SHORT TIME EXISTENCE AND BOREL SUMMABILITY IN THE NAVIER-STOKES EQUATION IN $\mathbb{R}^3$

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Abstract. We consider the Navier-Stokes initial value problem,

$$v_t - \Delta v = -\mathcal{P}\left[v \cdot \nabla v\right] + f \ , \ v(x,0) = v_0(x), \ x \in \mathbb{R}^3$$

where $\mathcal{P}$ is the Hodge-Projection to divergence free vector fields in the assumption that $\|f\|_{\mu,\beta} < \infty$ and $\|v_0\|_{\mu+2,\beta} < \infty$ for $\beta \geq 0, \mu > 3$, where

$$\|\hat{f}(k)\|_{\mu,\beta} = \sup_{k \in \mathbb{R}^3} \ e^{\beta|k|}(1+|k|)^{\mu}|\hat{f}(k)|$$

and $\hat{f}(k) = \mathcal{F}[f(\cdot)](k)$ is the Fourier transform in x.

By Borel summation methods we show that there exists a classical solution in the form

$$v(x,t) = v_0 + \int_0^\infty e^{-p/t} U(x,p) dp$$

$t \in \mathbb{C}$, $\mathrm{Re}\frac{1}{t} > \alpha$, and we estimate α in terms of $\|\hat{v}_0\|_{\mu+2,\beta}$ and $\|\hat{f}\|_{\mu,\beta}$. We show that $\|\hat{v}(\cdot;t)\|_{\mu+2,\beta} < \infty$. Existence and t-analyticity results are analogous to Sobolev spaces ones.

An important feature of the present approach is that continuation of v beyond $t = \alpha^{-1}$ becomes a growth rate question of $U(\cdot,p)$ as $p \to \infty$, U being is a known function. For now, our estimate is likely suboptimal.

A second result is that we show Borel summability of v for v_0 and f analytic. In particular, Borel summability implies a the Gevrey-1 asymptotics result: $v \sim v_0 + \sum_{m=1}^{\infty} v_m t^m$, where $|v_m| \leq m! A_0 B_0^m$, with A_0 and B_0 are given in terms of to v_0 and f and for small t, with $m(t) = \lfloor B_0^{-1} t^{-1} \rfloor$,

$$\left| v(x,t) - v_0(x) - \sum_{m=1}^{m(t)} v_m(x) t^m \right| \leq A_0 \ m(t)^{1/2} \ e^{-m(t)}$$

1. Introduction and main results

We consider the Navier-Stokes (NS) initial value problem

$$v_t - \Delta v = -\mathcal{P}[v \cdot \nabla v] + f(x) \ , \ v(x,0) = v_0(x), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}^+ \tag{1.1}$$

where v is the fluid velocity and $\mathcal{P} = I - \nabla \Delta^{-1}(\nabla \cdot)$ is the Hodge-Projection operator to the space of divergence free vector fields. We rescale v, x and t so that the viscosity is one. The initial condition v_0 and the forcing $f(x)$ are chosen to be divergence free. We assume f to be time-independent for simplicity, but a time dependent f could be treated similarly. Moreocver, from the analysis presented here, it will be clear that similar results can be obtained for the corresponding periodic problem, *i.e.* $v(.,t) \in \mathbb{T}^3$.

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We first write the equation in the Fourier space. We denote by $\mathcal{F}$ or simply $\hat{}$ the Fourier transform and $\hat{*}$ is the Fourier convolution. Since $\nabla \cdot v = 0$ we get

$$\hat{v}_t + |k|^2 \hat{v} = -ik_j P_k \left[\hat{v}_j \hat{*} \hat{v} \right] + \hat{f} \ , \ \ \hat{v}(k,0) = \hat{v}_0, \tag{1.2}$$

where as usual a repeated index j denotes summation over j $(= 1, 2, 3)$. If $P_k = \mathcal{F}(\mathcal{P})$ we get

$$P_k \equiv \left(1 - \frac{k(k \cdot)}{|k|^2} \right), \tag{1.3}$$

Definition 1.1. *We introduce the norm* $\| \cdot \|_{\mu,\beta}$ *by*

$$\| \hat{v}_0 \|_{\mu,\beta} = \sup_{k \in \mathbb{R}^3} (1 + |k|)^{\mu} e^{\beta |k|} |\hat{v}_0(k)| \ , \ \text{ where } \hat{v}_0(k) = \mathcal{F}[v_0(\cdot)](k), \tag{1.4}$$

We assume $\| \hat{v}_0 \|_{2+\mu,\beta} < \infty$, $\| \hat{f} \|_{\mu,\beta} < \infty$ for some $\beta \geq 0$ and $\mu > 3$. Clearly, if $\beta > 0$, then v_0 and f are analytic in a strip of width at least β.

There is considerable mathematical literature for Navier-Stokes equation, starting with Leray's papers in the 1930s [18], [19], [20]. Global existence and uniqueness are known in 2d (see for instance [4] and reference therein). However, this is not the case in 3d. It is not known whether classical solutions exist globally in time for arbitrary sized smooth or even analytic initial data. While weak solutions in the space of distributions are known to exist since Leray, it is not known if they are unique or not without additional assumptions. Only local existence and uniqueness of classical solutions is known, with a time of existence inversely proportional to a Sobolev norm of v_0. There are sufficient conditions that guarantee existence for all times [3], [8], but of course it is unknown whether they are satisfied. The solution, as long as it exists, is known to be analytic in part of the right half complex t-plane [21], [17], [12]. If space-periodic conditions are imposed, for $v_0 \in H^1(\mathbb{T}^3)$, and f analytic, then the solution v becomes analytic in space as well [13], [10].

The purpose of this paper is twofold. One is to introduce Borel transform techniques (the notions are explained in the sequel) in time for nonlinear evolution PDEs. The Borel transform in $1/t$, $\hat{U} = \mathcal{B}\hat{v}$ solves an integral equation (see (2.18) below); the Laplace transform of $\hat{U}$ is a classical solution of (1.2). The integral equation (2.18) is shown to have a unique solution in an exponentially weighted space, $L^1(dp\, e^{-\alpha p})$ for some $\alpha > 0$. An important advantage of this formulation is that existence in t of the evolution PDE is transformed into finding the large p-asymptotics of a known solution to an integral equation (finding α). In this paper we do not obtain the optimal value of α, but only a rough bound which implies existence for $t < \alpha^{-1}$. The question of asymptotic estimates is addressed in a more general setting in [14].

Moreover, given a specific initial condition, the p-space integral equation provides a basis for numerical investigation of global existence since there is no blow-up in p. Furthermore, the solution has to be evaluated for small and moderate values of p, since for large p the equation is naturally contractive. This will be the subject of a different paper.

A second purpose is to show Borel summability of the formal power series in small t of NS, when initial v_0 and f are analytic. This corresponds to $\beta > 0$ in the norm defined in 1.1. Borel summability is a subject of interest in its own right and has been the subject of much investigation in the context of nonlinear ODEs [11], [6], difference equations [5] as well as some particular type of PDEs [22], [23]; there are no results applying to initial value problems for nonlinear PDEs, as considered in this paper. Borel summability implies in particular that the formal expansion in powers of t,

$$\tilde{v}(x,t) = v_0(x) + tv_1(x) + \cdots$$

where v_j can be found algorithmically, is actually Gevrey-1 asymptotic to v. Borel summability also implies that $\|v_m\|_\infty \le m! A_0 B_0^m$, where A_0 and B_0 are determined by v_0 and f.

Borel summability methods have been used by the authors [22] to prove complex sectorial existence of solutions of a rather general class of nonlinear PDEs in $\mathbb{C}^d$ for arbitrary d. This is in some sense a generalization of the classical Cauchy-Kowalewski theorem to PDEs written as systems that are first order in time and higher order in space[(1)]

The main results in this paper are given by the following two theorems. The results in the first theorem are similar to classical ones, with $\|\cdot\|_{\mu,\beta}$ replacing Sobolev norms.

Theorem 1.1. *If $\|\hat{v}_0\|_{\mu+2,\beta} < \infty$, $\mu > 3$, $\beta \ge 0$, NS has a unique solution $v(\cdot,t)$ such that $\|\hat{v}(\cdot,t)\|_{\mu,\beta} < \infty$ for $\mathrm{Re}\frac{1}{t} > \alpha$. Here α depends on $\hat{v}_0$ through (2.39).*

Furthermore, $\hat{v}(\cdot,t)$ is analytic for $\mathrm{Re}\ \frac{1}{t} > \alpha$ and $\|\hat{v}(\cdot,t)\|_{\mu+2,\beta} < \infty$ for $t \in [0,\alpha^{-1})$. If $\beta > 0$, this implies that v is analytic in x with the same analyticity width as v_0 and f.

Remark 1.2. The main significance of Theorem 1.1 is not in the local existence results, which are classical. Rather, it stems from the relation of v to the the solution $\hat{U}$ of an integral equtation whose solution in known *a priori*. If refined asymptotics of $\hat{U}(k,p)$ as $p \to \infty$ show subexponential growth in p, then global existence follows. In [14], we discuss approaches to this asymptotic problem and using more general Borel transform methods. It is also shown in [14] that numerical calculation of $\hat{U}$ can be done with rigorously controlled errors unlike classical numerical techniques (in NS). Hence the integral equation approach has a useful computational aspect as well.

Remark 1.3. Sobolev space methods give local existence of solutions in H^m for $t \in [0,T)$, where T is proportional to $1/\|v_0\|_{H^m}$. In particular, for $m > \frac{7}{2}$, these solutions are classical solutions (the second derivatives are continuous). The result in Theorem 1.1 is similar, but in a different space. The existence time, $t = \alpha^{-1}$, involves $\|\hat{v}_0\|_{j+\mu}$ for $j = 0,1,2$ (see (2.39)). This solution is classical since $\|\hat{v}(.,t)\|_{\mu+2,\beta} < \infty$ for $\mu > 3$ implies $v(.,t) \in C^2(\mathbb{R}^3)$. Other norms, requiring (and providing) lower regularity can be used as well, see [14].

Remark 1.4. If v_0 has finite suitable Sobolev norms, it was known that v is analytic in t in a region in the right half complex t plane. In our setting, v is analytic in $\{t : \mathrm{Re}\ \frac{1}{t} > \alpha\}$ if $\|\hat{v}_0\|_{\mu+2,\beta} < \infty$.

Remark 1.5. Previous results [13] show that for space-periodic boundary conditions, analytic f and $v_0 \in H^1$, the solution $v(\cdot,t)$ becomes analytic in space, with an analyticity strip improving with time for small time. Moreover, for $f = 0$, a

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[(1)]Also, Cauchy-Kowalewski theorem usually requires a local expansion in all *all* independent variables. Our methods accommodate series type expansion in just one variable.

uniform estimate on the analyticity strip width for large time exists under the hypothesis that the local dissipation $\nu\|\nabla v(.,t)\|^2_{L^2(\mathbb{T}^3)}$ is bounded [10]. However, we are not aware of similar results in $\mathbb{R}^3$, as is the case in this paper. For $\beta > 0$, our results of Theorem 1.1 show that the analyticity width is preserved for $t \in [0, \frac{1}{\alpha})$.

Theorem 1.2. *For $\beta > 0$ (analytic initial data) and $\mu > 3$, the solution v is Borel summable in $1/t$, i.e. there exists $U(x,p)$, analytic in a neighborhood of $\mathbb{R}^+$, exponentially bounded, and analytic in x for $|\text{Im } x| < \beta$ so that*

$$v(x,t) = v_0(x) + \int_0^\infty U(x,p)e^{-p/t}dp$$

Therefore, in particular, as $t \to 0$,

$$v(x,t) \sim v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$$

with

$$|v_m(x)| \le m! A_0 B_0^m,$$

where A_0 and B_0 depend on v_0 and f, through (3.55), (3.57) and (3.58)

Remark 1.6. Borel summability and classical Gevrey-asymptotic results [2] imply for small t that

$$\left| v(x,t) - v_0(x) - \sum_{m=1}^{m(t)} v_m(x) t^m \right| \le A_0 \; m(t)^{1/2} e^{-m(t)}$$

where $m(t) = \lfloor B_0^{-1} t^{-1} \rfloor$. Our bounds on B_0 are likely suboptimal. Formal arguments in the recurrence relation of v_{m+1} in terms of v_m, v_{m-1},...,v_1, indicate that B only depends on β, but not on $\|\hat{v}_0\|_{\mu,\beta}$. Indeed, for periodic boundary conditions, with a finite number of nonzero initial modes, we have proved [15] that the radius of convergence of the series of $\hat{U}$ in powers of p does not depend on the size of the initial data.

Remark 1.7. For $\beta > 0$ the assumption $\mu > 3$ is not restrictive if β is consistent with the analyticity strips of v_0 and f. This is because $(1 + |k|)^\mu e^{-\tilde{\beta}|k|}$ is bounded in k for $\tilde{\beta} > 0$.

2. Formulation of Navier Stokes equation: Borel transform

We define $\hat{w}$ by

$$\hat{v}(k,t) = \hat{v}_0(k) + t\hat{v}_1(k) + \hat{w}(k,t) \; , \tag{2.5}$$

where

$$\hat{v}_1(k) = \left(-|k|^2 \hat{v}_0 - i k_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{v}_0\right]\right) + \hat{f}(k) \tag{2.6}$$

From (1.2) we get for $\hat{w}$

$$\begin{aligned} \hat{w}_t + |k|^2 \hat{w} = -ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{w} + \hat{w}_j \hat{*} \hat{v}_0 + t\hat{v}_{1,j} \hat{*} \hat{w} + t\hat{w}_j \hat{*} \hat{v}_1 + \hat{w}_j \hat{*} \hat{w}\right] \\ - t|k|^2 \hat{v}_1 - ik_j t P_k \left[\hat{v}_{0,j} \hat{*} \hat{v}_1 + \hat{v}_{1,j} \hat{*} \hat{v}_0 + t\hat{v}_{1,j} \hat{*} \hat{v}_1\right] \end{aligned} \tag{2.7}$$

We seek a solution as a Laplace transform

$$\hat{w}(k,t) = \int_0^\infty \hat{W}(k,p) e^{-p/t} dp \tag{2.8}$$

with the property $\lim_{p\to 0^+} \hat{W}(k,p) = 0$ and $\lim_{p\to 0^+} p\hat{W}_p(k,p) = 0$. The Borel transform of (2.7), which is the same as the formal inverse-Laplace transform in $1/t$ gives in the dual variable $p > 0$,

(2.9)
$$p\hat{W}_{pp} + 2\hat{W}_p + |k|^2\hat{W} + ik_jP_k\left[\hat{v}_{0,j}\hat{*}\hat{W} + \hat{W}_j\hat{*}\hat{v}_0 + \hat{v}_{1,j}\hat{*}(1*\hat{W}) + (1*\hat{W}_j)\hat{*}v_1\right]$$
$$+ ik_jP_k\hat{W}_j\overset{*}{*}\hat{W} + |k|^2\hat{v}_1 + ik_jP_k\left[\hat{v}_{0,j}\hat{*}\hat{v}_1 + \hat{v}_{1,j}\hat{*}\hat{v}_0 + p\hat{v}_{1,j}\hat{*}\hat{v}_1\right] = 0,$$

where $\overset{*}{*}$ denotes Laplace convolution in p, followed by Fourier convolution in k.

