On optimal truncation of divergent series solutions of nonlinear differential systems; Berry smoothing.

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Abstract

We prove that for divergent series solutions of nonlinear (or linear) differential systems near a generic irregular singularity, the common prescription of summation to the least term is, if properly interpreted, meaningful and correct, and we extend this method to transseries solutions. In every direction in the complex plane at the singularity (Stokes directions *not* excepted) there exists a nonempty set of solutions whose difference from the "optimally" (i.e., near the least term) truncated asymptotic series is of the same (exponentially small) order of magnitude as the least term of the series.

There is a family of generalized Borel summation formulas $\mathcal{B}$ which commute with the usual algebraic and analytic operations (addition, multiplication, differentiation, etc). We show that there is exactly one of them, $\mathcal{B}_0$, such that for any formal series solution $\tilde{f}$, $\mathcal{B}_0(\tilde{f})$ differs from the optimal truncation of $\tilde{f}$ by at most the order of the least term of $\tilde{f}$.

We show in addition that the Berry (1989) smoothing phenomenon is universal within this class of differential systems. Whenever the terms "beyond all orders" *change* in crossing a Stokes line, these terms vary smoothly on the Berry scale $\arg(x) \sim |x|^{-1/2}$ and the transition is always given by the error function; under the same conditions we show that Dingle's rule of signs for Stokes transitions holds.

1 Introduction

Summation to the least term (optimal truncation of series) can be traced back to Cauchy's study of the Gamma function (see Cauchy (1882)). The method was applied by Stokes (see Stokes (1904)) to solutions of differential equations and was instrumental in his theory of Stokes' phenomenon. Optimal truncation

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techniques have been greatly improved by Dingle (1973), and in recent years as a result of the new ideas and methods of hyperasymptotics introduced by Berry (1989), who also discovered a surprisingly universal transition in hyperasymptotics, often called Berry smoothing . Rigorous results on optimal truncation and smoothing under various assumptions were proved by McLeod (1992) (second order linear equations), Paris (1992) (linear systems with power coefficients), Olver (1995) (Gamma function), Olver and Olde Daalhuis (1995a,b) (linear second order systems) , Olde Daalhuis (1995, 1996) (linear second order systems), the authors (1996) (first order nonlinear and second order linear) and others. A wide range of interesting applications were considered by Berry and Howls (1990, 1993) (saddle point integrals), Berry (1994, 1995) and Berry and Keating (1992) (asymptotics of the zeta function, applications to dynamical systems, and PDE's). Other techniques of exponentially improved asymptotics have found interesting applications in dynamical systems (see Neishtadt 1984, Ramis and Schäfke 1996).

In this paper we prove that asymptotic series of solutions of generic analytic (linear or nonlinear) differential systems near irregular singularities are summable to the least term, with errors of the order of magnitude of the least term with respect to the functions associated to the series by a specific generalized Borel summation (the balanced average). In fact we show that the same is true for all series components of the transseries solutions of such differential systems.

We show that Berry smoothing extends to this class of asymptotic series.

We deal with equations of the form

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n \tag{1.1}$$

for large x in some direction, and study solutions that go to zero in this limit. The assumptions on (1.1) are

(a1) The function $\mathbf{f}$ is analytic at $(\infty, 0)$.

(a2) The eigenvalues λ_i of the linearization

$$\hat{\Lambda} := -\left(\frac{\partial f_i}{\partial y_j}(\infty, 0)\right)_{i,j=1,2,\dots n} \tag{1.2}$$

are different from zero and nonresonnant (cf. §1.1).

It is worth mentioning that a large class of differential systems are not presented in the form described above but can be brought to it by suitable changes of variables. For example, under the change of variables $2x^{3/2}/3 = t$, the Airy equation $y'' - xy = 0$ becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{1}{3t}\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$$

Also, the Painlevé I equation $y'' = 6y^2 + x$ becomes

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} + \frac{1}{t}\frac{\mathrm{d}h}{\mathrm{d}t} - h - \frac{3}{2}h^2 - \frac{392}{1875t^4} = 0$$

by taking $y(x) = i\sqrt{6x}/2[1/3 - 4t^{-2}/75 + h(t)]$ and $t = (-24x)^{5/4}/30$. In this form, written as systems, both equations satisfy our assumptions (see §1.1 and the notes therein). The transformations themselves, when they exist, are readily obtained by comparing the complete formal solution (transseries) of the original equation with the transseries of the normalized equation, (1.4) below.

It is convenient to pull out the inhomogeneous and the linear terms (relevant to leading order asymptotics) and rewrite the system in the form

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} - \frac{1}{x}\hat{B}\mathbf{y} + \mathbf{g}(x, \mathbf{y}) \tag{1.3}$$

Under the assumptions (a1) and (a2) equation (1.3) admits, in normalized form, an $n-$parameter family of formal exponential series solutions

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\geq 0; |\mathbf{k}|>0} C_1^{k_1}\cdots C_n^{k_n} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}} \tag{1.4}$$

(see §1.2) where $\tilde{\mathbf{y}}_{\mathbf{k}} = x^{-\mathbf{k}(\boldsymbol{\beta}+\mathbf{m})} \sum_{l=0}^{\infty} \mathbf{a}_{\mathbf{k};l} x^{-l}$ are formal power series.

We study those formal solutions (1.4) which are at the same time *asymptotic expansions* (or proper transseries, in the sense of Écalle 1993). In our context, only *finitely* many powers can be present, of any exponential which is *not small* in the given direction. Note however that any exponential will decrease in *some* direction, and our analysis eventually covers an n parameter family of formal solutions.

For *linear* equations, any formal solutions (1.4) contain finitely many exponentials and are proper transseries. For nonlinear equations we require, for large x in a given direction d in $\mathbb{C}$, that $C_i = 0$ for any i such that $\mathrm{e}^{-\lambda_i x} \not\to 0$ (more details in §1.2).

We start by illustrating the various concepts on a very simple example. The equation $f' + f = 1/x$ has the general solution $y(x) = e^{-x}\mathrm{Ei}(x) + C\mathrm{e}^{-x}$ ($\mathrm{Ei}(x) := PV\int_0^x t^{-1}e^t \mathrm{d}t$). The general *formal* solution for large x is the elementary transseries $\tilde{y}_C = \sum_{k=0}^{\infty} k! x^{-k-1} + C\mathrm{e}^{-x}$ with $C \in \mathbb{C}$ arbitrary. If $\arg(x) \in (\pi/2, 3\pi/2)$, the exponential term is large and classical asymptotics gives full meaning to the family $\tilde{y}_C$: for each $\tilde{y}_C$ there is a unique actual solution y_C of the equation such that $y_C \sim \tilde{y}_C$ for large x. But as $\arg(x)$ crosses an antistokes line $\arg(x) = (k+1/2)\pi$ into a sector where e^{-x} decays, e^{-x} becomes small *beyond all orders* of the divergent series and is undefined (as part of $\tilde{y}_C$) in classical asymptotics. Classical asymptotics gives up the exponential and retains the information that all solutions are asymptotic to $\tilde{y}_0$. In this framework there is no natural way for asymptotically choosing any privileged solution as corresponding to $\tilde{y}_0$; $\tilde{y}_0$ becomes more of a property of the differential equation as a whole, shared by all its particular solutions.

In terms of estimating solutions out of $\tilde{y}_0$, classical asymptotics provides *polynomial* precision: $|y(x) - \tilde{y}_0^{[N]}| \le \text{Const.}|x|^{-N-2}$ for large x, where $\tilde{y}_0^{[N]} := \sum_{k=0}^{N} k! x^{-k-1}$.

One of the most convenient and frequently used techniques going beyond Poincaré asymptotics is summation to the least term. When x is very large, the terms of $\tilde{y}_0$ start by decreasing rapidly ($k! x^{-k-1} \ll (k-1)! x^{-k}$ if $k \ll x$). It is tempting to keep adding these terms as long as they continue to decrease. The least term, for $k = k(x) = \lfloor |x| \rfloor$, evaluates to $k(x)!/x^{k(x)+1} \propto x^{-1/2}\mathrm{e}^{-|x|}$. Even taking into account the slight ambiguity as to where to stop adding the terms, the numerical precision attained at this stage appears to be high enough to measure the small exponential beyond all orders. It turns out that in any direction towards infinity there exists (see e.g. Costin and Kruskal 1996) an actual solution of the equation which lies within $O(x^{-1/2}\mathrm{e}^{-|x|})$ of the optimally truncated $\tilde{y}_0$ (a nearest solution); this solution, y_0, is then necessarily unique for any fixed $\arg(x)$, since different solutions are separated by $\text{Const.}\mathrm{e}^{-x}$.

As a manifestation of the Stokes phenomenon, the association between $\tilde{y}_0$ and y_0 changes with $\xi = \arg x$. In our example $y_0 = \mathrm{e}^{-x}\mathrm{Ei}(x) \pm \pi \mathrm{i}\mathrm{e}^{-x}$ if $\pm\xi > 0$ and $y_0 = e^{-x}\mathrm{Ei}(x)$ when $\xi = 0$.

Since the solutions y_0 do not have singularities for large x, nothing abrupt should happen to the terms beyond all orders, either. Michael Berry discovered that there is indeed an intermediate range $\arg x \sim |x|^{-1/2}$ where the constant C changes smoothly, in a surprisingly universal way.

We prove that Berry smoothing applies to all nonlinear systems satisfying (a1) and (a2), and the transition function is the same for all these systems. Technically, we show that Berry smoothing is governed by the dominant behavior at the singularities in Borel space nearest to the origin, and these singularities have the same essential features for all equations within our class. Going beyond nonresonance, the picture changes. However, properly interpreted, Berry smoothing might persist in many resonant cases. See Appendix 2 for a discussion.

As in the toy model above, in some cases (linear homogeneous second order and first order equations notably, see Costin and Kruskal 1996) there is *only one* solution of the original differential equation nearest to the divergent series. In general there is a restricted family of such solutions. Olver's (1964) example illustrates very well that these special solutions might not be the obvious ones. At least for this reason it is important to identify a method of recovering them from their transseries.

Hyperasymptotics as introduced by Berry leads to a substantial improvement in precision over optimal truncation but nevertheless ends up with nonzero errors. Recent very interesting developments of the original ideas of Berry due to Olde Daalhuis (1996), suggest that for some linear differential equations hyperasymptotics may eliminate all errors, but unfortunately at the price of giving up the convenience of optimal truncation by adding up vastly more terms beyond the least one in an extended family of expansions.

At present the only technique which at the same time gives full account of the complete formal solutions and is widely applicable is generalized Borel summation (see Écalle 1981 and 1993 , and also Costin 1995 and 1997 for rigorous results in the context of (1.3)). In this paper we show that there is a natural connection between Borel summation and optimal truncation, and that optimal truncation is compatible to a unique summation procedure, the balanced averaging introduced in Costin (1997).

1.1 Nonresonance

Let $\theta \in [0, 2\pi)$ and $\tilde{\boldsymbol{\lambda}} = (\lambda_{i_1}, ..., \lambda_{i_p})$ where $\left|\arg \lambda_{i_j} - \theta\right| \in (-\pi/2, \pi/2)$ (those eigenvalues contained in the *open* half-plane H_θ centered along $\mathrm{e}^{\mathrm{i}\theta}$). We require that for *any* θ:

(1) $\lambda_{i_1}, \ldots, \lambda_{i_p}$ are $\mathbb{Z}$-linearly independent.

(2) The complex numbers in the set $\{\tilde{\lambda}_i - \mathbf{k} \cdot \tilde{\boldsymbol{\lambda}} \in H_\theta : \mathbf{k} \in \mathbb{N}^p,\ i = 1, ..., p\}$ (note: the set is *finite*) have *distinct* directions. These are the Stokes lines $d_{i;\mathbf{k}}$.

That the set of $\boldsymbol{\lambda}$ which satisfy (1) and (2) has full measure follows from the fact that (1) and (2) follow from the condition:

$$\Big(\mathbf{m}, \mathbf{m}' \in \mathbb{Z}^n,\ \alpha \in \mathbb{R} \text{ and } (\mathbf{m} - \alpha\mathbf{m}') \cdot \boldsymbol{\lambda} = 0\Big) \Rightarrow \Big(\mathbf{m} = \alpha\mathbf{m}'\Big) \tag{1.5}$$

Indeed, if (1.5) fails, one of $\Re\lambda_j, \Im\lambda_j$ is a rational function with rational coefficients of the other $\Re\lambda_j$ and $\Im\lambda_j$, corresponding to a zero measure set in $\mathbb{R}^{2n}$.

Notes. (i) Since only small exponentials are allowed in a proper transseries, our conditions only restrict those (sets of) eigenvalues which are associated, through (1.4), to exponentials that can be simultaneously *small* in some direction.

