Global reconstruction of analytic functions from local expansions and a new general method of converting sums into integrals

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Abstract. A new summation method is introduced to convert a relatively wide family of Taylor series and infinite sums into integrals.

Global behavior such as analytic continuation, position of singularities, asymptotics for large values of the variable and asymptotic location of zeros thereby follow, through the integral representations, from the Taylor coefficients at a point, say zero.

The method can be viewed in some sense as the inverse of Cauchy's formula.

It can work in one or several complex variables.

There is a duality between the global analytic structure of the reconstructed function and the properties of the coefficients as a function of their index.

Borel summability of a class of divergent series follow as a byproduct.

Contents

1. Introduction

Finding the global behavior of an analytic function in terms of its Taylor coefficients is a notoriously difficult problem. In fact, there cannot exist a general solution to this problem, since undecidable questions can be quite readily formulated in such terms.

Obviously too, mere estimates on the coefficients do not provide enough information for global description. But constructive, detailed knowledge of the coefficients does. Simple examples are

$$f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}}, \quad f_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\pi + \ln n},$$
$$f_3(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{n+1}}, \quad f_4 = \sum_{n=1}^{\infty} n^n z^n \tag{1.1}$$

(Certainly, as presented, f_4 does not define a function; the question for f_4 is whether the series is Borel summable.)

Coefficients with a different type of growth, say $e^{\sqrt{n}}$ can be accommodated too, as seen below.

The example f_1 is the classical polylog$(1/2, z)$ but f_2, f_3 or f_4 satisfy no obvious relation from which analytic control can be otherwise gained. Yet they are particularly simple, in that the coefficients have an explicit formula. Integral representations however can be obtained in the much more general case when $f^{(n)}(0)$ is *analyzable* (cf. §1.3) in n [(1)]. Solutions to very general differential or partial differential equations, difference equations are known to be analyzable, and this class of functions is closed under many operations occurring in analysis. In fact, analyzable functions are obtained by an isomorphism from *transseries*, which are indeed constructed as the closure of series under a wide class of operations [3].

In particular it will follow from the results below that

$$f_1(z) = \frac{z}{\sqrt{\pi}} \int_0^\infty \frac{dt}{(1+t)\sqrt{\ln(1+t)}(t-(z-1))} \tag{1.2}$$

On the first Riemann sheet f_1 has only one singularity, at $z = 1$, of logarithmic type, and $f_1 = O(z^{-1})$ for large z. General Riemann surface information and monodromy follow straightforwardly (cf. (2.18)). A similar complex analytic structure is shared by f_2, which has one singularity at $z = 1$ where it is analytic in $\ln(1-z)$ and $(1-z)$; the singularity structure is that of the function

$$\phi(z) = \oint_0^\infty \frac{e^{-u\ln(z-1)}}{(-u)^\pi + \ln(-u)} du \tag{1.3}$$

(1) After developing these methods, it has been brought to our attention that a duality between resurgent functions and resurgent Taylor coefficients has been noted in an unpublished manuscript by Écalle.

where the notation $\oint_0^\infty$ is explained after (2.12) below. An explicit formula for the singular structure can be obtained in all cases, and their is a duality between the properties of the coefficients and the global structure, for instance monodromy, of the reconstructed function.

The function f_3 is entire; questions answered regard say the behavior for large negative z or the asymptotic location of zeros. It will follow that f_3 can be written as

$$f_3(z) = e^{-1} \int_0^\infty (1+u)^{-1} G(\ln(1+u)) \left[\exp\left(\frac{ze^{-1}}{1+u} \right) - 1 \right] du \quad (1.4)$$

where $G(p) = s_2'(1+p) - s_1'(1+p)$ and $s_{1,2}$ are two branches of the functional inverse of $s - \ln s$, cf. §3. Detailed behaviour for large z can be obtained from (1.4) by standard asymptotics methods; in particular, for large negative z, f_3 behaves like a constant plus $z^{-1/2}e^{-z/e}$ times a factorially divergent series (whose terms can be calculated).

It is often convenient to work with a series given in terms of the coefficients, even when an underlying generating problem exists [20, 14].

A reconstruction procedure was known in the context of nonlinear ODEs for which information about location of singularities of solutions can be "read" in their transseries representations [13].