Since the equation $\mathcal{D}y := [p\partial_p^2 + 2\partial_p + |k|^2]y = 0$ has explicit independent solutions in terms of Bessel functions, $y = J_1(z)/z$ and $y = Y_1(z)/z$, where $z = 2|k|\sqrt{p}$ which do not vanish at zero, we formally obtain from (2.9) by inverting $\mathcal{D}$ the Duhamel formulation

$$\hat{W}(k,p) = \frac{ik_j\pi}{2|k|\sqrt{p}}\int_0^p \mathcal{G}(z,z')\hat{H}^{[j]}(k,p')dp' \ , \text{ where} \tag{2.10}$$
$$\mathcal{G}(z,z') = z'\left(-J_1(z)Y_1(z') + Y_1(z)J_1(z')\right) \ , \ z = 2|k|\sqrt{p} \ , \ z' = 2|k|\sqrt{p'} \ ,$$

and

$$\hat{H}^{[j]} = -P_k\left[\hat{v}_{0,j}\hat{*}\hat{W} + \hat{W}_j\hat{*}\hat{v}_0 + \hat{v}_{1,j}\hat{*}(1*\hat{W}) + (1*\hat{W}_j)\hat{*}v_1\right] \tag{2.11}$$
$$- P_k\left[\hat{W}_j\overset{*}{*}W\right] + ik_j\hat{v}_1 - P_k\left[\hat{v}_{0,j}\hat{*}\hat{v}_1 + \hat{v}_{1,j}\hat{*}\hat{v}_0 + p\hat{v}_{1,j}\hat{*}\hat{v}_1\right]$$

Remark 2.1. $|\mathcal{G}(z,z')|$ is bounded for all real nonnegative $z' \le z$. This follows from standard properties of Bessel functions [1]. (The approximate bound is about 0.6.)

To obtain stronger results with less regularity of v_0, it is convenient to introduce $\hat{U}(k,p)$ by:

$$\hat{W}(k,p) = \hat{U}(k,p) - \hat{v}_1(k) \tag{2.12}$$

Substituting (2.12) into (2.11), we obtain

$$\hat{H}^{[j]}(k,p) = \hat{G}^{[j]}(k,p) + ik_j\hat{v}_1 \ , \text{ where } \hat{G}^{[j]} = -P_k\left[\hat{v}_{0,j}\hat{*}\hat{U} + \hat{U}_j\hat{*}\hat{v}_0 + \hat{U}_j\overset{*}{*}\hat{U}\right] \tag{2.13}$$

We can further simplify the integral $\int_0^p \mathcal{G}(z,z')\hat{H}^{[j]}(k,p')dp'$ by noting that the only solution to

$$\mathcal{D}y = -|k|^2\hat{v}_1, \tag{2.14}$$

satisfying $y(k,0) = 0$, as it is easy to check, is

$$y(k,p) = -\hat{v}_1(k)\left(1 - 2\frac{J_1(z)}{z}\right) \ , \text{ where } z = 2|k|\sqrt{p}, \tag{2.15}$$

where we used the fact that $J_1(z)/z$ is a solution to the associated homogeneous differential equation and that $\lim_{z\to 0} J_1(z)/z = 1/2$. On the other hand, inversion of $\mathcal{D}$ with zero boundary condition at $p = 0$ involves the same kernel $\mathcal{G}(z,z')$. Writing $-|k|^2\hat{v}_1 = ik_j[ik_j\hat{v}_1]$, it follows that

$$y(k,p) = \frac{ik_j\pi}{2|k|\sqrt{p}}\int_0^p \mathcal{G}(z,z')\left[ik_j\hat{v}_1(k)\right]dp' \tag{2.16}$$

Therefore

$$\frac{ik_j\pi}{2k\sqrt{p}}\int_0^p \mathcal{G}(z,z')\left[ik_j\hat{v}_1(\hat{k})\right] = \hat{v}_1(k)\left(2\frac{J_1(z)}{z}-1\right) \tag{2.17}$$

From (2.12),(2.13) and (2.17) we get

(2.18)

$$\hat{U}(k,p) \;=\; \frac{ik_j\pi}{2|k|\sqrt{p}}\int_0^p \mathcal{G}(z,z')\hat{G}^{[j]}(k,p')dp' \,+\, 2\hat{v}_1\frac{J_1\left(2|k|\sqrt{p}\right)}{2|k|\sqrt{p}} \;=:\; \mathcal{N}[\hat{U}](k,p),$$

where $\hat{G}^{[j]}(k,p)$ is given by (2.13).

We will show that $\mathcal{N}$ is contractive in a suitable space, and hence $\hat{U}=\mathcal{N}[\hat{U}]$ has a unique solution. The solution satisfies $\hat{U}(0,k)=\hat{v}_1(k)$, $\hat{U}$ and $\hat{U}_p$ are bounded for $p\in\mathbb{R}^+$ and exponentially bounded at ∞. Then, $\hat{W}(k,p)=\hat{U}(k,p)-\hat{v}_1(k)$ satisfies the integral equation (2.10) and hence the differential equation (2.9) is satisfied, with $\lim_{p\to 0} p\hat{W}_p(k,p)=0$, $\lim_{p\to 0}\hat{W}(k,p)=0$, and $\hat{W}$ and $\hat{W}_p$ are exponentially bounded at ∞. Thus the Laplace transform $\hat{w}(k,t) = \int_0^\infty e^{-p/t}\hat{W}(k,p)dp$ will indeed satisfy (2.7) for sufficiently large Re $\frac{1}{t}$, and because of the continuity of $\hat{W}$ at $p=0$ we have $\lim_{t\to 0^+}\hat{w}(k,t)=0$. Thus,

$$\hat{v}(k,t)=\hat{v}_0+t\hat{v}_1+\int_0^\infty e^{-p/t}\hat{W}(k,p)dp=\hat{v}_0+\int_0^\infty e^{-p/t}\hat{U}(k,p)dp \tag{2.19}$$

solves the NS equation (1.2) in the Fourier space, with the given initial condition. Furthermore, the sufficiently rapid decay in k of $\hat{U}$ implies that $v(x,t)=\mathcal{F}^{-1}[\hat{v}(\cdot,t)](x)$ is indeed a classical solution to (1.1). It is known (See *e.g.* [24]) that classical solutions are unique; thus $\hat{v}$ is the only solution to (1.1).

2.1. Existence of a solution to (2.18).

First, we prove some preliminary lemmas.

Lemma 2.2. *If* $\|\hat{v}\|_{\mu,\beta}$ and $\|\hat{w}\|_{\mu,\beta}<\infty$, *then we have*

$$\|\hat{v}\hat{*}\hat{w}\|_{\mu,\beta}\le C_0\|\hat{v}\|_{\mu,\beta}\|\hat{w}\|_{\mu,\beta} \tag{2.20}$$

where $\hat{*}$ *denotes Fourier convolution,*

$$C_0(\mu)=2^{\mu+2}\int_{k\in\mathbb{R}^3}\frac{1}{(1+|k|)^\mu}dk=\frac{32\pi 2^\mu}{(\mu-1)(\mu-2)(\mu-3)}$$

Proof. From the definition of $\|\cdot\|_{\mu,\beta}$, we get

$$\begin{aligned}|\hat{v}\hat{*}\hat{w}|\le \|\hat{v}\|_{\mu,\beta}\|\hat{w}\|_{\mu,\beta}&\int_{k'\in\mathbb{R}^3}\frac{e^{-\beta(|k'|+|k-k'|)}dk'}{(1+|k'|)^\mu(1+|k-k'|)^\mu}\\ &\le \|\hat{v}\|_{\mu,\beta}\|\hat{w}\|_{\mu,\beta}e^{-\beta|k|}\int_{k'\in\mathbb{R}^3}\frac{dk'}{(1+|k'|)^\mu(1+|k-k'|)^\mu}\end{aligned}$$

For large $|k|$, we break the integral range at $|k'|=|k|/2$. In the inner ball $|k'|<|k|/2$, we have

$$\frac{1}{(1+|k'|)^\mu(1+|k-k'|)^\mu}\le\frac{1}{(1+|k'|)^\mu(1+|k|/2)^\mu}\le\frac{2^\mu}{(1+|k|)^\mu(1+|k'|)^\mu}$$

while, in its complement,

$$\frac{1}{(1+|k'|)^\mu(1+|k-k'|)^\mu}\le\frac{1}{(1+|k|/2)^\mu(1+|k-k'|)^\mu}\le\frac{2^\mu}{(1+|k|)^\mu(1+|k-k'|)^\mu}$$

Using these estimates, we get for $\mu > 3$,

$$\int_{k' \in \mathbb{R}^3} \frac{dk'}{(1+|k'|)^\mu (1+|k-k'|)^\mu} \leq \frac{C_0}{2(1+|k|)^\mu} \tag{2.21}$$

∎

Lemma 2.3.

$$\| P_k \left[\hat{w}_j \hat{*} \hat{v} \right] \|_{\mu,\beta} \leq 2 C_0 \| \hat{w}_j \|_{\mu,\beta} \| \hat{v} \|_{\mu,\beta}$$

Proof. It is easily seen from the representation of P_k in (1.3) that

$$|P_k \hat{g}(k)| \leq 2 |\hat{g}(k)| \tag{2.22}$$

Therefore, using (1.4),

$$\| P_k \hat{g} \|_{\mu,\beta} \leq 2 \| g \|_{\mu,\beta}$$

Using Lemma 2.2, with $g = w_j v$, the proof follows. ∎

Lemma 2.4. *For $C_2 = 2\pi C_0 \sup_{z \in \mathbb{R}^+, 0 \leq z' \leq z} |\mathcal{G}(z,z')|$ (2), with C_0 as defined in Lemma 2.2,*

$$\| \mathcal{N}[\hat{U}](\cdot,p) \|_{\mu,\beta} \leq \frac{C_2}{\sqrt{p}} \int_0^p \left\{ \| \hat{U}(\cdot,p') \|_{\mu,\beta} * \| U(\cdot,p') \|_{\mu,\beta} + \| v_0 \|_{\mu,\beta} \| \hat{U}(\cdot,p') \|_{\mu,\beta} \right\} dp' + \| v_1 \|_{\mu,\beta} \tag{2.23}$$

$$\begin{aligned} & \| \mathcal{N}[\hat{U}^{[1]}](\cdot,p) - \mathcal{N}[\hat{U}^{[2]}](\cdot,p) \|_{\mu,\beta} \\ & \leq \frac{C_2}{\sqrt{p}} \int_0^p \left\{ \left(\| \hat{U}^{[1]}(\cdot,p') \|_{\mu,\beta} + \| \hat{U}^{[2]}(\cdot,p') \|_{\mu,\beta} \right) * \| \hat{U}^{[1]}(\cdot,p') - \hat{U}^{[2]}(\cdot,p') \|_{\mu,\beta} \right. \\ & \qquad \left. + \| v_0 \|_{\mu,\beta} \| \hat{U}^{[1]}(\cdot,p') - \hat{U}^{[2]}(\cdot,p') \|_{\mu,\beta} \right\} dp' \end{aligned} \tag{2.24}$$

Proof. From [1], $|J_1(z)/z| \leq 1/2$ for $z \in \mathbb{R}^+$ and therefore

$$\| 2 \hat{v}_1(k) J_1(z)/z \|_{\mu,\beta} \leq \| \hat{v}_1 \|_{\mu,\beta}$$

From Lemma 2.3, we have

$$|\mathcal{P}_k \left\{ \hat{U}_j \overset{*}{*} \hat{U} \right\} (k,p)| \leq 2 C_0 \| \hat{U}(\cdot,p) \|_{\mu,\beta} * \| \hat{U}(\cdot,p) \|_{\mu,\beta} \frac{e^{-\beta|k|}}{(1+|k|)^\mu}$$

Applying Lemma 2.3, we get

$$\left| \mathcal{P}_k \left\{ \hat{v}_{0_j} \hat{*} \hat{U}(\cdot,p) + \hat{U}_j(\cdot,p) \hat{*} \hat{v}_0 \right\} \right| \leq 4 C_0 \| \hat{v}_0 \|_{\mu,\beta} \| \hat{U}(\cdot,p) \|_{\mu,\beta} \frac{e^{-\beta|k|}}{(1+|k|)^\mu}$$

By Remark 2.1 and the definition of $\mathcal{N}$ in (2.18), it follows that for $C_2 \geq 2\pi C_0 |\mathcal{G}(z,z')|$ (2.23) holds.

The second part of the lemma follows by noting that

$$\hat{U}_j^{[1]} \overset{*}{*} \hat{U}^{[1]} - \hat{U}_j^{[2]} \overset{*}{*} \hat{U}^{[2]} = \hat{U}_j^{[1]} \overset{*}{*} \left(\hat{U}^{[1]} - \hat{U}^{[2]} \right) + \left(\hat{U}_j^{[1]} - \hat{U}_j^{[2]} \right) \overset{*}{*} \hat{U}^{[2]} \tag{2.25}$$

(2) Since sup of $|\mathcal{G}| \approx 0.6$, we get $C_2 \approx \frac{32(1.2)\pi^2 2^\mu}{(\mu-1)(\mu-2)(\mu-3)}$

Applying Lemma 2.3 to (2.25), we obtain

$$\left\| \mathcal{P}_k \left\{ \hat{U}_j^{[1]} {}^*_* \hat{U}^{[1]}(\cdot,p) - \hat{U}_j^{[2]} {}^*_* \hat{U}^{[2]}(\cdot,p) \right\} \right\|_{\mu,\beta}$$
$$\leq 2C_0 \|\hat{U}^{[1]}(\cdot,p)\|_{\mu,\beta} * \|\hat{U}^{[1]}(\cdot,p) - \hat{U}^{[2]}(\cdot,p)\|_{\mu,\beta}$$
$$+ 2C_0 \|\hat{U}^{[2]}(\cdot,p)\|_{\mu,\beta} * \|\hat{U}^{[1]}(\cdot,p) - \hat{U}^{[2]}(\cdot,p)\|_{\mu,\beta},$$

from which (2.24) follows easily. ∎

It is convenient to define a number of different norms for functions of (k,p) on $\mathbb{R}^3 \times (\mathbb{R}^+ \cup \{0\})$

Definition 2.5. *For $\alpha \geq 1$, we define*

$$\|\hat{f}\|^{(\alpha)} = \sup_{p\geq 0} (1+p^2) e^{-\alpha p} |\hat{f}(\cdot,p)|_{\mu,\beta} \tag{2.26}$$

We define $\mathcal{A}^\alpha$ to be the Banach-space of continuous functions of (k,p) for $k \in \mathbb{R}^3$ and $p \in [0,\infty)$ for which $\|.\|^{(\alpha)} < \infty$. It is also convenient to consider the Banach space $\mathcal{A}_1^\alpha$ of locally integrable (L^1_{loc}) functions for $p \in [0,L)$ on $\mathbb{R}^+$, and continuous in $k \in \mathbb{R}^3$ such that

$$\|\hat{f}\|_1^{(\alpha)} = \int_0^L e^{-\alpha p} \|\hat{f}(\cdot,p)\|_{\mu,\beta} dp \ < \infty \ , \tag{2.27}$$

where L is allowed to be finite or ∞. It is also convenient to define $\mathcal{A}_L^\infty$ to be the Banach space of continuous functions of (k,p) on $\mathbb{R}^3 \times [0,L]$ such that

$$\|\hat{f}\|_L^{(\infty)} = \sup_{p\in[0,L]} \|\hat{f}(\cdot,p)\|_{\mu,\beta} < \infty \tag{2.28}$$