(ii) If condition (1) fails, then higher terms in the transseries (which are present for nonlinear systems) may *resonate*, that is $\mathbf{k} \cdot \boldsymbol{\lambda} = \mathbf{k}' \cdot \boldsymbol{\lambda}$ for some $\mathbf{k} > 0, \mathbf{k}' > 0$ ($\mathbf{k} \neq \mathbf{k}'$). This affects the equations characterizing $\mathbf{y_k}$. However, we think that our results should not change in any substantial way unless in fact $\lambda_i = \lambda_j$ for some $i \neq j$. See Appendix 2 for examples of this latter situation.

(iii) In the Airy and P1 equations discussed in the introduction we have, after normalization, $\lambda_{1,2} = \pm 1$ and the assumptions of nonresonnance are satisfied.

(iv) For linear systems, a weaker nonresonnance condition would suffice for our analysis, namely that $\arg \lambda_i = \arg \lambda_j$ for $i \neq j$ (see Costin 1995). This condition, as well as (2) above, insures that the Stokes directions are distinct.

1.2 Normalization and notation.

By means of normal form calculations (changes of variables), assuming

(a1) and (a2), it is possible to prepare (1.3) so that (see Wasow 1968 and Tovbis 1992)

(n1) $\hat{\Lambda} = \mathrm{diag}(\lambda_i)$ and

(n2) $\hat{B} = \mathrm{diag}(\beta_i)$

(To this end, it is *not* necessary that the original matrix $\hat{B}$ in (1.3) is diagonalizable).

For convenience, we rescale x and reorder the components of $\mathbf{y}$ so that:

(n3) $|\lambda_i| \geq 1$ for $i = 1...n$, $\lambda_1 = 1$, and $\phi_i < \phi_j$ (cf. (a2)) if $i < j$, where $\phi = \arg(\lambda_i)$. (To simplify notations, we will formulate some of our results relative to λ_1; they can be easily adapted to any other λ_i, $|\lambda_i| = 1$.)

To unify the treatment, by taking $\mathbf{y} = \mathbf{y}_1 x^{-N}$ for some $N > 0$, we ensure that

(n4) $\Re(\beta_j) < 0,\ j = 1, 2, \ldots, n$.

(There is an asymmetry at this point: the opposite inequality cannot be achieved, in general, as simply and without violating analyticity at infinity.) Finally, through a transformation of the form $\mathbf{y} \leftrightarrow \mathbf{y} - \sum_{k=1}^{M} \mathbf{a}_k x^{-k}$ we arrange that

(n5) $\mathbf{f}_0 = O(x^{-M-1})$ and $\mathbf{g}(x, \mathbf{y}) = O(\mathbf{y}^2, x^{-M-1}\mathbf{y})$. We choose $M > 1 + \max_i \Re(-\beta_i)$.

Under the assumptions (a1) and (a2) and after the preparation (n1) through (n5), the system (1.3) admits an n–parameter family of formal exponential series solutions, (1.4) (see Iwano 1957 and Wasow 1968; in Costin 1997, a brief derivation in the context of (1.3) is given). In (1.4), $\mathbf{C} \in \mathbb{C}^n$ is an arbitrary vector of constants, and $m_i = 1 - \lfloor \beta_i \rfloor$.

The terms of an *asymptotic* expansion are well-ordered with respect to $\ll$. Thus, (1.4) is asymptotic in a given direction iff any *ascending* chain $\Re(-\mathbf{k}_1 \cdot \boldsymbol{\lambda} x) \leq \Re(-\mathbf{k}_2 \cdot \boldsymbol{\lambda} x) \leq \ldots$, $\mathbf{k}_i \neq \mathbf{k}_j$, in (1.4) is *finite* (agreeing to omit the terms with $C_i = 0$). For (1.3) the condition reads:

(n6) $\xi + \phi_i \in (-\pi/2, \pi/2)$ for all i such that $C_i \neq 0$. This selects the eigenvalues that lie in a half-plane centered at $\overline{x}$. Without loss of generality, we let $\arg(x)$ vary around $\arg(\lambda_1) = 0$. From now on, $\boldsymbol{\lambda} = (\lambda_{i_1}, \ldots, \lambda_{i_{n_1}})$, $\boldsymbol{\beta} = (\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_{n_1}})$, $\mathbf{m} = (m_{i_1}, m_{i_2}, \ldots, m_{i_{n_1}})$ and $\boldsymbol{\beta}' = \boldsymbol{\beta} + \mathbf{m}$ where the index is selected by (n6).

We will henceforth consider that (1.3) is presented in prepared form, and use the designation transseries only for those formal solutions satisfying (n6).

With respect to Borel summation we are using the results and notation of Costin (1995 and 1997). (A summary is presented in Appendix 1.)

2 Main results

All the analysis in this paper is based on the structure of singularities in Borel space of the solutions of (1.1) and does not otherwise use (1.1). The results and proofs can be easily adapted to other functions than solutions of differential systems, having a similar singularity structure.

To simplify the notation, as mentioned in the introduction, we formulate all results relative to $\lambda_1 = 1$; they can be reformulated without any difficulty relative to any other λ_i of modulus one. We can reduce the discussion to the case when

$$S_1 \neq 0$$

(cf. theorem 7; see §2.2 for a discussion).
(a) Behavior for large $r \in \mathbb{N}$ of the coefficients $a_{r,0}$ in the asymptotic series $\tilde{\mathbf{y}}_0$.
Let

$$\Gamma_{r,j} = \frac{\Gamma(r - \beta_j + 1)}{2\pi \mathrm{i}\, \mathrm{e}^{\mathrm{i}(r+1-\beta_j)\phi_j}} \tag{2.1}$$

Proposition 1 *For $r \to \infty$ with $r \in \mathbb{N}$ we have*

$$\mathbf{a}_{r+1,0} = \mathbf{Y}_0^{(r)}(0) = \sum_{j;|\lambda_j|=1} (\mathbf{e}_j S_j + \mathbf{h}_j(r))\, \Gamma_{r,j} \tag{2.2}$$

where $\mathbf{h}_j(r) \sim r^{-1} \sum_{k=0}^{\infty} \mathbf{h}_{j;k} r^{-k}$ for large r. The minimum of $|\mathbf{a}_{r,0} x^{-r}|$ is reached for $|r - |x|| \le Const.$ where the constant does not depend on x, r.

Remark. We can write $\mathbf{a}_{r,0} = \sum_{j;|\lambda_j|=1} \mathbf{a}_{r,0;[j]}$ with $\mathbf{a}_{r,0;[j]} \sim \mathbf{e}_j \Gamma_{r,j}$ and thus, defining $\tilde{\mathbf{y}}_{0;[j]} = \sum_r \mathbf{a}_{r,0;[j]} x^{-r}$ we have

$$\tilde{\mathbf{y}}_0 = \sum_{j;|\lambda_j|=1} \tilde{\mathbf{y}}_{0;[j]}$$

The terms of each series $\tilde{\mathbf{y}}_{0;[j]}$ have the same argument (phase) iff x belongs to the j-th Stokes line, i.e. iff $\arg(x) = \phi_j$. This is a precise formulation of *Dingle's rule of signs* (cf. Dingle 1973, pp.7). For resonant differential systems the rule needs to be reinterpreted. See also the Appendix for some further comments.
(b) Difference between solutions and their optimally truncated asymptotic series.

Theorem 2 *(i) Let $\mathbf{y}$ be a solution of (1.3) and $x = r\mathrm{e}^{\mathrm{i}\xi}$, with $r \in \mathbb{N}$ and $\xi \in [0, 2\pi)$. Then the difference between $\mathbf{y}$ and the optimal truncation of $\tilde{\mathbf{y}}_0$ is of the order of magnitude of the least term,*

$$\mathbf{y}(x) - \tilde{\mathbf{y}}_0^{[r]}(x) = O(\mathbf{a}_{0,r} r^{-r}) \quad (\textit{for fixed } \xi, \textit{ as } r \to \infty) \tag{2.3}$$

if and only if: (1) $\mathbf{y}$ is the balanced Borel sum (see Eq. (4.9) and theorem 8) of the formal solution $\tilde{\mathbf{y}}_0$, i.e.,

$$\mathbf{y} = \mathcal{L}\mathcal{B}_{\frac{1}{2}} \tilde{\mathbf{y}}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda} x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathcal{B}_{\frac{1}{2}} \tilde{\mathbf{y}}_{\mathbf{k}} \tag{2.4}$$

and (2) $C_j = 0$ for all j such that $|\lambda_j| = 1$.

(ii) A similar estimate holds for the higher series in the transseries: For any $\mathbf{k}$ and large r we have

$$\mathcal{L}\mathcal{B}_{\frac{1}{2}} \tilde{\mathbf{y}}_{\mathbf{k}}(x) - \tilde{\mathbf{y}}_{\mathbf{k}}^{[r]} = O(\mathbf{a}_{\mathbf{k},r} r^{-r}) \quad (\textit{as } r \to \infty) \tag{2.5}$$

(c) Berry smoothing. We now focus on the thin parabolic region (the Berry scale) $|x| \gg 1$, $\arg(x) - \arg(\lambda_j) = \Omega |x|^{-1/2}$ near a Stokes line, say $\mathbb{R}^+$ (the Stokes line for λ_1).

Theorem 3 shows that on the Berry scale near a Stokes line, the constant C ("beyond all orders") of a given solution changes, and the transition is given, as predicted by Berry, by an error function.

Theorem 3 *Let $x = r e^{\mathrm{i}\Omega r^{-1/2}}$, with $\Omega \in \mathbb{R}$ fixed and $r \in \mathbb{N}$; as $r \to \infty$ we have*

$$\mathcal{L}\mathcal{B}_{\frac{1}{2}} \tilde{\mathbf{y}}_0(x) - \tilde{\mathbf{y}}_0^{[r]} = \frac{1}{2} S_1 \operatorname{erf}\left(\frac{\Omega}{\sqrt{2}}\right) \mathbf{y}_{\mathbf{e}_1} + o(e^{-x} x^{-\beta'}) = \\ \frac{1}{2} S_1 \operatorname{erf}\left(\frac{\Omega}{\sqrt{2}}\right) e^{-x} x^{-\beta'} \mathbf{e}_1 + o(e^{-x} x^{-\beta'}) \quad (2.6)$$

We note here that there always exist directions along which the smoothing is visible (namely the directions ϕ_i such that $|\lambda_i| = 1$ in our normalization). For other Stokes directions, $O(\mathbf{a}_{0,r} r^{-r})$ may be much larger than the contribution of erfc. For the directions of λ_j with $|\lambda_j| > 1$ the smallest term of the series is too large for the Berry transition to be seen. The change in the terms beyond all orders when crossing the Stokes line is always measurable by Borel summation (cf. theorem 9 (ii) and Costin 1997) but might be too small for optimal truncation.

(d) Connection with Borel summation.

In the context of generic nonlinear systems there is a one parameter family of *distinct* generalized Borel summation formulas which are compatible with all usual operations (addition, multiplication, differentiation, composition and their inverses, when meaningful) and which preserve reality and asymptotic inequalities (see Costin 1997). In other words, the summation operators cannot be intrinsically distinguished, in terms of their algebraic and/or classical asymptotics properties.

Nevertheless, proposition 4 shows that only the generalized Borel summation with $\alpha = 0$, the balanced average, is compatible with least term truncation. Balanced average is thus the only Borel summation method that guarantees optimal least term truncation properties (incidentally, the balanced average is also the most symmetric and "natural" in its family). We note that in our context it can be shown that the balanced average equals the (apparently) more complicated median average of Ecalle.

Proposition 4 *Let $\mathbf{y} = \mathcal{L}\mathcal{B}_\alpha \tilde{\mathbf{y}}_0$. Then*

$$\mathbf{y}(r e^{\mathrm{i}\xi}) - \tilde{\mathbf{y}}_0^{[r]}(r e^{\mathrm{i}\xi}) = O(\mathbf{a}_{0,r} r^{-r}) \quad (2.7)$$

for large r and all ξ if and only if $\alpha = 0$.

Note: with $\tilde{\mathbf{y}}_0$ fixed, $\mathbf{y}$ is a different solution in different Stokes sectors, see theorem 9.

2.1 Proofs and further results

To simplify the exposition, detailed proofs are provided for the principal series $\tilde{\mathbf{y}}_0$. There are only minor adjustments necessary to cover all other components $\tilde{\mathbf{y}}_{\mathbf{k}}$ of the transseries; we mention them at the end. The notation is the same as in Costin (1997) and is explained in §4.1. We let $\mathbf{Y} = \mathbf{Y}_0$.