As it will be clear from the proofs, the method and results would apply, with minor adaptations to functions of several complex variables.

1.1. Evaluating series.

There are many other questions amenable to this method. For instance, we get that

$$\lim_{z \to -1^+} \sum_{n=1}^{\infty} e^{\sqrt{n}} z^n = -\frac{1}{4\sqrt{\pi}} \int_{C_1} \frac{e^{1/p} dp}{p^{-3/2}(e^p + 1)} \quad (1.5)$$

where C_1 starts along $\mathbb{R}^+$, loops clockwise once around the origin and ends up at $+\infty$. There is also a practical side to (1.5): while the sum is numerically unwieldy, the integral can be evaluated accurately by standard means. Likewise, we get

$$\sum_{n=0}^{\infty} \frac{e^{i\sqrt{n}}}{n^a} = \frac{2^{a-1/2}}{\sqrt{\pi}} \int_C dp \frac{e^{\frac{1}{8p}} U(2a + 1/2; \frac{-i}{\sqrt{2p}})}{p^{a-1}(e^p + 1)} \quad (1.6)$$

for $a > 1/2$ (convergence of the sum follows, *e.g.* by comparing it to an integral and estimating the remainder). Here C is a contour starting along $\mathbb{R}^-$, encircling the origin clockwise and ending up at $+\infty$, and U is the parabolic cylinder function [1]. These sums are obtained in §2.1.

1.2. Global description from local expansions

A first class of problems is finding the location and type of singularities in $\mathbb{C}$ and the behaviour for large values of the variable of functions given by series with finite radius of convergence, such as those in (1.1).

The second class of problems amenable to the techniques presented concerns the behaviour at infinity (growth, decay, asymptotic location of zeros etc.) of entire functions presented as Taylor series, such as f_3 above.

The third type of class of problems is to determine Borel summability of series with zero radius of convergence such as

$$\tilde{f}_4 = \sum_{n=0}^{\infty} n^{n+1} z^n \tag{1.7}$$

in which the coefficients of the series are analyzable ((1.7) is Borel summable).

1.3. Transseries and analyzable functions

In the early 1980's, Écalle discovered and extensively studied a broad class of functions, analyzable functions, closed under algebraic operations, composition, function inversion, differentiation, integration and solution of suitably restricted differential equations [2,3,4,5]. They are described as generalized sums of "transseries", the closure of power series under the same operations. The latter objects are surprisingly easy to describe; roughly, they are ordinal length, asymptotic expansions involving powers, iterated exponentials and logs, with at most power-of-factorially growing coefficients.

In view of the closure of analyzable functions to a wide class of operations, reconstructing functions from series with arbitrary *analyzable* coefficients would make the reconstruction likely applicable to series occurring in problems involving any combination of these many operations.

This paper deals with analyzable coefficients having finitely many singularities after a suitable EB transform. The methods however are open to substantial extension. In particular, we allow for general singularities, while analyzable and resurgent functions have singularities of a controlled type [3].

1.4. Classical and generalized Borel summation

A series $\tilde{f} = \sum_{n=1}^{\infty} c_n x^{-n}$ is Borel summable if its Borel transform, *i.e.* the formal inverse Laplace transform (2) converges to a function F analytic in a neighborhood of $\mathbb{R}^+$, and F grows at most exponentially at infinity. The Laplace transform of F is by definition the Borel sum of $\tilde{f}$. Since Borel summation is formally the identity, it is an extended isomorphism between functions and series, much as convergent Taylor series associate to their sums.

However expansions occurring in applications are often not *classically* Borel summable, sometimes for the relatively manageable reason that the expansions are not simple integer power series, or often, more seriously, because F is singular on $\mathbb{R}^+$, as is the case of $\sum n! x^{-n-1}$ where $F = (1-p)^{-1}$, or because F grows superexponentially.

To address the latter difficulties, Écalle defined averaging and cohesive continuation to replace analytic continuation, and acceleration to deal with superexponential growth [2,3,4,5].