Lemma 2.6. *For $\hat{f}, \hat{g} \in \mathcal{A}^\alpha, \mathcal{A}_1^\alpha$ or $\mathcal{A}_L^\infty$, we have the following the following Banach algebra properties:*

$$\|\hat{f} {}^*_* \hat{g}\|^{(\alpha)} \leq M_0 \|\hat{f}\|^{(\alpha)} \|\hat{g}\|^{(\alpha)}, \text{ where } M_0 \approx 3.76\cdots$$

$$\|\hat{f} {}^*_* \hat{g}\|_1^{(\alpha)} \leq \|\hat{f}\|_1^{(\alpha)} \|\hat{g}\|_1^{(\alpha)},$$

$$\|\hat{f} {}^*_* \hat{g}\|_L^{(\infty)} \leq L \|\hat{f}\|_L^{(\infty)} \|\hat{g}\|_L^{(\infty)}$$

Proof. In the following, we take $u(p) = \|\hat{f}(\cdot,p)\|_{\mu,\beta}$ and $v(p) = \|\hat{g}(\cdot,p)\|_{\mu,\beta}$. We observe that

$$\int_0^L u(s)v(p-s)ds \leq e^{\alpha p} \left(\sup_{p\in\mathbb{R}^+} (1+p^2) e^{-\alpha p} u(p) \right) \left(\sup_{p\in\mathbb{R}^+} (1+p^2) e^{-\alpha p} v(p) \right) \times$$
$$\int_0^p \frac{ds}{(1+s^2)[1+(p-s)^2]}$$

The first part of the lemma follows since [22]

$$\int_0^p \frac{ds}{(1+s^2)[1+(p-s)^2]} \leq \frac{M_0}{1+p^2}$$

with $M_0 = 3.76\cdots$. For the second part note that

$$(2.29)\quad \int_0^L e^{-\alpha p}\int_0^p u(s)v(p-s)ds$$
$$= \int_0^L\int_0^p e^{-\alpha s}e^{-\alpha(p-s)}u(s)v(p-s)ds \le \int_0^L e^{-\alpha s}u(s)ds\int_0^L e^{-\alpha\tau}v(\tau)$$

The third part follows from the fact that for $p\in[0,L]$

$$\int_0^p |u(s)||v(p-s)| \le \left\{\sup_{p\in[0,L]}|u(p)|\right\}\left(\sup_{p\in[0,L]}|v(p)|\right)L$$

∎

Lemma 2.7. *On $\mathcal{A}_1^\alpha$, the operator $\mathcal{N}$, defined in (2.18), satisfies the following inequalities, with C_2 defined in Lemma 2.4:*

$$(2.30)\quad \|\mathcal{N}[\hat U]\|_1^{(\alpha)} \le C_2\sqrt{\pi}\alpha^{-1/2}\left\{\left(\|\hat U\|_1^{(\alpha)}\right)^2 + \|\hat v_0\|_{\mu,\beta}\|\hat U\|_1^{(\alpha)}\right\} + \alpha^{-1}\|\hat v_1\|_{\mu,\beta}$$

$$(2.31)\quad \|\mathcal{N}[\hat U^{[1]}]-\mathcal{N}[\hat U^{[2]}]\|_1^{(\alpha)}$$
$$\le C_2\sqrt{\pi}\alpha^{-1/2}\left\{\left(\|\hat U^{[1]}\|_1^{(\alpha)}+\|\hat U^{[2]}\|_1^{(\alpha)}\right)\|\hat U^{[1]}-\hat U^{[2]}\|_1^{(\alpha)} + \|\hat v_0\|_{\mu,\beta}\|\hat U^{[1]}-\hat U^{[2]}\|_1^{(\alpha)}\right\}$$

while in $\mathcal{A}_L^\infty$, we have

$$(2.32)\quad \|\mathcal{N}[\hat U]\|_L^{(\infty)} \le C_2L^{1/2}\left\{L\left(\|\hat U\|_L^{(\infty)}\right)^2 + \|\hat v_0\|_{\mu,\beta}\|\hat U\|_L^{(\infty)}\right\} + \|\hat v_1\|_{\mu,\beta}$$

$$(2.33)\quad \|\mathcal{N}[\hat U^{[1]}]-\mathcal{N}[\hat U^{[2]}]\|_L^{(\infty)}$$
$$\le C_2L^{1/2}\left\{L\left(\|\hat U^{[1]}\|_L^{(\infty)}+\|\hat U^{[2]}\|_L^{(\infty)}\right)\|\hat U^{[1]}-\hat U^{[2]}\|_L^{(\infty)} + \|\hat v_0\|_{\mu,\beta}\|\hat U^{[1]}-\hat U^{[2]}\|_L^{(\infty)}\right\}$$

Proof. For the space $\mathcal{A}_1^\alpha$, for any $L>0$, including $L=\infty$, we note that

$$\int_0^L e^{-\alpha p}\|\hat v_1\|_{\mu,\beta}dp \le \alpha^{-1}\|\hat v_1\|_{\mu,\beta},$$

while

$$\int_0^L p^{-1/2}e^{-\alpha p}dp \le \Gamma\left(\frac{1}{2}\right)\alpha^{-1/2} = \sqrt{\pi}\alpha^{-1/2}$$

Furthermore, we note that for $u(p')\ge 0$ we have

(2.34)
$$\int_0^L e^{-\alpha p}p^{-1/2}\left(\int_0^p u(p')dp'\right) = \int_0^L u(p')e^{-\alpha p'}\left(\int_{p'}^L p^{-1/2}e^{-\alpha(p-p')}dp\right)dp'$$
$$\le \int_0^L e^{-\alpha p'}u(p')\int_0^L s^{-1/2}e^{-\alpha s}dsdp'$$

Therefore, it follows from (2.23) that

(2.35)
$$\int_0^L e^{-\alpha p}\|\mathcal{N}[\hat U](\cdot,p)\|_{\mu,\beta}dp \le C_2\sqrt{\pi}\alpha^{-1/2}\left(\left[\|\hat U\|_1^{(\alpha)}\right]^2 + \|v_0\|_{\mu,\beta}\|\hat U\|_1^{(\alpha)}\right) + \alpha^{-1}\|v_1\|_{\mu,\beta}$$

Furthermore, from (2.24), it follows that

$$\int_0^L \|\mathcal{N}[\hat{U}^{[1]}] - \mathcal{N}[\hat{U}^{[2]}]\|_{\mu,\beta} e^{-\alpha p} dp \\ \le C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ \left(\|\hat{U}^{[1]}\|_1^{(\alpha)} + \|\hat{U}^{[2]}\|_1^{(\alpha)} \right) \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_1^{(\alpha)} \right. \\ \left. + \|\hat{v}_0\|_{\mu,\beta} \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_1^{(\alpha)} \right\}$$

Hence the first part of the lemma follows.
For the second part, we first note that for any $p \in [0, L]$ we have

$$\left| p^{-1/2} \int_0^p u(p') dp' \right| \le \sup_{p \in [0,L]} |u(p)| \sqrt{L} \tag{2.36}$$

We note that

$$\left| \int_0^p y_1(s) y_2(p-s) ds \right| \le L \left(\sup_{p\in[0,L]} |y_1(p)| \right) \left(\sup_{p\in[0,L]} |y_2(p)| \right) \tag{2.37}$$

Taking

$$u(p) = \|\hat{U}(\cdot,p)\|_{\mu,\beta} * \|\hat{U}(\cdot,p)\|_{\mu,\beta} + \|v_0\|_{\mu,\beta} \|\hat{U}(\cdot,p)\|_{\mu,\beta}$$
$$y_1(p) = y_2(p) = \|\hat{U}(\cdot,p)\|_{\mu,\beta}$$

(2.32) follows from (2.23). To bound $\mathcal{N}[\hat{U}^{[1]}] - \mathcal{N}[\hat{U}^{[2]}]$ in $\mathcal{A}_L^\infty$, we take

$$u(p) = \left(\|\hat{U}^{[1]}(\cdot,p)\|_{\mu,\beta} + \|\hat{U}^{[2]}(\cdot,p)\|_{\mu,\beta} \right) * \|\hat{U}^{[1]}(\cdot,p) - \hat{U}^{[2]}(\cdot,p)\|_{\mu,\beta} \\ + \|v_0\|_{\mu,\beta} \|\hat{U}^{[1]}(\cdot,p) - \hat{U}^{[2]}(\cdot,p)\|_{\mu,\beta} \tag{2.38}$$

$$y_1(p) = \left(\|\hat{U}^{[1]}(\cdot,p)\|_{\mu,\beta} + \|\hat{U}^{[2]}(\cdot,p)\|_{\mu,\beta} \right); \; y_2(p) = \|\hat{U}^{[1]}(\cdot,p) - \hat{U}^{[2]}(\cdot,p)\|_{\mu,\beta}$$

in (2.36) and (2.37). The proof now follows from (2.24). ∎

Lemma 2.8. *Equation (2.18) has a unique solution in $\mathcal{A}_1^\alpha$ for any $L > 0$ (including $L = \infty$) in a ball of size $2\alpha^{-1}\|\hat{v}_1\|_{\mu,\beta}$, for α large enough to ensure*

$$2C_2 \sqrt{\pi} \alpha^{-1/2} \left(\|v_0\|_{\mu,\beta} + 2\alpha^{-1} \|v_1\|_{\mu,\beta} \right) < 1, \tag{2.39}$$

where $C_2 \approx \frac{32(1.2)\pi^2 2^\mu}{(\mu-1)(\mu-2)(\mu-3)}$ is the same as in Lemma 2.4. Furthermore, this solution belongs to $\mathcal{A}_L^\infty$ for L small enough so that

$$2C_2 L^{1/2} \left(\|v_0\|_{\mu,\beta} + 2L \|v_1\|_{\mu,\beta} \right) < 1, \tag{2.40}$$

In particular, $\lim_{p\to 0} \hat{U}(k,p) = \hat{v}_1(k)$. Also, $\hat{W}(k,p) = \hat{U}(k,p) - \hat{v}_1(k)$ is the unique solution to (2.9) which is zero at $p = 0$.

Proof. The estimates of Lemma 2.7 imply that $\mathcal{N}$ maps a ball of size $2\alpha^{-1}\|v_1\|_{\mu,\beta}$ in $\mathcal{A}_1^\alpha$ back to itself and that $\mathcal{N}$ is contractive in that ball when α satisfies (2.39). From Lemma 2.7 in space $\mathcal{A}_L^\infty$, it follows that $\mathcal{N}$ maps a ball of size $2\|v_1\|_{\mu,\beta}$ to itself and that $\mathcal{N}$ is also contractive in this ball if L is small enough to ensure (2.40). Thus, there is a unique solution in this ball. Since $\mathcal{A}_L^\infty \subset \mathcal{A}_1^\alpha$, it follows that the solutions are in fact the same.

Using Lemma 2.7, with $\hat{U}^{[1]} = \hat{U}$ and $\hat{U}^{[2]} = 0$, we obtain from (2.18),

$$\left\| \hat{U}(k,p) - \hat{v}_1(k) \frac{2J_1(z)}{z} \right\|_L^{(\infty)} \le C_2 L^{1/2} \left(L \left[\|\hat{U}\|_L^{(\infty)} \right]^2 + \|\hat{v}_0\|_{\mu,\beta} \|\hat{U}\|_L^{(\infty)} \right)$$

Since $\|\hat{U}\|_L^{(\infty)} < 2\|\hat{v}_1\|_{\mu,\beta}$, it follows that as $L \to 0$,

$$\|\hat{U}(k,p) - 2\hat{v}_1(k) J_1(z)/z\|_L^{(\infty)} \to 0$$

Since $\lim_{z\to 0} 2J_1(z)/z = 1$, it follows that for fixed k, $\lim_{p\to 0} \hat{U}(k,p) = \hat{v}_1(k)$. By construction, $\hat{U}$ satisfies (2.18) iff $\hat{W} = \hat{U} - \hat{v}_1$ satisfies (2.10). From the properties of $\mathcal{G}$ and $\hat{H}^{[j]}$, it follows that $\hat{W}$ will indeed satisfy (2.9) and that it is the only solution which is zero at $p = 0$. ▮

Proposition 2.9. *If α is large enough so that (2.39) holds, then for an absolute constant $C_3 > 0$, the solution $\hat{U}(k,p)$ in Lemma 2.8 and its p-derivative satisfy*

$$|\hat{U}(k,p)| \le \frac{2e^{-\beta|k|+\alpha p}\|\hat{v}_1\|_{\mu,\beta}}{(1+|k|)^\mu}$$

$$|\hat{U}_p(k,p)| \le \frac{C_3 e^{-\beta|k|}\|\hat{v}_1\|_{\mu,\beta}}{(1+|k|)^\mu} \left\{ \frac{\sqrt{\alpha}}{C_2} |k| e^{\alpha p} + |k|^2 \right\}$$

In particular, $\hat{U} \in \mathcal{A}^{\alpha'}$ for any $\alpha' > \alpha$, and

$$|\hat{U}(k,p)| \le \left(\sup_{p\in\mathbb{R}^+} (1+p^2) e^{-(\alpha'-\alpha)p} \right) \frac{2e^{-\beta|k|+\alpha' p}\|\hat{v}_1\|_{\mu,\beta}}{(1+p^2)(1+|k|)^\mu}$$

Proof. With $L = L_0 = \alpha^{-1}$, then (2.40) holds, and therefore $\hat{U} \in \mathcal{A}^\infty_{L_0}$. For $p \in [0, L_0]$, we obtain

$$e^{-\alpha p}\|\hat{U}(\cdot,p)\|_{\mu,\beta} < 2e^{-\alpha p}\|\hat{v}_1\|_{\mu,\beta} \tag{2.41}$$

We now consider $p \in [L_0, \infty)$. We define

$$y(p) = \|\hat{U}(\cdot,p)\|_{\mu,\beta} * \|\hat{U}(\cdot,p)\|_{\mu,\beta} + \|\hat{v}_0\|_{\mu,\beta}\|\hat{U}(\cdot,p)\|_{\mu,\beta}$$

We note that

$$\left| \frac{1}{\sqrt{p}} e^{-\alpha p} \int_0^p y(p')dp' \right| \le L_0^{-1/2} \left| \int_0^p e^{-\alpha p'} y(p')dp' \right| \le \alpha^{1/2} \|y\|_1^{(\alpha)} \tag{2.42}$$

From (2.18) and (2.39), it follows that for $p \in [L_0, \infty)$

(2.43)

$$|\hat{U}(k,p)| \le \frac{e^{-\beta|k|+\alpha p}}{(1+|k|)^\mu} \left\{ C_2 \alpha^{1/2} \left(\|\hat{U}\|_1^{(\alpha)} \right)^2 + C_2 \alpha^{1/2} \|\hat{v}_0\|_{\mu,\beta} \|\hat{U}\|_1^{(\alpha)} + e^{-\alpha p}\|\hat{v}_1\|_{\mu,\beta} \right\}$$
$$\le \frac{2e^{-\beta|k|+\alpha p}}{(1+|k|)^\mu} \|\hat{v}_1\|_{\mu,\beta}$$

By (2.41), (2.43) holds for $p \in [0, L_0]$ as well; hence the bound for $|\hat{U}|$ follows. For $\alpha' > \alpha$, $\|\hat{U}\|^{(\alpha')} < \infty$ because $e^{-(\alpha'-\alpha)p}(1+p^2)$ is bounded if $\alpha' > \alpha$.