Taking

$$\mathbf{y} = \int_{\mathcal{C}} \mathbf{Y}(p) \mathrm{e}^{-xp} dp \tag{2.8}$$

where $\mathcal{C}$ is a contour in $\mathcal{R}$ such that $\Re(xp) > 0$ (or a linear combination of such contours as in the balanced average), we get by successive integrations by parts

$$\mathbf{y}(x) = \sum_{k=0}^{r} \mathbf{Y}^{(k)}(0) x^{-k-1} + x^{-r-1} \int_{\mathcal{C}} \mathbf{Y}^{(r+1)}(p) \mathrm{e}^{-px} dp \tag{2.9}$$

Taking $x = r\mathrm{e}^{-\mathrm{i}\varphi}$ we get

$$\mathbf{y}(x) - \sum_{k=0}^{r} \frac{\mathbf{Y}^{(k)}(0)}{x^{k+1}} = r^{-r-1} \mathrm{e}^{-\mathrm{i}(r+1)\varphi} \int_{\mathcal{C}} \mathbf{Y}^{(r+1)}(p) \mathrm{e}^{-pr\mathrm{e}^{-\mathrm{i}\phi}} \mathrm{d}p \tag{2.10}$$

Equation (2.10) represents the error in summation near the least term; we will show that for large r

$$\int_{\mathcal{C}} \mathbf{Y}^{(r+1)}(p) \mathrm{e}^{-pr\mathrm{e}^{-\mathrm{i}\phi}} \mathrm{d}p = O\left(\mathbf{Y}^{(r)}(0)\right) \tag{2.11}$$

2.2 Asymptotic behavior of $\mathbf{Y}^{(r)}$ for large $r \in \mathbb{N}$

(1) If, exceptionally, all Stokes constants are zero (i.e. $S_j = 0$ (cf. (4.5)) for $j = 1, \ldots, n$), then $\mathbf{Y}$ is entire and $|\mathbf{Y}(p)| \le \exp(\nu|p|)$ for some ν (theorems 7 and contour,

$$|\mathbf{Y}^{(r)}(0)| = \left| \frac{r!}{2\pi\mathrm{i}} \oint \frac{\mathbf{Y}(s)}{s^{r+1}} ds \right| \le \frac{r!}{2\pi} \frac{\exp(\nu r)}{r^{r+1}} 2\pi r \sim \sqrt{2\pi r}\mathrm{e}^{r(\nu-1)} \tag{2.12}$$

for large r. This implies at once that $\tilde{\mathbf{y}}_0$ is *convergent* in $1/x$. This case is thus trivial: summation to the least term is nothing else than convergent summation, and the sum is a true solution of the equation.

(2) By theorem 7 for all i such that $S_i = 0$, $\mathbf{Y}$ is analytic along $\mathbb{R}^+\lambda_i$. We thus only count the λ_i with $S_i \ne 0$ and rescale variables and change notations so that $\min\{|\lambda_i| : S_i \ne 0\} = 1$ and $\beta_0 = \Re(\beta_1) = \min\{\Re(\beta_i) : |\lambda_i| = 1, \quad S_i \ne 0\}$.

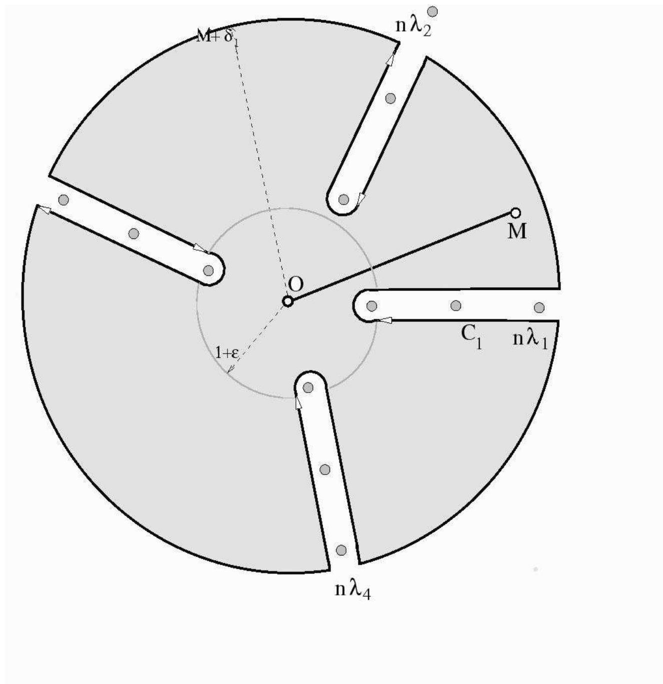

Fig. 1

Proposition 5 *For large $r \in \mathbb{N}$ we have*

$$\mathbf{Y}^{(r)}(0) = \sum_j \frac{S_j(r+m_j)!}{2\pi\mathrm{i}} \int_{\lambda_j}^{\lambda_j(1+\epsilon)} \frac{\mathbf{Y}_{\mathbf{e}_j}(s)}{s^{r+m_j+1}} ds + O(r!(1+\epsilon)^{-r}) \qquad (2.13)$$

Proof. Let $m = 1 + \max_j m_j$. Then $\mathcal{P}^m \mathbf{Y}(p\mathrm{e}^{\mathrm{i}\phi})$ is continuous in ϕ in $[0, 2\pi] \backslash \{\phi_i, i = 1...n\}$ and has at most jump discontinuities at $\{\phi_i, i = 1...n\}$ (theorem 8). Writing $2\pi\mathrm{i}(r+m)!\mathbf{Y}^{(r)}(0) = \oint \mathcal{P}^m \mathbf{Y}(s) s^{-r-m-1} ds$, pushing the contour of integration towards the circle of radius $1+\epsilon$ (Fig. 1), and letting $K = \max_\phi \mathcal{P}^m \mathbf{Y}((1+\epsilon)\mathrm{e}^{\mathrm{i}\phi})$ and $N_1 = \#\{i : S_i \neq 0\}$ we have

$$\left| \oint \mathcal{P}^m \mathbf{Y}(s) s^{-r-m-1} ds - \sum_{i=1}^{N_1} \int_{C_i} \mathcal{P}^m \mathbf{Y}(s) s^{-r-m-1} ds \right|$$

$$\leq 2\pi K(1+\epsilon)^{-(r+m+1)} \tag{2.14}$$

Using theorem 9 we have, with $E_{1,2} = O(r!(1+\epsilon)^{-r})$,

$$\begin{aligned}
\mathbf{Y}^{(r)}(0) &= \frac{(m+r)!}{2\pi\mathrm{i}} \sum_{j=1}^{N_1} \int_{\lambda_j}^{\lambda_j(1+\epsilon)} S_j \mathcal{P}^{m-m_j} \mathbf{Y}_{j\mathbf{e}_j}(s-\lambda_j) s^{-r-m-1} ds + E_1 \\
&= \sum_{j=1}^{N_1} \frac{(m_j+r)!}{2\pi\mathrm{i}} \int_{\lambda_j}^{\lambda_j(1+\epsilon)} S_j \mathbf{Y}_{j\mathbf{e}_j}(s-\lambda_j) s^{-r-m_j-1} ds + E_2 \qquad (2.15)
\end{aligned}$$

where the j-th term has been integrated by parts $m - m_j$ times and we used theorem 7 to show that the boundary terms vanish at $s = \lambda_j$, and are $O(r!(1+\epsilon)^{-r})$ at $\lambda_j(1+\epsilon)$. By theorem 7 (ii) we have for small p

$$\mathbf{Y}_{j\mathbf{e}_j}(p) = p^{\beta'_j - 1} \mathbf{A}_j(p) \tag{2.16}$$

and $\mathbf{A}_j(0) = \mathbf{e}_j / \Gamma(\beta'_j)$. Changing variables to $s = \lambda_j(1+z)$ in (2.15) we get, by Laplace's method,

$$\begin{aligned}
&\frac{(m_j+r)!}{2\pi\mathrm{i}} S_j \lambda_j^{\beta'_j - m_j - r - 1} \int_0^{\epsilon} z^{\beta'_j - 1} (\mathbf{e}_j + \mathbf{G}_j(z)) \mathrm{e}^{-(m_j+r+1)\ln(1+z)} \mathrm{d}z \\
&\qquad = \frac{S_j (r+m_j)!}{2\pi\mathrm{i}(r+m_j+1)^{\beta'_j} \lambda_j^{r+1-\beta_j}} (\mathbf{e}_j + \mathbf{H}_j(r)) \qquad (2.17)
\end{aligned}$$

where $\mathbf{G}_j(z) = O(z)$ are analytic and $O(z)$ for small z and thus the $\mathbf{H}_j(r)$ have an asymptotic series in r^{-1}, and $\mathbf{H}_j(r) = O(r^{-1})$. Combining all these estimates we get

$$\mathbf{Y}^{(r)}(0) = \sum_{j=1}^{N_1} \frac{\Gamma(r - \beta_j + 1)}{2\pi\mathrm{i}\, \mathrm{e}^{\mathrm{i}(r+1-\beta_j)\phi_j}} (S_j \mathbf{e}_j + O(r^{-1})) \tag{2.18}$$

so that, with $x = r\mathrm{e}^{-\mathrm{i}\varphi}$,

$$\frac{\mathbf{Y}^{(r)}(0)}{x^{r+1}} \quad = \quad r^{-r-1} \mathrm{e}^{\mathrm{i}(r+1)\varphi} \sum_{j;|\lambda_j|=1} \frac{S_j \Gamma(r - \beta_j + 1)}{2\pi\mathrm{i}\, \mathrm{e}^{\mathrm{i}(r+1-\beta_j)\phi_j}} (\mathbf{e}_j \ + \ o(1/r)) \tag{2.19}$$

Because of the $\mathbf{e}_j$, the factors inside the sum cannot cancel each–other. The rest of proposition 5 is immediate.

3 Asymptotics of Laplace Integrals

We distinguish two cases:

3.1 Regular directions

If $\arg(x) = \xi$ then, by convention, the direction of the Laplace transform integral is $\arg p = \varphi = -\xi$.

If $\varphi \neq \phi_i$ then $\mathbf{Y}$ is analytic and exponentially bounded along $\arg p = \varphi$ (theorems 7 and 8). We may assume $\varphi \in (0, \phi_2)$ (cf. §2.2). Let $M \in \mathbb{N}$ be large, subject to the following conditions:

(c_1) $\varphi \pm \arcsin(M^{-1}) \in (0, \phi_2)$.

(c_2) $\mathrm{dist}(\{t\mathrm{e}^{\mathrm{i}\varphi} : t \geq M\}, \{\mathbb{R}\lambda_i\}_{i=1..n}) > \alpha > 1$

(c_3) $M\cos(\varphi) > 2$.

Let $\delta_1 \geq 2\mathrm{e}$ be such that $M + \delta_1 \notin \{\mathbb{N}|\lambda_i| : i = 1, 2, \ldots, n\}$ (in particular, there are no singular points of $\mathbf{Y}$ on the circle of radius $M + \delta_1$). Let $K_1 = \max_{t \leq M+\delta_1} |\mathbf{Y}(t\mathrm{e}^{\mathrm{i}\varphi})|$ and $K_2 = \max_{|s|=M+\delta_1} |\mathbf{Y}(s)|$.

Let $N = M + \lfloor \delta_1 \rfloor$. By theorem 7, $\mathbf{Y}(p)$ is analytic in $\mathcal{J}_M = \{|p| < M + \delta_1, \arg(p) \neq \phi_i\}$ and by theorem 8 (i), $\mathcal{P}^{N|\mathbf{m}|}\mathbf{Y}$ is uniformly continuous in the interior of $\mathcal{J}_M$; let $K_3 = \max_{|s|=M+\delta_1; j \leq |\mathbf{m}|} |\mathcal{P}^{Nj}\mathbf{Y}(s)| < (K_1 + K_2)(2N|\mathbf{m}|)^{N|\mathbf{m}|}$. By theorems 7 and 8 (ii) we can choose ν large enough so that for $\arg(p) \pm \arcsin(M^{-1}) \in (0, \phi_2)$

$$|\mathbf{Y}(p)| \leq \exp(\nu(|p| + 1)) \tag{3.1}$$

We note that M, ν, K_3, N, δ_1 are chosen independently of r and of x.

