $\overline{\Delta}$), and $\lim_{\Delta\ni t\to 0} h_0(t) = 0$. We assume $t = 0$ is a singularity of h_0.
6. *There exists a sequence $\{t_n\}_n$ in Δ with $t_n \to 0$ such that $\lim_{n\to\infty} h_0(t_n)/t_n = L$ with $2L^2 - LB_1(0) - B_2(0) = 0$.*

We first show that $h_0 \to 0$, then prove h_0/t is bounded below and above, and finally that h_0/t has a limit.

(a) We estimate

$$\mathcal{M}_t = \max_{|s-t|\le|t|} |h_0(s)|$$

for $t \in (0,t_0)$ from (160) written as

$$h_0^2(t) = \int_0^t B_3(s)h_0(s)ds + t^2B_3(t)$$

where $B_{3,4}$ are analytic in Δ and continuous on $\overline{\Delta}$. Then

$$\mathcal{M}_t^2 \leq |t|\mathcal{M}_t \max_{\overline{\Delta}} |B_3| + |t^2| \max_{\overline{\Delta}} |B_4|$$

and thus $\mathcal{M}_t \leq K_1|t|$, for some $K_1 > 0$ and all $t \in [0, t_0]$.

(b) Cauchy's formula on the circle $|s - t| = |t|$ implies $|h_0'(t)| \leq K_1$.

(c) Equation (160) written in the form $t/h_0 = (2h_0' - B_1)B_2^{-1}$ implies now $|t/h_0| \leq K_2^{-1} < \infty$. In conclusion,

$$\left|\frac{t}{h_0}\right| \in [K_2, K_1] \text{ for all } t \in [0, t_0]$$

To conclude the proof of the statement at the beginning of this part, the function $y = h_0/t$, which is analytic in Δ and continuous on $\overline{\Delta}\backslash\{0\}$ satisfies the equation

$$2ty' = -2y + B_1 + B_2/y$$

which can be written as

$$\frac{1}{2t} = \frac{y'}{P_0(y) + t\,f(t, y)} \tag{161}$$

with $P_0(y) = -2y + B_1(0) + B_2(0)/y$. Assume, to get a contradiction, that for some $\epsilon > 0$ we had $|P_0(y(t))| > \epsilon$ for $t \in (0, t_1] \subset (0, t_0]$. Since

$$\frac{1}{P_0(y) + t\,f(t, y)} = \frac{1}{P_0(y)} + t\,f_1(t, y) \tag{162}$$

with $f_1(t, y)$ bounded on $(0, t_1]$ if t_1 is small, we get by integrating (161) on $[t, t_1] \subset (0, t_1]$

$$\frac{1}{2}\ln(t/t_1) = F_0(y(t)) - F_0(y(t_1)) + \int_{t_1}^{t} sy'(s)\,f_1(s, y(s))ds$$

where F_0 is a primitive of $1/P_0$, and thus $F_0(y(t))$ is bounded for $t \in (0, t_1)$. Hence the r.h.s. of (162) is uniformly bounded for $t \in (0, t_1)$, which is a contradiction, given the l.h.s. of (162).

7. With $\delta(t)$ defined by $h_0(t) = Lt(1 + \delta(t))$ we have $\delta(t) \to 0$ as $t \to 0$, $t \in (0, t_0]$. We show this by a contraction argument. δ satisfies the equation

$$t\delta' = \left(-2 + \frac{B_1(0)}{2L}\right)\delta + B(t, \delta) \tag{163}$$

where

$$B(t, \delta) = \left(1 - \frac{B_1(0)}{2L}\right)\frac{\delta^2}{1 + \delta} + t\left[\frac{B_1(t) - B_1(0)}{2Lt} + \frac{B_2(t) - B_2(0)}{2L^2t(1 + \delta)}\right]$$

or in integral form,

$$\delta(t) = (J\delta)(t) = \left(\frac{t}{t_n}\right)^{b_1} \delta(t_n) + t^{b_1} \int_{t_n}^{t} s^{-b_1-1} B(s, \delta(s)) ds \tag{164}$$

where $\{t_n\}_n$ is the sequence found in **6**, such that $\delta(t_n) \to 0$. We have two cases, according to whether $\Re(b_1)$ is positive or negative ($\Re(b_1) = 0$ is nongeneric).

