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Groups generated by reflections and aspherical manifolds not covered by Euclidean space

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Introduction

A *Coxeter system* (Γ, V) is a group Γ (a "Coxeter group") together with a set of generators V such that each element of V has order two and such that all relations in Γ are consequences of relations of the form $(vw)^{m(v,w)} = 1$, where $v, w \in V$ and $m(v, w)$ denotes the order of vw. The $m(v, w)$'s (which are positive integers or ∞) obviously determine the Coxeter system up to isomorphism. The list of finite Coxeter groups is short and well-known; however, general Coxeter groups are much more flexible. In fact, the set V and the $m(v, w)$'s can be specified arbitrarily (at least if V is finite) subject only to the conditions: $m(v, v) = 1$ and $m(v, w) = m(w, v) \geq 2$ if $v \neq w$.

Suppose that (Γ, V) is a Coxeter system, that X is a Hausdorff space and that $(X_v)_{v \in V}$ is a locally finite family of closed subspaces indexed by V. (The X_v are called the "panels" of X.[2]) There is a classical method for constructing a transformation group from these data. For each $x \in X$, let $V(x)$ denote the set of v in V such that $x \in X_v$. For each subset S of V, let Γ_S be the subgroup generated by S and let X_S be the "face" of X defined by

$$X_S = \bigcap_{v \in S} X_v$$

(By convention, $\Gamma_\varnothing = \{1\}$ and $X_\varnothing = X$.) Thus, $V(x)$ is the index of the smallest face containing x. Give Γ the discrete topology and let $\mathfrak{A}$ $(= \mathfrak{A}(\Gamma, X))$ denote the quotient space of $\Gamma \times X$ by the equivalent relation $\sim$ defined by $(g, x) \sim (h, y) \Leftrightarrow x = y$ and $g^{-1}h \in \Gamma_{V(x)}$. The natural Γ-action on $\Gamma \times X$ induces a Γ-action on $\mathfrak{A}$. The isotropy group at the equivalence class of $(1, x)$ is $\Gamma_{V(x)}$. Moreover, every isotropy group of Γ on $\mathfrak{A}$ is conjugate to one of this form. It is easily seen that the Γ-action is proper (i.e., properly discontinuous) if and only if

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(2) "Panel" is A. Borel's translation of "cloison" in [B, p. 39].

each isotropy group is finite (cf. Lemma 13.4). Thus, the Γ-action on $\mathfrak{A}$ is proper if and only if the following condition holds:

(*) For each subset S of V such that the face X_S is nonempty, the subgroup Γ_S is finite.

The main result of this paper is that the following condition (cf. Theorems 10.1 and 13.5) is necessary and sufficient for $\mathfrak{A}$ to be contractible:

(**) X is contractible and for each subset S of V such that Γ_S is finite, the face X_S is acyclic.

(Note: acyclic implies nonempty.)

There is a nice way to rephrase conditions (*) and (**) in terms of simplicial complexes. Denote by $K_0(\Gamma, V)$ or K_0 (resp. $D_0(X)$ or D_0), the abstract simplicial complex with vertex set V and with simplices, those nonempty subsets S of V such that Γ_S is finite (resp. such that X_S is nonempty). Thus, D_0 is the nerve of the covering of ∂X $(= \bigcup_{v \in V} X_v)$ by its panels. Condition (*) is then equivalent to the statement that D_0 is a subcomplex of K_0, while condition (**) implies that K_0 is a subcomplex of D_0. Therefore, if $\mathfrak{A}$ is contractible and the Γ-action is proper, then $K_0 = D_0$. In other words, the combinatorial properties of the panel structure on X are completely determined by the Coxeter system.

This leads to the question of which simplicial complexes can be realized as the complex associated to some Coxeter system. Although the answer to this is unclear, for many purposes the following easy result suffices: the barycentric subdivision of any finite simplicial complex is the associated complex of some Coxeter system (cf. Lemma 11.3).

The construction of $\mathfrak{A}$ arises naturally in the theory of groups generated by reflections. Classically, Γ is a discrete group of rigid motions of Euclidean space generated by affine orthogonal reflections, a "chamber" X is a convex polyhedron[(3)] cut out by the reflecting hyperplanes, the panels X_v are the codimension one faces of X, and V is the set of reflections through the hyperplanes supported by the panels of X. It is then well-known that

1) V generates Γ,

2) (Γ, V) is a Coxeter system, and

3) $\mathfrak{A}$ is Γ-equivariantly homeomorphic to the ambient Euclidean space.

It may be helpful to keep a simple example in mind: X is a rectangle in the plane, the panels of X are its sides, Γ is the group generated by the orthogonal

[(3)] In this classical case, Coxeter [C] proved that X is a Cartesian product of simplices, simplicial cones, and a Euclidean space.

reflections through the sides of the rectangle; the images of X under Γ then give a tiling of the plane by rectangles.

A *reflection* on a connected manifold M is an involution whose fixed point set separates M into two components. An effective and proper action $\Gamma \times M \to M$ of a discrete group Γ is a *reflection group* if it is locally smooth and if Γ is generated by reflections. In this more general situation a chamber X will be a convex polyhedron only locally; i.e., X will be a "manifold with faces." This means that X is a "nice" manifold with corners and that its panels are strata of codimension one. (The precise definitions are given in Section 6.) Let V be the set of reflections through the panels of X. As before,

1)′ V generates Γ (cf. Proposition 1.2),

2)′ (Γ, V) is a Coxeter system (cf. Theorem 4.1), and

3)′ $\mathfrak{A}$ is Γ-equivariantly homeomorphic to M (cf. Proposition 15.1).

Conversely, if (Γ, V) is a Coxeter system and X is a connected manifold with faces with panel structure $(X_v)_{v \in V}$ as above and satisfying (*), then $\mathfrak{A}$ is a manifold and Γ is a reflection group on $\mathfrak{A}$ (cf. Theorem 15.2). Thus, all reflection groups on manifolds can be constructed as above.

We are mainly interested in reflection groups on contractible manifolds with compact quotient. Since the orbit space $\mathfrak{A}/\Gamma$ is homeomorphic to X, this means that we are interested in the case where X is a compact contractible manifold with faces, each nonempty face is acyclic, and $D_0 = K_0$. From the point of view of homology, X "resembles" a convex polyhedron with its faces in general position. If X actually is a convex polyhedron with its faces in general position, then the simplicial complex D_0 can be identified with the boundary of the dual polyhedron. Even with the weaker hypotheses above, it should still be possible to conclude that D_0 "resembles" the boundary of a convex polyhedron. This is in fact the case: if S is a simplex in D_0, then the link of S is the nerve of the covering of the boundary of X_S by its codimension one faces; since X_S is a compact acyclic manifold, its boundary is a homology sphere; since each face is acyclic, it then follows from the Acyclic Covering Lemma that D_0 is a "generalized homology sphere" in the sense that it and the links of all its simplices have homology isomorphic to that of spheres of appropriate dimensions. Conversely, if a simplicial complex L is a generalized homology sphere, then one can take its "dual" to obtain a compact manifold with faces X with each of its faces contractible and with L as its nerve (cf. Theorem 12.2).

The results outlined above give a procedure for constructing many cocompact reflection groups on contractible manifolds. Start with a simplicial complex L which is a generalized homology sphere. Choose a Coxeter system (Γ, V) with $K_0(\Gamma, V) = L$ (it may be necessary first to subdivide L). Finally, "dualize" L to obtain a compact manifold with faces X with panel structure satisfying conditions (*) and (**). The resulting manifold $\mathfrak{A}$ is contractible and Γ acts as a cocompact reflection group.

In dimensions ≥ 4 a necessary and sufficient condition for a contractible manifold to be homeomorphic to Euclidean space is that it be "simply connected at infinity."(4) A contractible manifold $\mathfrak{A}$ of the type constructed above is simply connected at infinity if and only if the complex K_0 $(= D_0)$ is simply connected (cf. Section 16). The fact that there exist nonsimply connected homology spheres in every dimension ≥ 3 then allows us to conclude that in every dimension ≥ 4 there exist cocompact reflection groups on contractible manifolds which are not homeomorphic to Euclidean space.

Next let us consider a concrete example. Let X be a compact contractible manifold of dimension ≥ 4 with nonsimply connected boundary (e.g. a Mazur manifold). Let L be a PL-triangulation of its boundary and let L' be the barycentric of L. There is a Coxeter system (Γ, V) with $K_0(\Gamma, V) = L'$. (We can take Γ to be the Coxeter group with a generator of order two for each vertex in L' and a relation of the form $(vw)^2 = 1$ for each edge in L'; cf. the proof of Lemma 11.3.) The dual polyhedron to L' is a decomposition of ∂X into cells and it gives X the structure of a manifold with faces. As we indicated above the resulting manifold $\mathfrak{A}$ is contractible and not simply connected at infinity. The proofs of both these facts in this case can be sketched as follows. The manifold $\mathfrak{A}$ is constructed by pasting together copies of X, one for each element of Γ. If we order these copies by using the "length" of the elements in Γ, then it follows from the combinatorial theory of Coxeter groups that we are successively adjoining the copies of X along disks of codimension 0 in ∂X (cf. Lemmas 7.12 and 8.2). Thus, $\mathfrak{A}$ is the boundary connected sum of an infinite number of copies of X. This implies that $\mathfrak{A}$ is contractible and that its fundamental group at infinity is not finitely generated (it is a "projective free product" of an infinite number of copies of $\pi_1(\partial X)$).

A manifold is *aspherical* if its universal cover is contractible. Aspherical manifolds arise naturally in a variety of contexts. For example:

(I) Let G be a non-compact Lie group and K a maximal compact subgroup. Then G/K is diffeomorphic to Euclidean space. Let Γ' be a discrete torsion free subgroup of G (e.g., a "lattice" in G). The natural Γ'-action on G/K is free and proper. Hence, $\Gamma' \backslash G/K$ is an aspherical manifold.

(4) This condition is obviously necessary in dimensions ≥ 3. The proof of sufficiency is due to Stallings [Sta] in dimensions ≥ 5. Recently, Freedman [F] has proved sufficiency in dimension 4.

(II) If a closed Riemannian manifold has a non-positive sectional curvature, then the exponential map at a point is well-defined on the entire tangent space and is a covering projection. Hence, the universal cover of the manifold is identified with $\mathbf{R}^n$.

On the basis of such examples it was conjectured that the universal cover of any closed aspherical manifold must be homeomorphic to Euclidean space (cf. [Jo]).