We call Écalle's technique Écalle-Borel (EB) summability and "EB transform" the inverse of EB summation. While it is an open, imprecisely formulated, and in fact conceptually challenging question, whether EB summable series are closed under all operations needed in analysis, general results have been proved for ODEs, difference equations, PDEs, KAM resonant expansions and other classes of problems [8,12,7,19,20]. EB summability seems for now quite general.

A function is analyzable if it is an EB transform of a transseries. This transseries is then unique [3]. Then the EB transform is the mapping that associates this unique transseries to the function. Simple examples of such transseries are

$$f_n = \frac{1}{\sqrt{n}}; \quad f_n = \frac{1}{n}\sum_{j=1}^{\infty}(-1)^n \frac{\ln^j n}{n^{j\pi}};$$

$$f_n = \sum_{j=1}^{\infty} \frac{(-1)^j j!}{n^j} + 2^{-n-1} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(j-1/2)}{\sqrt{\pi}} \qquad (1.8)$$

The first two are convergent and correspond to the first two series in (1.1); the last one is divergent but Borel summable.

EB summation consists, in the simplest cases, in replacing the series in the transseries by their Borel sum. In a first stage one takes the Borel transform in n (cf. §1.4) of each component series. In (1.8)

(2) $\displaystyle\sum_{n=1}^{\infty} c_n \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{px} x^{-n} dx = \sum_{n=0}^{\infty} \frac{c_n p^{n-1}}{(n-1)!} = F(p)$

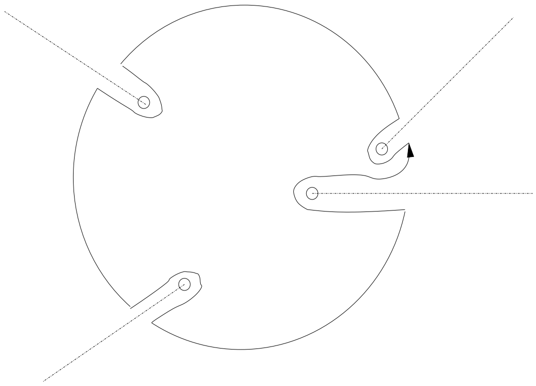

Fig. 1. Singularities of f, cuts, direction of integration and Cauchy contour deformation.

the Borel transforms are correspondingly,

$$F(p) = \frac{1}{\sqrt{p\pi}}, \quad F_1(p), \quad \left[\frac{1}{1+p}, \sqrt{p+1}\right] \tag{1.9}$$

and F_1 given in (2.27). The last pair represents the separate EB transforms of the two series in the last example in (1.8).

The EB sum of these transseries are, with $(\mathcal{L}F)(p) = \int_0^\infty e^{-pn}F(p)dp$ the usual Laplace transform,

$$\begin{aligned} f_n = \mathcal{L}\frac{1}{\sqrt{p\pi}} = \frac{1}{\sqrt{n}}; \ f_n = \mathcal{L}F_1; \ \ \mathcal{L}\frac{1}{1+p} + 2^{-n-1}\mathcal{L}\sqrt{1+p} \\ = e^n\mathrm{Ei}(-n) + 2^{-n-1}\left(\frac{1}{n} + \frac{\sqrt{\pi}e^n\mathrm{erfc}(\sqrt{n})}{n^{3/2}}\right) \end{aligned} \tag{1.10}$$

2. (i) Series with finite radius of convergence

Let $\mathcal{M}$ be the functions analytic at zero, algebraically bounded at infinity and which have finitely many, possibly branched, singularities in $\mathbb{C}$. We choose to make cuts from z_j to infinity which do not intersect.

$\mathcal{M}$ would be a proper subclass of *resurgent* functions [2] if the singularities are of the form described in Note 23 (b).

Note 21 (Generalizations) 1. It will be seen from the proofs that it is sufficient to have at most that algebraic growth in *some* sector, or with slight modifications, at most exponential growth.

2. Also, one can allow for infinitely many singularities, if some estimates for their strength is available. Écalle averaging would allow for singularities on the line of summation.

3. Exponential growth of F_j (see below) can also be accommodated, provided a sufficient number of initial terms of the series are left out.
4. Several complex variables can be treated very similarly, as it will become clear.
5. Other types of decay/growth of coefficients can be accommodated, cf. §2.1
   Some further generalizations are transparent, but to simplify the notation and proofs we will not pursue them.

