4-DIMENSIONAL LOCALLY CAT(0)-MANIFOLDS WITH NO RIEMANNIAN SMOOTHINGS

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Abstract. We construct examples of 4-dimensional manifolds M supporting a locally CAT(0)-metric, whose universal covers $\tilde{M}$ satisfy Hruska's isolated flats condition, and contain 2-dimensional flats F with the property that $\partial^\infty F \cong S^1 \hookrightarrow S^3 \cong \partial^\infty \tilde{M}$ are nontrivial knots. As a consequence, we obtain that the group $\pi_1(M)$ cannot be isomorphic to the fundamental group of any Riemannian manifold of nonpositive sectional curvature. In particular, if K is any locally CAT(0)-manifold, then $M \times K$ is a locally CAT(0)-manifold which does not support any Riemannian metric of nonpositive sectional curvature.

1. Introduction

Riemannian manifolds of nonpositive sectional curvature are a class of manifolds featuring a rich interplay between their geometry, their topology, and their dynamics. In the broader setting of geodesic metric spaces, we have the notion of a locally CAT(0)-metric. These provide a metric space analogue of nonpositively curved Riemannian manifolds, and many classic results concerning Riemannian manifolds of nonpositive sectional curvature have now been shown to hold more generally for locally CAT(0)-spaces. We are interested in understanding the difference, within the class of closed manifolds, between (1) supporting a Riemannian metric of nonpositive sectional curvature, and (2) supporting a locally CAT(0) metric. A closed topological manifolds equipped with a locally CAT(0)-metric will be called a *locally CAT(0)-manifold*.

In low dimensions, there is no difference between these two classes. In two dimensions, this follows easily from the classification of surfaces, while in three dimensions, this follows from Thurston's geometrization theorem (recently established by Perelman). In contrast, Davis and Januszkiewicz [DJ] have constructed examples, in all dimensions ≥ 5, of locally CAT(0)-manifolds which do *not* support any Riemannian metric of nonpositive sectional curvature. In this paper, we deal with the remaining open case.

Main Theorem: There exists a 4-dimensional closed manifold M with the following four properties:

(1) M supports a locally CAT(0)-metric,
(2) M is smoothable, and $\tilde{M}$ is diffeomorphic to $\mathbb{R}^4$,
(3) $\pi_1(M)$ is **not** isomorphic to the fundamental group of any Riemannian manifold of nonpositive sectional curvature.
(4) if K is any locally CAT(0)-manifold, then $M \times K$ is a locally CAT(0)-manifold which does not support any Riemannian metric of nonpositive sectional curvature.

Let us briefly outline the idea behind the proof of our main result. First of all, we introduce the notion of a triangulation of S^3 to have *isolated squares.* Any such triangulation has a well-defined *type*, which is the isotopy class of an associated link in S^3. In Section 3, we provide a proof that any given link in S^3 can be realized as the type of a suitable flag triangulation of S^3 with isolated squares. In Section 4, we start with a flag triangulation L of S^3 with isolated squares, whose type is a nontrivial knot, and use it to construct the desired 4-manifold. This is done by considering the right angled Coxeter group Γ_L associated to the triangulation L, and defining M to be the quotient of the corresponding Davis complex by a torsion free finite index subgroup $\Gamma \leq \Gamma_L$. Standard properties of the triangulation L ensure that M is smoothable, and that the Davis complex is CAT(0) and diffeomorphic to $\mathbb{R}^4$. The isolated squares condition on the flag triangulation L ensures the Davis complex satisfies Hruska's *isolated flats* condition. The fact that the type of L is a nontrivial knot ensures that the Davis complex contains a periodic 2-dimensional flat F which is *knotted at infinity.* But now if M supported a Riemannian metric g of nonpositive sectional curvature, the flat torus theorem ensures that one could find a corresponding flat F' (in the g-metric) which is Γ-equivariantly homotopic to F, and the isolated flats condition then forces F' to also be knotted at infinity. However, in the Riemannian setting, it is easy to see that a codimension two flat must be unknotted at infinity, yielding a contradiction.

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2. Previously known obstructions.

Our Main Theorem provides a new obstruction to the problem of finding a *Riemannian smoothing* on a manifold M supporting a locally CAT(0)-metric. More precisely, we say that such a manifold supports a Riemannian smoothing provided one can find a smooth Riemannian manifold (N, g), with g a Riemannian metric of nonpositive sectional curvature, and a homeomorphism $f : N \to M$. In this section, we briefly summarize the known obstructions to Riemannian smoothing.

2.1. Example: no smooth structure.

Given a Riemannian smoothing $f : N \to M$ of a locally CAT(0)-manifold M, one can forget the Riemannian structure and simply view N as a smooth manifold. This immediately tells us that, if M has a Riemannian smoothing, then it must be homeomorphic to a smooth manifold, i.e. the topological manifold M must be *smoothable.* The first examples of aspherical topological manifolds not homotopy equivalent to smooth manifolds were constructed (in all dimensions ≥ 13) by Davis and Hausmann [DH] by using the reflection group trick. Non-smoothable aspherical PL-manifolds were constructed (in all dimensions ≥ 8) in the same paper. For the sake of completeness, we now sketch out a (slightly different) construction of a closed 8-dimensional locally CAT(-1)-manifold M^8 which is not homotopy equivalent to any smooth 8-manifold.

