Davis, M. W. *et al.*. (0000) “Cohomology of hyperplane complements with group ring coefficients,”
International Mathematics Research Notices, Vol. 0000, Article ID rnn000, 6 pages.
doi:10.1093/imrn/rnn000

Cohomology of hyperplane complements with group ring coefficients

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We compute the cohomology with group ring coefficients of the complement of a finite collection of affine hyperplanes in $\mathbf{C}^n$. It is nonzero in exactly one degree, namely, the degree equal to the rank of the arrangement.

A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine hyperplanes in $\mathbf{C}^n$. A *subspace* of $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$. Denote by $L(\mathcal{A})$ the poset of subspaces, ordered by inclusion. Put $\overline{L}(\mathcal{A}) := L(\mathcal{A}) \cup \{\mathbf{C}^n\}$. An arrangement is *central* if $L(\mathcal{A})$ has a unique minimum element. In general, the minimal elements of $L(\mathcal{A})$ are a family of parallel subspaces. The *rank* of $\mathcal{A}$ is the codimension in $\mathbf{C}^n$ of a minimal element. $\mathcal{A}$ is *essential* if $\mathrm{rk}(\mathcal{A}) = n$. Given $G \in \overline{L}(\mathcal{A})$, put

$$\mathcal{A}_G := \{H \in \mathcal{A} \mid H \supseteq G\}.$$

It is a central arrangement of rank $\rho(G) = n - d(G)$, where $d(G) = \dim_{\mathbf{C}} G$.

The *singular set* $\Sigma(\mathcal{A})$ of the arrangement is the union of hyperplanes in $\mathcal{A}$ (so, $\Sigma(\mathcal{A})$ is a subset of $\mathbf{C}^n$). The complement of $\Sigma(\mathcal{A})$ in $\mathbf{C}^n$ is denoted $M(\mathcal{A})$. Similarly, the complement of $\Sigma(\mathcal{A}_G)$ in $\mathbf{C}^n$ is $M(\mathcal{A}_G)$.

We now state our main result.

Theorem 1. Suppose $\mathcal{A}$ is an arrangement of rank l. Let $\pi = \pi_1(M(\mathcal{A}))$. Then $H^*(M(\mathcal{A}); \mathbf{Z}\pi)$ is concentrated in degree l and is free abelian. □

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Corollary 2. The right $\mathbf{Z}\pi$-module $H^l(M(\mathcal{A});\mathbf{Z}\pi)$ is type *FL*. □

Proof. It is known that $M(\mathcal{A})$ is homotopy equivalent to a finite complex X of dimension l. Using the cellular cochains of the universal cover of X we get a free resolution of length l:

$$0 \to C^0(X;\mathbf{Z}\pi) \to \cdots \to C^l(X;\mathbf{Z}\pi) \to H^l(X;\mathbf{Z}\pi) \to 0.$$

■

Note that $H^l(M(\mathcal{A});\mathbf{Z}\pi) = H^*(X;\mathbf{Z}\pi)$ can be identified with the compactly supported cohomology of the universal cover of the finite complex X.

A group π is a *duality group* if it is type *FP* and $H^*(\pi;\mathbf{Z}\pi)$ is concentrated in a single degree and is torsion-free.

Corollary 3 (cf. [6])**.** Suppose $\mathcal{A}$ is a $K(\pi,1)$ arrangement (i.e., $M(\mathcal{A})$ is a $K(\pi,1)$ with $\pi = \pi_1(M(\mathcal{A}))$). Then π is a duality group. □

The next lemma is well-known.

Lemma 4 (cf. [3, Prop. 2.1])**.** Suppose $\mathcal{A}$ is a hyperplane arrangement of rank l. Then $\Sigma(\mathcal{A})$ is homotopy equivalent to a wedge of $(l-1)$-spheres. □

For each $G \in \overline{L}(\mathcal{A})$, $\mathcal{A}\cap G$ denotes the hyperplane arrangement in G consisting of all elements of $L(\mathcal{A})$ which are subspaces of codimension-one in G. Then $\mathcal{A}\cap G$ is an arrangement of rank $l(G) = d(G) - n_0$, where n_0 is the rank of a minimal element of $L(\mathcal{A})$. We note that

$$l(G) + \rho(G) = n - n_0 = l. \tag{1}$$

Let $\beta(\mathcal{A}\cap G)$ denote the reduced Betti number of $G \cap \Sigma$ in degree $l(G)-1$, i.e.,

$$\beta(\mathcal{A}\cap G) := \mathrm{rk}(H^{l(G)}(G, \Sigma(\mathcal{A}\cap G))). \tag{2}$$