Since $\hat{U}$ is a solution to (2.18), differentiation with respect to p implies that

$$\hat{U}_p(k,p) = \hat{v}_1(k) \left(\frac{J_1(z)}{z} \right)' \frac{4|k|^2}{z} + \frac{ik_j\pi}{p} \int_0^p \left\{ \mathcal{G}_z(z,z') - \frac{\mathcal{G}(z,z')}{z} \right\} \hat{G}^{[j]}(k,p')dp'$$

Since the functions $\mathcal{G}_z(z,z')$, $\mathcal{G}(z,z')/z$ and $z^{-1}\left(J_1(z)/z\right)'$ are easily checked to be bounded for $z \ge z' \in \mathbb{R}^+$, there exists $C_3 > 0$, independent of any parameter, so that

$$|\hat{U}_p(k,p)| \le \frac{C_3|k|}{p}\left|\int_0^p |\hat{G}|(k,p')|dp' + C_3|k|^2|\hat{v}_1(k)\right| \le \frac{C_3|k|e^{-\beta|k|}}{(1+|k|)^\mu}$$
$$\times \left[\frac{1}{p}\int_0^p \left(\|U(\cdot,p')\|_{\mu,\beta} * \|U(\cdot,p')\|_{\mu,\beta} + \|\hat{v}_1\|_{\mu,\beta}\|U(\cdot,p')\|_{\mu,\beta}\right) dp' + |k|\|\hat{v}_1\|_{\mu,\beta}\right]$$

For $p \in [0, L_0]$, with $L = L_0 = \frac{1}{\alpha}$ satisfying (2.40), we have

$$|\hat{U}_p(k,p)| \le \frac{C_3|k|e^{-\beta|k|}}{(1+|k|)^\mu}\left[\left\{L_0\left(\|U\|_{L_0}^{(\infty)}\right)^2 + \|\hat{v}_0|_{\mu,\beta}\|U\|_{L_0}^{(\infty)}\right\} + |k|\|\hat{v}_1\|_{\mu,\beta}\right]$$
$$\le \frac{C_3e^{-\beta|k|}}{(1+|k|)^\mu}\left(\frac{\sqrt{\alpha}}{C_2}|k| + |k|^2\right)\|\hat{v}_1\|_{\mu,\beta}$$

For $p \in [L_0, \infty)$ and α satisfying (2.39), we have

$$|\hat{U}_p(k,p)| \le \frac{C_3|k|e^{-\beta|k|+\alpha p}}{L_0(1+|k|)^\mu}\left[\left\{\left(\|U\|_1^{(\alpha)}\right)^2 + \|\hat{v}_0|_{\mu,\beta}\|U\|_1^{(\alpha)}\right\} + |k|L_0e^{-\alpha p}\|\hat{v}_1\|_{\mu,\beta}\right]$$
$$\le \frac{C_3e^{-\beta|k|}\|\hat{v}_1\|_{\mu,\beta}}{(1+|k|)^\mu}\left\{\frac{\sqrt{\alpha}}{C_2}|k|e^{\alpha p} + |k|^2\right\}$$

Continuity of $\hat{U}$ in p follows from the boundedness of $\hat{U}_p$ for $p \in \mathbb{R}^+$ for fixed k. ∎

Lemma 2.10. *Let $\|\hat{v}_0\|_{\mu+2,\beta} < \infty$ and $\|\hat{f}\|_{\mu,\beta} < \infty$, with $\mu > 3$, $\beta \ge 0$. Then NS has a unique solution with $\|\hat{v}(\cdot,t)\|_{\mu,\beta} < \infty$ and $\hat{v}(\cdot,t)$ analytic in t for $\mathrm{Re}\frac{1}{t} > \alpha$, where α depends on the initial data (see (2.39)). For $\beta > 0$, this implies v is analytic in x in the same analyticity strip as v_0, f.*

Proof. From (2.6) we see that $\|\hat{v}_1\|_{\mu,\beta} < \infty$, since

$$\|\hat{v}_1\|_{\mu,\beta} \le \|\hat{v}_0\|_{\mu+2,\beta} + 2C_0\|\hat{v}_0\|_{\mu,\beta}\|\hat{v}_0\|_{\mu+1,\beta} + \|\hat{f}\|_{\mu,\beta} \tag{2.44}$$

Therefore, when α is large enough to ensure (2.39), it follows that $\hat{U}(k,\cdot)$ and $\hat{W}(k,.) \equiv \hat{U}(k\cdot,.) - \hat{v}_1(k)$ are in $L^1(e^{-\alpha p}dp)$. From Lemma 2.8, it follows that $\lim_{p\to 0}\hat{W}(k,p) = 0$ and Proposition 2.9 implies $\hat{W}_p(k,p)$ (same as $\hat{U}_p(k,p)$) is bounded for $p \in \mathbb{R}^+$ and hence $\lim_{p\to 0^+} p\hat{W}_p = 0$. Since $\hat{U}$ satisfies (2.18), it follows that $\hat{W}$ will satisfy (2.10) and hence (2.9). For $\mathrm{Re}\, t^{-1} > \alpha$, we take the Laplace transform of (2.9) in p, using the fact $\partial_p[p\hat{W}]$ and $p\hat{W}$ vanish at $p = 0$. There is no contribution at ∞ because of boundedness of $e^{-\alpha p}\left(|\hat{W}| + \hat{W}_p\right)$ which follows from Proposition 2.9. It can be checked that $\hat{w}(k,t) = \int_0^\infty \hat{W}(k,p)e^{-p/t}dp$ satisfies (2.7). Therefore,

$$\hat{v}(k,t) = \hat{v}_0 + t\hat{v}_1 + \int_0^\infty \hat{W}(k,p)e^{-p/t}dp = \hat{v}_0 + \int_0^\infty \hat{U}(k,p)e^{-p/t}dp$$

satisfies NS in Fourier space. Since $\|\hat{U}(\cdot,p)\|_{\mu,\beta} < \infty$, it follows that $\|\hat{v}(\cdot,t)\|_{\mu,\beta} < \infty$ if $\mathrm{Re}\ \frac{1}{t} > \alpha$. ∎

Proposition 2.11 (Bounds on $\|\hat{v}(.,t)\|_{\mu+2,\beta}$). *For the solution $\hat{v}(k,t)$ given in Lemma 2.10 for $t \in [0,\alpha^{-1}]$, we have*

$$\sup_{t\leq T} \|\hat{v}(\cdot,t)\|_{\mu+2,\beta} < C\left(\|\hat{v}_0\|_{\mu+2,\beta}, T\right) < \infty$$

Proof. We note from (1.2) that if we define $V = \nabla v$, then $\hat{V} = \mathcal{F}[V] = ik\hat{v}$ satisfies,

$$\hat{V}_t + |k|^2 \hat{V} = -ik\mathcal{P}\left[\hat{v}_j \hat{*} \hat{V}^{[j]}\}\right] + ik\hat{f} \ , \ \hat{V}_0(k) = \mathcal{F}\left[\nabla v_0\right] \tag{2.45}$$

where $\hat{V}^{[j]} = ik_j \hat{v}_j$. Therefore,

$$\hat{V}(k,t) = e^{-|k|^2 t}\hat{V}_0(k) - ik\int_0^t e^{-|k|^2(t-\tau)}\left\{\mathcal{P}\left[\hat{v}_j \hat{*} \hat{V}^{[j]}\right](k,\tau)\} - \hat{f}(k)\right\} \tag{2.46}$$

Therefore,

$$|\hat{V}(\cdot,t)| \leq \frac{e^{-\beta|k|}}{(1+|k|)^\mu}\left\{\|\hat{V}_0\|_{\mu,\beta} + |k|\int_0^t e^{-|k|^2(t-\tau)}\left(\|\hat{f}\|_{\mu,\beta} + 2C_0\|\hat{v}(\cdot,\tau)\|_{\mu,\beta}\|\|\hat{V}(\cdot,\tau)\|_{\mu,\beta}\right)d\tau\right\} \tag{2.47}$$

Let $\mathcal{V}_{T_1}$ be the Banach space of continuous functions g of $k \in \mathbb{R}^3$ and $t \in [0,T_1]$ for which

$$\|g\|_{T_1} = \sup_{t\in[0,T_1]} \|g(\cdot,t)\|_{\mu,\beta} < \infty$$

Then, the estimates in (2.47), together with the fact that for any $t \in [0,T]$, $2C_0\|\hat{v}(\cdot,t)\|_{\mu,\beta} \leq \tilde{C}\left(T, \|\hat{v}_0\|_{\mu+2,\beta}\right)$ imply there exists $C_1(T,\|v_0\|_{\mu+2,\beta}) > 0$ so that

$$\|\hat{V}\|_{T_1} \leq C_1\left\{\sqrt{T_1}\|\hat{V}_1\|_{T_1} + \|\hat{V}_0\|_{\mu,\beta} + \sqrt{T_1}\|\hat{f}\|_{\mu,\beta}\right\}, \tag{2.48}$$

where we have used the fact that

$$|k|\int_0^t e^{-|k|^2(t-\tau)}d\tau = \frac{1-e^{-|k|^2 t}}{|k|} \leq \sqrt{T_1}\sup_{\gamma\in\mathbb{R}^+}\frac{1-e^{-\gamma}}{\gamma^{1/2}} \leq C_*\sqrt{T_1},$$

for some $C_* > 0$. Thus, thinking of $\hat{v}$ as given in (2.45), the estimates in (2.48) and similar estimates on $\hat{V}^{[1]} - \hat{V}^{[2]}$ show that for $C_1\sqrt{T_1} < 1$ the right hand side of (2.45) is contractive in $\mathcal{V}_{T_1}$. We choose $T_1 \leq T$. Therefore, $\sup_{t\in[0,T_1]}\|\hat{V}(\cdot,t)\|_{\mu,\beta} < \infty$. Since the choice of T_1 depends on C_1, which is independent of $\|\hat{V}_0\|_{\mu,\beta}$, we can repeat the same argument in another interval $[T_1, 2T_1]$ and so on until we span the whole interval $[0,T]$ over which $\|\hat{v}_1(\cdot,t)\|_{\mu,\beta}$ is uniformly bounded.

We can take additional derivative and repeat the same type argument for $\mathcal{F}[D^2\hat{v}] = -kk\hat{v}$ to show that in $\||k|^2\hat{v}(\cdot,t)\|_{\mu,\beta}$ is also bounded uniformly for $t \in [0,T]$. In this part of the argument, we use the prior knowledge that both $\|\hat{v}(\cdot,t)\|_{\mu,\beta}$ and $\|k\hat{v}(\cdot,t)\|_{\mu,\beta}$ are uniformly bounded in $[0,T]$ and that

$$|k|^2\int_0^t e^{-|k|^2(t-\tau)}\|\hat{f}\|_{\mu,\beta}d\tau = \|\hat{f}\|_{\mu,\beta}\left(1-e^{-|k|^2 t}\right) \leq \|\hat{f}\|_{\mu,\beta}\sup_{\gamma\in\mathbb{R}^+}[1-e^{-\gamma}] \leq C\|\hat{f}\|_{\mu,\beta}$$

Combining all the results, it follows that $\|\hat{v}(\cdot,t)\|_{\mu+2,\beta}$ is bounded for $t \in [0,T]$ ∎

Proof of Theorem 1.1. This follows from Lemma 2.10 and Proposition 2.11, noting that $\|\hat{v}(k,t)\|_{\mu+2,\beta} < \infty$ implies $v(x,t) = \mathcal{F}^{-1}[\hat{v}(\cdot,t)](x) \in C^2(\mathbb{R}^3)$ and so v is a classical solution to (1.1) for $\mathrm{Re}\,\frac{1}{t} > \alpha$, which is known to be unique. From the definition of $\|\cdot\|_{\mu,\beta}$ it follows that $\|\hat{v}_0\|_{\mu+2,\beta} < \infty$ and $\|\hat{f}\|_{\mu,\beta} < \infty$ for $\beta > 0$ imply $\|\hat{v}(.,t)\|_{\mu+2,\beta} < \infty$. Thus v preserves the analyticity strip width for $t \in [0, \frac{1}{\alpha})$.

3. Analyticity of $\hat{U}(k,p)$ at $p = 0$

We now consider the case $\beta > 0$. We note that by Remark 1.7 we can choose $\mu > 3$. The starting point of this section is (2.9), which is satisfied by $\hat{W}(k,p) = \hat{U}(k,p) - \hat{v}_1(k)$. From Lemma 2.8, this is the only solution to (2.9) satisfying $\hat{W}(k,0) = 0$. We seek an potentially alternate solution to (2.9) as a power series,

$$\hat{W}(k,p) = \sum_{l=1}^{\infty} \hat{W}^{[l]}(k) p^l \tag{3.49}$$

Substituting (3.49) into (2.9) and identifying the coefficients of $p^l, l = 0, 1$ we get

$$2\hat{W}^{[1]} = -|k|^2 \hat{v}_1 - ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{v}_1 + \hat{v}_{1,j} \hat{*} \hat{v}_0\right], \tag{3.50}$$

$$6\hat{W}^{[2]} = -k^2 \hat{W}^{[1]} - ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{W}^{[1]} + \hat{W}_j^{[1]} \hat{*} \hat{v}_0 + \hat{v}_{1,j} \hat{*} \hat{v}_1\right] \tag{3.51}$$

It follows from (3.50) and Lemma (2.3) that

$$|\hat{W}^{[1]}(k,p)| \le \frac{e^{-\beta|k|}}{2(1+|k|)^{\mu}} \left(|k|^2 \|v_1\|_{\mu,\beta} + 4C_0 |k| \|v_0\|_{\mu,\beta} \|v_1\|_{\mu,\beta}\right) \tag{3.52}$$

The coefficient of p^l for $l \ge 2$ in (2.9) can be computed as well, using $p^{l_1} * p^{l_2} = p^{l_1+l_2+1} l_1! l_2! / (l_1+l_2+1)!$. Interpreting $\hat{W}^{[0]} = 0$, we get

(3.53)

$$(l+1)(l+2)\hat{W}^{[l+1]} = -k^2 \hat{W}^{[l]} - ik_j P_k \left[\sum_{l_1=1}^{l-2} \frac{l_1!(l-1-l_1)!}{l!} \hat{W}_j^{[l_1]} \hat{*} \hat{W}^{[l-1-l_1]}\right]$$
$$- ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{W}^{[l]} + \hat{W}_j^{[l]} \hat{*} \hat{v}_0 + \frac{1}{l} \hat{v}_{1,j} \hat{*} \hat{W}^{[l-1]} + \frac{1}{l} \hat{W}_j^{[l-1]} \hat{*} \hat{v}_1\right]$$

Definition 3.1. *It is convenient to define the n-th order polynomial Q_n:*

$$Q_n(y) = \sum_{j=0}^{n} 2^{n-j} \frac{y^j}{j!}$$

Lemma 3.2. *If $\|v_0\|_{\mu+2,\beta} < \infty$, for $\mu > 3$, $\beta > 0$, then there exist positive constants $A_0, B_0 > 0$ independent of l and k so that for any $l \ge 1$ we have*

$$|\hat{W}^{[l]}(k)| \le e^{-\beta|k|} A_0 B_0^l (1+|k|)^{-\mu} \frac{Q_{2l}(|\beta k|)}{(2l+1)^2} \tag{3.54}$$

and

$$|W^{[l]}(x)| \le \frac{8\pi A_0 (4B_0)^l}{(2l+1)^2}, \quad |DW^{[l]}(x)| \le \frac{8\pi A_0 (4B_0)^l}{\beta(2l+1)^2}, \quad |D^2 W^{[l]}(x)| \le \frac{16\pi A_0 (4B_0)^l}{\beta^2 (2l+1)^2}$$

Furthermore, the solution in Lemma 2.8, §2 has a convergent series representation in p: $\hat{U}(k,p) = \hat{v}_1(k) + \sum_{l=1}^{\infty} \hat{W}^{[l]} p^l$ for $|p| < (4B_0)^{-1}$.