Note 22 *By taking sufficiently many derivatives we can assume that* $f \in \mathcal{M}' = \{f \in \mathcal{M} : f = O(z^{-1-\epsilon})$ *as* $z \to \infty\}$.

Consider a series with finite radius of convergence

$$f = \sum_{n=0}^{\infty} f_n z^n \tag{2.11}$$

In the following, if g is analytic on $\mathbb{R}^+$ and has a singularity at zero we denote by

$$\oint_0^{\infty} g(s)ds \tag{2.12}$$

the integral of g around $\mathbb{R}^+$, traversed towards $+\infty$ on the upper side.

While providing integral formulas, in terms of functions with known singularities, which are often rather explicit, the following result can also be interpreted as a *duality of resurgence.*

Theorem 21 *(i) Assume that* f_n *have Borel sum-like representations of the form*

$$f_n = \sum_{j=1}^{N} a_j^{-n} \oint_0^{\infty} e^{-np} F_j(p)ds \tag{2.13}$$

(2.13) where F_j *are analytic and bounded on* $\mathbb{R}^+$ *and singular at zero.*
Then, f *is given by*

$$f(z) = f(0) + z \oint_0^{\infty} \sum_{j=1}^{N} \frac{F_j(\ln(1+s))ds}{(1+s)(sa_j + a_j - z)} \tag{2.14}$$

(ii) Furthermore, $f \in \mathcal{M}$ *and the singularities, located at* a_j *and are of the same type as the singularities of* $F_j(\ln(1+s))$.
(iii) Conversely, let $f \in \mathcal{M}'$*, with singularities at* $\{a_1, ..., a_N\}$*:*

$$f(z) = \sum_{n=0}^{\infty} f_n z^n; \quad |z| < r \tag{2.15}$$

Then f_n have Borel sum-like representations of the form

$$f_n = \frac{1}{2\pi i}\sum_{j=1}^{N} a_j^{-n} \oint_0^{\infty} e^{-np}\delta_{a_j}F(a_j e^s)ds \tag{2.16}$$

where $\delta_{a_j}F(a_j e^s)$ is the difference between the values of F on the left of and right the cut from a_j to ∞.

The behaviour at a_j and at ∞ will follow from the proof.

Note 23 (a) The singularities presented are on the first Riemann sheet. A more global information requires like information on F_j.

(b) **Classically Borel summable series.** The integral representations in (2.13) would be true Borel sums if $F(p) = (2\pi i)^{-1}\ln p\, H(p)$ with H analytic at zero, as it can be easily checked.

(c) **EB Borel summable series.** More generally, f_n are EB summable if F_j are analytic at 0 in $p^{\beta_j}\ln p^{\alpha_j}$, $j = 1, ..., N$ with $\mathrm{Re}(\beta_j) > 0$. If such is the case, the singularity of F_j at zero is of the same type as that of $F_j(\ln(1+p))$ since $\ln(1+p)$ is analytic at zero. Strictly speaking, duality of resurgence only applies to these cases.

If F has an exponential or worse type singularity at zero, the representation is *not* a generalized Borel sum in any sense. Écalle acceleration might bring it to the simpler case above. But we allow for any singularities since the result goes through.

(d) Equation (1.2) follows straightforwardly from (2.14). To obtain analytic information it suffices to take first say, the imaginary part of z sufficiently large. We rewrite (1.2) in the form

$$f_1(z) = \frac{z}{\sqrt{\pi}2}\oint_0^{\infty} \frac{dt}{(1+t)\sqrt{\ln(1+t)}(t-(z-1))} \tag{2.17}$$

and expand the contour of integration in (2.17) up, down and to the left by $2\pi - \epsilon$, to the boundary of a strip. Say the new integral is I_2. Then, upon bringing z inside the contour we collect a residue

$$\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\ln z}} \tag{2.18}$$

Now z can be moved around 0 or -1, on curves staying inside the strip and I_2 remains analytic. The monodromy around 0 and 1 is thus solely contained in the term (2.18). This analysis can be extended to general polylogs, and more detailed information can be obtained. This will be the subject of a different paper.