By (c_2) and (3.1) for $t \geq M$ we have

$$|\mathbf{Y}^{(r)}(t\mathrm{e}^{\mathrm{i}\varphi})| \leq \frac{r!}{2\pi} \exp(\nu(t + 2)) \tag{3.2}$$

For $|x| > M\nu(M - 2)^{-1}$ we have

$$\left| \int_{M\mathrm{e}^{\mathrm{i}\varphi}}^{\infty\mathrm{e}^{\mathrm{i}\varphi}} \mathrm{e}^{-px}\mathbf{Y}^{(r)}(p)\mathrm{d}p \right| \leq \frac{r!}{2\pi} \frac{\mathrm{e}^{2\nu}}{\nu - |x|} e^{M(\nu-|x|)} \leq K_4 r! \mathrm{e}^{-2|x|} \tag{3.3}$$

where K_4 depends only on ν and M. By construction, for $t \leq M$ we have $|t\mathrm{e}^{\mathrm{i}\varphi} - s| > 2\mathrm{e}$ if $|s| = M + \delta_1$. We then have (see Fig 1.),

$$\begin{aligned}
\mathbf{Y}^{(r)}(t\mathrm{e}^{\mathrm{i}\varphi}) &= \frac{(N|\mathbf{m}| + r)!}{2\pi\mathrm{i}} \oint_{\partial\mathcal{J}_M} \frac{\mathcal{P}^{N|\mathbf{m}|}\mathbf{Y}(s)}{(s - \mathrm{t}\mathrm{e}^{\mathrm{i}\varphi})^{r+N|\mathbf{m}|+1}} \mathrm{d}s \\
&= \frac{(N|\mathbf{m}| + r)!}{2\pi\mathrm{i}} \sum_i \int_{C_i} \frac{\mathcal{P}^{N|\mathbf{m}|}\mathbf{Y}(s)}{(s - \mathrm{t}\mathrm{e}^{\mathrm{i}\varphi})^{r+N|\mathbf{m}|+1}} \mathrm{d}s + r!\mathrm{e}^{-2r} E_2 \\
&= \sum_i f_i \int_{C_i} \frac{\mathcal{P}^{Nm_i}\mathbf{Y}(s)}{(s - \mathrm{t}\mathrm{e}^{\mathrm{i}\varphi})^{r+Nm_i+1}} \mathrm{d}s + r!\mathrm{e}^{-2r} E_3
\end{aligned} \tag{3.4}$$

where

$$f_i = (Nm_i + r)!/(2\pi \mathrm{i})$$

E_2 contains the contribution of the outer circle, and in the last equality we integrated by parts $|\mathbf{m}| - m_i$ times and included the boundary terms in E_3. We have $|E_2| + |E_3| \le E_4$ for some E_4 depending only on K_3, M and $|\mathbf{m}|$. In conclusion,

$$\begin{aligned}\int_0^{\infty e^{\mathrm{i}\varphi}} e^{-px}\mathbf{Y}^{(r)}(p)dp &= \sum_{j=1}^{n} f_j \int_0^{M e^{\mathrm{i}\varphi}} e^{-px}\mathrm{d}p \int_{C_j} \frac{\mathcal{P}^{m_j N}\mathbf{Y}(s)}{(s - p e^{\mathrm{i}\varphi})^{Nm_j+r+1}}\mathrm{d}s \\ + r!(e^{-2r} + e^{-2|x|})E_5 &= \sum_{j=1}^{n} f_j \int_{C_j} \mathrm{d}s\mathcal{P}^{m_j N}\mathbf{Y}(s) \int_0^{M e^{\mathrm{i}\varphi}} \frac{e^{-px}\mathrm{d}p}{(s - p e^{\mathrm{i}\varphi})^{Nm_j+r+1}} \\ &\qquad + r!(e^{-2r} + e^{-2|x|})E_5 \end{aligned} \tag{3.5}$$

for large enough r, x where E_5 is independent of r, x.

We have, with $p = t e^{\mathrm{i}\varphi}, x = r e^{-\mathrm{i}\varphi}$, $R = m_j N + 1$:

$$\begin{aligned}\int_0^{M e^{\mathrm{i}\varphi}} \frac{e^{-px}}{(s-p)^{r+R}}dp &= \int_0^{\infty e^{\mathrm{i}\varphi}} \frac{e^{-px}}{(s-p)^{r+R}}dp + O(e^{-M(r+1)}) \\ &= e^{\mathrm{i}\varphi}\int_0^{\infty} \frac{e^{-tr}}{(s - t e^{\mathrm{i}\varphi})^{r+R}}dt + O(e^{-M(r+1)}) \end{aligned} \tag{3.6}$$

The behavior for large r of the last integral is obtained by standard steepest descent: The function $f(t) = e^{-t}(\kappa - t)^{-1}$ has a saddle point at $t = \kappa - 1$ and, for $\kappa \notin [0, \infty]$ the steepest descent path originating at zero starts in a direction opposite to the pole κ and continues to $+\infty$. For $\arg(s) \neq \varphi$, the contour in (3.6) can thus be deformed to a steepest descent curve without crossing the pole or the saddle. The main contribution to the integral comes then from a region where t is small. We have the estimate for (3.6):

$$\begin{aligned}\int_0^{\infty} \frac{e^{-tr}}{(s - t e^{\mathrm{i}\varphi})^{r+R}}dt &= \int_0^{\infty} \exp\left(-(r+R)\ln(s - t e^{\mathrm{i}\varphi}) - tr\right) dt \\ &= \frac{1}{s^{r+R}}\int_0^{\infty} \exp(-t(r - (r+R)e^{\mathrm{i}\varphi}s^{-1}) + O(t^{-2}))dt \\ &= [rs^{r+R}(e^{\mathrm{i}\varphi}s^{-1} - 1)]^{-1}(1 + O(r^{-1})) \end{aligned} \tag{3.7}$$

As $\mathcal{P}^{Nm_j}\mathbf{Y}^{\pm}$ are continuous along C_j, we obtain

$$\left| \int_{\lambda_j(1+\epsilon)}^{(M+\delta_1)e^{\mathrm{i}\phi_j}} s^{-r-R}\frac{\mathcal{P}^{Nm_j}\mathbf{Y}^{\pm}(s)}{e^{\mathrm{i}\varphi}s^{-1} - 1}\mathrm{d}s \right| \le (1+\epsilon)^{-r-R}E_6 \tag{3.8}$$

for some E_6 independent of r and x. Since $\mathcal{P}^{m_j}\mathbf{Y}^{\pm}$ are continuous for small ϵ on the segment between λ_j and $\lambda_j(1+\epsilon)$ we have, using (4.13), (3.8) and integration by parts, with c_j being the part of C_j within ϵ distance of λ_j,

$$
\begin{aligned}
f_j \int_{C_j} \mathrm{d}s \mathcal{P}^{m_j N}\mathbf{Y}(s) \int_0^{M\mathrm{e}^{\mathrm{i}\varphi}} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p\mathrm{e}^{\mathrm{i}\varphi})^{Nm_j+r+1}} \\
= f_j \int_{c_j} \mathrm{d}s \mathcal{P}^{m_j N}\mathbf{Y}(s) \int_0^{M\mathrm{e}^{\mathrm{i}\varphi}} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p\mathrm{e}^{\mathrm{i}\varphi})^{Nm_j+r+1}} + O((1+\epsilon)^{-r}) \\
= f_j \int_{c_j} \mathcal{P}^{m_j}\mathbf{Y}(s)(-\mathcal{P})^{m_j(N-1)} \int_0^{M\mathrm{e}^{\mathrm{i}\varphi}} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p\mathrm{e}^{\mathrm{i}\varphi})^{Nm_j+r+1}} + O((1+\epsilon)^{-r}) \\
= e_j \int_{\lambda_j}^{\lambda_j(1+\epsilon)} S_j \mathbf{Y}_{\mathbf{e}_j}(s-\lambda_j) \int_0^{M\mathrm{e}^{\mathrm{i}\varphi}} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p\mathrm{e}^{\mathrm{i}\varphi})^{m_j+r+1}} + O((1+\epsilon)^{-r}) \\
= e_j \int_{\lambda_j}^{\lambda_j(1+\epsilon)} \mathrm{d}s \frac{S_j \mathbf{Y}_{\mathbf{e}_j}(s-\lambda_j)}{rs^{r+m_j+1}(\mathrm{e}^{\mathrm{i}\varphi}s^{-1}-1)}(1+O(r^{-1})) + O((1+\epsilon)^{-r}) \qquad (3.9)
\end{aligned}
$$

with

$$e_j = (m_j + r)!/(2\pi\mathrm{i})$$

The last integral is very similar to (2.15) and is estimated in the same way, giving

$$
\begin{aligned}
e_j \int_{\lambda_j}^{\lambda_j(1+\epsilon)} \mathrm{d}s \frac{S_j \mathbf{Y}_{\mathbf{e}_j}(s-\lambda_j)}{rs^{r+m_j+1}(\mathrm{e}^{\mathrm{i}\varphi}s^{-1}-1)} = \\
= \frac{S_j(r+m_j)!}{2\pi\mathrm{i}r^2(r+m_j+1)^{\beta_j'-1}\lambda_j^{r+1-\beta_j}(\mathrm{e}^{\mathrm{i}\varphi}\lambda_j^{-1}-1)}(\mathbf{e}_j + O(r^{-1})) \qquad (3.10)
\end{aligned}
$$

where, in view of (2.10) we need to take $r+1$ instead of r. Noting that $(r+m_j)!r^{-\beta_j'-1} = \Gamma(r-\beta_j+1)(1+O(r^{-1}))$ and comparing (3.10) to (2.18), (2.11) is proven.

□

3.2 Stokes directions; balanced averages; Berry smoothing

This section deals with the special and important case of exponential asymptotics on and near the Stokes line corresponding to one of the λ_i of largest module ($|\lambda_i| = 1$ in our normalization; without loss of generality we take $i = 1$). Berry's smoothing formula is proved, and we show that the balanced average is the only Borel summation process compatible with optimal truncation. The generalized Borel summations that have good algebraic and analytic properties are given by (4.9). Using theorem 7 (i) which shows that for $p \in (0,1)$ we have

$\mathbf{Y}_0^+(p) = \mathbf{Y}_0^-(p)$, formula (4.13), and the estimates in theorem 8 (i) for $(\mathcal{L}\mathbf{Y}_\mathbf{k})_\gamma$ we see that

$$\mathbf{y} := \mathbf{y}_\alpha = \mathcal{L}\mathcal{B}_\alpha \tilde{\mathbf{y}}_0 = (1-\alpha)\mathcal{L}\mathbf{Y}_0^+ + \alpha\mathcal{L}\mathbf{Y}_0^- + O(\mathrm{e}^{-(2-\epsilon)x}) \tag{3.11}$$

Choosing $0 < \delta < \min_i |\arg(\lambda_1) - \arg(\lambda_i)|$, relation (2.10) reads, in view of theorem 8,

$$\mathbf{y}(x) - \sum_{k=0}^{r} \frac{\mathbf{Y}^{(k)}(0)}{x^{k+1}} = r^{-r-1}\mathrm{e}^{-\mathrm{i}(r+1)\varphi} \int_{<\alpha;\infty>} \mathbf{Y}^{(r+1)}(p)\mathrm{e}^{-pr\mathrm{e}^{-\mathrm{i}\varphi}}\mathrm{d}p + O(\mathrm{e}^{-(2-\epsilon)x}) \tag{3.12}$$

where $\int_{<\alpha;a>}$ means

$$\alpha \int_0^{a\mathrm{e}^{-\mathrm{i}\delta}} + (1-\alpha)\int_0^{a\mathrm{e}^{\mathrm{i}\delta}}$$

In order to prove theorem 3 and proposition 4, we need to estimate (3.12) near the Stokes line. Replacing everywhere $\int_0^{M\mathrm{e}^{\mathrm{i}\varphi}}$ by $\int_{<\alpha;M>}$ and taking $\varphi = 0$, the calculations leading to (3.5) work without any other change and we get:

$$\begin{aligned} \int_{<\alpha;\infty>} \mathrm{e}^{-px}\mathbf{Y}^{(r)}(p)dp &= \sum_{j=1}^{n} f_j \int_{<\alpha;M>} \mathrm{e}^{-px}\mathrm{d}p \int_{C_j} \frac{\mathcal{P}^{m_jN}\mathbf{Y}(s)}{(s-p)^{Nm_j+r+1}}\mathrm{d}s \\ + r!E_5(e^{-2r} + e^{-2|x|}) &= \sum_{j=1}^{n} f_j \int_{C_j} \mathrm{d}s\mathcal{P}^{m_jN}\mathbf{Y}(s) \int_{<\alpha;M>} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p)^{Nm_j+r+1}} \\ &\quad + r!E_5(e^{-2r} + e^{-2|x|}) \end{aligned} \tag{3.13}$$

For $j \neq 1$, $s - p$ does not vanish for $s \in C_j$, $|\arg(p)| < \delta$, and thus

$$\begin{aligned} &\sum_{j=2}^{n} f_j \int_{C_j} \mathrm{d}s\mathcal{P}^{m_jN}\mathbf{Y}(s) \int_{<\alpha;M>} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p)^{Nm_j+r+1}} \\ &\qquad = \sum_{j=2}^{n} f_j \int_{C_j} \mathrm{d}s\mathcal{P}^{m_jN}\mathbf{Y}(s) \int_0^M \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p)^{Nm_j+r+1}} \end{aligned} \tag{3.14}$$

and the estimates leading to (3.10) are valid if we take $\varphi = 0$. We get, for $j \neq 1$,

$$\begin{aligned} &f_j \int_{C_j} \mathrm{d}s\mathcal{P}^{m_jN}\mathbf{Y}(s) \int_0^M \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p)^{Nm_j+r+1}} \\ &\quad = e_j \int_{\lambda_j}^{\lambda_j(1+\epsilon)} \mathrm{d}s \frac{S_j\mathbf{Y}_{\mathbf{e}_j}(s-\lambda_j)}{rs^{r+m_j+1}(s^{-1}-1)}(1+O(r^{-1})) + O((1+\epsilon)^{-r}) \end{aligned} \tag{3.15}$$

and

$$
\begin{aligned}
e_j \int_{\lambda_j}^{\lambda_j(1+\epsilon)} & \frac{S_j \mathbf{Y}_{\mathbf{e}_j}(s-\lambda_j)}{r s^{r+m_j+1}(s^{-1}-1)}(1+O(r^{-1})) = \\
&= \frac{S_j (r+m_j)!}{2\pi \mathrm{i} r (r+m_j+1)^{\beta_j'} \lambda_j^{r+1-\beta_j}(\lambda_j^{-1}-1)}(\mathbf{e}_j + O(r^{-1})) \quad (3.16)
\end{aligned}
$$

We are therefore left with the problem of estimating

$$
J := \int_{C_1} \mathrm{d}s \mathcal{P}^{m_1 N} \mathbf{Y}(s) \int_{<\alpha;M>} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p)^{Nm_1+r+1}} \quad (3.17)
$$

We consider the case $\Omega \le 0$, the other case being treated symmetrically.