(a) In the case $\Re(b_1) > 0$ we define J on the space $\mathcal{A}$ of analytic functions in B_n and continuous in $\overline{B_n}$, where B_n is the ball having $[0, t_n]$ as a diameter. Let $\mathcal{A}_r = \{\delta \in \mathcal{A} : \|\delta\|_\infty \le r\}$ and $r_n = 2|\delta(t_n)|$.

(b) For $\Re(b_1) < 0$ we instead define J on the space $\mathcal{A}'$ of analytic functions in B'_n and continuous in $\overline{B'_n}$, where B'_n is the ball having $[t_n, t_{n-1}]$ as a diameter. Let $\mathcal{A}'_r = \{\delta \in \mathcal{A}' : \|\delta\|_\infty \le r\}$ and $r'_n = 2|\delta(t_n)|+2(t_{n-1}-t_n)$.

Lemma 23 *(a) If $\Re(b_1) > 0$, for n large enough, $J : \mathcal{A}_{r_n} \mapsto \mathcal{A}_{r_n}$ is contractive. Therefore, as n is large, we have $|\delta(t)| \le 2|\delta(t_n)|$ on $[0, t_n]$ so that $\delta(t) \to 0$ as $t \to 0$ in $(0, t_0]$.*

(b) If $\Re(b_1) < 0$, for n large enough, $J : \mathcal{A}'_{r'_n} \mapsto \mathcal{A}'_{r'_n}$ is contractive. Therefore, as n is large, we have $|\delta(t)| \le 2|\delta(t_n)| + t_{n-1} - t_n$ on $[t_n, t_{n-1}]$ so that, again, $\delta(t) \to 0$ as $t \to 0$ in $(0, t_0]$.

Proof. A straightforward calculation.

8. Now we bootstrap the information that $\delta(t) \to 0$ to sharpen the characterization of $\delta(t)$ for small t.

Lemma 24 *(a) If $\Re(b_1) < 0$ then $\delta(t)$ is analytic in t at zero, and $\delta(0) = 0$, thus $\delta(t) = O(t)$.*

(b) If $\Re(b_1) > 0$ then $\delta(t) = O(t^{b_1}) + O(t)$ for small t.

Proof. (a) Since $\delta(t) \to 0$ we have

$$\delta(t) = \int_0^1 \big(tzb_3(\delta(tz), tz) + b_2\delta^2(zt)\big) z^{-b_1-1} \mathrm{d}z \tag{165}$$

which for small t is manifestly contractive in a small sup ball in a space of analytic functions in a neighborhood of $t = 0$.

(b) Fixing some n, from (164) and Lemma 23 we see that

$$|\delta(t)| \le \frac{|\delta(t_n)|}{t_n} t^{b_1} + (t_n - t)t\|B\|_\infty$$

$\square$

(In fact it is not difficult to show that in case (b), δ can be written as an analytic function in the two variables t and t^{b_1}.)

With $u = \frac{dh_0}{dA_2}$ we get the equation in variations

$$uh'_0 + h_0u' = \big(2\lambda_2\xi^{-1} + a_2\big)u + 1 \tag{166}$$

whence u is analytic as long as $h_0 \neq 0$. This equation being linear we can use classical Frobenius theory and, in the only interesting case $\Re(b_1) > 0$ we have $u(\xi_0) = (A_{1,2} - 2\lambda_2\xi_0^{-1} - a_2)^{-1} \neq 0$. Thus an arbitrarily small variation of A_2 makes $h_0(\xi_0) \neq 0$.

6.9. The recursive system for $\mathbf{F}_m$

In applications it is usually more convenient to determine the functions $\mathbf{F}_m$ recursively, from their differential equation. Formally the calculation is the following.