Proof of Theorem 21 If $f \in \mathcal{M}'$ we write the Taylor coefficients in the form

$$f_n = \frac{1}{2\pi i}\oint \frac{ds f(s)}{s^{n+1}} \tag{2.19}$$

where the contour of integration is a small circle of radius r around the origin. We attempt to increase r without bound. In the process, the contour will hang around the singularities of f as shown in Figure 1. The integrals converge by the decay assumptions and the contribution of the arcs at large r vanish.

In the opposite direction, we let z be sufficiently small, and such that $za_j \notin \mathbb{R}$ for all j. We choose the contour in such close enough to zero so that $|za_j e^{-p}| < \alpha < 1$ all along the contour. By dominated convergence we have

$$\sum_{n=1}^{\infty} f_n z^n = \sum_{j=1}^{N} \oint_0^{\infty} \sum_{n=1}^{\infty} z^n a_j^{-n} e^{-np} F_j(p) ds$$
$$= \sum_{j=1}^{N} \oint_0^{\infty} \frac{z a_j^{-1} e^{-p}}{1 - z a_j^{-1} e^{-p}} F_j(p) dp = \sum_{j=1}^{N} z \oint_0^{\infty} \frac{F_j(\ln(1+s)) ds}{(1+s)(s a_j + a_j - z)} \tag{2.20}$$

as stated. Now the integral can be analytically continued in z. The nature of the singularities of f is examined in §2.2

2.1. Other growth rates; examples of special sums

Other growth rates can be accommodated, for instance by analytic continuation. We have for positive γ,

$$e^{-\gamma\sqrt{n}} = \frac{\gamma}{2\sqrt{\pi}} \int_0^{\infty} p^{-3/2} e^{-\frac{\gamma^2}{4p}} e^{-np} dp \tag{2.21}$$

which can be analytically continued in γ. We note first that the contour cannot be, for this function, detached from zero. Instead, we keep one endpoint at infinity and, near the origin, simultaneously rotate γ and p to maintain $-\gamma/p$ real and negative. We get

$$e^{\sqrt{n}} = -\frac{1}{4\sqrt{\pi}} \int_{C_1} p^{-3/2} e^{\frac{1}{4p}} e^{-np} dp \tag{2.22}$$

where C_1 is described in the introduction, and (1.5) follows. Eq. (1.6) is obtained in a similar way.

Obviously, if the behavior of the coefficients is of the form $A^n f_n$ where f_n satisfies the conditions in the paper, one simply changes the independent variable to $z' = Az$.

2.2. Singularity formula. Duality.

The problem of the type of singularities of the resummed series reduces to finding the singularity type of a Hilbert-transform-like integral of the form

$$g(t) = \oint_0^\infty \frac{G(s)}{s-t} ds \tag{2.23}$$

The singularity of g at $t = 0$ is the same as the singularity of G at $s = 0$ as follows from a simple calculation.

Lemma 24 (Analytic structure at $t = 0$.) *For small t we have*

$$\oint_0^\infty \frac{G(s)}{s-t} ds = 2\pi i G(t) + G_2(t) \tag{2.24}$$

where $G_2(t)$ is analytic for small t.

Proof. We take t, $|t| = \epsilon$ small, outside the contour of integration. Around $s = 0$ we deform the contour into a circle of radius 2ϵ in the process collecting a residue

$$2\pi i G(t) \tag{2.25}$$

The new integral is manifestly analytic for $|t| < \epsilon$.

Note 25 In the particular case of Borel summable series, leading to expressions of the form $\int_0^\infty (p-t)^{-1} H(p) dp$ with p analytic at zero, either by converting them to the form $(2\pi i)^{-1} \oint_0^\infty (p-t)^{-1} H(p) \ln p \, dp$ or simply writing near zero $H(p) = H(t) + (H(p) - H(t))$ we see that the behaviour for small t is of the form $H(t) \ln t + h(t)$ with h analytic.

Example: For the function f_2 in the introduction, the inverse Laplace transform is

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^{xp}}{x^\pi + \ln x} dx \tag{2.26}$$

where the contour can be bent backwards, to hang around $\mathbb{R}^-$. Then, with the change of variable $x = -u$ (2.26) becomes

$$F_1(p) = \oint_0^\infty \frac{e^{-up}}{(-u)^\pi + \ln(-u)} du \tag{2.27}$$

for which the singularity, at one, according to (2.25) is (1.3).