Proposition 6 *Let $\delta \in (0,\pi/2)$, $r_1 \in \mathbb{N}$, $x = r\mathrm{e}^{\mathrm{i}\Omega r^{-1/2}}$, $\Omega \le 0$, $s \ge 1$ and $R = r + r_1$.*

i) For large $r \in \mathbb{N}$

$$
\begin{aligned}
g_\alpha &= \alpha \int_0^{M\mathrm{e}^{\mathrm{i}\delta}} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p)^R} + (1-\alpha)\int_0^{M\mathrm{e}^{-\mathrm{i}\delta}} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p)^R} \\
&= \frac{\mathrm{i}r^{-1/2}}{s^{R-1}}\left(\int_0^\infty \exp\left\{-\frac{1}{2}t^2 + \left[\Omega_r - \mathrm{i}\sigma + \frac{1}{\sqrt{r}}(\sigma\Omega_r + \mathrm{i}r_1)\right]t\right\}\mathrm{d}t + O(r^{-1})\right) \\
&\qquad - (1-\alpha)\frac{2\pi \mathrm{i}}{(R-1)!}x^{R-1}\mathrm{e}^{-xs} \quad (3.18)
\end{aligned}
$$

with

$$
\Omega_r = -\mathrm{i}\sqrt{r}(\mathrm{e}^{\mathrm{i}\Omega r^{-1/2}} - 1); \quad \sigma = \sqrt{r}(s-1) \quad (3.19)
$$

ii) When σ is small we have

$$
g_\alpha = \mathrm{i}\mathrm{e}^{\Omega^2/2}\sqrt{\frac{\pi}{2r}}\mathrm{e}^{-\sqrt{r}\sigma}\left(\mathrm{erf}\left(\frac{\Omega}{\sqrt{2}}+1\right)E_1 - 2(1-\alpha)E_2 + \mathrm{e}^{-\Omega^2/2}O(r^{-1})\right) \quad (3.20)
$$

where $E_{1,2} = (1 + O(r^{-1/2},\sigma))$, $\mathrm{erf}(x) = 2\pi^{-1/2}\int_0^x \exp(-t^2)\mathrm{d}t$.

Proof.

$$
\begin{aligned}
\int_0^{M\mathrm{e}^{\pm\mathrm{i}\delta}} \frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p)^R} &= \frac{1}{s^{R-1}}\int_0^{\frac{M}{s}\mathrm{e}^{\pm\mathrm{i}\delta}} \frac{\mathrm{e}^{-psx}\mathrm{d}p}{(1-p)^R} \\
&= \frac{1}{s^{R-1}}\left(\int_0^{\infty\mathrm{e}^{\mathrm{i}\delta}} \frac{\mathrm{e}^{-psx}\mathrm{d}p}{(1-p)^R} + O(\mathrm{e}^{-\frac{xM}{s}\mathrm{e}^{\pm\mathrm{i}\delta}})\right) \quad (3.21)
\end{aligned}
$$

Furthermore

$$\int_0^{\infty e^{\pm i\delta}} \frac{e^{-px}dp}{(s-p)^R} + \frac{\pi i(1\mp 1)}{(R-1)!}x^{R-1}e^{-xs} = \int_0^{i\infty} \frac{e^{-px}dp}{(s-p)^R} = i\int_0^{\infty} \frac{e^{-ipx}dp}{(s-ip)^R} \tag{3.22}$$

Taking $a = 1/2 - \epsilon$ for small ϵ we have

$$\int_0^{\infty} \frac{e^{-ipsx}dp}{(1-ip)^R} = \int_0^{r^{-a}} \frac{e^{-ipsx}dp}{(1-ip)^R} + O\left(\exp\left(-\frac{1}{2}r^{2\epsilon}\right)\right) \tag{3.23}$$

and

$$\begin{aligned} &\int_0^{r^{-a}} e^{-isxt - R\ln(1-it)}dt = \int_0^{r^{-a}} e^{-isxt+Rit-Rt^2/2}dt + O(\frac{1}{r}) \\ = &\int_0^{\infty} e^{-isxt+Rit-Rt^2/2}dt + O(\frac{1}{r}) = r^{-1/2}\left(\int_0^{\infty} e^{ir^{-1/2}z(R-sx)-\frac{R}{2r}z^2}dz + O\left(\frac{1}{r}\right)\right) \end{aligned} \tag{3.24}$$

Part (i) of proposition 6 follows by combining (3.19...3.24)). Part (ii) is a straightforward calculation from (i), using Stirling's formula and Laplace method. $\square$

We note that with Ω, r_1 held constant the integral in (3.18) is bounded for $s \geq 1$ uniformly in $r \in \mathbb{N}$. Thus, for $s \geq 1+\epsilon$ we have $g_\alpha \leq K_2 e^{-r\epsilon}$ for some K_2 and thus we obtain from (3.17), proceeding as for (3.9),

$$\begin{aligned} J = &\int_{C_1;|s|<1+\epsilon} ds\mathcal{P}^{m_1N}\mathbf{Y}(s)\int_{<\alpha;M>} \frac{e^{-px}dp}{(s-p)^{Nm_1+r+1}} + E_3 \\ &= \frac{e_1}{f_1}\int_1^{1+\epsilon} dsS_1\mathbf{Y}_{\mathbf{e}_1}(s-1)\int_{<\alpha;M>} \frac{e^{-px}dp}{(s-p)^{m_1+r+1}} + E_4 \\ = &\frac{e_1S_1}{f_1}\int_1^{1+\epsilon} ds(s-1)^{\beta_1'-1}(\mathbf{e}_1 + (s-1)h(s))\int_{<\alpha;M>} \frac{e^{-px}dp}{(s-p)^{m_1+r+1}} + E_4 \end{aligned} \tag{3.25}$$

where $E_{3,4} = O(e^{-r\epsilon})$, and $h(s)$ is smooth on $[1, 1+\epsilon]$. Taking δ small and r correspondingly large we have, in view of proposition 6 (i),

$$J=\frac{e_1S_1}{f_1}\int_1^{1+r^{-1+\delta}}\mathrm{d}s(s-1)^{\beta_1'-1}(\mathbf{e}_1+(s-1)h(s))\int_{<\alpha;M>}\frac{\mathrm{e}^{-px}\mathrm{d}p}{(s-p)^{m_1+r+1}}+E_5 \tag{3.26}$$

where $E_5=O(\mathrm{e}^{-r^\delta})$. Using now proposition 6 (ii) to evaluate the inner integral we obtain

$$f_1J=\mathrm{i}\frac{(m_1+r)!}{2\pi\mathrm{i}}S_1\mathrm{e}^{\Omega^2/2}\sqrt{\frac{\pi}{2r}}\frac{1}{r^{\beta_1'+1}}\Big(\mathrm{erf}(\Omega/\sqrt{2})-1+2\alpha \\ +(\mathrm{erf}(\Omega/\sqrt{2})+1)E_6-2(1-\alpha)E_7+\mathrm{e}^{-\Omega^2/2}O(r^{-1})\Big) \tag{3.27}$$

where $E_{6,7}=O(r^{-1/2})$, where, as in (3.10) we need to take $r+1$ instead of r (cf. (2.11)). Taking $\lambda=1$ in the expression (3.10), which was shown to be of the order of the least term, we see that the least term is $O(r^{-1/2})$ *smaller* than (3.27) unless

$$\mathrm{erf}(\Omega/\sqrt{2})-1+2\alpha=0 \tag{3.28}$$

(a) Asymptotics along straight lines: If $\Omega=0$ ($\arg(x)=0$) we see that only

$$\alpha=1/2$$

(corresponding to the balanced average, which on the interval $(0,2)$ is the half sum of the upper and lower continuations) ensures errors in optimal truncation of the order of the least term.

(b) Asymptotics along parabolas and Berry smoothing. Theorem 3 is straightforward application of Stirling's formula to (3.27). (Note also that by theorem 8 (ii) and theorem 7 (ii) $\mathbf{y}_{\mathbf{e}_1}\sim x^{-\beta'}e^{-x}\mathbf{e}_1$.) Convergence of the Puiseux series near the singularities of $\mathbf{Y}_0$ in Borel space was the key in proving universality of the Berry transition, since using this convergence, the calculation reduced in effect to the case of one particular function, $\mathrm{Ei}(x)$.

*

The same line of proof as for $\tilde{\mathbf{y}}_0$ works for $\tilde{\mathbf{y}}_{\mathbf{k}}$ as well. Indeed, we note that $x^{\mathbf{k}\cdot(\boldsymbol{\beta}+\mathbf{m})}\tilde{\mathbf{y}}_{\mathbf{k}}$ have the same singularity structure in Borel space as $\tilde{\mathbf{y}}_0$. This follows from theorem 7 (ii) and the fact that convolution with a locally analytic function preserves the convergence of Puiseux series (cf. Example (3a) in appendix 1). □

4 Appendix 1.

4.1 Results on Borel summation

In this paper we make use of some of the results in Costin (1997); for convenience we summarize them below.
We use the convention $\mathbb{N} = \mathbb{N} \cup \{0\}$. Let

$$\mathcal{W} = \{p \in \mathbb{C} : p \neq k\lambda_i \,, \forall k \in \mathbb{N}, i = 1, 2, \ldots, n\} \tag{4.1}$$

The directions $d_j = \{p : \arg(p) = \phi_j\}, j = 1, 2, \ldots, n$ are the *Stokes lines* (Note: sometimes known as *anti*-Stokes lines!). We construct over $\mathcal{W}$ a surface $\mathcal{R}$, consisting of homotopy classes of smooth curves in $\mathcal{W}$ starting at the origin, moving away from it, and crossing at most one Stokes line, at most once (Fig. 2):

$$\mathcal{R} := \Big\{\gamma : (0,1) \mapsto \mathcal{W} : \ \gamma(0_+) = 0; \ \frac{\mathrm{d}}{\mathrm{d}t}|\gamma(t)| > 0; \ \arg(\gamma(t)) \text{ monotonic}\Big\} \tag{4.2}$$

The Laplace transform along a direction ϕ of a function F $\mathcal{L}_\phi F$ will depend in general on ϕ; the usual convention is to choose ϕ so that $xp \in \mathbb{R}^+$. Thus, the Borel sum of $\tilde{f}$ in the direction x, if it exists, is defined as $\mathcal{L}_{\phi(x)}\mathcal{B}f$ with $\phi(x) = -\arg(x)$. By (n6) and the agreed association between p and x, and since Laplace integrals will not depend on the direction of p until one of the Stokes lines is crossed, we may as well assume that the direction of integration is either d_j or is arbitrarily close to it. Define $\mathcal{R}_1$ as the restriction of $\mathcal{R}$ to $\arg(\gamma) \in (\psi_n - 2\pi, \psi_2)$ where $\psi_n = \max\{-\pi/2, \phi_n - 2\pi\}$ and $\psi_2 = \min\{\pi/2, \phi_2\}$.