Reflection groups provide a new method for constructing closed aspherical manifolds. Let Γ be a cocompact reflection group on a contractible manifold $\mathfrak{A}$. Since finitely generated Coxeter groups have faithful linear representations (cf. [B] and Section 5), they are virtually torsion-free (by Selberg's Lemma). Hence there is a torsion-free subgroup Γ' of finite index in Γ. Since each Γ-isotropy group is finite, each Γ'-isotropy group is trivial. Hence, Γ' acts freely on $\mathfrak{A}$ and consequently, $\mathfrak{A}/\Gamma'$ is aspherical. It is closed since the index of Γ' in Γ is finite. The universal cover of $\mathfrak{A}/\Gamma'$ is $\mathfrak{A}$. Since in dimensions ≥ 4 we can choose $\mathfrak{A}$ to be not simply connected at infinity, it follows that there exist closed aspherical manifolds which are not covered by Euclidean space. Thus, the above conjecture is false in every dimension ≥ 4.[5]

At this point there are three general remarks which can be made concerning discrete, proper and cocompact transformation groups on contractible manifolds and Euclidean spaces. First, cocompact reflection groups and their torsion-free subgroups form a much larger class than has been previously recognized. Secondly, these cocompact reflection groups are easy to understand (perhaps easier than lattices in Lie groups) in that most of the classical methods go over with little change. Thirdly, although the intersection of the class of reflection groups with the class of cocompact lattices is nonempty (there are some geometric reflection groups on flat and hyperbolic spaces of the latter type), these two classes appear to be essentially disjoint. Even when the ambient manifold $\mathfrak{A}$ is homeomorphic to $\mathbf{R}^n$, the reflection group Γ will generally not be equivalent to a cocompact lattice (e.g. if $K_0(\Gamma, V)$ is the suspension of a nontrivial homology sphere). Also, it is almost certainly true that a general torsion-free subgroup of Γ will not be equivalent to a cocompact lattice. The same remarks should apply even if the chamber X is required to be combinatorially equivalent to a convex polyhedron. For example, the Coxeter groups, which arise when X is combinatorially an n-cube, undoubtedly contain many new and interesting examples.

[5] The conjecture is, of course, true in dimensions ≤ 2. In dimension 3 it remains open.

I was led to condition (**) after listening to William Thurston discuss Andreev's Theorem on reflection groups on hyperbolic 3-space (cf. [A], [Th]). Condition (**) is also suggested by results of Vinberg [V]. Similar homological results for finite Coxeter groups had been obtained by the Hsiangs [H]. The proof of the necessity of (**) uses arguments somewhat similar to theirs. The proof of the sufficiency of (**) is similar to Serre's proof in [S] of the contractibility of Coxeter complexes. The argument uses the combinatorial theory of Coxeter groups developed in [B]. I would also like to thank Wu-chung Hsiang and John Morgan for several helpful conversations.

1. Reflection groups: Basic definitions

Let G be a discrete group acting on a Hausdorff space X. The action is *proper* if the following three conditions hold:

(a) the orbit space X/G is Hausdorff,

(b) for each $x \in X$ the isotropy subgroup G_x is finite,

(c) each $x \in X$ has a G_x-invariant open neighborhood U_x such that $gU_x \cap U_x = \varnothing$ whenever $g \notin G_x$.

(This is equivalent to the usual definition.) Next suppose that X is an n-dimensional manifold and that G acts properly. The action is *locally smooth* if:

(d) For each $x \in X$ there is an open neighborhood U_x as in (c) and a faithful representation $G_x \to O(n)$ so that U_x is G_x-homeomorphic to $\mathbf{R}^n$ with the linear G_x-action given by the representation.

Such a neighborhood U_x is called a *linear neighborhood* of x.

A *reflection* on a connected manifold M is a locally smooth involution $r\colon M \to M$ such that the fixed point set M_r separates M. The space M_r is called the *wall* associated to r. It follows easily that M_r is a locally flat submanifold of M of codimension 1 and that $M - M_r$ has exactly two components which are interchanged by r. The closure of such a component is a *half-space bounded by* M_r. If A is a subset of M such that $A - M_r$ is nonempty and contained in a single component of $M - M_r$, then $M_r^+(A)$ denotes the half-space containing A and $M_r^-(A)$ denotes the other one.

Two such subsets A and A' are on the *same side* (resp. *opposite sides*) of M_r if $M_r^+(A) = M_r^+(A')$ (resp. $M_r^+(A) = M_r^-(A)$).

Suppose that Γ is a discrete group acting properly, locally smoothly and effectively on a connected manifold M and that Γ is generated by reflections. Then Γ is a *reflection group* on M.

We suppose for the remainder of this section that Γ is a reflection group on M. Let R denote the set of all reflections in Γ. For each $x \in M$, let $R(x)$ be the set of all r in R such that x belongs to M_r. A point x is *nonsingular* if $R(x) = \varnothing$;

otherwise it is *singular*. A *chamber* of Γ on M is the closure of a connected component of the set of nonsingular points.

Let Q be a chamber. Denote by V_Q (or simply by V) the set of reflections v such that $R(x) = \{v\}$ for some $x \in Q$. If $v \in V$, then

$$Q_v = M_v \cap Q$$

is a *panel of* Q. Also, M_v is *the wall supported by* Q_v and V is the *set of reflections through the panels of* Q. As a convenient shorthand, we shall say that (Γ, V) is a *reflection system on* M *with fundamental chamber* Q. A reflection system is *cocompact* if its fundamental chamber is compact.

For any $x \in Q$ denote by $V(x)$ the intersection of $R(x)$ with V. In other words, $V(x)$ is the set of reflections through the panels of Q which contain x.

For any subset T of R let Γ_T denote the subgroup of Γ generated by T.

The following lemma is obvious.

LEMMA 1.1. *Let* $x \in Q$ *and* U_x *be a linear neighborhood of* x *in* M. *Put* $C_x = U_x \cap Q$. *Then* $\Gamma_{R(x)}$ *is equivalent to a finite linear reflection group*; C_x *is a closed chamber for* $\Gamma_{R(x)}$ *on* U_x; *and the set of reflections through the panels of* C_x *is* $V(x)$.

The next proposition is well known. The proof given below is adapted from [B; p. 73].

PROPOSITION 1.2. (i) $\Gamma Q = M$.

(ii) Γ *acts transitively on the set of chambers.*

(iii) R *is the set of conjugates of* V.

(iv) V *generates* Γ.

Proof. We shall prove that:

(a) $\Gamma_V Q = M$.

(b) Γ_V acts transitively on the set of chambers.

(c) For any $r \in R$ there is a $v \in V$ and $g \in \Gamma_V$ with $r = gvg^{-1}$.

Clearly, (a) $\Rightarrow$ (i), (b) $\Rightarrow$ (ii), (c) $\Rightarrow$ (iii), and since R generates Γ, (c) $\Rightarrow$ (iv). Hence it suffices to establish (a), (b), (c).

(a) Let $x \in Q$ and let U_x be a linear neighborhood of x. Property (a) holds for finite reflection groups acting linearly and orthogonally on $\mathbf{R}^n$ ([B; Lemma 2, p. 72]); hence, by Lemma 1.1, $\Gamma_{V(x)}(Q \cap U_x) = U_x$. This implies that $\Gamma_V Q$ is open in M; it is clearly closed since it is a locally finite union of closed chambers.

(b) Let C be the interior of Q, C' the interior of another chamber and $y \in C'$. By (a) there is a $g \in \Gamma_V$ with $g^{-1}y \in Q$. Since y is nonsingular, $g^{-1}y \in C$. Hence, $gC = C'$.

(c) Let $r \in R$. The wall M_r is the wall supported by a panel of at least one chamber; call it Q'. By the previous paragraph, there is $g \in \Gamma_V$ with $Q' = gQ$. Hence, the wall $g^{-1}M_r$ ($= M_{g^{-1}rg}$) is supported by a panel of Q; consequently, $g^{-1}rg \in V$.

2. Coxeter systems and their associated graphs

Suppose that Γ is a group and V is a set of generators, each element of which has order two. For any pair of elements (v, w) of V let $m(v, w)$ denote the order of vw in Γ. Since $vw = (wv)^{-1}$, we have $m(v, w) = m(w, v)$. Let E be the set of unordered pairs $\{v, w\}$ of distinct elements in V such that $m(v, w) \neq \infty$. The pair (Γ, V) is a *Coxeter system* if the set of generators V together with the relations

$$v^2 = 1, \qquad v \in V,$$

$$(vw)^{m(v,w)} = 1, \qquad \{v, w\} \in E$$

form a presentation for Γ ([B; p. 11]). The *associated graph* $G(\Gamma, V)$ is the graph with vertex set V and edge set E. Let $m\colon E \to \{2, 3, \ldots\}$ be the labelling of the edges defined by $m(\{v, w\}) = m(v, w)$. The labelled graph $(G(\Gamma, V), m)$ clearly determines the Coxeter system up to isomorphism.(6)

Conversely, if G is any finite graph and $m\colon E \to \{2, 3, \ldots\}$ is a labelling of its edge set, then there is a Coxeter system with (G, m) as its associated labelled graph ([B; Prop. 4, p. 92]).

3. Length and a characterization of Coxeter systems

In this section we shall begin a review of the combinatorial theory of Coxeter systems developed in the first section and exercises of [B]. We shall continue this review in more detail in Section 7.

Suppose that Γ is a group and that V is a set of generators each element of which has order two. For any $g \in \Gamma$, the *length* of g (with respect to V), denoted by $l_V(g)$ (or simply by $l(g)$), is the smallest integer n such that g is the product of n elements of V. If $d\colon \Gamma \times \Gamma \to \mathbf{R}$ is defined by $d(g, h) = l(g^{-1}h)$, then d is a metric on Γ ([B; pp. 1–2]).

(6)The associated graph is *not* the "Coxeter graph" of (Γ, V). The *Coxeter graph* has the same vertex set V; however, the edge set is obtained by discarding the edges of E labelled 2 and then by adjoining an edge for each pair $\{v, w\}$ of distinct elements not in E. The new edges are labelled ∞. The associated graph and the Coxeter graph together with their labellings obviously carry exactly the same information.

LEMMA 3.1 ([B, p. 18]). *Suppose that* (Γ, V) *is a Coxeter system. For each* $v \in V$, *let*

$$P_v = \{g \in \Gamma \mid l(vg) > l(g)\}.$$

Then the family of subsets $(P_v)_{v\in V}$ *has the following properties*:

(A) $\bigcap_{v\in V} P_v = \{1\}$.

(B) *For each* $v \in V$, P_v *and* vP_v *form a disjoint partition of* Γ.

(C) *Let* $v, w \in V$ *and* $g \in \Gamma$. *If* $g \in P_w$ *and* $gv \notin P_w$, *then* $wg = gv$.

Conversely, we have the following characterization of Coxeter systems.

LEMMA 3.2 ([B; Prop. 6, p. 18]). *Suppose that* Γ *is a group and that* V *is a set of generators, each element of which has order two. Let* $(P_v)_{v\in V}$ *be a family of subsets of* Γ *satisfying condition* (C) *of Lemma* 3.1 *and the following additional conditions*:

(A′) *For each* $v \in V$, $1 \in P_v$.

(B′) *For each* $v \in V$, $P_v \cap vP_v = \varnothing$.

Then (Γ, V) *is a Coxeter system. Moreover, for each* $v \in V$, $P_v = \{g \in \Gamma \mid l(vg) > l(g)\}$.

4. The relationship between reflection groups and Coxeter systems

In the following theorem we state the basic properties of reflection groups. Its proof is taken from [B; pp. 74–75] where the results are stated only for groups generated by affine orthogonal reflections on Euclidean space.

THEOREM 4.1. *Let* (Γ, V) *be a reflection system on a manifold* M *with fundamental chamber* Q. *The following statements are true.*

(i) (Γ, V) *is a Coxeter system.*

(ii) *For each* $v \in V$ *and each* $g \in \Gamma$ *the relation* $l(vg) > l(g)$ *means that* Q *and* gQ *are on the same side of the wall* M_v.

(iii) Γ *acts freely and transitively on the set of chambers in* M.

(iv) $Q = \bigcap_{v\in V} M_v^+(Q)$.