Remark 3.3. Lemma 3.2 is proved by induction on l. For $l=1$, by (3.52) we just choose

$$A_0 B_0 \geq \frac{18}{\beta^2} \|\hat{v}_1\|_{\mu,\beta} (1 + \beta C_0 \|v_0\|_{\mu,\beta}) \tag{3.55}$$

Let now $l \geq 2$. For the induction step, we will estimate each term on the right of (3.53).

Lemma 3.4. *If for $l \geq 1$, $W^{[l]}$ satisfies (3.54), then*

$$\frac{|k|^2 |\hat{W}^{[l]}|}{(l+1)(l+2)} \leq \frac{6 A_0 B_0^l e^{-\beta|k|}}{\beta^2 (1+|k|)^\mu} \frac{Q_{2l+2}(\beta|k|)}{(2l+3)^2}$$

Proof. The proof simply follows from the (3.54) and noting that for $y \geq 0$

$$\frac{y^2}{(2l+2)(2l+1)} Q_{2l}(y) \leq Q_{2l+2}(y) \ , \ \frac{(2l+3)^2}{(2l+1)(l+2)} \leq 3$$

∎

Lemma 3.5. *If $W^{[l]}$ satisfies (3.54), then for $l \geq 1$,*

$$\frac{1}{(l+1)(l+2)} |k_j P_k \hat{u}_{0,j} \hat{*} \hat{W}^{[l]}| \leq 2^\mu \|v_0\|_{\mu,\beta} \frac{9\pi A_0 B_0^l e^{-\beta|k|}}{\beta^3 (2l+3)^2 (1+|k|)^\mu} Q_{2l+2}(\beta|k|)$$

$$\frac{1}{(l+1)(l+2)} |k_j P_k \hat{W}_j^{[l]} \hat{*} \hat{u}_{0,j}| \leq 2^\mu \|v_0\|_{\mu,\beta} \frac{9\pi A_0 B_0^l e^{-\beta|k|}}{\beta^3 (2l+3)^2 (1+|k|)^\mu} Q_{2l+2}(\beta|k|)$$

Proof. We use the estimate (3.54) on $\hat{W}^{[l]}$. From Lemma 6.7 for $n=0$, we obtain

$$|k_j \hat{W}_j^{[l]} \hat{*} \hat{u}_0| \leq \|v_0\|_{\mu,\beta} \frac{A_0 B_0^l}{(2l+1)^2} \left(|k| \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta(|k'|+|k-k'|)}}{(1+|k'|)^\mu [1+|k-k'|]^\mu} Q_{2l}(\beta|k'|) dk' \right)$$

$$\leq \frac{\|v_0\|_{\mu,\beta} A_0 B_0^l}{(2l+1)^2} \sum_{m=0}^{2l} \frac{2^{2l-m}}{m!} |k| \int_{k' \in \mathbb{R}^3} e^{-\beta(|k'|+|k-k'|)} (1+|k'|)^{-\mu} (1+|k-k'|)^{-\mu} |k'|^{2m} dk'$$

$$\leq \frac{2\pi \|v_0\|_{\mu,\beta} A_0 B_0^l 2^\mu e^{-\beta|k|}}{(2l+1)^2 \beta^3 (1+|k|)^\mu} \sum_{m=0}^{2l} 2^{2l-m} (m+2) Q_{m+2}(\beta|k|)$$

$$\leq \frac{2^{\mu+1}\pi}{(2l+1)\beta^3 (1+|k|)^\mu} \|v_0\|_{\mu,\beta} A_0 B_0^l e^{-\beta|k|} (l+2) Q_{2l+2}(\beta|k|)$$

The first part of the lemma follows by using (1.4) and checking that $\frac{2(2l+3)^2}{(2l+1)(l+1)} \leq 9$ for $l \geq 1$. The proof of the second part is essentially the same since $|\hat{W}_j^{[l]}| \leq |\hat{W}^{[l]}|$.
∎

Lemma 3.6. *If $W^{[l-1]}$ satisfies (3.54) for any $l \geq 2$, then*

$$\frac{1}{l(l+1)(l+2)} |k_j P_k \left[\hat{u}_{1,j} \hat{*} \hat{W}^{[l-1]} \right]| \leq 2^\mu \|v_1\|_{\mu,\beta} 9\pi A_0 B_0^l (1+|k|)^{-\mu} e^{-\beta|k|} \frac{Q_{2l}(\beta|k|)}{\beta^3 (l+2)(2l+1)^2}$$

$$\frac{1}{l(l+1)(l+2)} |k_j P_k \left[\hat{W}_j^{[l-1]} \hat{*} \hat{u}_{1,j} \right]| \leq 2^\mu \|v_1\|_{\mu,\beta} 9\pi A_0 B_0^{l-1} (1+|k|)^{-\mu} e^{-\beta|k|} \frac{Q_{2l}(\beta|k|)}{\beta^3 (l+2)(2l+1)^2}$$

Proof. The proof is identical to that of Lemma 3.5 with l replaced by $l-1$ and v_0 by v_1. ∎

Lemma 3.7. *If for $l \geq 3$, $\hat{W}^{[l_1]}$ and $\hat{W}^{[l-1-l_1]}$ for $l_1 = 1, ..., (l-2)$ satisfy (3.54), then*

$$\left| \frac{k_j}{(l+1)(l+2)} P_k \left[\sum_{l_1=1}^{l-2} \frac{l_1!(l-1-l_1)!}{l!} \hat{W}_j^{[l_1]} \hat{*} \hat{W}^{[l-1-l_1]} \right] \right| \\ \leq 2^\mu 36 A_0^2 B_0^{l-1} (1+|k|)^{-\mu} e^{-\beta|k|} \frac{Q_{2l}(\beta|k|)}{\beta^3 (2l+3)^2}$$

Proof. First note that if we define $l_2 = l-1-l_1$, then for $l \geq 3$, Lemma 6.9 implies

$$\frac{l_1! l_2!}{l!} \left| k_j \hat{W}_j^{[l_1]} \hat{*} \hat{W}^{[l_2]} \right| \leq A_0^2 B_0^{l-1} \frac{(l_1)!(l_2)!}{l!(2l_1+1)^2(2l_2+1)^2} \times \\ |k| \int_{k' \in \mathbb{R}^3} e^{-\beta(|k'|+|k-k'|)} (1+|k'|)^{-\mu} (1+|k-k'|))^{-\mu} Q_{2l_1}(\beta|k'|) Q_{2l_2}(\beta|k-k'|) dk' \\ \leq \frac{2^{\mu+1} \pi A_0^2 B_0^{l-1} e^{-\beta|k|}}{3\beta^3 (1+|k|)^\mu} \frac{(2l-1)(2l)(2l+1) l_1! l_2!}{l!(2l_1+1)^2(2l_2+2)^2} Q_{2l}(\beta|k|)$$

Therefore,

$$\sum_{l_1=1}^{l-2} \frac{l_1! l_2!}{l!(l+1)(l+2)} |k_j \hat{W}_j^{[l_1]} \hat{*} \hat{W}^{[l_2]}| \\ \leq \frac{2^{\mu+2} \pi e^{-\beta|k|} Q_{2l}(\beta|k|)(2l-1)(2l+1)}{3(l+1)(l+2)\beta^3(1+|k|)^\mu} \sum_{l_1=1}^{l-2} \frac{l_1! l_2!}{(l-1)!(2l_1+1)^2(2l_2+1)^2}$$

and the proof follows noting that $\frac{l_1! l_2!}{(l-1)!} = \frac{l_1! l_2!}{(l_1+l_2)!} \leq 1$ and checking $\frac{4(2l-1)(2l+1)}{(l+1)(l+2)} \leq 16$; by breaking up the sum in the ranges: $l_1 \leq (l-1)/2$ and $l_1 > (l-1)/2$ (in which $l_2 \leq (l-1)/2$) it is easily seen that for some $C_* > 0$ and any $l \geq 3$ we have

$$\sum_{l_1=1}^{l-2} \frac{1}{(2l_1+1)^2(2l_2+1)^2} \leq \frac{C_*}{(2l+3)^2},$$

where $C_* = 1.07555\cdots$ (the upper-bound being achieved at $l = 4$). ∎

Lemma 3.8.

$$|\hat{W}^{[2]}| \leq \frac{e^{-\beta|k|}}{(1+|k|)^\mu} \frac{Q_4(\beta|k|)}{7^2} \left(\frac{A_0 B_0}{\beta^2} + A_0 B_0 \|v_0\|_{\mu,\beta} \frac{2^\mu 36\pi}{\beta^2} + \|v_1\|_{\mu,\beta}^2 \right) \tag{3.56}$$

and therefore $\hat{W}^{[2]}$ satisfies (3.54) if

$$A_0 B_0^2 \geq \frac{3 A_0 B_0}{\beta^2} + A_0 B_0 \|v_0\|_{\mu,\beta} \frac{2^\mu 36\pi}{\beta^2} + \frac{C_0}{\beta} \|v_1\|_{\mu,\beta}^2 \tag{3.57}$$

Proof. We use Lemmas 3.4, 3.5 and 2.2 to estimate different terms on the right hand side of (3.53) for $l = 1$. ∎

Proof of Lemma 3.2

We use Lemmas 3.4, 3.5 3.6 and 3.7 to estimates the terms on the right hand side of (3.53) and note that $Q_{2l}(y) \leq \frac{1}{4} Q_{2l+2}(y)$. Hence, combining all the estimates, we obtain for $l \geq 2$,

$$|\hat{W}^{[l+1]}| \leq A_0 B_0^{l-1} \frac{Q_{2l+2}(\beta|k|)e^{-\beta|k|}}{(2l+3)^2(1+|k|)^\mu}$$

$$\times \left\{ \frac{6}{\beta^2} B_0 + 2^\mu \frac{18\pi}{\beta^3} B_0 \|v_0\|_{\mu,\beta} + \frac{2^\mu 18\pi(2l+3)^2}{(l+2)(2l+1)^2\beta^3} \|v_1\|_{\mu,\beta} + \frac{9A_0 2^\mu}{\beta^3} \right\}$$

$$\leq \frac{A_0 B_0^{l+1} e^{-\beta|k|}}{(1+|k|)^\mu (2l+3)^2} Q_{2l+2}(\beta|k|)$$

for large enough B_0 so that

$$\left\{ \frac{6}{\beta^2} B_0 + 2^\mu \frac{18\pi}{\beta^3} B_0 \|v_0\|_{\mu,\beta} + \frac{2^\mu 18\pi}{\beta^3} \|v_1\|_{\mu,\beta} + \frac{9A_0 2^\mu}{\beta^3} \right\} \leq B_0^2 \tag{3.58}$$

Combining (3.58) with (3.55) and (3.57), we that (3.54) is satisfied for any $l \geq 1$. Therefore, it follows that $\sum_{l=1}^\infty \hat{W}^{[l]}(k)p^l$ is convergent for $|p| < \frac{1}{4B_0}$. The recurrence relations (3.50),(3.51) and (3.53) imply that $\sum_{l=1}^\infty \hat{W}^{[l]}(k)p^l$ is indeed a solution to (2.9), which is zero at $p = 0$. However, from §2 Lemma 2.8, we know that there is a unique $\hat{W} = \hat{U}(k,p) - \hat{v}_1(k)$ with this property in $\mathcal{A}_L^\infty$, which for sufficiently small L includes analytic functions at the origin. Therefore

$$\hat{U}(k,p) = \hat{v}_1(k) + \sum_{l=1}^\infty \hat{W}^{[l]}(k)p^l$$

Moreover, from the well-known relation between a function and its Fourier transform, $\|f\|_{L^\infty(\mathbb{R}^3)} \leq \|\hat{f}\|_{L^1(\mathbb{R}^3}$, the inequalities involving $W^{[l]}(x)$ and its x-derivatives follow.

4. Estimates on $\partial_p^l \hat{W}(k,p)$ and proof of Theorem 1.2

In this section, we find inductively (in l) that $\hat{W}^{[l]} := \partial_p^l \hat{W}/l!$ exists for any l and $\hat{W}^{[l]}$ generate power series (4.72) with p_0- independent radius of convergence. This does not necessarily imply in itself that the series converges to $\hat{W}$. The fact that these objects do coincide locally will be shown in Lemma 4.13. This leads to proof of Theorem 1.2.

Definition 4.1. *It is convenient to define for $l \geq 1$,*

$$\hat{W}^{[l]}(k,p) = \frac{1}{l!} \partial_p^l \hat{W}(x,p) \ ,$$

It is also convenient to define $\hat{W}^{[0]}(k,p) = \hat{W}(k,p) = \hat{U}(k,p) - \hat{v}_1(k)$.

The proof therefore reduces to finding appropriate bounds on $\hat{W}^{[l]}(k,p)$. The main result proved in this section is the Lemma 4.2, which, using Lemma 4.13, leads directly to the proof of Theorem 1.2.

Proposition 2.9 implies that $\hat{U} \in \mathcal{A}^{\alpha'}$ for $\alpha' > \alpha$, with α chosen large enough to satisfy (2.39). In particular, if we choose $\alpha' = \alpha + 1$, it follows that $\hat{W}^{[0]}(k,p) =$

$\hat{U}(k,p) - \hat{v}_k(k)$ satisfies

$$|\hat{W}^{[0]}(k,p)| \leq \frac{3e^{-\beta|k|+\alpha' p}\|\hat{v}_1\|_{\mu,\beta}}{(1+p^2)(1+|k|)^\mu} \tag{4.59}$$

In the rest of this section, with some abuse of notation, we will replace α' by α.

Lemma 4.2. *If $\|v_0\|_{\mu+2,\beta} < \infty$, $\mu > 3$, there exists positive constants A, B independent of l, k and p so that for any $l \geq 0$*

$$|\hat{W}^{[l]}(k,p)| \leq \frac{e^{\alpha p} e^{-\beta|k|}}{(1+p^2)(1+|k|)^\mu} A B^l \frac{Q_{2l}(|\beta k|)}{(2l+1)^2} \tag{4.60}$$

The series (4.72) converges uniformly for any $p_0 \geq 0$ for $|p - p_0| < \frac{1}{4B}$.

Remark 4.3. The proof requires some further lemmas. We will use induction on l. Clearly, from (4.59), the conclusion is valid for $l = 0$, when

$$A = 3\|\hat{v}_1\|_{\mu,\beta} \tag{4.61}$$

We assume (4.60) for $l \geq 0$ and then establish it for $l+1$. We obtain a recurrence relation for $\hat{W}^{[l+1]}(k,.)$ for any $k \in \mathbb{R}^3$ in terms of $\hat{W}^{[j]}(k,.)$ for $j \leq l$.