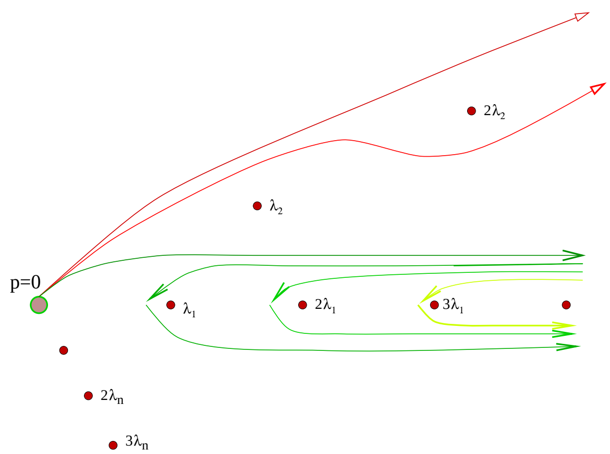

Fig 2. *The paths near λ_2 belong to $\mathcal{R}$.*
The paths near λ_1 relate to the balanced average

We denote the analytic continuation of f along a curve γ by $AC_\gamma(f)$. For the analytic continuations near a Stokes line d_i we use notations similar to Écalle's: f^- is the branch of f along a path γ with $\arg(\gamma) < \phi_i$, while f^{-k+} denotes the branch along a path that crosses the Stokes line between $k\lambda_i$ and $(k+1)\lambda_i$. We use the notations $\mathcal{P}f$ for $\int_0^p f(s)\mathrm{d}s$ and $\mathcal{P}_\gamma f$ if integration is along a curve γ.

We write $\mathbf{k} \geq \mathbf{k}'$ if $k_i \geq k_i'$ for all i and $\mathbf{k} \succ \mathbf{k}'$ if $\mathbf{k} \geq \mathbf{k}'$ and $\mathbf{k} \neq \mathbf{k}'$. The relation $\succ$ is a well ordering on $\mathbb{N}^{n_1}$. We let $\mathbf{e}_j$ be the unit vector in the j^{th} direction.

Formal expansions are denoted with a tilde and capital letters $\mathbf{Y}, \mathbf{V}\ldots$ will usually denote Borel transforms or functions otherwise associated to Borel space. For notational convenience, we will not however distinguish between $\tilde{\mathbf{Y}}_k = \mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$, which turn out to be convergent series, and the sums of these series $\mathbf{Y}_{\mathbf{k}}$ as germs of ramified analytic functions.

We have

$$\mathbf{g}(x,\mathbf{y}) = \sum_{|\mathbf{l}|\ge 1} \mathbf{g}_{\mathbf{l}}(x)\mathbf{y}^{\mathbf{l}} = \sum_{s\ge 0;|\mathbf{l}|\ge 1} \mathbf{g}_{s,\mathbf{l}} x^{-s}\mathbf{y}^{\mathbf{l}} \quad (|x|>x_0, |\mathbf{y}|<y_0) \tag{4.3}$$

where $\mathbf{y}^{\mathbf{l}} = y_1^{l_1}\cdots y_n^{l_n}$ and $|\mathbf{l}| = l_1+\cdots+l_n$. By construction $\mathbf{g}_{s,\mathbf{l}} = 0$ if $|\mathbf{l}|=1$ and $s\le M$.

Theorem 7 *(i) $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$ is analytic in $\mathcal{R}\cup\{0\}$. The singularities of $\mathbf{Y}_0$ (which are contained in the set $\{l\lambda_j : l\in\mathbb{N}^+, j=1,2,\ldots,n\}$) are described as follows. For $l\in\mathbb{N}^+$ and small z*

$$\mathbf{Y}_0^{\pm}(z+l\lambda_j) = \pm\Big[(\pm S_j)^l \ln(z)^{0,1}\mathbf{Y}_{l\mathbf{e}_j}(z)\Big]^{(lm_j)} + \mathbf{B}_{lj}(z) =$$
$$\Big[z^{l\beta'_j-1}\ln z^{0,1}\,\mathbf{A}_{lj}(z)\Big]^{(lm_j)} + \mathbf{B}_{lj}(z)\ (l=1,2,\ldots) \tag{4.4}$$

where the power of $\ln(z)$ is one iff $l\beta_j\in\mathbb{Z}$, and $\mathbf{A}_{lj},\mathbf{B}_{lj}$ are analytic for small z. The functions $\mathbf{Y}_{\mathbf{k}}$ are, in addition, analytic at $p=l\lambda_j$, $l\in\mathbb{N}^+$, iff, exceptionally,

$$S_j = r_j\Gamma(\beta'_j)\left(\mathbf{A}_{1,j}\right)_j(0) = 0 \tag{4.5}$$

where $r_j = 1-\mathrm{e}^{2\pi\mathrm{i}(\beta'_j-1)}$ if $l\beta_j\notin\mathbb{Z}$ and $r_j=-2\pi\mathrm{i}$ otherwise. The S_j are Stokes constants, see theorem 10.

(ii) $\mathbf{Y}_{\mathbf{k}} = \mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$, $|\mathbf{k}|>1$, are analytic in $\mathcal{R}\backslash\{-\mathbf{k}'\cdot\boldsymbol{\lambda}+\lambda_i : \mathbf{k}'\le\mathbf{k}, 1\le i\le n\}$. For $l\in\mathbb{N}$ and p near $l\lambda_j$, $j=1,2,\ldots,n$ there exist $\mathbf{A}=\mathbf{A}_{\mathbf{k}jl}$ and $\mathbf{B}=\mathbf{B}_{\mathbf{k}jl}$ analytic at zero so that (z is as above)

$$\mathbf{Y}_{\mathbf{k}}^{\pm}(z+l\lambda_j) = \pm\Big[(\pm S_j)^l\binom{k_j+l}{l}\ln(z)^{0,1}\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(z)\Big]^{(lm_j)} + l\mathbf{B}_{\mathbf{k}lj}(z) =$$
$$\Big[z^{\mathbf{k}\cdot\boldsymbol{\beta}'+l\beta'_j-1}(\ln z)^{0,1}\,\mathbf{A}_{\mathbf{k}lj}(z)\Big]^{(lm_j)} + l\mathbf{B}_{\mathbf{k}lj}(z)\ (l=0,1,2,\ldots) \tag{4.6}$$

where the power of $\ln z$ is 0 iff $l=0$ or $\mathbf{k}\cdot\boldsymbol{\beta}+l\beta_j-1\notin\mathbb{Z}$ and $\mathbf{A}_{\mathbf{k}0j} = \mathbf{e}_j/\Gamma(\beta'_j)$. Near $p\in\{-\mathbf{k}'\cdot\boldsymbol{\lambda} : 0\prec\mathbf{k}'\le\mathbf{k}\}$, (where $\mathbf{Y}_0$ is analytic) $\mathbf{Y}_{\mathbf{k}}$, $\mathbf{k}\ne 0$ have convergent Puiseux series.

Let $\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$ be extended along d_j by the "balanced average" of analytic continuations

$$\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}} = \mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}^{+} + \sum_{j=1}^{\infty}\frac{1}{2^j}\left(\mathbf{Y}_{\mathbf{k}}^{-} - \mathbf{Y}_{\mathbf{k}}^{-(j-1)+}\right) \tag{4.7}$$

The sum above coincides with the one in which $+$ is exchanged with $-$, accounting for the reality-preserving property. Clearly, if $\mathbf{Y}_{\mathbf{k}}$ is analytic along d_j, then the terms in the infinite sum vanish and $\mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}$; we also let $\mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}$ if $d \neq d_j$, where again $\mathbf{Y}_k$ is analytic. It follows from (4.7) and theorem 8 below that the Laplace integral of $\mathbf{Y}_{\mathbf{k}}^{ba}$ along $\mathbb{R}^+$ can deformed into contours as those depicted in Fig. 2, with weight 2^{-k} for a contour turning around $(k+1)\lambda_1$. More generally, we consider the averages

$$\mathcal{B}_\alpha \tilde{\mathbf{y}}_{\mathbf{k}} = \mathbf{Y}_{\mathbf{k}}^{\alpha} = \mathbf{Y}_{\mathbf{k}}^{+} + \sum_{j=1}^{\infty} \alpha^j \left(\mathbf{Y}_{\mathbf{k}}^{-} - \mathbf{Y}_{\mathbf{k}}^{-(j-1)+} \right) \tag{4.8}$$

and correspondingly

$$(\mathcal{LB})_\alpha \tilde{\mathbf{y}}_{\mathbf{k}} := \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{\alpha} \tag{4.9}$$

With $\alpha \in \mathbb{R}$, this represents the most general family of averages of Borel summation formulas which commute with complex conjugation, with the algebraic and analytic operations and have good continuity properties (see Costin 1995). The value $\alpha = 1/2$ is special in that it is the only one compatible with optimal truncation.

Theorem 8 *(i) The branches of $(\mathbf{Y}_{\mathbf{k}})_\gamma$ in $\mathcal{R}_1$ have limits in a C^*-algebra of distributions, $\mathcal{D}'_{m,\nu}(\mathbb{R}^+) \subset \mathcal{D}'$ (cf. § 4.3) Their Laplace transforms in $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ $\mathcal{L}(\mathbf{Y}_{\mathbf{k}})_\gamma$ exist simultaneously and with $x \in \mathcal{S}_x$ and for any $\delta > 0$ there is a constant K and an x_1 large enough, so that for $\Re(x) > x_1$ we have $|\mathcal{L}(\mathbf{Y}_{\mathbf{k}})_\gamma(x)| \leq K\delta^{|\mathbf{k}|}$.*

In addition, $\mathbf{Y}_{\mathbf{k}}(p\mathrm{e}^{\mathrm{i}\phi})$ are continuous in ϕ with respect to the $\mathcal{D}'_{m,\nu}$ topology, (separately) on $[\psi_n - 2\pi, 0]$ and $[0, \psi_2]$.

If $m > \max_i(m_i)$ and $l < \min_i |\lambda_i|$ then $\mathbf{Y}_0(p\mathrm{e}^{\mathrm{i}\phi})$ is continuous in $\phi \in [0, 2\pi]\backslash\{\phi_i : i \leq n\}$ in the $\mathcal{D}'_{m,\nu}(\mathbb{R}^+, l)$ topology and has (at most) jump discontinuities for $\phi = \phi_i$. For each $\mathbf{k}$, $|\mathbf{k}| \geq 1$ and any K there is an $l > 0$ and an m such that $\mathbf{Y}_k(p\mathrm{e}^{\mathrm{i}\phi})$ are continuous in $\phi \in [0, 2\pi]\backslash\{\phi_i; -\mathbf{k}' \cdot \boldsymbol{\lambda} + \lambda_i : i \leq n, \mathbf{k}' \leq \mathbf{k}\}$ in the $\mathcal{D}'_{m,\nu}((0, K), l)$ topology and have (at most) jump discontinuities on the boundary.

