Cohomology computations for relatives of Coxeter groups

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(joint work with Boris Okun)

We compute the group cohomology $H^*(G; \mathbf{Z}G)$ or the reduced L^2-cohomology, $\mathcal{H}^*(G; L^2G)$ when G is either

- a graph product of infinite groups,
- an Artin group, or
- a Bestvina-Brady group.

The proofs will appear in [8].

Suppose (W, S) is a Coxeter system. A subset $T \subset S$ is *spherical* if it generates a finite subgroup of W. Let $\mathcal{S}$ denote the poset of spherical subsets of S. The *nerve* of (W, S) is the simplicial complex L with vertex set S and with one simplex for each nonempty $T \in \mathcal{S}$. Let $K := \mathrm{Flag}(\mathcal{S})$ be the simplicial complex of all flags in $\mathcal{S}$ (the *geometric realization of* $\mathcal{S}$). Note that K is isomorphic to the cone on the barycentric subdivision of L. For each $s \in S$, put $K_s := \mathrm{Flag}(\mathcal{S}_{\geq \{s\}})$ and for each $T \leq S$, put

$$K_T := \bigcap_{s \in T} K_s \quad \text{and} \quad K^T := \bigcup_{s \in T} K_s.$$

(K is the *Davis chamber* and K_s is a *mirror* of K.) Also, for each $T \in \mathcal{S}$, $\partial K_T := \mathrm{Flag}(\mathcal{S}_{>T})$ (which is isomorphic to the barycentric subdivision of the link of T in L).

Previous results. The following theorem was proved in [2] (also see [3]).

Theorem A. ([2, 5]).

$$H^*(W; \mathbf{Z}W) = \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes A_T,$$

where A_T is a certain nontrivial free abelian group.

Essentially the same result holds for the compactly supported cohomology of any locally finite building of type (W, S) (except that the free abelian group A_T is larger), cf. [4]. In particular, since any graph product of finite groups acts properly and cocompactly on a right-angled building, Theorem A also holds for graph products of finite groups.

Theorem B. (Davis-Leary [7]). *Suppose A is the Artin group associated to (W, S) and that X is its Salvetti complex. Then*

$$\mathcal{H}^*(X; L^2W) = H^*(K, \partial K) \otimes L^2(A).$$

When the $K(\pi, 1)$-Conjecture holds for A, this formula computes $\mathcal{H}^*(A; L^2W)$. (Recall that the $K(\pi, 1)$-Conjecture asserts that $X = BA$.)

The von Neumann dimension of $\mathcal{H}^k(X; L^2A)$ is called the k^{th} *L^2-Betti number*, and denoted $L^2b^k(X; A)$. So, Theorem B gives: $L^2b^k(X; A) = b^k(K, \partial K)$, where $b^k(K, \partial K)$ denotes the ordinary Betti number of $(K, \partial K)$.

Theorem C. (Jensen-Meier [9]). *Suppose A is the right-angled Artin group (the RAAG) associated to the right-angled Coxeter system (a RACS) (W, S). Then*

$$H^n(A; \mathbf{Z}A) = \bigoplus_{T \in \mathcal{S}} H^{n-|T|}(K_T, \partial K_T) \otimes B,$$

where B is a certain free abelian group.

Computations. Suppose that Γ is a graph with vertex set S and that (W, S) is the associated RACS. Let $(G_s)_{s \in S}$ be a family of groups and $G = \prod_\Gamma G_s$, the corresponding graph product. For each spherical subset T, define G_T to be direct product $\prod_{s \in T} G_s$ (it is a subgroup of G). The proofs of following computations use a spectral sequence and in the end, only an associated graded module to a cohomology group is computed which we denote $\operatorname{Gr} H^*(\;)$.