Taking ∂_p^l in (2.9) and dividing by $l!$, we obtain

$$\begin{aligned} &p\partial_p^2 \hat{W}^{[l]} + (l+2)\partial_p \hat{W}^{[l]} + |k|^2 \hat{W}^{[l]} = \\ &-ik_j P_k \left[\int_0^p \left\{ \hat{W}_j^{[l]}(\cdot, p-s) \hat{*} \hat{W}^{[0]}(\cdot, s) \right\} ds + \sum_{l_1=1}^{l-1} \frac{l_1!(l-1-l_1)!}{l!} \hat{W}_j^{[l_1]}(\cdot, 0) \hat{*} \hat{W}^{[l-1-l_1]}(\cdot, p) \right] \\ &\quad - ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{W}^{[l]} + \hat{v}_0 \hat{*} \hat{W}_j^{[l]} + \frac{1}{l} \hat{v}_{1,j} \hat{*} \hat{W}^{[l-1]} + \frac{1}{l} \hat{v}_1 \hat{*} \hat{W}_j^{[l-1]} + \hat{v}_{1,j} \hat{*} \hat{v}_1 \delta_{l,1} \right] \\ &\qquad - ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{v}_1 + \hat{v}_{1,j} \hat{*} \hat{v}_0 \right] \delta_{l,0} \equiv \hat{R}^{(l)}(k,p) \end{aligned} \tag{4.62}$$

Lemma 4.4. *For any $l \geq 0$, for some absolute constant $C_6 > 0$, if $\hat{W}^{[l]}(k,p)$ satisfies (4.62) and is bounded at $p = 0$, then $\hat{W}^{[l+1]}(k,p)$ is bounded in terms of $\hat{R}^{(l)}(k,p)$, defined in (4.62):*

$$|\hat{W}^{[l+1]}(k,p) \leq \frac{C_6}{(l+1)^{5/3}} \sup_{p' \in [0,p]} |\hat{R}^{[l]}(k,p)| + \frac{|k|^2 |\hat{W}^{[l]}(k,0)|}{(l+1)(l+2)}$$

Proof. We invert the operator on the left hand side of (4.62). With the requirement that $\hat{W}^{[l]}$ is bounded at $p = 0$, we obtain

$$\begin{aligned} \hat{W}^{[l]}(k,p) &= \int_0^p \mathcal{Q}\left(z(p), 2|k|\sqrt{p'}\right) \hat{R}^{(l)}(k,p') dp' \\ &\quad + 2^{(l+1)}(l+1)! \hat{W}^{[l]}(k,0) \frac{J_{l+1}(z)}{z^{l+1}} \text{ , where } z = 2|k|\sqrt{p} \end{aligned} \tag{4.63}$$

and

$$\mathcal{Q}(z,z') = \pi z^{-(l+1)} \left[-J_{l+1}(z) {z'}^{(l+1)} Y_{l+1}(z') + {z'}^{(l+1)} J_{l+1}(z') Y_{l+1}(z) \right] \tag{4.64}$$

On taking the first derivative with respect to p, we obtain

(4.65) $$(l+1)\hat{W}^{[l+1]}(k,p)$$
$$= \frac{|k|}{\sqrt{p}} \int_0^p \mathcal{Q}_z \left(2|k|\sqrt{p}, 2|k|\sqrt{p'}\right) \hat{R}^{(l)}(k,p')dp' - 2^{l+2}(l+1)!|k|^2 \frac{J_{l+2}(z)}{z^{l+2}} \hat{W}^{[l]}(k,0)$$

Using again the properties of Bessel functions [1] we get

(4.66)
$$\frac{1}{z}\mathcal{Q}_z(z,z') = \frac{\pi}{z}\left[-\left(\frac{J_{l+1}(z)}{z^{l+1}}\right)' z'^{(l+1)} Y_{l+1}(z') + z'^{(l+1)} J_{l+1}(z') \left(\frac{Y_{l+1}(z)}{z^{l+1}}\right)' \right]$$
$$= \pi \left[\frac{J_{l+2}(z)}{z^{l+2}} z'^{(l+1)} Y_{l+1}(z') - z'^{(l+1)} J_{l+1}(z') \frac{Y_{l+2}(z)}{z^{l+2}} \right]$$

It is also known [1] that

$$2^{l+2}(l+1)! \left| \frac{J_{l+2}(z)}{z^{l+2}} \right| \leq \frac{1}{(l+2)}$$

Using (4.66) and the known uniform asymptotics of Bessel functions for large l [1], it is easily to see that C_* independent of l so that

$$\int_0^z \frac{z'}{z} |Q_z(z,z')| dz' \leq \frac{C_*}{(l+1)^{2/3}}$$

It follows that
(4.67)
$$(l+1)|\hat{W}^{[l+1]}(k,p)| \leq \sup_{p' \in [0,p]} |\hat{R}^{(l)}(k,p')| \int_0^z \frac{z'}{z} |Q_z(z,z')| dz' + \frac{|k|^2}{(l+2)} |\hat{W}^{[l]}(k,0)|$$

Therefore, it follows that

(4.68) $$\left| \hat{W}^{[l+1]}(k,p) \right| \leq \frac{C_6}{(l+1)^{5/3}} \sup_{p' \in [0,p]} |\hat{R}^{(l)}(k,p')| + \frac{|k|^2 |\hat{W}^{[l]}(k,0)|}{(l+1)(l+2)}$$

∎

Remark 4.5. We now find bounds on the different terms in $\hat{R}^{(l)}(k,p)$.

Lemma 4.6. *If $W^{[l]}$ satisfies (4.60), for $l \geq 0$ then*

$$|k_j P_k \left(\hat{v}_{0,j} \hat{*} \hat{W}^{[l]} \right)| \leq C_1 \|\hat{v}_0\|_{\mu,\beta} \frac{(l+1)^{2/3} A B^l e^{-\beta|k| + \alpha p} Q_{2l+2}(\beta|k|)}{(2l+1)(1+|k|)^\mu (1+p^2)}$$
$$|k_j P_k \left(\hat{W}_j^{[l]} \hat{*} \hat{v}_{0,j} \right)| \leq C_1 \|\hat{v}_0\|_{\mu,\beta} \frac{(l+1)^{2/3} A B^l e^{-\beta|k| + \alpha p} Q_{2l+2}(\beta|k|)}{(2l+1)(1+|k|)^\mu (1+p^2)}$$

Proof. We use (4.60). From Lemma 6.10, we obtain

$$(1+p^2) e^{-\alpha p} |k_j \hat{W}_j^{[l]} \hat{*} \hat{v}_0| \leq \|\hat{v}_0\|_{\mu,\beta} \frac{AB^l}{(2l+1)} |k| \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta(|k'| + |k-k'|)}}{(1+|k'|)^\mu [1+|k-k'|)]^\mu} Q_{2l}(\beta|k'|) dk'$$
$$\leq C_1 (l+1)^{2/3} \|\hat{v}_0\|_{\mu,\beta} \frac{AB^l}{(2l+1)} \frac{e^{-\beta|k|}}{(1+|k|)^\mu} Q_{2l+2}(\beta|k|)$$

The first part of the Lemma follows. The proof of the second part is essentially the same since $|\hat{W}_j^{[l]}| \le |\hat{W}^{[l]}|$. ∎

Lemma 4.7. *If $W^{[l-1]}$ satisfies (4.60) for $l \ge 1$, then*

$$\left| \frac{k_j}{l} P_k \left[\hat{v}_{1,j} \hat{*} \hat{W}^{[l-1]} \right] \right| \le C_1 \|v_1\|_{\mu,\beta} A B^{l-1} \frac{e^{-\beta|k|+\alpha p}}{(1+p^2)(1+|k|)^\mu} \frac{l^{2/3} Q_{2l}(\beta|k|)}{l(2l-1)}$$

$$\left| \frac{k_j}{l} \left(1 - \frac{k(k\cdot)}{|k|^2} \right) \hat{v}_1 \hat{*} \hat{W}_j^{[l-1]} \right| \le C_1 \|v_1\|_{\mu,\beta} A B^{l-1} \frac{e^{-\beta|k|+\alpha p}}{(1+p^2)(1+|k|)^\mu} \frac{l^{2/3} Q_{2l}(\beta|k|)}{l(2l-1)}$$

Proof. The proof is identical to Lemma 4.6 replacing l by $l-1$ and $\hat{v}_0$ by $\hat{v}_1$. ∎

Lemma 4.8. *If $W^{[l]}$ satisfies (4.60), then for $l \ge 1$,*

$$\left| \frac{k_j}{l} P_k \left[\hat{W}^{[l-1]}(\cdot, 0) \hat{*} \hat{W}^{[0]}(\cdot, p) \right] \right| \le C_1 \frac{(l+1)^{2/3} A^2 B^{l-1} e^{-\beta|k|+\alpha p} Q_{2l}(\beta|k|)}{l(2l-1)(1+|k|)^\mu (1+p^2)}$$

Proof. Noting that

$$|\hat{W}^{[0]}(k,p)| \le A \frac{e^{-\beta|k|+\alpha p}}{(1+|k|)^\mu (1+p^2)}$$

and

$$|\hat{W}^{[l-1]}(k,0) \le \frac{e^{-\beta|k|}}{(2l-1)^2 (1+|k|)^\mu} A B^{l-1} Q_{2l-2}(\beta|k|)$$

the rest of the proof is very similar to the proof of Lemma 4.6 ∎

Lemma 4.9. *If $\hat{W}^{[l_1]}$ and $\hat{W}^{[l-1-l_1]}$ for $l_1 = 1, ..(l-2)$ for $l \ge 2$ satisfy (4.60), then*

$$\left| k_j P_k \left[\sum_{l_1=1}^{l-2} \frac{l_1!(l-1-l_1)!}{l!} \hat{W}_j^{[l_1]}(\cdot, 0) \hat{*} \hat{W}^{[l-1-l_1]}(\cdot, p) \right] \right|$$

$$\le C_8 2^{\mu+1} \pi A^2 B^{l-1} \frac{e^{-\beta|k|+\alpha p}}{3\beta^3 (1+p^2)(1+|k|)^{-\mu}} \frac{l Q_{2l}(\beta|k|)}{(2l+3)^2} \text{ ; where } C_8 = 82$$

Proof. First note that if we define $l_2 = l - 1 - l_1$, then for $l \ge 2$, using Lemma 6.9, we get

$$\frac{l_1! l_2!}{l!} |k_j \hat{W}_j^{[l_1]}(\cdot, 0) \hat{*} \hat{W}^{[l_2]}(\cdot, p)| \le \frac{e^{\alpha p}}{(1+p^2)} A^2 B^{l-1} \frac{l_1! l_2!}{l!(2l_1+1)^2 (2l_2+1)^2}$$

$$\times |k| \int_{k' \in \mathbb{R}^3} e^{-\beta(|k'|+|k-k'|)} (1+|k'|)^{-\mu} (1+|k-k'|))^{-\mu} Q_{2l_1}(\beta|k'|) Q_{2l_2}(\beta|k-k'|) dk'$$

$$\le \frac{A^2 B^{l-1} 2^{\mu+1} \pi e^{-\beta|k|+\alpha p}}{3\beta^3 (1+p^2)(1+|k|)^\mu} \frac{l_1! l_2! (2l)(2l-1)(2l+1)}{l!(2l_1+1)^2 (2l_2+1)^2} Q_{2l}(\beta|k|)$$

Therefore,

$$\sum_{l_1=1}^{l-2} \frac{l_1! l_2!}{l!} |k_j \hat{W}_j^{[l_1]}(.;0) \hat{*} \hat{W}^{[l_2]}(.;p)|$$

$$\le \frac{2^\mu A^2 B^l e^{-\beta|k|+\alpha p} Q_{2l}(\beta|k|)}{\beta^3 (1+|k|)^\mu (1+p^2)} \sum_{l_1=1}^{l-2} \frac{l_1! l_2! (2l)(2l+1)(2l-1)}{l!(2l_1+1)^2 (2l_2+1)^2}$$

We claim that for $l \geq 2$, with $l_1 \geq 1$, $l_2 = l - l_1 - 1 \geq 1$,

$$\sum_{l_1=1}^{l-1} \frac{l_1! l_2! (2l)(2l-1)(2l+1)}{l!(l-1)(2l_1+1)^2(2l_2+1)^2} \leq \frac{C_8 l}{(2l+3)^2}$$

for some C_8 independent of l; C_8 is bounded by 82.

Proving the above bound only requires consideration for sufficiently large l. We will therefore assume $l \geq 5$. Further, consider summation terms other than $l_1 = 1$ and $l_2 = 1$. So, we may assume $l_1, l_2 \geq 2$. Then, we claim that

$$\frac{l_1! l_2! 2l(2l+1)(2l-1)}{l!(2l_1+1)^2(2l_2+1)^2} = \left(\frac{(l_1-2)!(l_2-2)!}{(l-5)!}\right)\left(\frac{l_1(l_1-1)l_2(l_2-1)}{(2l_1+1)^2(2l_2+2)^2}\right) \times \left(\frac{2l(2l-1)(2l+1)}{l(l-1)(l-2)(l-3)(l-4)}\right) \leq \frac{12}{(2l+3)^2} \tag{4.69}$$

This follows since the first two parenthesis term on the right of (4.69) is clearly bounded, while the last term is a cubic in l divided by fifth order polynomial, and simple estimates give the upperbound of 12. Therefore, for $l \geq 5$,

$$\sum_{l_1=2}^{l-3} \frac{l_1! l_2! 2l(2l+1)(2l-1)}{l!(2l_1+1)^2(2l_2+1)^2} \leq 12 \frac{(l-4)}{(2l+3)^2}$$

For $l_1 = 1$ or $l_2 = 1$, clearly

$$\frac{l_1! l_2! 2l(2l+1)(2l-1)}{l!(2l_1+1)^2(2l_2+1)^2} = \frac{(l-2)! 2l(2l+1)(2l-1)}{9l!(2l-3)^2} = \frac{2l(2l+1)(2l-1)}{9l(l-1)(2l-3)^2} \leq 82 \frac{l}{(2l+3)^2}$$

∎

Lemma 4.10. *If $\hat{W}^{[l]}$ satisfies (4.60), then for $l \geq 0$,*

$$\left| k_j \left(1 - \frac{k(k\cdot)}{|k|^2}\right) \int_0^p \hat{W}_j^{[l]}(.;p-s) \hat{*} \hat{W}^{[0]}(.;s) ds \right| \leq C_1 (l+1)^{2/3} A^2 B^l \frac{e^{-\beta|k| + \alpha p} Q_{2l+2}(\beta|k|)}{(1+p^2)(1+|k|)^\mu (2l+1)}$$

Proof. We note that Lemma 6.10 implies

$$\left| k_j \int_{k' \in \mathbb{R}^3} \int_0^p \hat{W}_j^{[l]}(k', p-s) \hat{W}^{[0]}(k-k';s) ds dk' \right| \leq \frac{A^2 B^l e^{\alpha p}}{(1+p^2)(2l+1)^2} \times |k'| \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta|k'| - \beta|k-k'|}}{(1+|k'|)^\mu (1+|k-k'|)^\mu} Q_{2l}(\beta k') dk' \leq \frac{C_1 (l+1)^{2/3} A^2 B^l e^{\alpha p - \beta|k|}}{(2l+1)(1+p^2)(1+|k|)^\mu} Q_{2l+2}(\beta|k|)$$