Suppose $\mathcal{A}$ is an essential, central arrangement in $\mathbf{C}^n$. Projectivizing we get a projective hyperplane arrangement $P\mathcal{A}$ in $\mathbf{C}P^{n-1}$. Choose a hyperplane in $P\mathcal{A}$ to regard as the hyperplane at infinity. Removing it, we obtain a hyperplane arrangement $\mathcal{A}'$ in $\mathbf{C}^{n-1}$, called an *associated affine arrangement*. We note that $M(\mathcal{A})$ is a $\mathbf{C}^*$-bundle over $M(\mathcal{A}')$; moreover, this bundle is trivial (since either $n = 1$ or $\mathcal{A}'$ is nonempty). Thus, $M(\mathcal{A}) \cong M(\mathcal{A}') \times \mathbf{C}^*$. Let C_∞ denote the fundamental group of $\mathbf{C}^*$ (i.e., C_∞ is the infinite cyclic group). From the above discussion we get the following.

Lemma 5. Suppose $\mathcal{A}$ is an essential, central arrangement in $\mathbf{C}^n$ and $\mathcal{A}'$ is an associated affine arrangement. Put $\pi = \pi_1(M(\mathcal{A}))$, $\pi' = \pi_1(M(\mathcal{A}'))$. Then $\pi = \pi' \times C_\infty$, and

$$H^*(M(\mathcal{A}); \mathbf{Z}\pi) = H^{*-1}(M(\mathcal{A}'); \mathbf{Z}\pi') \otimes \mathbf{Z},$$

where C_∞ acts trivially on $\mathbf{Z}$. □

Proof.

$$H^i(\mathbf{C}^*; \mathbf{Z}C_\infty) = H^i(S^1; \mathbf{Z}C_\infty) = \begin{cases} \mathbf{Z}, & \text{if } i = 1; \\ 0, & \text{if } i \neq 1. \end{cases}$$

So, the equation in the lemma follows from the Künneth Formula. ■

Suppose $\mathcal{A}$ is a hyperplane arrangement in $\mathbf{C}^n$. An open convex subset U in $\mathbf{C}^n$ is *small* (with respect to $\mathcal{A}$) if $\{G \in \overline{L}(\mathcal{A}) \mid G \cap U \neq \emptyset\}$ has a unique minimum element $\mathrm{Min}(U)$. The intersection of two small convex open sets is also small; hence, the same is true for any finite intersection of such sets.

Now let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $\mathbf{C}^n$ by small convex sets. We may suppose that $\mathcal{U}$ is finite and that it is closed under taking intersections. For each $G \in \overline{L}(\mathcal{A})$, put

$$\begin{aligned} \mathcal{U}_G &:= \{U \in \mathcal{U} \mid \mathrm{Min}(U) \subseteq G\}, \\ \mathcal{U}_G^{\mathrm{sing}} &:= \{U \in \mathcal{U} \mid \mathrm{Min}(U) \subsetneq G\} = \{U \in \mathcal{U}_G \mid U \cap \Sigma(\mathcal{A} \cap G) \neq \emptyset\}. \end{aligned}$$

The open cover $\mathcal{U}$ restricts to an open cover $\widehat{\mathcal{U}} = \{U - \Sigma(\mathcal{A})\}_{U \in \mathcal{U}}$ of $M(\mathcal{A})$. Any element $\widehat{U} = U - \Sigma(\mathcal{A})$ of the cover is homotopy equivalent to the complement of a central arrangement $M(\mathcal{A}_G)$, where $G = \mathrm{Min}(U)$.