Theorem 1. *Suppose each G_s is infinite. Then*

$$\operatorname{Gr} H^n(G; \mathbf{Z}G) = \bigoplus_{T \in \mathcal{S}} \bigoplus_{p+q=n} H^p(K, \partial K_T; H^q(G_T; \mathbf{Z}G)).$$

Similarly, for L^2-cohomology, we have,

$$L^2 b^n(G) = \sum_{T \in \mathcal{S}} \sum_{p+q=n} b^p(K_T, \partial K_T) L^2 b^q(G_T).$$

We note that $H^q(G_T; \mathbf{Z}G)$ and $L^2 b^q(G_T)$ can be calculated from their values for the G_s by using the Künneth Formula. We also note that if each $G_s = \mathbf{Z}$, then G is a RAAG. Moreover, $G_T = \mathbf{Z}^T$, so $H^q(\mathbf{Z}^T; \mathbf{Z}\mathbf{Z}^T)$ is nonzero only in degree $q = |T|$ (where it is $= \mathbf{Z}$) and we recover Theorem C. Since all L^2-Betti numbers of $\mathbf{Z}^T$ vanish for $T \neq \emptyset$ we also recover Theorem B in the right-angled case.

It is known that any Artin groups A_T of spherical type is a $|T|$-dimensional duality groups, i.e., $H^*(A_T; \mathbf{Z}A_T)$ is concentrated in degree $|T|$ and is free abelian.

Theorem 2. *Suppose A is the Artin group associated to (W, S) and that X is its Salvetti complex. Then*

$$\operatorname{Gr} H^n(A; \mathbf{Z}A) = \bigoplus_{T \in \mathcal{S}} H^{n-|T|}(K_T, \partial K_T) \otimes (\mathbf{Z}A \otimes_{A_T} H^{|T|}(A_T; \mathbf{Z}A_T)).$$

Suppose A is the RAAG associated to the RACS (W, S) with nerve L. Let BB_L denote the kernel of the map $A \to \mathbf{Z}$ which sends each standard generator to 1. If L is acyclic, then BB_L is called a *Bestvina-Brady group* (in which case Bestvina and Brady proved it was type FP).

Theorem 3. *Suppose BB_L is a Bestvina-Brady group. Then its cohomology with group ring coefficients is the same as that of the corresponding Artin group, shifted in degree by* 1,

$$\operatorname{Gr} H^n(BB_L; \mathbf{Z}BB_L) = \bigoplus_{T \in \mathcal{S}_{>\emptyset}} H^{n-|T|+1}(K_T, \partial K_T) \otimes \mathbf{Z}(BB_L/(BB_L \cap A_T)).$$

Similarly, for L^2-cohomology, we have,

$$L^2b^n(BB_L) = \sum_{s\in S} b^n(K_s, \partial K_s).$$

A spectral sequence. For all of these computations the proofs involve the following lemma concernng a Mayer-Vietoris type spectral sequence. Suppose a CW-complex X is a union of a collection of subcomplexes $\{X_a\}_{a\in\mathcal{P}}$ indexed by a poset $\mathcal{P}$ giving it the structure of a "poset of spaces" as in [8]. There is a spectral sequence converging to $H^*(X)$ with E_1-page:

$$E_1^{p,q} = C^p(\mathrm{Flag}(\mathcal{P}); \mathcal{H}^q),$$

where $\mathcal{H}^q$ denotes the constant coefficient system $\sigma \mapsto X_{\min\sigma}$, where $\min\sigma$ denotes the minimum element of the flag, σ. For each $a \in \mathcal{P}$, put $X_{<a} := \bigcup_{b<a} X_b$. Consider the following hypothesis:

(*) For each $a \in \mathcal{P}$, the map induced by the inclusion, $H^*(X_a) \to H^*(X_{<a})$ is the 0-homomorphism.

Lemma. *Suppose (*) holds. Then the spectral sequences decomposes into a direct sum at E_2 and $E_2 = E_\infty$:*

$$E_\infty^{p,q} = E_2^{p,q} = \bigoplus_{a\in\mathcal{P}} H^p(\mathrm{Flag}(\mathcal{P}_{\geq a}), \mathrm{Flag}(\mathcal{P}_{>a}); H^q(X_a)).$$

In each case in which we are interested, $\mathcal{P} = \mathcal{S}$; moreover, the appropriate space associated to the group in question is covered by subcomplexes indexed by $\mathcal{S}$ and condition (*) is easily verified.

References

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