∎

Lemma 4.11.

$$\left| k_j \left(1 - \frac{k(k\cdot.)}{|k|^2}\right) (\hat{v}_{0,j} \hat{*} \hat{v}_1 + \hat{v}_{1,j} \hat{*} \hat{v}_0) \right| \leq 4C_0 |k| \frac{e^{-\beta|k|}}{(1+|k|)^\mu} \|\hat{v}_0\|_{\mu,\beta} \|\hat{v}_1\|_{\mu,\beta}$$

$$\left| k_j \left(1 - \frac{k(k\cdot.)}{|k|^2}\right) \hat{v}_{1,j} \hat{*} \hat{v}_1 \right| \leq 2C_0 |k| \frac{e^{-\beta|k|}}{(1+|k|)^\mu} \|\hat{v}_1\|_{\mu,\beta}^2$$

Proof. This follows simply from the observation that

$$\left| k_j \left(1 - \frac{k(k\cdot.)}{|k|^2} \right) \hat{v}_{0,j} \hat{*} \hat{v}_1 \right| \leq 2|k| \|\hat{v}_1\|_{\mu,\beta} \|\hat{v}_0\|_{\mu,\beta} \int_{k'\in\mathbb{R}^3} \frac{e^{-\beta|k'|-\beta|k-k'|}}{(1+|k'|)^\mu (1+|k-k'|)^\mu} dk'$$

and using (2.21) to bound the convolution. Other parts of the Lemma follow similarly. ∎

Lemma 4.12.

$$|\hat{W}^{[1]}(\cdot,p)| \leq \frac{e^{-\beta|k|+\alpha p}}{(1+|k|)^\mu (1+p^2)} ABQ_2(\beta|k|)$$

with

$$AB \geq \left(2C_1 \|v_0\|_{\mu,\beta} A + C_1 A^2 + \frac{2C_0}{\beta} \|\hat{v}_0\|_{\mu,\beta} \|v_1\|_{\mu,\beta} \right) \tag{4.70}$$

Proof. Combining Lemmas 4.6, 4.10 and 4.11 with (4.68) for $l = 0$, we obtain

$$|\hat{W}^{[1]}(\cdot,p)| \leq \frac{e^{-\beta|k|+\alpha p}}{(1+|k|)^\mu (1+p^2)} Q_2(\beta|k|) \left(2C_1 \|v_0\|_{\mu,\beta} A + C_1 A^2 + \frac{2C_0}{\beta} \|\hat{v}_0\|_{\mu,\beta} \|\hat{v}_1\|_{\mu,\beta} \right)$$
$$\leq \frac{e^{-\beta|k|+\alpha p}}{(1+|k|)^\mu (1+p^2)} ABQ_2(\beta|k|)$$

∎

Proof of Lemma 4.2

From Lemmas 4.6-4.10 and 4.11 (the latter is only needed for $l = 1$), it follows that $\hat{R}^{(l)}$ (cf. (4.60)) satisfies

$$|\hat{R}^{(l)}| \leq AB^{l-1} \frac{e^{-\beta|k|+\alpha p}}{(2l+3)^2 (1+p^2)(1+|k|)^\mu} Q_{2l+2}(\beta|k|)$$
$$\times \left[ABC_1 \frac{(l+1)^{2/3}(2l+3)^2}{(2l+1)} + \frac{AC_1 (l+1)^{2/3} (2l+3)^3}{4l(2l-1)} + \frac{82\ 2^{\mu+1} \pi A l}{12\beta^3} + \frac{2C_1 \|\hat{v}_1\|_{\mu,\beta} (2l+3)^3}{4l^{1/3}(2l-1)} \right.$$
$$\left. + \frac{2C_0 (l+1)^{2/3} (2l+3)^2 \|\hat{v}_0\|_{\mu,\beta}}{(2l+1)} + \frac{25C_0}{\beta} (1+p^2) e^{-\alpha p} \|\hat{v}_1\|^2_{\mu,\beta} \delta_{l,1} \right]$$

Noting that $e^{-\alpha p}(1+p^2) \leq 1$ and

$$\sup_{p'\in[0,p]} \frac{e^{\alpha p'}}{1+p'^2} = \frac{e^{\alpha p}}{1+p^2}$$

for $\alpha \geq 1$, it follows from Lemma 4.4 and the above bounds that (4.60) holds when l is replaced by $l+1$, provided B is chosen large enough to satisfy (4.70) and

$$C_6 \left[ABC_1 \frac{(2l+3)^2}{(l+1)(2l+1)} + \frac{AC_1 (2l+3)^3}{4l(l+1)(2l-1)} + \frac{(82) 2^{\mu+1} \pi A l}{12\beta^3 (l+1)^{5/3}} + \frac{2C_1 \|\hat{v}_1\|_{\mu,\beta} (2l+3)^3}{4l^{1/3}(l+1)^{5/3}(2l-1)} \right.$$
$$\left. + \frac{2C_0 (2l+3)^2 \|\hat{v}_0\|_{\mu,\beta}}{(l+1)(2l+1)} + \frac{25C_0}{\beta} (1+p^2) e^{-\alpha p} \|\hat{v}_1\|^2_{\mu,\beta} \delta_{l,1} \right] + \frac{100B}{9\beta^2} \leq B^2, \tag{4.71}$$

for any $l \geq 1$, with A given by (4.61). From the asymptotic behavior of the left hand side of (4.71) as $l \to \infty$ and recalling that constants C_0, C_1 and C_6 are independent of l, it follows that B can be chosen independent of l. Therefore, by induction, (4.60) follows for all l. The proof of Lemma 4.2 is complete.

From (4.60), after noting that that $Q_{2l}(|q|) \leq 4^l e^{-|q|/2}$, it follows that

$$\tilde{W}(k,p;p_0) = \sum_{l=0}^{\infty} \hat{W}^{[l]}(k,p_0)(p-p_0)^l := \hat{W}_1(k,p) \tag{4.72}$$

is convergent for $|p-p_0| < \frac{1}{4B}$ for B independent of $p_0 \in \mathbb{R}^+$. The following Lemma shows that $\tilde{W}(k,p;p_0)$ is indeed the local representation of the solution $\hat{W}(k,p)$ to (2.9).

Lemma 4.13. *The unique solution to (2.9) satisfying $\hat{W}(k,0) = 0$, given by $\hat{W}(k,p) = \hat{U}(k,p) - \hat{v}_1(k)$, where $\hat{U}(k,p)$ is determined in §2 in Lemma 2.8, has the local representation $\tilde{W}(k,p;p_0)$ in a neighborhood of $p_0 \in \mathbb{R}^+$. Therefore, $\hat{W}(k,.)$ (and therefore $\hat{U}(k,.)$) is analytic in $\mathbb{R}^+ \cup \{0\}$.*

Proof. First, by permanence of relations (for analyticity of convolutions, see e.g., [6]), it follows that if $\hat{V}$ is an analytic solution of an equation of the form (2.9) on an interval $[0,L]$ and $\hat{V}$ has analytic continuation on $[0,L']$ with $L' > L$, then the equation is automatically satisfied in the larger interval. Therefore, if we analytically continue $\hat{W}$ to $\mathbb{R}^+$, the analytic continuation will automatically satisfy (2.9) and will therefore be the same as $\hat{W}(k,p)$.

From §3, Lemma 3.2, we know that the actual solution to (2.9) satisfying $\hat{W}(k,0) = 0$, is unique, and given by

$$\hat{W}(k,p) = \tilde{W}(k,p;0)$$

for $|p| < (4B)^{-1}$.

We now choose a sequence of $\{p_{0,j}\}_{j=0}^{\infty}$, with $p_{0,j} = j/(8B)$ and define the intervals $\mathcal{I}_j = (p_{0,j} - 1/(4B), p_{0,j} + 1/(4B))$. Consider the sequence of analytic functions $\left\{\tilde{W}(k,p;p_{0,j})\right\}_{j=0}^{\infty}$. Since $p_{0,1} \in \mathcal{I}_0 \cap \mathcal{I}_1$, it follows from (4.72) that $\hat{W}(k,p)$ has analytic continuation to $\mathcal{I}_1$, namely $\tilde{W}(k,p;p_{0,1})$. Again $p_{0,2} \in \mathcal{I}_1 \cap \mathcal{I}_2$. Hence $\tilde{W}(k,p;p_{0,2})$ provides analytic continuation of $\hat{W}(k,p)$ to the interval $\mathcal{I}_2$. We can continue this process to obtain analytic continuation of $\hat{W}$ to any interval $\mathcal{I}_j$. Since the union of $\{\mathcal{I}_j\}_{j=0}^{\infty}$ contains $\mathbb{R}^+ \cup \{0\}$, it follows that $\hat{W}(k,.)$ is analytic in $\mathbb{R}^+$. In particular, (4.72) provides the local Taylor series representation of $\hat{W}(k,p)$ near $p = p_0$. ∎

Proof of Theorem 1.2

Using Lemma 4.2, it follows from the inequality $\|W^{[l]}(\cdot,p_0)\|_\infty \leq \|\hat{W}^{[l]}(\cdot,p_0)\|_{\mathbb{L}^1}$ by integration in k that

$$|W^{[l]}(x,p_0)| \leq \frac{8\pi A(4B)^l e^{\alpha p_0}}{\beta(2l+1)^2(1+p_0^2)}$$

$$|DW^{[l]}(x,p_0)| \leq \frac{8\pi A(4B)^l e^{\alpha p_0}}{\beta(2l+1)^2(1+p_0^2)}$$

$$|D^2W^{[l]}(x,p_0)| \leq \frac{16\pi A(4B)^l e^{\alpha p_0}}{\beta^2(2l+1)^2(1+p_0^2)}$$

and therefore, the series (4.72) converges for $|p-p_0| < B^{-1}/4$ and, from Lemma 4.13 it is the local representation of the solution $\hat{W}(k,p)$ to (2.9) satisfying $\hat{W}(k,0) = 0$ for any $p_0 \geq 0$. These estimates on W in terms of $\hat{W}$, and the fact that $W(x,p)$ is analytic in a neighborhood of for $p \in \{0\} \cup \mathbb{R}^+$ and is exponentially bounded in p for large p (recall $\hat{W} \in \mathcal{A}^\alpha$) implies Borel summability of v in $1/t$. Watson's Lemma [25] implies $w(x,t) = \int_0^\infty e^{-p/t} W(x,p)dp \sim \sum_{m=2}^\infty v_m(x) t^m$, implying

$$v(x,t) = v_0(x) + tv_1(x) + \sum_{m=2}^{\infty} v_m(x) t^m,$$

where $v_m(x) = m! W^{[m-1]}(x;0) = m! U^{[m-1]}(x;0)$ for $m \geq 2$. It follows from the bounds on $\hat{W}^{[m-1]}(k)$ in §3, that for $m \geq 2$, $|W^{[m-1]}(x;0)| \leq A_0 B_0^m$, where A_0 and B_0[(3)] are chosen to ensure (3.55), (3.57) and (3.58).

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6. Appendix

6.1. **Some Fourier convolution inequalities.**

The following lemmas are relatively straightforward.

Definition 6.1. *Consider the polynomial*

$$P_n(z) = \sum_{j=0}^{n} \frac{n!}{j!} z^j$$

Remark 6.2. Integration by parts yields

$$\int_0^z e^{-\tau} \tau^n d\tau = -e^{-z} P_n(z) + n! \tag{6.73}$$

Lemma 6.3. *For all $y \geq 0$ and nonnegative integers $m, n \geq 0$ we have*

$$y^{m+1} \int_0^1 \rho^m P_n(y(1-\rho)) d\rho = m! n! \sum_{j=0}^{n} \frac{y^{m+j+1}}{(m+j+1)!}$$

Proof. This follows from a simple computation:

$$y^{m+1} \int_0^1 \rho^m P_n(y(1-\rho)) d\rho = \sum_{j=0}^{n} \frac{n!}{j!} y^{j+m+1} \int_0^1 (1-\rho)^j \rho^m d\rho = m! n! \sum_{j=0}^{n} \frac{y^{j+m+1}}{(m+j+1)!}$$

∎

(3) We may express it in terms of A and B as well, however, the estimates A_0 and B_0 found in §3, are better.

Lemma 6.4. *For all $y \geq 0$ and nonnegative integers $n \geq m \geq 0$ we have*

$$y^{m+1} \int_1^\infty e^{-2y(\rho-1)} \rho^m P_n(y(\rho-1)) d\rho \leq 2^{-m} (m+n)! \sum_{j=0}^m \frac{y^j}{j!}$$

Proof. First we note that

$$y^{m+1+l} \int_1^\infty e^{-2y(\rho-1)} \rho^m (\rho-1)^l d\rho = y^{m+1+l} \int_0^\infty e^{-2y\rho} (1+\rho)^m \rho^l d\rho$$
$$= y^{m+1+l} \sum_{j=0}^m \frac{m!}{j!(m-j)!} \int_0^\infty e^{-2y\rho} \rho^{l+j} d\rho = 2^{-l-1} \sum_{j=0}^m \frac{y^{m-j} m!(l+j)!}{j!(m-j)! 2^j}$$
$$= 2^{-l-1} \sum_{j=0}^m \frac{y^j m!(l+m-j)!}{(m-j)! j! 2^{m-j}}$$

Therefore, from the definition of P_n, it follows that

$$y^{m+1} \int_1^\infty e^{-2y(\rho-1)} \rho^m P_n(y(\rho-1)) d\rho$$
$$= m!n! \sum_{j=0}^m \frac{y^j}{j!(m-j)! 2^{m-j}} \left(\sum_{l=0}^n \frac{(l+m-j)!}{2^{l+1} l!} \right)$$

Taking the ratio of two consecutive terms we see that $(l+m-j)!/l!$ is nondecreasing with l since $m-j \geq 0$. Therefore the $l=n$ term is the largest term in the summation over l. Further, $\sum_{l=0}^n 2^{-l-1} \leq 1$. Therefore, $\sum_{l=0}^n 2^{-l-1} (l+m-j)!/l! \leq (m-j+n)!/n!$, and hence

$$y^{m+1} \int_1^\infty e^{-2y(\rho-1)} \rho^m P_n(y(\rho-1)) d\rho \leq 2^{-m} m!n! \sum_{j=0}^m \frac{y^j}{j!} \frac{2^j (m-j+n)!}{n!(m-j)!}$$

The ratio of two consecutive (in j) terms in $2^j(m-j+n)!/(m-j)!$ is ≤ 1 for $m \leq n$, hence the largest value is attained at $j=0$ and thus

$$y^{m+1} \int_1^\infty e^{-2y(\rho-1)} \rho^m P_n(y(\rho-1)) d\rho \leq 2^{-m} (m+n)! \sum_{j=0}^m \frac{y^j}{j!}$$

∎

Lemma 6.5. *For all $y \geq 0$ and nonnegative integers $n \geq m \geq 0$ we have*

$$y^{m+1} \int_0^\infty e^{-y(\rho-1)[1+sgn(\rho-1)]} \rho^m P_n(y|1-\rho|) d\rho \leq m!n! Q_{m+n+1}(y)$$

Proof. By breaking up the integral range into $\int_0^1$ and $\int_1^\infty$ and using the two previous Lemmas, we obtain

$$y^{m+1} \int_0^\infty e^{-y(\rho-1)[1+\mathrm{sgn}(\rho-1)]} \rho^m P_n(y|1-\rho|) d\rho \leq m!n! \Bigg(\sum_{j=m+1}^{m+n+1} \frac{y^j}{j!}$$
$$+ 2^{-m-n} \frac{(m+n)!}{m!n!} \sum_{j=0}^m 2^n \frac{y^j}{j!} \Bigg) \leq m!n! \sum_{j=0}^{m+n+1} 2^{m+n+1-j} \frac{y^j}{j!} = m!n! Q_{m+n+1}(y)$$

where we used $2^{-m-n}\frac{(m+n)!}{m!n!} \le 1$. ∎

Lemma 6.6. *If m and n, are integers no less that -1 we obtain*

$$|q|\int_{q'\in\mathbb{R}^3} e^{|q|-|q'|-|q-q'|}|q'|^m|q-q'|^n dq' \le 2\pi(m+1)!(n+1)!Q_{m+n+3}(|q|)$$

Proof. We note that we may assume $m \le n$ without loss of generality since changing variable $q' \mapsto q - q'$ switches the roles of m and n.