(v) *If* $x, y \in Q$ *and* $gx = y$ *for some* $g \in \Gamma$, *then* $x = y$ *and* $g \in \Gamma_{V(x)}$.

(vi) Q *is a closed fundamental domain*; *i.e., if* π: $M \to M/\Gamma$ *denotes the orbit map, then* $\pi \mid Q$ *is a homeomorphism.*

(vii) *The isotropy group at* $x \in Q$ *is* $\Gamma_{V(x)}$.

Proof. Each element of V has order two and V generates Γ (Proposition 1.2 (iv)). For each $v \in V$ let P_v denote the set of $g \in \Gamma$ such that Q and gQ are on the same side of M_v. We shall verify conditions (A′), (B′) and (C) of Lemma 3.2. Conditions (A′) and (B′) hold trivially. Condition (C) is the following:

(C) Let $v, w \in V$ and $g \in \Gamma$. If $g \in P_w$ and $gv \notin P_w$, then $wg = gv$.
By definition of P_w, gQ is on the same side of M_w as is Q, while gvQ is on the opposite side. Therefore, M_w separates gvQ and gQ. Hence, $g^{-1}M_w$ separates vQ and Q. Let $x \in Q_v$ be a point with $V(x) = \{v\}$. The point x $(= vx)$ is on the boundary of Q and of vQ. The interiors of these chambers lie in different components of $M - g^{-1}M_w$; hence, $x \in g^{-1}M_w$. Therefore, $M_v = g^{-1}M_w$ $(= M_{g^{-1}wg})$. Hence $gv = wg$. Since conditions (A′), (B′), (C) hold, assertions (i) and (ii) follow from Lemma 3.2. The group Γ is transitive on the set of chambers (Proposition 1.2 (ii)). If $gQ = Q$ for some $g \in \Gamma$, then $g \in P_v$ for all $v \in V$ and consequently, $g = 1$ (Lemma 3.1 (A)). Hence, Γ also acts freely on the set of chambers, i.e., (iii) holds. Let $A = \bigcap_{v \in V} M_v^+(Q)$. Then A is clearly a union of chambers, one of which is Q. If gQ is another chamber in A, then $g = 1$ (by Lemma 3.1 (A), again); i.e., (iv) holds. The proof of (v) is exactly the same as the proof of [B; assertion (I), p. 75]. Assertions (vi) and (vii) are immediate consequences of (v).

Remark. The above result has been known for many years (cf. [K], [W]) but only under the additional hypothesis that M be simply connected. It does not seem to have been previously recognized that the arguments in [B] go over, essentially without change, to reflection groups on manifolds without the assumption of simple connectivity. The results in the above theorem are also proved in [St].

5. Linear reflection groups

THEOREM 5.1 (Tits, [B; p. 93]). *Let (Γ, V) be a Coxeter system with V a finite set. Let C be the standard simplicial cone in $\mathbf{R}^V$ defined by the linear inequalities, $x_v \geq 0$, for $v \in V$. There exists a faithful representation $\rho: \Gamma \to GL(\mathbf{R}^V)$ with discrete image, called the "canonical representation" such that*

(a) *For each $v \in V$, $\rho(v)$ is a linear reflection through the hyperplane $x_v = 0$.*

(b) *$gC = C$ if and only if $g = 1$.*

According to Selberg's Lemma (cf. [Sel]), every finitely generated subgroup of $GL(n)$ has a torsion-free subgroup of finite index. Therefore, we have the following corollary.

COROLLARY 5.2 ([S; p. 107]). *A finitely generated Coxeter group is virtually torsion-free.*

Note that if Γ is infinite then $\rho(\Gamma)$ does not act properly on $\mathbf{R}^V$ (the isotropy group at the origin is infinite). However, there is a $\rho(\Gamma)$-stable open convex set on which the action is proper (and a reflection group). This is explained by the following result of Vinberg.

THEOREM 5.3 ([V, p. 1092]). *Let Γ be a discrete linear group generated by a set V of linear reflections through the walls of a convex polyhedral cone C. Let $C^f = \{x \in C \mid \Gamma_{V(x)}$ is finite$\}$. The following statements are true.*

(i) *ΓC is a convex cone.*

(ii) *Let Ω be the interior of this cone. Then Γ is a reflection group on Ω, $\Omega \cap C = C^f$, and C^f is a closed chamber for Γ on Ω.*

Note that when Γ is finite, $C^f = C$ and Ω is the entire vector space. In this case there is the following classical result.

LEMMA 5.4 ([B; p. 85]). *Let Γ be a finite linear reflection group on $\mathbf{R}^n$, C a closed chamber, and V the set of reflections through the panels of C. Then $\mathrm{Card}(V) \leq n$ and the family of hyperplanes supported by the panels of C intersect in general position. Thus, C is the product of a simplicial cone with the subspace fixed by Γ.*

6. Panel structures

A *panel structure* on a space X is a locally finite family of closed subspaces $(X_v)_{v\in V}$ indexed by some set V. The X_v are the *panels* of X. A space together with a panel structure is a *space with faces*. For each $x \in X$, let $V(x)$ denote the set of v in V such that $x \in X_v$. For each subset S of V denote by X_S (resp. ∂X_S) the set of x in X such that $V(x)$ contains (resp. properly contains) S. Also, set $\partial X = \partial X_\varnothing$ (note that $X_\varnothing = X$). The X_S are the *faces* of X. For each nonempty subset S of X let $X_{\sigma(S)}$ denote the union of panels which are indexed by S. Thus,

$$X_S = \bigcap_{v\in S} X_v; \qquad \text{while}$$

$$X_{\sigma(S)} = \bigcup_{v\in S} X_v.$$

If X and X' have panel structures indexed by V, then a map $f\colon X \to X'$ is *face-preserving* if $V(f(x)) = V(x)$ for all $x \in X$.

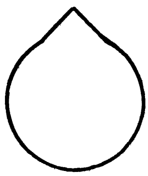
Next we want to define the notion of a "manifold with faces".

Let C^n be the standard simplicial cone in $\mathbf{R}^n$ defined by the linear inequalities $x_i \geq 0$, $1 \leq i \leq n$. For any $x = (x_1, \ldots, x_n) \in C^n$, its *codimension* $c(x)$ is the number of x_i which are equal to 0.

An *n-manifold with corners* Q is a Hausdorff space together with a maximal atlas of local charts onto open subsets of C^n so that the overlap maps are homeomorphisms which preserve codimension. For any $x \in Q$, its codimension

$c(x)$ is then well defined. An *open pre-face of Q of codimension m* is a connected component of $c^{-1}(m)$. A *closed pre-face* is the closure of an open pre-face. For any $x \in Q$, let $\Sigma(x)$ be the set of closed pre-faces of codimension one which contain x. The manifold with corners Q is *nice* if $\mathrm{Card}(\Sigma(x)) = 2$ for any x with $c(x) = 2$.

For example, the manifold-with-corners structure on D^2 pictured below is *not* nice.



Suppose that Q is a nice n-manifold with corners and that F is a closed pre-face of codimension m. Then it is easy to see that F is naturally a nice $(n-m)$-manifold with corners. Moreover, for each $x \in \mathring{F}$, $\mathrm{Card}(\Sigma(x)) = m = c(x)$.

A *manifold with faces* is a nice manifold with corners Q together with a panel structure on Q such that:

(a) Each panel is a pairwise disjoint union of closed pre-faces of codimension one, and

(b) Each closed pre-face of codimension one is contained in exactly one panel.

It is then clear that if S is a subset of V such that $Q_S \neq \varnothing$, then Q_S is a disjoint union of closed pre-faces of Q each of which has codimension $\mathrm{Card}(S)$.

Suppose (Γ, V) is a Coxeter system and that a space X has a panel structure indexed by V. The panel structure is *Γ-finite* if the subgroup $\Gamma_{V(x)}$ is finite for all $x \in X$. (This is condition $(*)$ of the introduction.)

PROPOSITION 6.1. *Suppose that Q is the fundamental chamber of a reflection system (Γ, V) on a manifold. Then Q together with its natural panel structure is a manifold with faces. Moreover, the panel structure is Γ-finite.*

Proof. The fact that the panel structure is Γ-finite is just the fact that each isotropy group of a proper action is finite (cf. the definition at the beginning of Section 1). The remainder of the proposition follows immediately from Lemmas 1.1 and 5.4.

7. Combinatorial properties of Coxeter systems

We continue our review of the results of [B]. Throughout this section (Γ, V) is a Coxeter system and R denotes the set of conjugates of V. For each finite sequence $\mathrm{v} = (v_1, \ldots, v_n)$ of elements of V, define a sequence $\Phi(\mathrm{v}) = (r_1, \ldots, r_n)$ of elements of R by

$$r_i = (v_1 \ldots v_{i-1}) v_i (v_{i-1} \ldots v_1).$$

For any sequence $\mathrm{v} = (v_1, \ldots, v_n)$ of elements of V and $r \in R$, let $n(\mathrm{v}, r)$ denote the number of integers j, $1 \leq j \leq n$, such that $r_j = r$.

LEMMA 7.1 ([B; Lemma 1, p. 13]). (i) *Let $g \in \Gamma$ and $r \in R$. For each sequence $\mathrm{v} = (v_1, \ldots, v_n)$ of elements of V with $g = v_1 \ldots v_n$, the number $(-1)^{n(\mathrm{v}, r)}$ has the same value $\eta(g, r)$.*

(ii) *For $g \in \Gamma$, let φ_g be the mapping from $\{\pm 1\} \times R$ to itself defined by $\varphi_g(\varepsilon, r) = (\varepsilon \eta(g^{-1}, r), grg^{-1})$. Then $g \to \varphi_g$ is a homomorphism from Γ into the group of permutations of $\{\pm 1\} \times R$.*

For each $g \in \Gamma$ let R_g be the set of $r \in R$ such that $\eta(g, r) = -1$.

A sequence $(v_1, \ldots, v_n)$ of elements of V is a *reduced decomposition of* g if $g = v_1 \ldots v_n$ and $l(g) = n$.

LEMMA 7.2 ([B; Lemma 2, p. 14]). *Let $\mathrm{v} = (v_1, \ldots, v_n)$, $\Phi(\mathrm{v}) = (r_1, \ldots, r_n)$ and $g = v_1 \ldots v_n$. In order for* v *to be a reduced decomposition of g it is necessary and sufficient that the r_i be distinct. One then has $R_g = \{r_1, \ldots, r_n\}$ and* $\mathrm{Card}(R_g) = l(g)$.

LEMMA 7.3 ([B; Ex. 16 and 22 a), pp. 40 and 43]). *Let $g \in \Gamma$ and $r \in R$; then $r \in R_g$ if and only if $l(rg) < l(g)$.*

Remark 7.4. These results have the following geometric interpretation. Let (Γ, V) be a reflection system with fundamental chamber Q. A sequence of chambers $(Q_0, \ldots, Q_n)$ is a *gallery of length* n if Q_{i-1} and Q_i are distinct and have a common panel, $1 \leq i \leq n$. Suppose $Q_0 = Q$ and $Q_n = gQ$. Let r_i be the reflection through the common panel of Q_{i-1} and Q_i and let $\mathrm{v} = (v_1, \ldots, v_n)$ be the sequence defined by $\Phi(\mathrm{v}) = (r_1, \ldots, r_n)$. Then $\mathrm{v} \leftrightarrow (Q_0 = Q, Q_1, \ldots, Q_n = gQ)$ sets up a $1-1$ correspondence between sequences $(v_1, \ldots, v_n)$ with $g = v_1 \ldots v_n$ and galleries from Q to gQ. Moreover, v is a reduced decomposition if and only if the gallery is of minimal length.