Suppose $N(\mathcal{U})$ is the nerve of $\mathcal{U}$ and $N(\mathcal{U}_G)$ is the subcomplex defined by $\mathcal{U}_G$. Since $N(\mathcal{U}_G)$ and $N(\mathcal{U}_G^{\mathrm{sing}})$ are nerves of covers of G and $\Sigma(\mathcal{A} \cap G)$, respectively, by contractible open subsets, we have that for each $G \in \overline{L}(\mathcal{A})$,

$$H^*(N(\mathcal{U}_G), N(\mathcal{U}_G^{\mathrm{sing}})) = H^*(G, \Sigma(\mathcal{A} \cap G)). \tag{3}$$

For each k-simplex $\sigma = \{i_0, \ldots, i_k\}$ in $N(\mathcal{U})$, let

$$U_\sigma := U_{i_0} \cap \cdots \cap U_{i_k}$$

denote the corresponding intersection.

Let $r : \widetilde{M}(\mathcal{A}) \to M(\mathcal{A})$ be the universal cover. The induced cover $\{r^{-1}(\widehat{U})\}$ of $\widetilde{M}(\mathcal{A})$ has the same nerve $N(\widehat{\mathcal{U}})$ $(= N(\mathcal{U}))$. We have the Mayer–Vietoris double complex,

$$C_{i,j} := \bigoplus_{\sigma \in N^{(i)}} C_j(r^{-1}(\widehat{U}_\sigma)),$$

where $N^{(i)}$ denotes the set of i-simplices in $N(\mathcal{U})$ (cf. [1, Ch. VII].) We get a corresponding double cochain complex,

$$E_0^{i,j} := \mathrm{Hom}_\pi(C_{i,j}, \mathbf{Z}\pi), \tag{4}$$

where $\pi = \pi_1(M(\mathcal{A}))$. The filtration on the double complex gives a spectral sequence converging to the associated graded module for cohomology:

$$\mathrm{Gr}\, H^m(M(\mathcal{A}); \mathbf{Z}\pi) = E_\infty := \bigoplus_{i+j=m} E_\infty^{i,j}.$$

Proof of Theorem 1. The proof is by induction on the rank l of $\mathcal{A}$. The result is trivial for $l = 0$ (for then the arrangement is empty). Lemma 5 shows that if we know the result for ranks less than l, then we also know it for any central arrangement of rank l. So, given a rank l arrangement $\mathcal{A}$, the inductive hypothesis implies that the theorem holds for each small open set in our cover $\mathcal{U}$. In other words, we can assume that for each $U \in \mathcal{U}$, for $G = \mathrm{Min}(U)$ and $\pi_G = \pi_1(M(\mathcal{A}_G))$, $H^*(U - \Sigma; \mathbf{Z}\pi_G) = H^*(M(\mathcal{A}_G); \mathbf{Z}\pi_G)$ is free abelian and is concentrated in degree $\rho(G) = l - l(G)$.

By first using the horizontal differential in (4), we get a spectral sequence with E_1-terms

$$E_1^{i,j} = C^i(N(\mathcal{U}); \mathcal{H}^j), \tag{5}$$

where $\mathcal{H}^j$ is the coefficient system on $N(\mathcal{U})$ defined by

$$\sigma \mapsto H^j(M(\mathcal{A}_G); \mathbf{Z}\pi),$$

for $G = \mathrm{Min}(U_\sigma)$. These coefficients are 0 for $j \neq \rho(G)$, i.e., for $l(G) \neq l - j$ (by (1)). Moreover, for any coface σ' of σ, if $G' := \mathrm{Min}(U_{\sigma'}) \subsetneq G$, then the coefficient homomorphism $H^j(M(\mathcal{A}_G); \mathbf{Z}\pi) \to H^j(M(\mathcal{A}_{G'}); \mathbf{Z}\pi)$ is the zero map. It follows that the E_1 page of the spectral sequence decomposes as a direct sum (cf. [5, Lemma 2.2]). For a fixed j, the $E_1^{i,j}$ term decomposes as

$$E_1^{i,j} = \bigoplus_{G \in \overline{L}_{n-j}(\mathcal{A})} C^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\mathrm{sing}}); H^j(M(\mathcal{A}_G); \mathbf{Z}\pi)),$$

where we have constant coefficients in each summand. Hence, at E_2 we have

$$\begin{aligned} E_2^{i,j} &= \bigoplus_{G\in \overline{L}_{n-j}(\mathcal{A})} H^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\mathrm{sing}}); H^j(M(\mathcal{A}_G);\mathbf{Z}\pi)) \\ &= \bigoplus_{G\in \overline{L}_{n-j}(\mathcal{A})} H^i(G, \Sigma(\mathcal{A}\cap G); H^j(M(\mathcal{A}_G);\mathbf{Z}\pi)), \end{aligned}$$