The series $\tilde{\mathbf{F}} = \sum_{m\geq 0} x^{-m}\mathbf{F}_m(\xi)$ is a formal solution of (2); substitution in the equation and identification of coefficients of x^{-m} yields the recursive system (56), (57). To determine the $\mathbf{F}_m$'s associated to $\mathbf{y}$ we first note that these functions are analytic at $\xi = 0$ (cf. Theorem 1). Denoting by $F_{m,j},\ j = 1, .., n$ the components of $\mathbf{F}_m$, a simple calculation shows that (56) has a unique analytic solution satisfying $F_{0,1}(\xi) = \xi + O(\xi^2)$ and $F_{0,j}(\xi) = O(\xi^2)$ for $j = 2, \dots, n$. For $m = 1$, there is a one parameter family of solutions of (57) having a Taylor series at $\xi = 0$, and they have the form $F_{1,1}(\xi) = c_1\xi + O(\xi^2)$ and $F_{1,j}(\xi) = O(\xi^2)$ for $j = 2, \dots, n$. The parameter c_1 is determined from the condition that (57) has an analytic solution for $m = 2$. For this value of c_1 there is a one parameter family of solutions $\mathbf{F}_2$ analytic at $\xi = 0$ and this new parameter is determined by analyzing the equation of $\mathbf{F}_3$. The procedure can be continued to any order in m, in the same way; in particular, the constant c_m is only determined at step $m + 1$ from the condition of analyticity of $\mathbf{F}_{m+1}$.

6.10. Sketch of a classical proof of Theorem 1

It is also interesting to mention a direct, classical proof (i.e. not involving results of exponential asymptotics) of Theorem 1. (Since we do not rely on this more involved approach, we only give a brief outline of this proof.)

Having determined the initial conditions for $\mathbf{F}_m$ as above, equations (56), (57) can be transformed to integral equations possessing unique analytic solutions $\mathbf{F}_m$ for small ξ.

To show (20) let $\mathbf{y}(x)$ be a solution of (2) such that $\mathbf{y}(x) \to 0$ in S_{trans}. Denote

$$\mathbf{R}_N(x) = x^{N+1}\left(\mathbf{y}(x) - \sum_{m=0}^{N} x^{-m}\mathbf{F}_m(\xi(x))\right)$$

(the remainder of $\mathbf{y}(x)$ with respect to the truncated expansion). Then $\mathbf{R}_N$ satisfies the differential equation

$$\frac{d\mathbf{R}_N}{dx} + \left[\hat{\Lambda} + x^{-1}(A - (N+1)I)\right]\mathbf{R}_N = \mathbf{E}_N(x, \mathbf{R}_N) \qquad (167)$$

where

$$\mathbf{E}_N(x,\mathbf{R}_N) = d_y g(0,\mathbf{F}_0)\mathbf{R}_N + \mathcal{F}(x)\mathbf{R}_N + \mathbf{u}_N(x) + \mathbf{E}^{[1]}{}_N(x,\mathbf{R}_N) \quad (168)$$

with $\mathbf{E}^{[1]}{}_N(x,\mathbf{R}_N) = O(\mathbf{R}_N^2)$.

Let $R_{N,j},\ j = 1,..n$ denote the components of $\mathbf{R}_N$ and $E_{N,j}$ be the components of $\mathbf{E}_N$.

Equation (167) can be written in the integral form

$$\mathbf{R}_{N,j}(x) = e^{-\lambda_j x} x^{N+1-\alpha_j} \int_{x_j^0}^{x} e^{\lambda_j s} s^{-N-1+\alpha_j} E_N(s,\mathbf{R}_N(s))\, ds \qquad (169)$$

(for $j = 1, \ldots, n$). For an appropriate (rather delicate) choice of the initial points x_j^0 and of the contours of integration in (169) the integral operators defined by the r.h.s. of (169) are contractive for N sufficiently large, hence (169) has a unique analytic solution $\mathbf{R}_N$.

Acknowledgements. We are grateful to Martin Kruskal for so many interesting discussions and to Sadjia Chettab for carefully examining the contents of our manuscript. The work of O.C. was partially supported by NSF grant 9704968 and of R.D.C. by NSF grant 0074924.

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