LEMMA 7.5 ([B; Prop. 7, p. 19]). *Let $g \in \Gamma$. There exists a subset V_g of V such that $V_g = \{v_1, \ldots, v_n\}$ for every reduced decomposition $(v_1, \ldots, v_n)$ of g.*

COROLLARY 7.6 ([B; pp. 19–20]). *Let S be a subset of V. Then*
(i) *Γ_S consists of the elements $g \in \Gamma$ such that $V_g \subset S$.*
(ii) *$\Gamma_S \cap V = S$.*
(iii) *If $g \in \Gamma_S$, then $l_S(g) = l_V(g)$.*
(iv) *(Γ_S, S) is a Coxeter system.*

Suppose S, T are subsets of V and $h \in \Gamma$. There is a unique element g in the double coset $\Gamma_S h \Gamma_T$ of minimum length and every element g' in this double coset can be written uniquely in the form $g' = sgt$ with $s \in \Gamma_S$, $t \in \Gamma_T$ and $l(g') = l(s) + l(g) + l(t)$. An element $g \in \Gamma$ is *(S, T)-reduced* if it is the element of minimum length in its double coset. Clearly, g is $(S, \varnothing)$-reduced if and only if $l(vg) = l(g) + 1$ for all $v \in S$. Each element of Γ can be written uniquely in the form sh where $s \in \Gamma_S$ and h is $(S, \varnothing)$-reduced. (This paragraph is taken from [B; Ex. 3, p. 37].)

LEMMA 7.7. *Let S be a subset of V and $r \in R - \Gamma_S$. Then $\eta(h, r) = 1$ for all $h \in \Gamma_S$.*

Proof. Let $(v_1, \ldots, v_n)$ be a reduced decomposition of h. Each $v_i \in S$ (Corollary 7.6(i)). If $\eta(h, r) = -1$, then $r = (v_1 \ldots v_{i-1}) v_i (v_{i-1} \ldots v_1)$ for some i, $1 \le i \le n$, and hence, $r \in \Gamma_S$. Thus, $r \neq \Gamma_S$ implies $\eta(h, r) = 1$ for all $h \in \Gamma_S$.

COROLLARY 7.8. *Let S be a subset of V. Then each element of Γ_S is $(V - S, \varnothing)$-reduced.*

COROLLARY 7.9. *Let S be a subset of V and let $g \in \Gamma$ and $r \in R$ be such that $g^{-1} rg \notin \Gamma_S$. Then $\eta(gh, r) = \eta(g, r)$ for all $h \in \Gamma_S$.*

Proof. By Lemma 7.1 (ii), $\eta(gh, r) = \eta(g, r)\eta(h, g^{-1}rg)$ for all $g, h \in \Gamma$ and $r \in R$. If $h \in \Gamma_S$ and $g^{-1}rg \notin \Gamma_S$, then Lemma 7.7 implies $\eta(h, g^{-1}rg) = 1$.

LEMMA 7.10 ([B; Ex. 22b, p. 43]). *Suppose Γ is finite. There is a unique element $g_0 \in \Gamma$ of longest length. It is characterized by any of the following properties:*
(a) *$l(vg_0) < l(g_0)$, for all $v \in V$.*
(b) *$l(rg_0) < l(g_0)$, for all $r \in R$.*
(c) *$l(g_0) = l(g_0 g^{-1}) + l(g)$, for all $g \in \Gamma$.*
Moreover, $(g_0)^2 = 1$, $g_0 V g_0 = V$, and $l(g_0) = \mathrm{Card}(R)$.

LEMMA 7.11. *Suppose there exists an element $g_0 \in \Gamma$ such that $l(vg_0) < l(g_0)$ for all $v \in V$. Then Γ is a finite group and g_0 is the element of longest length.*

Proof. We use the canonical representation $\Gamma \to GL(\mathbf{R}^V)$ and the notation of Theorems 5.1 and 5.3. Let $\mathring{C}$ be the interior of the standard simplicial cone in $\mathbf{R}^V$ (so that $\mathring{C}$ is an open chamber for Γ on Ω). Let $x \in g_0\mathring{C}$. By Theorem 4.1 (ii), $\mathring{C}$ and $g_0\mathring{C}$ lie on opposite sides of the wall $x_v = 0$ for all $v \in V$. Hence, each coordinate of x is negative, i.e., $-x \in \mathring{C}$. Since Ω is convex this implies that $0 \in \Omega$ and consequently, that Γ is finite and $\Omega = \mathbf{R}^V$. It then follows from the previous lemma that g_0 is the element of longest length.

For each $g \in \Gamma$ define subsets $A(g)$, $B(g)$ of V by

$$A(g) = \{v \in V \mid l(vg) < l(g)\},$$

$$B(g) = \{v \in V \mid l(gv) < l(g)\}.$$

LEMMA 7.12. *For each $g \in \Gamma$ the subgroups $\Gamma_{A(g)}$ and $\Gamma_{B(g)}$ are finite.*

Proof. Since $l(g^{-1}) = l(g)$, we have that $B(g) = A(g^{-1})$; hence, it suffices to prove the lemma for $A(g)$. Write g in the form $g = ah$, where $a \in \Gamma_{A(g)}$, h is $(A(g), \varnothing)$-reduced, and $l(g) = l(a) + l(h)$. Let $v \in A(g)$. Since h is $(A(g), \varnothing)$ reduced, $l(vg) = l(va) + l(h)$. Since $l(vg) < l(g)$, this implies $l(va) < l(a)$. Hence, by the previous lemma, $\Gamma_{A(g)}$ is finite and a is the element of longest length.

8. Intersections of chambers

In this section (Γ, V) is a reflection system on M with fundamental chamber Q.

LEMMA 8.1. *Let $g \in \Gamma$ and V_g be the subset of V defined in Lemma 7.5. Then $Q \cap gQ = Q_{V_g}$. Thus, any two distinct chambers intersect in a face (possibly empty).*

Proof. Let $x \in Q \cap gQ$. By Theorem 4.1 (v), $g \in \Gamma_{V(x)}$ and by Corollary 7.6 (i), $V_g \subset V(x)$. Hence, $x \in Q_{V_g}$. Conversely, suppose $x \in Q_{V_g}$. Let $(v_1, \ldots, v_n)$ be a reduced decomposition of g. Since $V_g = \{v_1, \ldots, v_n\}$, $x \in Q_{v_i}$ for all i, $1 \le i \le n$. Therefore, each v_i fixes x as does g. This implies $x \in Q \cap gQ$.

Order the elements of Γ,

$$g_1 = 1, g_2, \ldots, g_n, \ldots$$

so that $l(g_{n+1}) \ge l(g_n)$.(7) Denote by $T_n(Q)$ (or simply by T_n) the union of the chambers g_iQ with $i \le n$. The proof of the following key lemma is similar to an argument of [S; p. 108].

(7)Although the notation makes sense only when V is finite, this assumption is not necessary.

LEMMA 8.2. *$g_nQ \cap T_{n-1} = g_nQ_{\sigma(B(g_n))}$, where $B(g_n)$ is the set of $v \in V$ such that $l(g_nv) < l(g)$.*

Thus, each T_n is a manifold with boundary and T_n is obtained from T_{n-1} by attaching a copy of Q along a union of panels of the form $Q_{\sigma(S)}$, where the subset S $(= B(g_n))$ of V is such that Γ_S is finite (cf. Lemma 7.12).

Proof. To simplify notation put $g = g_n$. It is obvious that $gQ_{\sigma(B(g))} \subset gQ \cap T_{n-1}$: for if $v \in B(g)$, then gQ_v is a common panel of gQ and of gvQ and $gvQ \subset T_{n-1}$. Suppose $gx \in gQ \cap T_{n-1}$ with $x \in Q$. Then there is an index $i < n$ such that $gx \in g_iQ$. This implies $g \in g_i\Gamma_{V(x)}$. Since $l(g_i) \le l(g)$, g is not $(\varnothing, V(x))$-reduced. Hence, we can write g in the form kh where k is $(\varnothing, V(x))$-reduced and $1 \neq h \in \Gamma_{V(x)}$. Let $(v_1, \dots, v_m)$ be a reduced decomposition of h and put $v = v_m$. By Corollary 7.6 (i), $v \in V(x)$. We have $l(hv) = l(h) - 1$ and since k is $(\varnothing, V(x))$-reduced, $l(gv) = l(k) + l(hv) = l(g) - 1$, i.e., $v \in B(g)$. Therefore, $gx \in gQ_v \subset gQ_{\sigma(B(g))}$.

9. Alternation

Let i: $Q \to M$ denote the inclusion, π: $M \to M/\Gamma$ the orbit map, and $p = (\pi \mid Q)^{-1} \circ \pi$: $M \to Q$. Then p is a retraction of M onto Q. This simple observation has the following consequences (which are well known).

LEMMA 9.1. *The map p_*: $\pi_1(M) \to \pi_1(Q)$ is onto.*

LEMMA 9.2. *The map i_*: $H_*(Q) \to H_*(M)$ takes $H_*(Q)$ isomorphically onto a direct summand.*

Suppose S is a subset of V such that Γ_S is finite. We next want to prove a relative version of the previous lemma: $H_*(Q, Q_{\sigma(S)})$ is isomorphic to a direct summand of $H_*(M)$. Define chain endomorphism Alt_S: $C_*(M) \to C_*(M)$ by

$$\text{Alt}_S(c) = \sum_{g \in \Gamma_S} \varepsilon(g) g \cdot c$$

where the homomorphism ε: $\Gamma \to \{\pm 1\}$ is defined by $\varepsilon(g) = (-1)^{l(g)}$. It is easy to see that Alt_S vanishes on $C_*(Q_{\sigma(S)})$; hence, there is a well defined chain map from $C_*(Q, Q_{\sigma(S)})$ to $C_*(M)$ inducing a homomorphism $(\text{Alt}_S)_*$: $H_*(Q, Q_{\sigma(S)}) \to H_*(M)$.

Let P be the closed chamber for Γ_{V-S} on M which contains Q. We have

$$P = \bigcap_{v \in V-S} M_v^+(Q).$$

(Clearly, P is contained in the intersection; the opposite inclusion follows as in the proof of Theorem 4.1 (iv).) This implies that P is the union of all chambers gQ where g is $(V - S, \varnothing)$-reduced. We want to establish that

$$(1) \qquad (\overline{P - Q}) \cap Q = Q_{\sigma(S)}.$$

Let $s \in S$. The chambers Q and sQ are adjacent and since s is $(V - S, \varnothing)$-reduced (Corollary 7.8), $sQ \subset P$. Hence, $Q_{\sigma(S)} = \bigcup_{s \in S} Q_s \subset (\overline{P - Q}) \cap Q$. Suppose $g \in \Gamma$ and $g \neq 1$. By Lemma 3.2 (A), there exists $v \in V$ with $l(vg) < l(g)$. Such a v belongs to V_g (cf. [B; Prop. 4, p. 15]). If g is $(V - S, \varnothing)$-reduced, then v must also belong to S. Hence, for any $g \neq 1$ which is $(V - S, \varnothing)$-reduced and any $v \in V$ with $l(vg) < l(g)$, we have $Q \cap gQ = Q_{V_g} \subset Q_v \subset Q_{\sigma(S)}$. This proves that $(\overline{P - Q}) \cap Q \subset Q_{\sigma(S)}$ and hence, (1).