where the second equation is by (3). By Lemma 4, $H^i(G,\Sigma(\mathcal{A}\cap G))$ is nonzero only for $i = l(G) = l - j$. So, the E_2 terms are nonzero only in total degree l. It follows that the spectral sequence collapses at E_2. Thus, for $k \neq l$, $H^k(M(\mathcal{A});\mathbf{Z}\pi) = 0$, while

$$\operatorname{Gr} H^l(M(\mathcal{A});\mathbf{Z}\pi) = \bigoplus_{G\in \overline{L}(\mathcal{A})} H^{l(G)}(G, \Sigma(G\cap\mathcal{A}); H^{\rho(G)}(M(\mathcal{A}_G);\mathbf{Z}\pi)), \tag{6}$$

and, therefore, is free abelian. It follows that the ungraded object, $H^l(M(\mathcal{A});\mathbf{Z}\pi)$ is also free abelian. ■

Remark 6. Here are some more comments about equation (6). Since $\mathbf{Z}\pi$ is a π-bimodule, $H^l(M(\mathcal{A});\mathbf{Z}\pi)$ is a right $\mathbf{Z}\pi$-module and $\operatorname{Gr} H^l(M(\mathcal{A});\mathbf{Z}\pi)$ is the associated graded $\mathbf{Z}\pi$-module. Similarly, each summand on the right hand side of (6) is a $\mathbf{Z}\pi$-module and the formula is an isomorphism of $\mathbf{Z}\pi$-modules.

The coefficients in the summand corresponding to G come from the induced representation,

$$H^{\rho(G)}(M(\mathcal{A}_G);\mathbf{Z}\pi) = H^{\rho(G)}(M(\mathcal{A}_G);\mathbf{Z}\pi_G) \otimes_{\pi_G} \mathbf{Z}\pi,$$

where $\pi_G := \pi_1(M(\mathcal{A}_G))$. So, the summand corresponding to G is a sum of $\beta(\mathcal{A}\cap G)$ copies of the induced representation $H^{\rho(G)}(M(\mathcal{A}_G);\mathbf{Z}\pi_G)\otimes_{\pi_G}\mathbf{Z}\pi$, where $\beta(\mathcal{A}\cap G)$ was defined in (2). If $\mathcal{A}_G \neq \emptyset$, $H^{\rho(G)}(M(\mathcal{A}_G);\mathbf{Z}\pi_G)$ is not a free $\mathbf{Z}\pi_G$-module. The reason is that if $M(\mathcal{A}'_G)$ is an associated affine arrangement to $\mathcal{A}_G$ and $\pi'_G = \pi_1(\mathcal{A}'_G)$, then, by Lemma 5,

$$H^{\rho(G)}(M(\mathcal{A}_G);\mathbf{Z}\pi_G) = H^{\rho(G)-1}(M(\mathcal{A}'_G);\mathbf{Z}\pi'_G)\otimes\mathbf{Z},$$

which is not free (unless $\pi_G = 1$). Hence, only one summand on the right hand side of (6) is a free $\mathbf{Z}\pi$-module, the one corresponding to $G = \mathbf{C}^n$. It is

$$H^l(\mathbf{C}^n, \Sigma(\mathcal{A}))\otimes\mathbf{Z}\pi,$$

which is a free of rank $\beta(\mathcal{A})$. Since this free module is the top summand of $\operatorname{Gr} H^l(M(\mathcal{A});\mathbf{Z}\pi)$, it must also be a direct summand of $H^l(M(\mathcal{A});\mathbf{Z}\pi)$. In [3, Theorem 6.2] we showed that the reduced ℓ^2-cohomology of $M(\mathcal{A})$ is $H^l(\mathbf{C}^n,\Sigma(\mathcal{A}))\otimes\ell^2\pi$. The free summand described above injects into ℓ^2-cohomology, while the other summands map to 0.

In [3, Theorem 5.3] we used a similar argument to reprove the well-known result that the cohomology of $M(\mathcal{A})$ with coefficients in a generic abelian local coefficient system vanishes except in degree l. □

Funding

The first and second named authors were partially supported by NSF grant DMS-0706259 and the third named author was partially supported by NSF grant DMS-0804226.

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