3. (ii) Entire functions

We restrict the analysis to entire functions of exponential order one, with complete information on the Taylor coefficients. Such functions include of course the exponential itself, or expressions such as f_3. It is useful to start with f_3 as an example. The analysis is brought to the case in §2.2 by first taking a Laplace transform. Note that

$$\int_0^\infty e^{-xz} f(z)dz = \frac{1}{x}\sum_{n=0}^\infty \frac{n!}{n^{n+1}x^n} \tag{3.28}$$

The study of entire functions of exponential order one likely involves the factorial, and then a Borel summed representation of the Stirling formula is needed.

3.1. The Gamma function and Borel summed Stirling formula

We have

$$\begin{aligned} n! = \int_0^\infty t^n e^{-t}dt = n^{n+1}\int_0^\infty e^{-n(s-\ln s)}ds \\ = n^{n+1}\int_0^1 e^{-n(s-\ln s)}ds + n^{n+1}\int_1^\infty e^{-n(s-\ln s)}ds \end{aligned} \tag{3.29}$$

On $(0,1)$ and $(1,\infty)$ separately, the function $s-\ln(s)$ is monotonic and we may write, after inverting $s-\ln(s)=t$ on the two intervals to get $s_{1,2}=s_{1,2}(t)$,

$$n! = n^{n+1}\int_1^\infty e^{-nt}(s_2'(t)-s_1'(t))dt = n^{n+1}e^{-n}\int_0^\infty e^{-np}G(p)dp \tag{3.30}$$

where $G(p) = s_2'(1+p) - s_1'(1+p)$. From the definition it follows that G is bounded at infinity and $p^{-1/2}G$ is analytic in $p^{1/2}$ at $p=0$. Using now (3.30) and Theorem 21 in (3.28) we get

$$\int_0^\infty e^{-xz} f(z)dz = \frac{1}{x^2}\int_0^\infty \frac{G(\ln(1+t))}{(te+(e-x^{-1}))(t+1)}dt \tag{3.31}$$

Upon taking the inverse Laplace transform we obtain (1.4).

More generally we obtain from Theorem 21, in the same way as above, the following.

Theorem 31 *Assume that the entire function f is given by*

$$f(z)=\sum_{n=1}^{\infty}\frac{f_n z^n}{n!} \tag{3.32}$$

with f_n as in Theorem 21. Then,

$$f(z)=\oint_0^{\infty}\sum_{j=1}^{N}\left[\left(e^{\frac{z}{s+a_j}}-1\right)\frac{F_j(\ln(1+s))}{(1+s)}\right]ds \tag{3.33}$$

As in the simple example, the behavior at infinity follows from the integral representation by classical means.

4. (iii) Borel summation

Theorem 41 *Consider the formal power series*

$$\tilde{f}(z)=\sum_{n=1}^{\infty}f_n n! z^{n+1} \tag{4.34}$$

with coefficients f_n as in Theorem 21. Then the series is (generalized) Borel summable to

$$\int_0^{\infty}dpe^{-p/z}p\sum_{j=1}^{N}\oint_0^{\infty}\frac{F_j(\ln(1+s))}{(1+s)(a_js+a_j-p)}ds$$
$$=-\sum_{j=1}^{N}\oint_0^{\infty}\frac{F_j(\ln(1+s))}{1+s}\left(z-a_j(s+1)e^{-\frac{a_j(s+1)}{z}}\mathrm{Ei}\left(\frac{a_j(s+1)}{z}\right)\right)ds \tag{4.35}$$

The proof proceeds as in the previous sections, taking now a Borel transform in z^{-1} followed by a Laplace transform.

4.1. Appendix: Borel summed version of $1/n!$

We can use the following representation [1]

$$\frac{1}{\Gamma(n)}=-\frac{ie^{-\pi iz}}{2\pi}\oint_0^{\infty}s^{-z}e^{-s}ds=-\frac{ie^{-\pi iz}z^{-z}}{2\pi}\oint_0^{\infty}s^{-z}e^{-zs}ds \tag{4.36}$$

with our convention of contour integration. From here, one can proceed as in §3.1.

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