First, we will show that

$$\frac{|q|}{2\pi}\int_{q'\in\mathbb{R}^3} e^{|q|-|q'|-|q-q'|}|q'|^m|q-q'|^n dq' \le |q|^{m+2}\int_0^\infty e^{-|q|(\rho-1)[1+\mathrm{sgn}(\rho-1)]}\rho^{m+1}P_{n+1}(|q|(|\rho-1|)d\rho \tag{6.74}$$

We scale q' with $|q|$ and use a polar representation (ρ,θ,ϕ) for $q'/|q|$, where θ is the angle between q and q'. As a variable of integration however, we prefer to use $z = \sqrt{1+\rho^2-2\rho\cos\theta}$ to θ. Then, it is clear that

$$|q-q'| = |q|\sqrt{1+\rho^2-2\rho\cos\theta} = |q|z, \text{ and } dz = \frac{\rho\sin\theta d\theta}{\sqrt{1+\rho^2-2\rho\cos\theta}}$$

Therefore,

$$|q|\int_{q'\in\mathbb{R}^3} e^{|q|-|q'|-|q-q'|}|q'|^m|q-q'|^n dq' = 2\pi|q|^{m+n+4}\int_0^\infty d\rho\rho^{m+1}e^{-|q|(\rho-1)}\left\{\int_{|\rho-1|}^{1+\rho} dz e^{-|q|z}z^{n+1}\right\}$$

$$= 2\pi|q|^{m+2}\int_0^\infty d\rho\rho^{m+1}e^{-|q|(\rho-1)}\left[e^{-|q||\rho-1|}P_{n+1}(|q||\rho-1|) - e^{-|q|(1+\rho)}P_{n+1}(|q|(1+\rho))\right]$$

Inequality (6.74) follows since $e^{-|q|(1+\rho)}P_{n+1}(|q|(1+\rho)) \ge 0$. The rest of the Lemma follows from Lemma 6.5, with $y = |q|$, and m replaced by $m+1$, n by $n+1$ respectively. ∎

Lemma 6.7. *For any $\mu \ge 1$, and nonnegative integers m, n we have*

$$|k|\int_{k'\in\mathbb{R}^3}\frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^\mu(1+|k-k'|)^\mu}|\beta k'|^m|\beta(k-k')|^n dk' \le \frac{\pi 2^{\mu+1}e^{-\beta|k|}m!n!}{\beta^3(1+|k|)^\mu}(m+n+2)Q_{m+n+2}(\beta|k|)$$

Proof. We break up the integral into two ranges:

$$\int_{|k'|\le|k|/2} + \int_{|k'|>|k|/2} \tag{6.75}$$

In the first integral we have

$$\frac{1}{(1+|k-k'|)^\mu(1+|k'|)^\mu} \le \frac{1}{(1+|k|/2)^\mu(1+|k'|)} \le \frac{\beta}{(1+|k|/2)^\mu|\beta k'|}$$

While in the second integral we have

$$\frac{1}{(1+|k-k'|)^\mu(1+|k'|)^\mu} \le \frac{1}{(1+|k|/2)^\mu(1+|k-k'|)} \le \frac{\beta}{(1+|k|/2)^\mu|\beta(k-k')|}$$

Introducing in the first integral $q=\beta k$ and $q'=\beta k'$, we obtain

$$|k|\int_{|k'|\le|k|/2} \frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^\mu(1+|k-k'|)^\mu}|\beta k'|^m|\beta(k-k')|^n dk' \\ \le \frac{2^\mu e^{-\beta|k|}}{\beta^3(1+|k|)^\mu}|q|\int_{q'\in\mathbb{R}^3} e^{-|q'|-|q-q'|+|q|}|q'|^{m-1}|q-q'||^n dq'$$

while in the second integral, with $q=\beta k$ and $q-q'=\beta k'$, we obtain

$$|k|\int_{|k'|>|k|/2} \frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^\mu(1+|k-k'|)^\mu}|\beta k'|^m|\beta(k-k')|^n dk' \\ \le \frac{2^\mu e^{-\beta|k|}}{\beta^3(1+|k|)^\mu}|q|\int_{q'\in\mathbb{R}^3} e^{-|q'|-|q-q'|+|q|}|q'|^{n-1}|q-q'|^m dq'$$

We now use Lemma 6.6 to bound the first integral, with m replaced by $m-1$. We also use Lemma 6.6 to bound the second integral, with $n-1$ replacing n. The proof is completed by adding the two bounds. ∎

Lemma 6.8. *For any $\mu\ge 2$, and $n\in\mathbb{N}\setminus\{0\}$ we have*

$$|k|\int_{k'\in\mathbb{R}^3} \frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^\mu(1+|k-k'|)^\mu}|\beta(k-k')|^n dk' \\ \le \frac{2^{\mu+1}\pi e^{-\beta|k|}}{\beta^2(1+|k|)^\mu}\left\{(n-1)!Q_{n+1}(|q|)+\frac{3(n+1)!|q|^{2/3}}{2\beta^{2/3}}\sum_{j=0}^{n+1}\frac{|q|^j}{j!}\right\}$$

Proof. We break up the integral into $\int_{|k'|<|k|/2}+\int_{|k'|\ge|k|/2}$. In the first integration range we have $[1+|k-k'|]^{-\mu}\le 2^\mu[1+|k|]^{-\mu}$, whereas in the second range $[1+|k'|]^{-\mu}\le 2^\mu[1+|k|]^{-\mu}$. Therefore, using Lemma 6.6,

$$|k|\int_{|k'|\ge|k|/2} \frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^\mu(1+|k-k'|)^\mu}|\beta(k-k')|^n dk' \\ \le \frac{2^\mu e^{-\beta|k|}}{\beta^2(1+|k|)^\mu}|q|\int_{q\in\mathbb{R}^3} e^{-|q'|-|q-q'|+|q|}\,|q-q'|^{n-2}dq' \le \frac{2^{\mu+1}\pi e^{-\beta|k|}}{\beta^2(1+|k|)^\mu}(n-1)!Q_{n+1}(|q|)$$

On the other hand, using $(1+|k'|)^{-\mu}\le(1+|k'|)^{-2+2/3}\le|k'|^{-2+2/3}$ we get

$$|k|\int_{|k'|<k|/2} \frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^\mu(1+|k-k'|)^\mu}|\beta(k-k')|^n dk' \\ \le \frac{2^\mu e^{-\beta|k|}}{\beta^{2+2/3}(1+|k|)^\mu}|q|\int_{|q'|\le|q|/2}|q'|^{-2+2/3}e^{-|q'|-|q-q'|+|q|}\,|q-q'|^n dq'$$

We note that

$$\frac{|q|}{2\pi}\int_{|q'|<|q|/2}|q'|^{-2+2/3}e^{-|q'|-|q-q'|+|q|}\ |q-q'|^n dq'$$

$$= |q|^{n+2+2/3}\int_0^{1/2}\rho^{-1+2/3}e^{-|q|(\rho-1)}\left\{\int_{1-\rho}^{1+\rho}dze^{-|q|z}z^{n+1}\right\}d\rho$$

$$\le |q|^{2/3}\int_0^{1/2}\rho^{-1+2/3}P_{n+1}(|q|(1-\rho)d\rho$$

$$\le |q|^{2/3}\sum_{j=0}^{n+1}\frac{(n+1)!}{j!}|q|^j\int_0^1\rho^{-1/3}(1-\rho)^j d\rho \le \frac{3}{2}|q|^{2/3}(n+1)!\sum_{j=0}^{n+1}\frac{|q|^j}{j!}$$

∎

Lemma 6.9. *For any $\mu\ge 1$ and nonnegative integers $l_1,l_2\ge 0$ we have*

$$(6.76)\quad |k|\int_{k'\in\mathbb{R}^3}\frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^\mu(1+|k-k'|)^\mu}Q_{2l_1}(|\beta k'|)Q_{2l_2}(|\beta(k-k')|)dk'$$
$$\le \frac{2^{\mu+1}\pi e^{-\beta|k}}{3\beta^3(1+|k|)^\mu}(2l_1+2l_2+1)(2l_1+2l_2+2)(2l_1+2l_2+3)Q_{2l_1+2l_2+2}(\beta|k|)$$

Proof. As before, we define $q=\beta k$. Also, for notational convenience, we define

$$k^m\bowtie k^n = |k|\int_{k'\in\mathbb{R}^3}\frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^\mu(1+|k-k'|)^\mu}|\beta k'|^m|\beta(k-k')|^n dk'$$

$$K=\frac{2^{\mu+1}\pi e^{-\beta|k|}}{\beta^3(1+|k|)^\mu}$$

Lemma 6.7 and $2^{2l_1+2l_2+2-(j+2)}Q_{j+2}(|q|)\le Q_{2l_1+2l_2+2}$ for $j\le 2l_1+2l_2$ imply that the left side of (6.76) is given by

$$\sum_{m=0}^{2l_1}\sum_{n=0}^{2l_2}\frac{2^{2l_1+2l_2-m-n}}{m!n!}k^m\bowtie k^n \le K\sum_{m=0}^{2l_1}\sum_{n=0}^{2l_2}2^{2l_1+2l_2-m-n}(m+n+2)Q_{m+n+2}(\beta|k|)$$

$$\le K\sum_{j=0}^{2l_1+2l_2}2^{2l_1+2l_2+2-(j+2)}(j+2)(j+1)Q_{j+2}(|q|)$$

$$\le KQ_{2l_1+2l_2+2}(|q|)\sum_{j=0}^{2l_2+2l_1}(j+1)(j+2)$$

$$\le \frac{K}{3}(2l_1+2l_2+1)(2l_1+2l_2+2)(2l_1+2l_2+3)Q_{2l_1+2l_2+2}(|q|),$$

which imply the result. ∎

Lemma 6.10. *If $\mu\ge 2$ and $l\ge 0$, then*

$$\frac{|k|}{(l+1)^{2/3}}\int_{k'\in\mathbb{R}^3}\frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^\mu(1+|k-k'|)^\mu}Q_{2l}(|\beta(k-k')|)dk'$$
$$\le \frac{C_1e^{-\beta|k}}{(1+|k|)^\mu}(2l+1)Q_{2l+2}(\beta|k|),$$

where

$$C_1 = 12\pi 2^{\mu}\beta^{-8/3} + 2\pi 2^{\mu}\beta^{-2} + \frac{1}{2}C_0(\mu)\beta^{-1}$$

Proof. The case $l = 0$ follows easily by using (2.21) and the fact that

$$|k| = \beta^{-1}|q| \leq \frac{1}{2}\beta^{-1}Q_2(|q|)$$

For $l \geq 1$, it is convenient to separate out the constant term 2^{2l} in Q_{2l} and note that from (2.21) and the definition of $Q_n(z)$ we have

$$|k|\int_{k'\in\mathbb{R}^3}\frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^{\mu}(1+|k-k'|)^{\mu}}2^{2l}dk' \leq \frac{C_0|k|e^{-\beta|k|}}{(1+|k|)^{\mu}}2^{2l} \leq \frac{C_0 e^{-\beta|k|}}{2\beta(1+|k|)^{\mu}}Q_{2l}(\beta|k|)$$

As in previous Lemma, for notational convenience, we define

$$k^m \bowtie k^n = |k|\int_{k'\in\mathbb{R}^3}\frac{e^{-\beta[|k'|+|k-k'|]}}{(1+|k'|)^{\mu}(1+|k-k'|)^{\mu}}|\beta k'|^m|\beta(k-k')|^n dk'$$

Then, it is clear from Lemma 6.8 that

$$\begin{aligned}
&[Q_{2l}(\beta|k|) - 2^{2l}] \bowtie k^0 = \sum_{n=1}^{2l}\frac{2^{2l-n}}{n!}k^n \hat{*} k^0 \\
&\leq \frac{2^{\mu+1}\pi e^{-\beta|k|}}{\beta^2(1+|k|)^{\mu}}\sum_{n=1}^{2l}\frac{2^{2l-n}}{n!}\left\{(n-1)!\sum_{j=0}^{n+1}\frac{2^{n+1-j}(\beta|k|)^j}{j!} + \frac{3(n+1)!|q|^{2/3}}{2\beta^{2/3}}\sum_{j=0}^{n+1}\frac{(\beta|k|)^j}{j!}\right\} \\
&\qquad \leq \frac{2^{\mu+1}\pi e^{-\beta|k|}}{\beta^2(1+|k|)^{\mu}}\left\{\sum_{j=0}^{2l+1}\frac{2^{2l+1-j}(\beta|k|)^j}{j!}\sum_{n'=\max\{j,2\}}^{2l+1}\frac{(n'-2)!}{(n'-1)!}\right. \\
&\qquad\qquad \left. + \frac{3|q|^{2/3}}{\beta^{2/3}2}\sum_{j=0}^{2l+1}\frac{2^{2l+1-j}(\beta|k|)^j}{j!}\sum_{n'=\max\{j,2\}}^{2l+1}2^{j-n'}n'\right\} \\
&\quad \leq \frac{2^{\mu+1}\pi e^{-\beta|k|}}{\beta^3(1+|k|)^{\mu}}\left[\beta Q_{2l+1}(\beta|k|)\log(2l+2) + 3(2l+1)\beta^{1/3}|\beta k|^{2/3}Q_{2l+1}(\beta|k|)\right]
\end{aligned}$$

The lemma follows since $\log(2l+2)/(2l+1) \leq 1$, while if $|\beta k| \leq (l+1)$,

$$\left(\frac{|\beta k|}{(l+1)}\right)^{2/3}Q_{2l+1}(\beta|k|) \leq Q_{2l+1}(\beta|k|) \leq \frac{1}{2}Q_{2l+2}(\beta|k|)$$

whereas for $\beta|k| \geq (l+1)$ we have

$$\left(\frac{|\beta k|}{(l+1)}\right)^{2/3}Q_{2l+1}(\beta|k|) \leq \frac{2\beta|k|}{2l+2}Q_{2l+1}(\beta|k|) \leq 2Q_{2l+2}(\beta|k|)$$

∎

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