Let $\pi'\colon M \to M/\Gamma_{V-S}$ be the orbit map and $p' = (\pi' | P)^{-1} \circ \pi'\colon M \to P$. Define $q_*\colon H_*(M) \to H_*(Q, Q_{\sigma(S)})$ by the following composition

$$\underbrace{H_*(M) \xrightarrow{p'_*} H_*(P) \xrightarrow{k_*} H_*(P, \overline{P - Q}) \xrightarrow{j_*} H_*(Q, Q_{\sigma(S)})}_{q_*}$$

where k_* is the natural map and j_* is the excision isomorphism (cf. (1)).

Lemma 9.3. *Let S be a subset of V such that Γ_S is finite. Then $q_* \circ (\mathrm{Alt}_S)_*$ is the identity map of $H_*(Q, Q_{\sigma(S)})$. Thus, $(\mathrm{Alt}_S)_*$ maps $H_*(Q, Q_{\sigma(S)})$ isomorphically onto a direct summand of $H_*(M)$.*

Proof. The image of $(\mathrm{Alt}_S)_*$ is contained in $H_*(\Gamma_S Q)$. By Corollary 7.8 each element of Γ_S is $(V - S, \varnothing)$-reduced; hence, $\Gamma_S Q \subset P$. Therefore, $q_* \circ (\mathrm{Alt}_S)_* = j_* \circ k_* \circ (\mathrm{Alt}_S)_*$ and the second composition is clearly the identity.

Remark. The alternation map Alt_S was defined by the Hsiangs in [H] and used for similar purposes; however, the use of the chamber for Γ_{V-S} and Lemma 9.3 seem to be new.

10. Necessary and sufficient conditions for M to be contractible

Let $\overline{H}_*(X)$ denote the reduced homology of a space X. The reduced homology of the empty set is defined to be zero in degrees ≥ 0 and to be $\mathbf{Z}$ in degree -1. A space X is *m-acyclic* (m an integer) if $\overline{H}_i(X) = 0$ for $-1 \leq i \leq m$. (Note that the empty set is not m-acyclic if $m \geq -1$.) Similarly, a pair (X, Y) is *m-acyclic* if $H_i(X, Y) = 0$ for $0 \leq i \leq m$.

THEOREM 10.1. *Let* (Γ, V) *be a reflection system on* M *with fundamental chamber* Q. *The following statements are equivalent.*

(i) M *is m-acyclic (resp. m-connected).*

(ii) Q *is m-acyclic (resp. m-connected) and* $(Q, Q_{\sigma(S)})$ *is m-acyclic for each nonempty subset* S *of* V *such that the subgroup* Γ_S *is finite.*

(iii) *For each* $n \geq 1$, *the union of chambers* $T_n(Q)$ *defined in Section* 8 *is m-acyclic (resp. m-connected).*

(iv) Q *is m-acyclic (resp. m-connected) and* Q_S *is* $(m\text{-Card}(S))$*-acyclic for each nonempty subset* S *of* V *such that* Γ_S *is finite.*

The special cases $m = 1$ and $m = \infty$ are the following corollaries.

COROLLARY 10.2. M *is simply connected if and only if the following three conditions hold*:

(a) Q *is simply connected,*

(b) *Each panel of* Q *is connected,*

(c) *The codimension two face* $Q_v \cap Q_w$ *is nonempty whenever* $m(v, w) < \infty$.

COROLLARY 10.3. M *is contractible if and only if* Q *is contractible and* Q_S *is acyclic for all* $S \subset V$ *with* Γ_S *finite.*

Example 10.4. Suppose that Q is a convex 3-cell, not a tetrahedron, and that $V = \{v_1, \ldots, v_n\}$. Put $Q_i = Q_{v_i}$ and $m_{ij} = m(v_i, v_j)$. Then Q satisfies the conditions of the above corollary if and only if the following two conditions hold:

(1) Distinct panels Q_i and Q_j are adjacent if and only if $m_{ij} < \infty$.

(2) If Q_i, Q_j, Q_k are three distinct panels which are pairwise adjacent, then the intersection $Q_i \cap Q_j \cap Q_k$ is nonempty (and equal to a vertex) if and only if

$$\frac{1}{m_{ij}} + \frac{1}{m_{ik}} + \frac{1}{m_{jk}} > 1.$$

This result is suggested by Andreev's theorem on convex cells in hyperbolic 3-space, [A], [Th].

Proof of Theorem 10.1. The implication (i) ⇒ (ii) is immediate from Lemmas 9.1, 9.2, and 9.3; (ii) ⇒ (iii) is immediate from Lemma 8.2; (iii) ⇒ (i) is obvious. To show (ii) ⇔ (iv) we need the following routine modification of the Acyclic Covering Lemma (Helly's lemma).

LEMMA 10.5. *Let* Y *be a space and* $(Y_s)_{s \in S}$ *a finite covering by closed subsets. For any* $T \subset S$, *put* $Y_T = \bigcap_{s \in T} Y_s$ *and* $d(T) = \text{Card}(T) - 1$. *Let* n *be an integer and put* $k = \min(n, d(S) - 2)$. *Suppose that for each proper nonempty subset* T *of* S, Y_T *is* $(n - d(T))$*-acyclic. Then the homology of* Y *in* *degrees* $\leq n$ *is determined as follows*:

(a) Y *is* k-*acyclic*.

(b) $\bar{H}_i(Y_s) \cong H_{i+d(S)}(Y)$ *for* $-1 \leq i \leq n - d(S)$.

Proof. There is a Grothendieck spectral sequence

$$E^2_{pg} = H_p(N; T \to H_q(Y_T)) \Rightarrow H_{p+q}(Y),$$

where N is the nerve of the covering (see [G; Appendix II]). The lemma follows easily.(8)

Proof that (ii) $\Leftrightarrow$ (iv). Suppose (ii) holds. Then $Q_{\sigma(T)}$ is $(m-1)$-acyclic for any $T \subset V$ with Γ_T finite. Suppose $S \subset V$ and Γ_S is finite. By induction on Card(S) we may assume Q_T is (m-Card(T))-acyclic for all proper nonempty subsets T of S. By the previous lemma, $\bar{H}_i(Q_S) \cong H_{i+\text{Card}(S)-1}(Q_{\sigma(S)}) = 0$ for $-1 \leq i \leq m - \text{Card}(S)$; i.e., Q_S is $(m - \text{Card}(S))$-acyclic. The other implication (iv) $\Rightarrow$ (ii) is immediate from the previous lemma.

Remark 10.5. The Hsiangs [H] proved Corollary 10.3 in the special case that Γ is finite. The proof of (i) $\Rightarrow$ (ii) is in the same spirit as the argument given there in that it depends on alternation. The proof of (ii) $\Leftrightarrow$ (iv) (which is routine) also essentially occurs in that paper. The proof that (ii) $\Rightarrow$ (i), which is based on Lemma 8.2, is similar to an argument of Serre [S, p. 108] showing that the "Coxeter complex" of Γ is contractible.

Remark 10.6. A *neighborhood of infinity* in a non-compact space Y is the complement of a compact subset. A space Y is *simply connected at infinity* if every neighborhood of infinity contains a simply connected neighborhood of infinity. Now suppose Y is a locally compact, second countable, Hausdorff space. Then Y is a countable increasing union $C_1 \subset C_2 \subset \ldots$, where for each n, the subspace C_n is compact. Put $G_n = \pi_1(Y - C_n, x_n)$ where $x_n \in Y - C_n$ is some base point. After choice of a path from x_{n+1} to x_n the inclusion $Y - C_{n+1} \subset Y - C_n$ induces a homomorphism $\varphi_{n+1}\colon G_{n+1} \to G_n$. The space Y is *semistable* if it is connected at infinity and if the inverse sequence $G_1 \leftarrow G_2 \leftarrow \ldots$ satisfies the Mittag-Leffler condition, i.e., if for every n there exists N such that Image $(G_{n+m} \to G_n) =$ Image $(G_{n+N} \to G_n)$ for all $m \geq N$. (This definition is independent of the choice of C_n's and of base points.) Define $\pi_1^\infty(Y) = \varprojlim G_n$. If Y is semistable then, up to isomorphism, this group is independent of all choices. Moreover, Y is simply connected at infinity if and only if it is semistable and $\pi_1^\infty(Y)$ is trivial. (The above material is basically contained in Siebenmann's thesis, as well as in [J] and [Jo].)

(8) I would like to thank Karen Vogtmann for telling me this argument.

Now suppose that (Γ, V) is a cocompact reflection system on a contractible manifold M of dimension ≥ 4 with fundamental chamber Q. Lemma 8.2 can also be used to show that M is semistable and to calculate its fundamental group at infinity. Since Q is compact contractible, ∂Q is a homology sphere. Put $G = \pi_1(\partial Q)$ and assume that $G \neq \{1\}$. To simplify the discussion suppose further that each proper face of Q is a cell. (This assumption will be removed in Section 16.) Under these hypotheses we claim that $\pi_1^\infty(M)$ is not trivial and hence, that M is not homeomorphic to Euclidean space. Let $S \subset V$ be such that Γ_S is finite. Consider the covering of $Q_{\sigma(S)}$ by its panels $(Q_s)_{s \in S}$. Since the panels intersect in general position and since each intersection of panels is a nonempty cell, it follows that $Q_{\sigma(S)}$ is homeomorphic to the cone on the boundary of a simplex; i.e., $Q_{\sigma(S)}$ is a disk of codimension 0 in ∂Q. Therefore, the union of chambers T_n is homeomorphic to the boundary connected sum of T_{n-1} and Q. Let $\dot{T}_n$ denote the interior of T_n. It follows that 1) ∂T_n is a deformation retract of $T_{n+1} - T_n$; 2) ∂T_n is homotopy equivalent to $M - \dot{T}_n$ (which is homotopy equivalent to $M - T_n$); and 3) ∂T_n is the connected sum of n copies of ∂Q. Put $G_n = \pi_1(M - T_n)$ and let $\varphi_{n+1}\colon G_{n+1} \to G_n$ be the map induced by the inclusion. It follows from 1), 2) and 3) that G_n is the free product of n copies of G and that $\varphi_{n+1}\colon G_{n+1} \to G_n$ is the natural projection onto the first n factors. Since each φ_{n+1} is onto, M is semistable. Also, $\pi_1^\infty(M) = \varprojlim G_n$ is the "projective free product" of an infinite number of copies of G. In particular it is not trivial. (It is not even finitely generated.)

11. The simplicial complexes associated to panel structures and Coxeter systems

A *poset* is a partially ordered set.

Let V be a set. Denote by $\sigma(V)$ the poset of finite nonempty subsets of V partially ordered by inclusion. A subposet K of $\sigma(V)$ is an *abstract simplicial complex with vertex set* V if for every $v \in V$, $\{v\} \in K$ and if $S \in K$ implies $\sigma(S) \subset K$. (We shall reserve the term "simplicial complex" for the usual geometric notion: a space made out of geometric simplices.) An element S of $\sigma(V)$ is a *simplex*; its *dimension* $d(S)$ is $\mathrm{Card}(S) - 1$.