(ii) The sum (4.7) converges in $\mathcal{D}'_{m,\nu}$ (and coincides with the analytic continuation of $\mathbf{Y}_{\mathbf{k}}$ when $\mathbf{Y}_{\mathbf{k}}$ is analytic along $\mathbb{R}^+$). For any δ there is a large enough x_1 independent of $\mathbf{k}$ *so that $\mathbf{Y}_{\mathbf{k}}^{ba}(p)$ with $p \in \mathcal{R}_1$ are Laplace transformable for $\Re(xp) > x_1$ and furthermore $|(\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba})(x)| \leq \delta^{|\mathbf{k}|}$. In addition, if $d \neq \mathbb{R}^+$, then for large ν, $\mathbf{Y}_{\mathbf{k}} \in L^1_\nu(d)$.*

The functions $\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba}$ are analytic for $\Re(xp) > x_1$. For any $\mathbf{C} \in \mathbb{C}^{n_1}$ there is an $x_1(\mathbf{C})$ large enough so that the sum

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^{ba} + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda} x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba} \tag{4.10}$$

converges uniformly for $\Re(xp) > x_1(\mathbf{C})$, and $\mathbf{y}$ is a solution of (1.3). When the direction of p is not the real axis then, by definition, $\mathbf{Y}_{\mathbf{k}}^{ba} = \mathbf{Y}_{\mathbf{k}}$, $\mathcal{L}$ is the usual Laplace transform and (4.10) becomes

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0 + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}} \tag{4.11}$$

In addition, $\mathcal{L}\mathbf{Y}_{\mathbf{k}}^{ba} \sim \tilde{\mathbf{y}}_{\mathbf{k}}$ *for large* x *in the half plane* $\Re(xp) > x_1$*, for all* $\mathbf{k}$*, uniformly.*

iii) The general solution of (1.3) that is asymptotic to $\tilde{\mathbf{y}}_0$ *for large* x *along a ray in* S_x *can be equivalently written in the form (4.10) or as*

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^{\pm} + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{-\mathbf{k}\cdot\boldsymbol{\beta}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{\pm} \tag{4.12}$$

for some $\mathbf{C}$ *(depending on the solution and chosen form). With the convention binding the directions of* x *and* p *and the representation form being fixed, (cf. the beginning of §4.1)) the representation of a solution is unique.*

Theorem 9 *i) For all* $\mathbf{k}$ *and* $\Re(p) > j, \Im(p) > 0$ *as well as in* $\mathcal{D}'_{m,\nu}$ *we have*

$$\mathbf{Y}_{\mathbf{k}}^{\pm j \mp}(p) - \mathbf{Y}_{\mathbf{k}}^{\pm(j-1)\mp}(p) = (\pm S_1)^j \binom{k_1 + j}{j} \left(\mathbf{Y}_{\mathbf{k}+j\mathbf{e}_1}^{\pm}(p-j)\right)^{(mj)} \tag{4.13}$$

and also,

$$\mathbf{Y}_{\mathbf{k}}^{\pm} = \mathbf{Y}_{\mathbf{k}}^{\mp} + \sum_{j\ge 1} \binom{j+k}{k} (\pm S_1)^j (\mathbf{Y}_{\mathbf{k}+j\mathbf{e}_1}^{\mp}(p-j))^{(mj)} \tag{4.14}$$

ii) Local Stokes transition.
Consider the expression of a fixed solution $\mathbf{y}$ *of (1.3) as a Borel summed transseries (4.10). As* $\arg(x)$ *varies, (4.10) changes only through* $\mathbf{C}$*, and that change occurs when the Stokes lines are crossed. We have, in the neighborhood of* $\mathbb{R}^+$*, with* S_1 *defined in (4.5):*

$$\mathbf{C}(\xi) = \begin{cases} \mathbf{C}^- = \mathbf{C}(-0) & \textit{for } \xi < 0 \\ \mathbf{C}^0 = \mathbf{C}(-0) + \frac{1}{2} S_1 \mathbf{e}_1 & \textit{for } \xi = 0 \\ \mathbf{C}^+ = \mathbf{C}(-0) + S_1 \mathbf{e}_1 & \textit{for } \xi > 0 \end{cases} \tag{4.15}$$

Remark 1 *In view of (4.13) the different analytic continuations of* $\mathbf{Y}_0$ *along paths crossing* $\mathbb{R}^+$ *at most once can be expressed in terms of* $\mathbf{Y}_{j\mathbf{e}_1}$*. The most general formal solution of (1.3) that can be formed in terms of* $\mathbf{Y}_{j\mathbf{e}_j}$ *with* $j \ge 0$ *is (1.4) with* $C_1 = \alpha$ *arbitrary and* $C_j = 0$ *for* $j \ne 1$*. Any true solution of (1.3) based on such a transseries is given in (4.12) with* $\mathbf{C}$ *as above. Any average* $\mathcal{A}\mathbf{Y}_0$ *along paths going forward in* $\mathbb{R}^+$ *such that* $\mathcal{L}\mathcal{A}\mathbf{Y}_0$ *is thus of the form (4.9).*

Theorem 10 *Assume only λ_1 lies in the right half plane. Let $\gamma^{\pm}$ be two paths in the right half plane, near the positive/ negative imaginary axis such that $|x^{-\beta_1+1}\mathrm{e}^{-x\lambda_1}| \to 1$ as $x \to \infty$ along $\gamma^{\pm}$. Consider the solution $\mathbf{y}$ of (1.3) given in (4.10) with $\mathbf{C} = C\mathbf{e}_1$ and where the path of integration is $p \in \mathbb{R}^+$. Then*

$$\mathbf{y} = (C \pm \frac{1}{2}S_1)\mathbf{e}_1 x^{-\beta_1+1}\mathrm{e}^{-x\lambda_1}(1+o(1)) \tag{4.16}$$

for large x along $\gamma^{\pm}$, where S_1 is the same as in (4.5), (4.15).

Proposition 11 *i) Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be solutions of (1.3) so that $\mathbf{y}_{1,2} \sim \tilde{\mathbf{y}}_0$ for large x in an open sector S (or in some direction d); then $\mathbf{y}_1 - \mathbf{y}_2 = \sum_j C_j \mathrm{e}^{-\lambda_{i_j} x} x^{-\beta_{i_j}}(\mathbf{e}_{i_j} + o(1))$ for some constants C_j, where the indices run over the eigenvalues λ_{i_j} with the property $\Re(\lambda_{i_j} x) > 0$ in S (or d). If $\mathbf{y}_1 - \mathbf{y}_2 = o(\mathrm{e}^{-\lambda_{i_j} x} x^{-\beta_{i_j}})$ for all j, then $\mathbf{y}_1 = \mathbf{y}_2$.*

ii) Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be solutions of (1.1) and assume that $\mathbf{y}_1 - \mathbf{y}_2$ has differentiable asymptotics of the form $\mathbf{K}a\exp(-ax)x^b(1+o(1))$ with $\Re(ax) > 0$ and $\mathbf{K} \neq 0$, for large x. Then $a = \lambda_i$ for some i.

iii) Let $\mathbf{U}_{\mathbf{k}} \in \mathcal{T}_{\{\cdot\}}$ for all $\mathbf{k}$, $|\mathbf{k}| > 1$. Assume in addition that for large ν there is a function $\delta(\nu)$ vanishing as $\nu \to \infty$ such that

$$\sup_{\mathbf{k}} \delta^{-|\mathbf{k}|} \int_d \left|\mathbf{U}_{\mathbf{k}}(p)\mathrm{e}^{-\nu p}\right| \mathrm{d}|p| < K < \infty \tag{4.17}$$

Then, if $\mathbf{y}_1, \mathbf{y}_2$ are solutions of (1.3) in S where in addition

$$\mathbf{y}_1 - \mathbf{y}_2 = \sum_{|\mathbf{k}|>1} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{m}\cdot\mathbf{k}} \int_d \mathbf{U}_{\mathbf{k}}(p)\exp(-xp)\mathrm{d}p \tag{4.18}$$

where $\boldsymbol{\lambda}, x$ are as in (n6), then $\mathbf{y}_1 = \mathbf{y}_2$, and $\mathbf{U}_{\mathbf{k}} = 0$ for all $\mathbf{k}$, $|\mathbf{k}| > 1$.

4.2 Focusing spaces and algebras

The proofs of the properties stated in this section are given in Costin (1997).

We say that a family of norms $\|\|_\nu$ depending on a parameter $\nu \in \mathbb{R}^+$ is **focusing** if for any f with $\|f\|_{\nu_0} < \infty$

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{4.19}$$

Let $\mathcal{E}$ be a linear space and $\{\|\|_\nu\}$ a family of norms satisfying (4.19). For each ν we define a Banach space $\mathcal{B}_\nu$ as the completion of $\{f \in \mathcal{E} : \|f\|_\nu < \infty\}$. Enlarging $\mathcal{E}$ if needed, we may assume that $\mathcal{B}_\nu \subset \mathcal{E}$. For $\alpha < \beta$, (4.19) shows that the identity is an embedding of $\mathcal{B}_\alpha$ in $\mathcal{B}_\beta$. Let $\mathcal{F} \subset \mathcal{E}$ be the projective limit of the $\mathcal{B}_\nu$. That is to say

$$\mathcal{F} := \bigcup_{\nu>0} \mathcal{B}_\nu \tag{4.20}$$

is endowed with the topology in which a sequence is convergent if it converges in *some* $\mathcal{B}_\nu$. We call $\mathcal{F}$ a **focusing space**.

Consider now the case when $(\mathcal{B}_\nu, +, *, \|\|_\nu)$ are commutative Banach algebras. Then $\mathcal{F}$ inherits a structure of a commutative algebra, in which $*$ ("convolution") is continuous. We say that $(\mathcal{F}, *, \|\|_\nu)$ is a **focusing algebra**.

4.3 Examples

For more details see Costin (1997). Let $K \in \mathbb{R}^+$ and $\mathcal{S} = \mathcal{S}_{K,\alpha_1,\alpha_2} = \{p : \arg(p) \in [\alpha_1, \alpha_2] \subset (-\pi/2, \pi/2), |p| \le K\}$ (or a finite union of such sectors) and $\mathcal{V}$ be a small neighborhood of the origin. $\overline{\mathcal{V}}$ will be the closure of $\mathcal{V}$, cut along the negative axis, and together with these upper and lower cuts.

(1). $L^1{}_\nu(\mathcal{K})$. Let $\mathcal{K} = \mathcal{S}_{K,\phi,\phi}$. The space $L^1{}_\nu(\mathcal{K})$ with the convolution $f * g := p \mapsto \int_0^p f(s)g(p-s)\mathrm{d}s$ is a commutative Banach algebra under each of the (equivalent) norms

$$\|f\|_\nu = \int_0^K \mathrm{e}^{-\nu t}|f(t\exp(\mathrm{i}\phi))|\mathrm{d}t \tag{4.21}$$

(2) If $K = \infty$ in example (1), then the norms (4.21) are not equivalent anymore for different ν, but convolution is still continuous in (4.21) and the projective limit of the $L^1_\nu(\mathbb{R}^+\mathrm{e}^{\mathrm{i}\phi})$, $\mathcal{F}(\mathbb{R}^+\mathrm{e}^{\mathrm{i}\phi}) \subset L^1_{loc}(\mathbb{R}^+\mathrm{e}^{\mathrm{i}\phi})$, is a focusing algebra.

(3a) $\mathcal{T}_\beta(\mathcal{S} \cup \overline{\mathcal{V}})$. For $\Re(\beta) > 0$ and $\phi_1 \ne \phi_2$, this space is given by $\{f : f(p) = p^\beta F(p)\}$, where F is analytic in the interior of $\mathcal{S} \cup \mathcal{V}$ and continuous in its closure. We take the family of (equivalent) norms

$$\|f\|_{\nu,\beta} = K \sup_{s\in\mathcal{S}\cup\overline{\mathcal{V}}} \left|\mathrm{e}^{-\nu p} f(p)\right| \tag{4.22}$$

It is clear that convergence of f in $\|\|_{\nu,\beta}$ implies uniform convergence of F on compact sets in $\mathcal{S} \cup \mathcal{V}$ (for p near zero, this follows from Cauchy's formula). $\mathcal{T}_\beta$ are thus Banach spaces and focusing spaces by (4.22). The spaces $\{\mathcal{T}_\beta\}_\beta$ are isomorphic to each-other. The application

$$(\cdot * \cdot) : \mathcal{T}_{\beta_1} \times \mathcal{T}_{\beta_2} \mapsto \mathcal{T}_{\beta_1+\beta_2+1} \tag{4.23}$$

is continuous:

A natural generalization of $\mathcal{T}_\beta$ is obtained taking $\beta_1, \ldots, \beta_N \in \mathbb{C}$ with positive real parts, no two of them differing by an integer. If $f_\beta = \sum_{i=1}^k p^{\beta_i} A_i(p)$ with A_i analytic, then $f_\beta \equiv 0$ iff $A_i \equiv 0$ for all i (e.g., by a Puiseux series argument). It is then natural to identify the space $\mathcal{T}_{\{\beta_1,\ldots,\beta_k\}}$ of functions of the form

f_β with $\oplus_{i=1}^k \mathcal{T}_{\beta_i}$. Convolution with analytic functions is defined on $\mathcal{T}_{\{\beta_1,\dots,\beta_k\}}$ while convolution of two functions in $\mathcal{T}_{\{\beta_1,\dots,\beta_k\}}$ takes values in $\mathcal{T}_{\{\beta_i+\beta_j \bmod 1\}}$. We write $\mathcal{T}_{\{\cdot\}}$ when the concrete values of $\beta_1,\dots,\beta_k$ do not matter.

(3b) A particular case of the preceding example is $\mathcal{A}_{z,l}(\mathcal{S}\cup\mathcal{V})$ consisting of analytic functions in the interior of $\mathcal{S}\cup\mathcal{V}$, continuous on its closure, and vanishing at the origin together with the first l derivatives. $\mathcal{A}_{z,l}$ can be identified with $\mathcal{T}_l$.