The *derived complex* of a poset A, denoted by A', is the subset of $\sigma(A)$ consisting of all finite chains in A. It is an abstract simplicial complex with vertex set A.

Example 11.1. Suppose that $(X_v)_{v \in V}$ is a panel structure on a space X (cf. Section 6). Its *nerve* $D_0(X)$ is the abstract simplicial complex consisting of those $S \in \sigma(V)$ such that $X_S \neq \varnothing$. (Strictly speaking, $D_0(X)$ is the nerve of the covering of $X_{\sigma(V)}$ ($= \partial X$) by the family of panels.) Its *associated poset* $D(X)$ is the poset $D_0(X) \cup \{\varnothing\}$.

Example 11.2. Suppose (Γ, V) is a *Coxeter system*. Its *associated complex* $K_0(\Gamma, V)$ is the abstract simplicial complex consisting of those $S \in \sigma(V)$ such that Γ_S is finite. (The associated graph of (Γ, V), defined in Section 2, is the 1-skeleton of $K_0(\Gamma, V)$.) The *associated poset* of (Γ, V), denoted by $K(\Gamma, V)$, is the poset $K_0(\Gamma, V) \cup \{\varnothing\}$.

The condition that the panel structure on X be Γ-finite (cf. Section 6) is equivalent to the condition that $D_0(X)$ be a subcomplex of $K_0(\Gamma, V)$. The combinatorial import of Theorem 10.1 is the following: if (Γ, V) is a reflection system on an m-acyclic manifold M with fundamental chamber Q, then the m-skeleton of $D_0(Q)$ must be equal to the m-skeleton of $K_0(\Gamma, V)$. In particular, if M is acyclic, then $D_0(Q) = K_0(\Gamma, V)$.

The question arises: which finite complexes L can occur as the associated complex of a Coxeter system? Label the edges of L by integers ≥ 2. This defines a Coxeter system (Γ, V) with associated graph equal to the 1-skeleton of L. The higher skeletons of L and $K_0(\Gamma, V)$ may be unrelated. If we label each edge by 2, then $\Gamma_S = (\mathbf{Z}_2)^S$ for all $S \in L$; and hence, in this case L is a subcomplex of $K_0(\Gamma, V)$. In general it is unclear if a labelling can be chosen to make $L = K_0(\Gamma, V)$. However, for our purposes the following result is sufficient.

LEMMA 11.3. *The barycentric subdivision of any finite abstract simplicial complex is the associated complex of some Coxeter system.*

Proof. Let L be an abstract simplicial complex and L' its derived complex. Label all edges of L' by 2 and let (Γ, V) be the associated Coxeter system (here $V = L$). If $\sigma \in K_0(\Gamma, V)$, then since each edge of σ belongs to L', σ is a totally ordered finite subset of L; i.e., $\sigma \in L'$. Hence, $L' = K_0(\Gamma, V)$.

Remark 11.4. There are subdivisions other than the barycentric subdivision which could have been used to prove the above lemma. For example, let L^* be the subdivision of L obtained by introducing a barycenter into each simplex of dimension ≥ 2. Label the new edges in L^* by 2 and the old edges by integers ≥ 4. Then it is easy to check that the resulting Coxeter system has associated complex equal to L^*.

Suppose A is a poset. For each $a \in A$, let $A_{\geq a}$ denote the subposet consisting of all elements $\geq a$. Similarly, define $A_{>a}$, $A_{\leq a}$ and $A_{<a}$.

A well known construction associates to any abstract simplicial complex K a simplicial complex Geom(K), called its *geometric realization*. The *geometric realization* of a poset A, denoted by $|A|$, is the simplicial complex Geom(A').

Suppose A_0 is an abstract simplicial complex with vertex set V and let A denote the poset $A_0 \cup \{\varnothing\}$. The *canonical panel structure* on $|A|$ is the panel structure $(|A|_v)_{v \in V}$ defined by $|A|_v = |A_{\geq \{v\}}|$. For any $S \in A_0$ we have that $\bigcap_{v \in S} A_{\geq\{v\}} = A_{\geq S}$ and therefore, that the face $|A|_S$ is equal to $|A_{\geq S}|$. (Actually, this is the natural way to define faces on $|A^{op}|$ $(=|A|)$, where A^{op} denotes the "dual poset" of A, that is to say, $|A_{\geq S}|$ is the natural "dual face" associated to S.)

Proposition 11.5. *Suppose (Γ, V) is a Coxeter system. Then $|K(\Gamma, V)|$ with its canonical panel structure has the following universal property. Given a Γ-finite panel structure $(X_v)_{v \in V}$ on a* CW*-complex X (so that the panels are subcomplexes), there is a face-preserving map $f\colon X \to |K(\Gamma, V)|$ unique up to a face-preserving homotopy.*

Proof. This is immediate from the fact that $|K(\Gamma, V)_{\geq S}|$ is the cone on $|K(\Gamma, V)_{>S}|$.

Note that if X also satisfies the universal property of the above proposition, then $f\colon X \to |K(\Gamma, V)|$ must be a face-preserving homotopy equivalence. Thus, the universal property is equivalent to the condition that X_S be contractible for each $S \in K(\Gamma, V)$. Any such X will be called "aspherical." (The analogy with Eilenberg-MacLane spaces will become clear in Section 14.)

12. The homology of the nerve

Let K be an abstract simplicial complex and let $S \in K$. The *link of S in K*, denoted by $\mathrm{Link}(S; K)$, is the abstract simplicial complex consisting of all simplices $T \in K$ such that $S \cap T = \varnothing$ and $S \cup T \in K$. An n-dimensional abstract simplicial complex K is a *generalized n-manifold* (or a "Cohen-Macaulay complex") if $H_*(\mathrm{Link}(S; K)) \cong H_*(S^{n-d(S)-1})$ for all $S \in K$. If, in addition, K has the homology of S^n, then it is a *generalized homology n-sphere*.

Let Q be a manifold with faces. The poset of faces of Q can be naturally identified with $D(Q)^{op}$. Any proper face Q_S is naturally a manifold with faces with panels indexed by $\mathrm{Vert}(\mathrm{Link}(S; D_0(Q)))$. Moreover, there is a canonical isomorphism $D_0(Q_S) \cong \mathrm{Link}(S; D_0(Q))$.

A compact manifold with faces is a *homology-cell* (resp. a *homotopy-cell*) if each face is acyclic (resp. contractible).

Proposition 12.1. *If Q is a homology-cell of dimension $n+1$, then $D_0(Q)$ is a generalized homology n-sphere.*

Proof. Since each face is acyclic, the homology of ∂Q is isomorphic to the homology of $D_0(Q)$. ($D_0(Q)$ is the nerve of the covering of ∂Q by $(Q_v)_{v \in V}$.)

Since Q is compact and acyclic, Lefschetz duality implies $H_*(\partial Q) \cong H_*(S^n)$. Similarly, for each $S \in D_0(Q)$,

$$H_*(\text{Link}(S; D_0(Q))) \cong H_*(\partial Q_S) \cong H_*(S^{n-d(S)-1}).$$

Conversely, we have the following result.

THEOREM 12.2. *Let K_0 be a generalized homology n-sphere. Then there is a homotopy $(n+1)$-cell Q with $D_0(Q) = K_0$.*

Basically this is a consequence of the following well-known result.

THEOREM 12.3 ([H′], [F]). *Let Σ^m be a nonsingular homology sphere (i.e., a closed topological manifold with the homology of S^m). Then there is a compact contractible manifold W^{m+1} with ∂W homeomorphic to Σ.*

Proof of Theorem 12.2. Put $K = K_0 \cup \{\varnothing\}$ and endow $|K|$ with its canonical panel structure. If K_0 is a PL-triangulation of S^n, then each face of $|K|$ is a cell and the panels intersect in general position (this follows from the fact that K_0 is an abstract simplicial complex). Hence, we can take $Q = |K|$ in this case. If K_0 is a PL-manifold but not S^n, then the only problem is that $|K|$ is not a manifold at the cone point. However, by Theorem 12.3 there is a contractible manifold Q with $\partial Q = |K_0|$. The panel structure on $|K_0|$ gives Q the structure of a manifold with faces. The general situation is not much more complicated. We "desingularize" $|K|$ by inductively replacing each face (a cone on a generalized homology sphere) by a contractible manifold with boundary. For $i \leq 3$ the face corresponding to an $(n-i)$-simplex $S \in K_0$ is an i-cell. The first problem occurs in filling in the 4-dimensional faces. We do this using Theorem 12.3 and continue in the same manner.

Remark 12.4. The fact that Theorem 12.3 holds when $m = 3$ is a corollary of Freedman's recent proof of the 4-dimensional Poincaré Conjecture. Theorem 12.3 is false in the smooth category. For $m \neq 3$ this can be remedied if one is allowed to vary the smooth structure on Σ^m; however, when $m = 3$ this does not work: any homology 3-sphere with nontrivial μ-invariant is a counterexample. Despite this, one can still prove a smooth version of Theorem 12.2 if one allows the 3- and 4-dimensional faces of Q to be nonsimply connected (but still requires them to be acyclic). The proof of this weaker version of Theorem 12.2, which is independent of Freedman's results, is given in Section 17.

THEOREM 12.5. *Suppose Q and Q' are homotopy cells (resp. homology cells) with $D_0(Q) = D_0(Q')$. Then there is a "stratified h-cobordism" (resp. "stratified homology h-cobordism") between Q and Q'.*

We leave it to the reader to supply definitions for the terms within quotation marks. The proof is entirely similar to that of Theorem 12.2.

13. The basic construction

Much of the discussion in this section follows [V] (see also [K], [T], [S]).

Let (Γ, V) be a Coxeter system and X a space with faces with walls indexed by V. Give Γ the discrete topology. Define an equivalence relation $\sim$ on $\Gamma \times X$ by $(g, x) \sim (h, y) \Leftrightarrow x = y$ and $g^{-1}h \in \Gamma_{V(x)}$. The natural Γ-action on $\Gamma \times X$ is compatible with the equivalence relation; hence, it passes to an action on the quotient space $(\Gamma \times X)/\sim$. Denote this quotient space by $\mathfrak{A}(\Gamma, X)$ (or simply by $\mathfrak{A}$) and call it the Γ-*space associated to* (Γ, X). Let $[g, x]$ denote the image of (g, x) in $\mathfrak{A}$. The proofs of the following lemmas are all completely straightforward.

Lemma 13.1 ([V; p. 1088]). *Let Y be a Γ-space and f: $X \to Y$ a map such that $vf(x) = f(x)$ for all $v \in V$ and $x \in X_v$. Then there is a unique Γ-equivariant map $\tilde{f}$: $\mathfrak{A} \to Y$ such that $\tilde{f}([1, x]) = f(x)$ for all $x \in X$.*

Lemma 13.2 ([V; p. 1088]). *Let π: $\mathfrak{A} \to \mathfrak{A}/\Gamma$ be the orbit map and define p: $X \to \mathfrak{A}/\Gamma$ by $p(x) = \pi([1, x])$. Then p is a homeomorphism.*

Lemma 13.3. *For each $g \in \Gamma$ and $r \in R$ let $\eta(g, r) \in \{\pm 1\}$ be the number defined in Lemma 7.1, and let $\mathfrak{A}_r$ be the fixed point set of r on $\mathfrak{A}$. Then the map η_r: $\mathfrak{A} - \mathfrak{A}_r \to \{\pm 1\}$ given by $[g, x] \to \eta(g, r)$ is well-defined, continuous, and onto.*

Proof. We have $\mathfrak{A}_r = \{[g, x] \mid g^{-1}rg \in \Gamma_{V(x)}\}$. Hence, Corollary 7.9 implies that η_r is well-defined. It is continuous and onto, since the map $\Gamma \times X \to \{\pm 1\}$ given by $(g, x) \to \eta(g, r)$ is continuous and onto.