(4) $\mathcal{D}'_{m,\nu}$, the "staircase distributions". Let $\mathcal{D}(0,x)$ be the test functions on $(0,x)$ and $\mathcal{D}=\mathcal{D}(0,\infty)$. Let $\mathcal{D}'_m \subset \mathcal{D}'$ be the distributions f for which $f = F_k^{(km)}$ on $\mathcal{D}(0,k+1)$ with $F_k \in L^1[0,k+1]$. There is a uniquely associated staircase decomposition, a sequence $\{\Delta_i(f)\}_{i\in\mathbb{N}} = \{\Delta_i\}_{i\in\mathbb{N}}$ such that $\Delta_i \in L^1(\mathbb{R}^+)$, $\Delta_i = \Delta_i \chi_{[i,i+1]}$ and

$$f = \sum_{i=0}^{\infty} \Delta_i^{(mi)} \tag{4.24}$$

With respect to the norm

$$\|f\|_{\nu,m} := \sqrt{2}\sum_{i=0}^{\infty} \nu^{im}\|\Delta_i\|_\nu \tag{4.25}$$

where $\|\Delta\|_\nu$ is computed from (4.21) with $K=\infty$ and with convolution defined as

$$\Delta_k(f*\tilde{f}) = \sum_{i+j=k} \Delta_i * \tilde{\Delta}_j - \mathcal{P}^m\left\{\sum_{i+j=k-1}\left(\Delta_i * \tilde{\Delta}_j\right)\chi_{[0,k+1]}\right\} \tag{4.26}$$

$(\mathcal{D}'_m,+,*)$ is a commutative Banach algebra. With respect to the family of norms $\|\|_{m,\nu}$, the projective limit of the $\mathcal{D}'_{m,\nu}$, $\mathcal{F}_m$ is a focusing algebra.

For any $f\in L^1_{\nu_0}(\mathbb{R}^+)$ there is a constant $C(\nu,\nu_0)$ such that $f\in\mathcal{D}'_{m,\nu}$ for all $\nu>\nu_0$ and

$$\|f\|_{\mathcal{D}'_{m,\nu}} \le C(\nu_0,\nu)\|f\|_{L^1_{\nu_0}} \tag{4.27}$$

and formula (4.26) is equivalent to the usual convolution in this case.

For $a\in\mathbb{R}^+$, $\mathcal{D}'_{m,\nu}(a,\infty) = \{f\in\mathcal{D}'_{m,\nu} : \Delta_i(x)=0 \text{ for } x<a\}$ is a closed ideal in $\mathcal{D}'_{m,\nu}$ (isomorphic to the restriction $\mathcal{D}'_{m,\nu}(a,\infty)$ of $\mathcal{D}'_{m,\nu}$ to $\mathcal{D}(a,\infty)$). The restrictions $\mathcal{D}'_{m,\nu}(a,b)$ of $\mathcal{D}'_{m,\nu}$ to $\mathcal{D}(a,b)$ are for $0<a<b<\infty$ Banach spaces with respect to the norm (4.25) restricted to (a,b).

The functions in $\mathcal{D}(\mathbb{R}^+\backslash\mathbb{N})$ are dense in $\mathcal{D}'_{m,\nu}$, with respect to the norm (4.25).

If we choose a different interval length $l>0$ instead of $l=1$ in the partition associated to (4.24), we then write $\mathcal{D}'_{m,\nu}(l)$. Obviously, dilation gives a natural isomorphism between these structures. If $d = \{te^{i\phi} : t \in \mathbb{R}^+\}$ is any ray, $\mathcal{D}'_{m,\nu}(d)$ and $\mathcal{F}_{m;\phi}$ are defined in an analogous way and have the same properties as their real counterpart.

Laplace transforms are naturally defined in $\mathcal{D}'_{m,\nu}$.

Lemma 12 *Laplace transform extends continuously from $\mathcal{D}(\mathbb{R}^+\backslash\mathbb{N})$ to $\mathcal{D}'_{m,\nu}(\mathbb{R}^+)$ by the formula*

$$(\mathcal{L}f)(x) := \sum_{k=0}^{\infty} x^{mk} \int_0^{\infty} \mathrm{e}^{-sx} \Delta_k(s) \mathrm{d}s \tag{4.28}$$

In particular, with $f, g, h' \in \mathcal{D}'_{m,\nu}$ we have

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$$
$$\mathcal{L}(h') = x\mathcal{L}(h) - h(0)$$
$$\mathcal{L}(pf) = -(\mathcal{L}(f))' \tag{4.29}$$

For $x \in S_\nu = \{x : \Re(x) > \nu\}$ the sum (4.28) converges absolutely. Laplace transform is, for fixed $x \in S_\nu$, a continuous functional (of norm less than one) on $\mathcal{D}'_{m,\nu}$.

$(\mathcal{L}f)(x)$ is analytic in S_ν.

Furthermore, $\mathcal{L}$ is injective *in $\mathcal{D}'_{m,\nu}$.*

4.4 Appendix 2. Two examples of resonant equations

Once resonant equations are allowed, Berry transitions tend to become more complicated: $y = \mathrm{e}^{-x}\mathrm{Ei}(x) + \exp(-x^2 - \mathrm{i}x^4)$ is a (least term summable for $x \to +\infty$) solution of a linear differential equation with rational coefficients:

$$\left[\frac{d}{dx} - \frac{P'}{P} + 2x + 4\mathrm{i}x^3\right]\left[x\frac{d^2}{dx^2} + (x+1)\frac{d}{dx} + 1\right] y = 0$$

where $P(x) = -16x^7 + 16\mathrm{i}x^5 - 4\mathrm{i}x^4 + (4 - 16\mathrm{i})x^3 - 2x^2 - 4x + 1$. The term $\exp(-x^2 - \mathrm{i}x^4)$ is not seen by least term truncation on the real axis but becomes much larger than the error function contribution before entering the Berry region $\arg(x) \sim |x|^{-1/2}$.

Borel summation using Ecalle acceleration (see Écalle 1993) gives an unambiguous description of the solutions of this type of equations and can be used to decompose solutions conveniently before analyzing transitions. On the other hand, imposing that all Stokes constants are nonzero suffices to exclude this and similar examples, but looks rather restrictive and hard to verify.

We next consider the Berry transition of a family of *resonant* equations with *nonzero* Stokes constants. The formal solutions of such equations depart from Dingle's rule, and also exhibit an interesting splitting of the Stokes rays, with two Berry transitions in the same region. (We are mainly aiming at illustration and the calculation is heuristic but a rigorous treatment along these same lines is not difficult). Let first $m \in \mathbb{R}$ and

$$L[y] = y'' + 2y' + \left(1 + \frac{m^2}{x}\right) y = \frac{1}{x} \tag{4.30}$$

(A) *Formal solutions; behavior of coefficients.*

Taking $\tilde{y} = \sum_{k=0}^{\infty} a_k x^{-k}$ we get for a_k:

$$a_{k+1} = (2k - m^2)a_k - k(k-1)a_{k-1} \tag{4.31}$$

With $a_k = (k-1)!b_k$ we have

$$b_{k+1} = (2 - \frac{m^2}{k})b_k - b_{k-1} \tag{4.32}$$

which we analyze for $k \gg 1$ by WKB (an explicit solution is also possible in this case). With $b_k = \mathrm{e}^{w_k}$ we get:

$$\begin{aligned}
&\mathrm{e}^{w+w'+\frac{1}{2}w''} \sim (2 - \frac{m^2}{k})\mathrm{e}^{w} - \mathrm{e}^{w-w'+\frac{1}{2}w''} \\
&\qquad \Rightarrow \mathrm{e}^{\frac{1}{2}w''}\cosh(w') \sim 1 - \frac{m^2}{2k}(1 + \frac{1}{2}w'')(1 + \frac{1}{2}w'^2) \sim 1 - \frac{m^2}{2k} \\
&\qquad \Rightarrow w' \sim \pm \mathrm{i}\sqrt{\frac{m^2}{k} + w''} \sim \pm \mathrm{i}\sqrt{\frac{m^2}{k} \mp \mathrm{i}\frac{m}{2k^{3/2}}} \sim \pm \mathrm{i}\frac{m}{\sqrt{k}} + \frac{1}{4k}
\end{aligned} \tag{4.33}$$

so that

$$b_k \sim k^{1/4}\left(A_+\mathrm{e}^{2\mathrm{i}m\sqrt{k}} + A_-\mathrm{e}^{-2\mathrm{i}m\sqrt{k}}\right) \text{ and } a_k = a_k^+ + a_k^- \\ \text{with } a_k^{\pm} \sim (k-1)!A_{\pm}k^{1/4}e^{\pm 2\mathrm{i}m\sqrt{k}} \tag{4.34}$$

and, in analogy with the nonresonant case we write $\tilde{y} = \tilde{y}^+ + \tilde{y}^-$ where $\tilde{y}^{\pm} = \sum_{k=0}^{\infty} a_k^{\pm} x^{-k}$. We first see that there is no curve $\arg(x) = f(|x|)$ for which the terms of $\tilde{y}^{\pm}$ (or the terms of any linear combination of $y^{\pm}$) have the same phase. There are instead two parabolic curves, $\pm\arg(x) = 2m|x|^{-1/2}$ along which for $n \sim |x|$ we have $[a_{n+1}^+/x^{n+1}]/[a_n^+/x^n] = 1 + o(1)$. This does not amount to the terms being in phase, but rather is the discrete equivalent of a stationary point.

To reinterpret Dingle's rule for this example, we will see that there exist two Stokes parabolas, each associated with one degree of freedom in the original equation along which the transitions in the constants beyond all orders, as measured by optimal truncation, are maximal.

(B) *Asymptotic solutions of the homogeneous equation.*

Taking $y = \mathrm{e}^w$ we obtain:

$$w'' + {w'}^2 + 2w' + 1 + \frac{m^2}{x} = 0 \Rightarrow w' = -1 \pm \mathrm{i}\sqrt{\frac{m^2}{x} + w''}$$
$$w \sim -x \pm 2\mathrm{i}m\sqrt{x} + \frac{1}{4}\ln x \Rightarrow y_\pm \sim x^{1/4}\mathrm{e}^{-x \pm 2\mathrm{i}m\sqrt{x}} \tag{4.35}$$

as before.

(C) *Berry smoothing.*

Let $y = \sum_{k=1}^{n-1} a_k x^{-k} + \sum_\pm C_\pm(x) y_\pm(x)$. Then $L[y] - x^{-1}$ gives

$$\sum_\pm y_\pm \left[C''_\pm + 2C'_\pm\left(1 + \frac{y'_\pm}{y_\pm}\right)\right] + \frac{n(n-1)a_{n-1}}{x^{n+1}} + \frac{(n-1)(n-2)a_{n-2}}{x^n}$$
$$+ \frac{m^2 - 2n - 2}{x^n} a_{n-1} = L_+C_+ + L_-C_- + \frac{n(n-1)a_{n-1}}{x^{n+1}} - \frac{a_n}{x^n}$$
$$= L_+C_+ + L_-C_- + \frac{n!}{x^n} b_n \left(\frac{1}{n} - \frac{b_{n-1}}{b_n x}\right) \tag{4.36}$$

with obvious notations. Changing variables to $x = n\mathrm{e}^{\mathrm{i}\beta n^{-1/2}}$ above we get, to leading order,

$$n^{1/4} \sum_\pm \mathrm{e}^{-n + \mathrm{i}\sqrt{n}(\pm 2m - \beta) + \frac{1}{2}\beta^2 \mp m\beta} \left(-\frac{C''_\pm}{n} \mp \frac{2mC'_\pm}{n}\right)$$
$$= -\sum_\pm \frac{n!}{n^n} \mathrm{e}^{-\mathrm{i}\sqrt{n}\beta} \left[n^{1/4} A_\pm \mathrm{e}^{\pm 2\mathrm{i}mn^{1/2}} n^{-3/2} \mathrm{i}(\pm m + \beta)\right] \tag{4.37}$$

Equating the coefficients of $\mathrm{e}^{2\mathrm{i}mn^{1/2}}$ we get, with $B_\pm = Const + o(1)$ the system

$$C''_\pm \pm \frac{2\mathrm{i}m}{x^{1/2}} C'_\pm = \mathrm{i}B_\pm n^{-1} \mathrm{e}^{-\frac{1}{2}\beta^2 \pm m\beta}(\pm m + \beta) \tag{4.38}$$

where we change variables to $x = n\exp(\mathrm{i}\beta n^{-1/2})$ and get to leading order:

$$-C''_\pm(\beta) \pm 2mC'_\pm(\beta) = \mathrm{i}B_\pm \mathrm{e}^{-\frac{1}{2}\beta^2 \pm m\beta}(\pm m + \beta) \tag{4.39}$$

with the bounded solutions:

$$C_\pm(\beta) = \mathrm{i}\sqrt{\frac{\pi}{2}} \mathrm{e}^{\frac{1}{2}m^2} \operatorname{erf}\left(\frac{1}{\sqrt{2}}\beta - \frac{1}{\sqrt{2}}m\right) B_\pm + \mathrm{Const.}_\pm \tag{4.40}$$

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