Lemma 13.4 ([V; p. 1089]). *Suppose X is Hausdorff. Then Γ acts properly on $\mathfrak{A}$ if and only if the panel structure on X is Γ-finite.*

Proof. The definition of a proper action is given at the beginning of Section 1, while the definition of a Γ-finite panel structure occurs at the end of Section 6. The isotropy group at $[g, x] \in \mathfrak{A}$ is equal to $g\Gamma_{V(x)}g^{-1}$. If Γ acts properly, then each isotropy group is finite; hence, the panel structure on X is Γ-finite. Conversely, suppose that the panel structure is Γ-finite. Since the orbit space of $\mathfrak{A}$ is X and since each isotropy group is conjugate to some $\Gamma_{V(x)}$, conditions (a) and (b) in the definition of proper obviously hold. It remains to verify condition (c). For each x in X, let W_x be a neighborhood of x in X which intersects only the panels indexed by $V(x)$. Put $U_x = \{[h, y] \mid h \in \Gamma_{V(x)}$ and $y \in W_x\}$. Then U_x is obviously an open $\Gamma_{V(x)}$-invariant neighborhood of $[1, x]$ in $\mathfrak{A}$. It is also clear that if $g \notin \Gamma_{V(x)}$, then $gU_x \cap U_x = \varnothing$; i.e., (c) holds.

If X is Hausdorff and if the panel structure is Γ-finite, then (Γ, V) should clearly be called a "reflection system" on $\mathfrak{A}$. Our previous notation and terminology go over to this situation in an obvious fashion: $(gX)_{g\in\Gamma}$ is the family of "chambers," $(\mathfrak{A}_r)_{r\in R}$ is the family of "walls," etc.

THEOREM 13.5. *If X is Hausdorff and the panel structure is Γ-finite, then Theorem* 10.1 *holds with M replaced by $\mathfrak{A}$ and Q replaced by X.*

The proof is the same as that of Theorem 10.1 (actually the hypothesis of Γ-finiteness is unnecessary).

14. The universal Γ-complex

Suppose (Γ, V) is a Coxeter system. Let Y denote the Γ-complex $\mathfrak{A}(\Gamma, |K(\Gamma, V)|)$. Since the canonical panel structure on $|K(\Gamma, V)|$ is Γ-finite, Γ acts properly on Y (Lemma 13.4). By Theorem 13.5, Y is contractible. Given a complex with faces X with Γ-finite panel structure $(X_v)_{v\in V}$, there is a face-preserving map $f\colon X \to |K(\Gamma, V)|$ (Proposition 11.3). By Lemma 13.1, f lifts to a Γ-isovariant map $\tilde{f}\colon \mathfrak{A}(\Gamma, X) \to Y$ defined by $\tilde{f}([g, x]) = [g, f(x)]$. In view of these facts, Y is called the *universal Γ-complex*.

One immediate consequence of the existence of the contractible and proper Γ-complex Y is the following proposition, which improves a result of Serre [S, p. 107].

PROPOSITION 14.1. *The virtual cohomological dimension of Γ is $\leq$ the dimension of $|K(\Gamma, V)|$.*

There is a close relationship between Y and the "Coxeter complex" of (Γ, V) (also called the "apartment associated to (Γ, V)," [S; p. 107], [T], [B; p. 40]). The Coxeter complex is constructed as follows. Let $P(V)$ be the poset of proper (but possibly empty) subsets S of V such that $\mathrm{Card}(V - S) < \infty$. (Note that $P(V) \cong \sigma(V)^{op}$ via $S \leftrightarrow V - S$.) Define a panel structure on $|P(V)|$ by $|P(V)|_v = |P(V)_{\geq\{v\}}|$. (If V is finite, then $|P(V)|$ may be identified with a simplex with panels the codimension one faces.) The associated Γ-complex $\mathfrak{A}(\Gamma, |P(V)|)$ is the *Coxeter complex*. If Γ is infinite, then $K(\Gamma, V)$ is a subposet of $P(V)$ and hence, $Y \subset \mathfrak{A}(\Gamma, |P(V)|)$.

There is a well-known relationship between the Coxeter complex and the canonical representation of Γ in $\mathbf{R}^V$ (cf. [S; p. 107], [B; Ex. 2, p. 130]). If C is the standard simplicial cone in $\mathbf{R}^V$, then the Coxeter complex can be identified with the intersection of the convex cone ΓC and the unit sphere. The complex Y is related to the interior of this cone (on which the action is properly discontinuous). More generally, suppose that Γ is represented as a discrete linear group generated by a set of linear reflections V through the walls of a convex polyhedral cone C. We shall use the notation of Theorem 5.3. It follows from Corollary 10.3 that the face $C_S^f \neq \emptyset$ for each $S \in K(\Gamma, V)$. (This result is also proved in [V, Thm. 7, p. 1114].) Hence, $K(\Gamma, V) = D(C^f)$. Barycentric subdivision provides a face-preserving embedding $|K(\Gamma, V)| \to C^f$ leading to an equivariant embedding $Y \to \Omega$. It is clear that Y is an isovariant deformation retract of Ω. Since V is finite, $|K(\Gamma, V)|$ is compact. Thus, Y is the "combinatorial cocompact core" of any representation of (Γ, V) as a linear reflection system.

15. Construction of reflection groups on manifolds

PROPOSITION 15.1. *Let (Γ, V) be a reflection system on a manifold M and let $i: Q \to M$ be the inclusion of its fundamental chamber. Then the induced map $\tilde{i}: \mathfrak{U}(\Gamma, Q) \to M$ (cf. Lemma 13.1) is a Γ-equivariant homeomorphism.*

Proof. This is essentially just a restatement of Theorem 4.1.

THEOREM 15.2. *Let (Γ, V) be a Coxeter system and let Q be a connected manifold with faces with Γ-finite panel structure indexed by V. Put $M = \mathfrak{U}(\Gamma, Q)$. Then M is a manifold and (Γ, V) is a reflection system on M.*

Proof. The Γ-action on M is proper (Lemma 13.4). To prove that M is a manifold and that the action is locally smooth it clearly suffices to find a linear neighborhood of $[1, x]$ in M for each $x \in Q$. The group $\Gamma_{V(x)}$ is finite by hypothesis and $(\Gamma_{V(x)}, V(x))$ is a Coxeter system (Corollary 7.6 (iv)). Let $\Gamma_{V(x)}$ act on $\mathbf{R}^{V(x)} \times \mathbf{R}^m$ via the canonical representation on $\mathbf{R}^{V(x)}$ and the trivial action on $\mathbf{R}^m$. (Here $m = \dim Q - \mathrm{Card}(V(x))$.) Let C be a chamber for this linear reflection group. Thus, C is the product of a simplicial cone in $\mathbf{R}^{V(x)}$ with $\mathbf{R}^m$ (Lemma 5.4). Since Q is a manifold with faces, we can find a neighborhood C_x of x in Q and a face-preserving homeomorphism $f: C_x \to C$. Let U_x denote the subset $\mathfrak{U}(\Gamma_{V(x)}, C_x)$ of M. By the previous proposition the induced map $\tilde{f}: U_x \to \mathbf{R}^{V(x)} \times \mathbf{R}^m$ is a $\Gamma_{V(x)}$-equivariant homeomorphism. Thus, U_x is the required linear neighborhood. By Lemma 13.3, $M - M_r$ is disconnected for all $r \in R$. Hence, Γ is a reflection group on M.

THEOREM 15.3. *Let (Γ, V) be a Coxeter system. Then (Γ, V) can be represented as a cocompact reflection system on a contractible manifold if and only if $K_0(\Gamma, V)$ is a generalized homology sphere.*

Proof. This is immediate from Corollary 10.3, Proposition 12.1, Theorem 12.2 and the previous theorem.

In a similar fashion using Proposition 12.5 one can deduce the following uniqueness result.

THEOREM 15.4. *Suppose that a Coxeter system* (Γ, V) *is represented as a cocompact reflection system on contractible manifolds* M *and* M'. *Then the actions are "concordant". That is to say, there is a representation of* (Γ, V) *as a reflection system on an h-cobordism* W *from* M *to* M' *such that its restriction to either end is given action.*

THEOREM 15.5. *Let* L *be a generalized homology sphere. Then there is a subdivision* L^* *of* L *and a cocompact reflection system* (Γ, V) *on a contractible manifold with* $K_0(\Gamma, V) = L^*$.

Proof. This is immediate from Lemma 11.3 and Theorem 15.3.

In the next section we shall prove the following result.

THEOREM 15.6. *Let* (Γ, V) *be a cocompact reflection system on a contractible manifold* M. *Then* M *is simply connected at infinity if and only if* $|K_0(\Gamma, V)|$ *is simply connected.*

COROLLARY 15.7. *In every dimension* ≥ 4 *there exist cocompact reflection systems on contractible manifolds not homeomorphic to Euclidean space.*

Proof. This follows from the previous two theorems and the fact that there exist nonsimply connected homology spheres in dimensions ≥ 3.

COROLLARY 15.8. *In every dimension* ≥ 4 *there exist closed aspherical manifolds not covered by Euclidean space.*

Remark 15.9. There are further variations of the above methods which can be used to construct more examples of closed aspherical manifolds. As an illustration, start by letting Q be any compact aspherical manifold with boundary (e.g. let Q be the product of a torus and a disk). Let L be a PL-triangulation of ∂X and assume (as we may, possibly after subdividing) that there is a Coxeter system (Γ, V) with $K_0(\Gamma, V) = L$. The dual polyhedron of L together with its canonical panel structure give Q with the structure of a manifold with faces with Γ-finite panel structure. The resulting manifold $M = \mathfrak{A}(\Gamma, Q)$ is not contractible (since Q is not); however, it is aspherical. The reason is as follows. For each $S \in K_0(\Gamma, V)$ we still have that $Q_{\sigma(S)}$ is a disk of codimension 0 in ∂Q; hence, the argument in Section 10 shows that M is an infinite boundary connected sum of copies of Q. The union of two aspherical spaces along a contractible subspace is again aspherical; the fundamental group of the union is the free product of the fundamental groups of each piece. Hence, M is aspherical; its fundamental group is the free product of an infinite number of copies of $\pi_1(Q)$ (one copy for each element of Γ). Now let Γ' be any torsion free subgroup of finite index in Γ. Then M/Γ' is a closed aspherical manifold; its fundamental group is, of course, an extension of Γ' by $\pi_1(M)$:

$$1 \to \pi_1(M) \to \pi_1(M/\Gamma') \to \Gamma' \to 1.$$

There are, clearly, further variations along the line. For example, in the above it was unnecessary that each proper face of Q be a cell. Rather we only need require that in the induced manifold-with-faces structure on the universal cover $\tilde{Q}$ each face be contractible. In other words, the proper faces of Q can also be aspherical manifolds provided that whenever $T \subset S \in K_0(\Gamma, V)$ we have that $\pi_1(X_S) \to \pi_1(X_T)$ is an injection. (The construction in this remark was also motivated by a similar construction for hyperbolic orbifolds in Chapter 5 of [Th].)

16. The fundamental group at infinity

In this section (Γ, V) is a Coxeter system such that its associated complex is a generalized homology N-sphere with $N \geq 2$. Let X be $|K(\Gamma, V)|$ with its canonical panel structure, and Y $(= \mathfrak{A}(\Gamma, X))$ the universal Γ-complex. It is clear that Y is connected at infinity. We wish to calculate $\pi_1^\infty(Y)$. The argument is similar to that of Remark 10.6.

For each $S \in K_0(\Gamma, V)$, $X_{\sigma(S)}$ is contractible (by the Acyclic Covering Lemma and Van Kampen's Theorem) and a generalized N-manifold with boundary. Hence, $\partial X_{\sigma(S)}$ has the homology of S^{N-1}; however, it need not be simply connected. Let $T_1 \subset T_2 \ldots \subset T_n \ldots$ be the increasing union of chambers defined in Section 8. Let $G_n = \pi_1(Y - \dot{T}_n)$ and let φ_n: $G_{n+1} \to G_n$ be the map induced by the inclusion. Since T_{n+1} is the union of T_n and X along a set of the form $X_{\sigma(S)}$ and since both X and $X_{\sigma(S)}$ are contractible, it follows that ∂T_n is homotopy equivalent to $Y - \dot{T}_n$. By Van Kampen's Theorem, $G_{n+1} = G'_n *_{K_n} H_n$, where

$$G'_n \cong \pi_1(\partial T_n - X_{\sigma(S)}), \qquad H_n = \pi_1(\partial X - X_{\sigma(S)}), \qquad K_n = \pi_1(\partial X_{\sigma(S)})$$

and where $G'_n/K_n = G_n$ and $H_n/K_n = \pi_1(\partial X)$. The map φ_n can be identified with the projection onto $G'_n/K_n \cong G_n$. In particular, φ_n is onto and if $\pi_1(\partial X) = \pi_1(|K_0(\Gamma, V)|)$ is nontrivial, it has nonzero kernel. Thus, the sequence of G_n is semistable and not stable (cf. [J]). Therefore, we have proved the following result.

THEOREM 16.1. *The complex Y is simply connected at infinity if and only if $|K_0(\Gamma, V)|$ is simply connected. If $\pi_1(|K_0(\Gamma, V)|) \neq 1$, then $\pi_1^\infty(Y)$ is not finitely generated.*

Theorem 15.6 is an immediate consequence of the following result.

THEOREM 16.2. *Suppose* (Γ, V) *is a cocompact reflection system on a contractible manifold* M *with fundamental chamber* Q. *Let* $f\colon Q \to X = |K(\Gamma, V)|$ *be a face-preserving map and* $\tilde{f}\colon M \to Y$ *the induced* Γ*-equivariant map. Then for each* $n \geq 1$, $\tilde{f}$ *takes* $M - \mathring{T}_n(Q)$ *to* $Y - \mathring{T}_n(X)$ *and this map is a homotopy equivalence. This implies, in particular, that* $\pi_1^\infty(M) \cong \pi_1^\infty(Y)$.

Proof. Let $\tilde{f}_n$ denote the restriction of $\tilde{f}$ to $M - \mathring{T}_n(Q)$. The space $M - \mathring{T}_n(Q)$ is covered by the family of chambers $(g_m Q)_{m>n}$ and $\tilde{f}_n$ takes this covering to a similar covering $(g_m X)_{m>n}$ of $Y - \mathring{T}_n(X)$. (Some of the notation is from Section 8.) Since all intersections of chambers are faces and since both spaces have the same set of faces, the nerves of the two coverings are equal. Since the chambers are contractible, $\tilde{f}_n$ induces an isomorphism on fundamental groups. Since all faces are acyclic, it is also an isomorphism on homology (by the Acyclic Covering Lemma). For the same reason the map of universal covering spaces which covers $\tilde{f}_n$ is an isomorphism on homology. Hence, $\tilde{f}_n$ is a homotopy equivalence.

17. Smoothness questions

Orbifolds. Much of the previous material can be translated into the language of "orbifolds." We shall assume that the reader has some familiarity with this language (cf. [Th; Chapter 5]). Roughly speaking, a locally smooth n-dimensional orbifold is a Hausdorff space which is locally modelled on orbit spaces of $\mathbf{R}^n$ by finite subgroups of $O(n)$. Each overlap map is required to have a local lift to an equivariant homeomorphism; a suitably defined equivalence class of this lift is part of the structure. The orbifold is *smooth* if the local lifts are equivariant diffeomorphisms. A locally smooth or smooth orbifold is of *reflection type* if the local models are finite reflection groups.

A manifold with faces Q with Γ-finite panel structure is naturally a locally smooth orbifold of reflection type. The $\Gamma_{V(x)}$, $x \in Q$, are the "local fundamental groups." If $\mathfrak{A}(\Gamma, Q)$ is simply connected, then it is the "universal cover" of the orbifold and Γ is the "fundamental group." (In general, $\mathfrak{A}(\Gamma, Q)$ is a "covering" and Γ is a quotient of the "fundamental group.") The fact that $\mathfrak{A}(\Gamma, Q)$ is a manifold means that Q is a "good" orbifold. (Recall that the basic reason that it is a manifold is Corollary 7.6 (iv), which translates as "the local fundamental groups inject into the fundamental group.") Thus, our notion of a Γ-finite manifold with faces is more or less equivalent to the notion of a locally smooth good orbifold of reflection type.

These notions are *not* equivalent in the smooth category. The underlying space of an orbifold of reflection type is canonically a smooth manifold with corners. However, a smooth orbifold structure is a finer notion than a smooth manifold-with-corners structure. Consider the following example. Regard $[0, \infty)$ as the fundamental chamber for $\mathbf{Z}_2$ acting on $\mathbf{R}$ via $x \to -x$. If f: $[0, \infty) \to [0, \infty)$ is any map with $f(0) = 0$, then there is an induced $\mathbf{Z}_2$-equivariant map $\tilde{f}$: $\mathbf{R} \to \mathbf{R}$. If $\tilde{f}$ is smooth at 0, then so is f; however, the reverse implication fails completely. If f is a diffeomorphism and if we allow variation by an isotopy (preserving 0), then the situation can be remedied: there is an isotopy of f to a linear map and for linear maps the induced equivariant map is also linear and hence, smooth.

If Q is a manifold with faces with Γ-finite panel structure, then (as the above example shows) a smooth manifold-with-corners structure on Q does not canonically induce a Γ-invariant smooth structure on $\mathfrak{A}(\Gamma, Q)$. However, if Q has the structure of a smooth orbifold, then its orbifold covering $\mathfrak{A}(\Gamma, Q)$ canonically does have such a smooth structure. The variation by isotopy in the above example has the following generalization: if Q has a smooth manifold-with-corners structure, then there exists a smooth orbifold structure on Q which induces the given smooth manifold with corners structure and which is unique up to a face-preserving isotopy. This can be seen by reducing to the case where Q is a tubular neighborhood of a face. The uniqueness statement follows from the uniqueness up to isotopy of tubular neighborhoods. In summary, the natural way to study smooth reflection groups is via smooth orbifolds of reflection type; however, for most purposes it is sufficient to use the simpler concept of smooth manifolds with faces, since each of these can be given a smooth orbifold structure unique up to isotopy.

The smooth version of Theorem 12.2. The only place in which the distinction between smooth and topological manifolds with faces enters in a serious way is in the construction of a homotopy cell with nerve a prescribed generalized homology sphere (cf. Remark 12.4). We shall now discuss what happens in the smooth category. Let θ_n (resp. θ_n^h) denote the abelian group of h-cobordism classes (resp. homology h-cobordism classes) of smooth oriented homotopy (resp. homology) n-spheres. We recall two facts from differential topology:

(a) For $n \neq 3$, if Σ^n is a smooth homology sphere smoothly bounding an acyclic manifold, then it also smoothly bounds a contractible manifold.

(b) For $n \neq 3$, every element of θ_n^h can be represented by a homotopy sphere.

It follows that for $n \neq 3$, $\theta_n^h = \theta_n$. (For $n = 3$, nothing is known.)

Now suppose that L is an abstract simplicial complex which is a generalized homology n-sphere. We shall construct a smooth homology $(n + 1)$-cell Q with nerve L. Suppose that we have constructed the faces of dimension $< m$, with

$0 \leq m \leq n$. For each $(n-m)$-simplex S in L let Σ_S be the smooth homology $(m-1)$-sphere given as $\bigcup Q_{T \cup S}$ where T ranges over Link(S; L). (The smooth structure on Σ_S is obtained by rounding the corners.) Choose an orientation for L which is an orientable generalized manifold. If the simplex S is oriented, then Link(S; L) and Σ_S inherit natural orientations. Let $[\Sigma_S]$ denote the class of Σ_S in θ^h_{m-1}. The correspondence $S \to [\Sigma_S]$ defines a chain $d_{n-m} \in C_{n-m}(L; \theta^h_{m-1})$, where $C_*(L; \theta^h_{m-1})$ denotes the oriented simplicial chains on L with coefficients in θ^h_{m-1}. The chain d_{n-m} has the following properties:

(1) We can fill in the m-dimensional faces with acyclic manifolds, relative to the faces of dimension $< m$, if and only if $d_{n-m} = 0$.

(2) If $d_{n-m} = \partial c$ for some $c \in C_{n-m+1}(L; \theta^h_{m-1})$, then we can use c to alter the $(m-1)$-dimensional faces (by taking connected sums with homology spheres) so that the new d_{n-m} becomes 0.

(3) If $n = m$, then $d_0 \in C_0(L; \theta^h_{n-1})$ is in the augmentation ideal.

(4) If $n - m > 0$, then d_{n-m} is a cycle.

Properties (1), (2), (3) are fairly obvious and (4) follows from (3) applied to the complexes Link(S; L), with $d(S) = n - m$. Since the reduced homology of L vanishes in degrees $< n$ (and since the coefficients are 0 in degree n), it follows that we can build the faces of dimension $\leq n$. We can fill in the final $(n+1)$-dimensional face in a similar fashion, possibly after altering a face of dimension n. In view of statement (b), the altering of faces of dimension $m-1$ may be done by taking connected sums with homotopy spheres, provided $m \neq 4$. Also, if $m \neq 4$, then it follows from (a) that the m-dimensional faces can be filled in with contractible manifolds. Thus, we have proved the following theorem.

THEOREM 17.1. *Let L be a generalized homology n-sphere. Then there exists a smooth homology $(n+1)$-cell Q whose nerve is equal to L. Furthermore, we may take each face of dimension $\neq 3, 4$ to be contractible. If for every $(n-4)$-simplex S, $|$Link(S; L)$|$ smoothly bounds a contractible 4-manifold, then every face may be taken to be contractible.*

COROLLARY 17.2. *If (Γ, V) is a (locally smooth) cocompact reflection system on a contractible N-manifold, $N \neq 4$, then it is concordant to a smooth cocompact reflection system on a contractible manifold.*

In dimension four, it is in general only possible to deduce the existence of such a smooth representative of (Γ, V) on an acyclic 4-manifold.

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