Recall that Milnor constructed [Mi] an 8-dimensional PL-manifold N^8 which is not homotopy equivalent to any smooth 8-manifold. Milnor's example had the property that the second rational Pontrjagin class $p_2(N^8)$ was *not* an integral class, and hence cannot be homeomorphic to a smooth manifold. Let us take N^8 equipped with a PL-triangulation. Charney and Davis [CD] developed a *strict hyperbolization* process, which inputs a triangulated manifold M and outputs a piecewise hyperbolic manifold $h(M)$ equipped with a locally CAT(-1)-metric. Furthermore, they showed that the hyperbolization process preserves rational Pontrjagin classes. In particular, applying their strict hyperbolization process to N^8, we obtain a locally CAT(-1)-manifold $h(N^8)$, having the property that $p_2(h(N^8))$ fails to be integral, and hence forcing $h(N^8)$ to be non-smoothable. Finally, we note that the Borel Conjecture is known to hold for this class of aspherical manifolds (see [BL]), so if $h(N^8)$ was homotopy equivalent to some smooth manifold, it would in fact be homeomorphic to the smooth manifold (contradicting non-smoothability). Similar examples can be constructed in all dimensions of the form $n = 4k$, with $k \geq 2$ (see also the discussion in [BLW, Section 5]).

2.2. **Example: no PL structure.** In a similar vein, it is also possible to construct (topological) locally CAT(0)-manifolds that do not even support any PL-structures. We recall such an example from [DJ, Section 5a]. We let $M^4(E_8)$ denote the E_8 homology manifold. Recall that this space is constructed by first plumbing together eight copies of the tangent disk bundle to S^2, according to the pattern given by the E_8 Dynkin diagram. This results in a smooth 4-manifold with boundary N^4, whose boundary ∂N^4 is homeomorphic to Poincaré's homology 3-sphere. Coning off the boundary gives the space $M^4(E_8)$, a simply connected homology manifold of signature 8 with one singular point. Taking a triangulation of N^4, one can extend it (by coning on the boundary) to a triangulation of $M^4(E_8)$, which we can then hyperbolize to obtain a space H^4.

The space H^4 is now a homology 4-manifold of signature 8 with one singular point, and comes equipped with a locally CAT(0)-metric. It follows from Edward's Double Suspension Theorem that $H^4 \times T^k$ is a topological $(4+k)$-manifold (where T^k denotes the k-torus and $k \geq 1$). The manifolds $H^4 \times T^k$ come equipped with a (product) locally CAT(0)-metric, but it follows from the arguments in [DJ, Section 5a] that they do not admit a PL structure. Thus, in each dimension ≥ 5 there is a locally CAT(0)-manifold with no PL structure.

2.3. **Example: universal cover distinct from $\mathbb{R}^n$.** For a third family of examples, we recall that the classic Cartan-Hadamard theorem asserts that the universal cover of a Riemannian manifold of nonpositive sectional curvature must be diffeomorphic to $\mathbb{R}^n$. In particular, a CAT(0)-manifold M with the property that $\tilde{M}$ is *not* diffeomorphic to $\mathbb{R}^n$ can not support a Riemannian smoothing. Davis and Januszkiewicz constructed (see [DJ, Thm. 5b.1]) examples of locally CAT(0)-manifolds M^n (for $n \geq 5$), with the property that their universal covers $\tilde{M}^n$ are *not* simply connected at infinity (and hence, not homeomorphic to $\mathbb{R}^n$). Further examples of this type are described in [ADG].

2.4. **Example: boundary at infinity distinct from S^{n-1}.** In the previous three families of examples, *topological* properties (smoothability, PL-smoothings, topology of universal cover) were used to obstruct the existence of a Riemannian metric of nonpositive sectional curvature. The next family of examples have obstructions that arise from the *large scale geometry* of the universal covers. Associated to a CAT(0)-space X, we have a topological space called the *boundary at infinity* $\partial^\infty X$. If X is Gromov hyperbolic, then the homeomorphism type of $\partial^\infty X$

is a quasi-isometry invariant of X. In particular, if X is the universal cover of a locally CAT(-1)-space Y, then $\partial^\infty X$ depends only on $\pi_1(Y)$. When X is the universal cover of an n-dimensional closed Riemannian manifold of nonpositive sectional curvature, the corresponding $\partial^\infty X$ is homeomorphic to the standard sphere S^{n-1}.

Now consider the locally CAT(-1) 5-manifold M^5 obtained by applying a strict hyperbolization procedure (from [CD]) to the double suspension of a triangulation of Poincaré's homology 3-sphere. Denote by X^5 its universal cover, and observe that, although $\partial^\infty X^5$ has the homotopy type of S^4, it is proved in [DJ, Section 5c] that $\partial^\infty X^5$ is *not* locally simply connected. So $\partial^\infty X^5$ cannot be homeomorphic to S^4 (in fact, is not even an ANR). Thus, M^5 is not homotopy equivalent to a Riemannian 5-manifold of strictly negative sectional curvature. The same argument applies to a strict hyperbolization of the manifold $M^4(E_8) \times S^1$ discussed in Section 2.2. There are similar examples in higher dimensions $n > 5$ obtained by strictly hyperbolizing double suspensions of homology $(n-2)$-spheres. Thus, in each dimension $n \geq 5$ there are closed locally CAT(-1) manifolds M^n with universal cover homeomorphic to $\mathbb{R}^n$ but which are not homotopy equivalent to any Riemannian n-manifold of strictly negative sectional curvature.

2.5. **Example: stability under products.** Finally, we point out one last method for producing manifolds which do not have Riemannian smoothings:

Proposition 1. *Let M^n be a locally CAT(0)-manifold which does not support any Riemannian smoothing, and assume that $n \geq 5$. Then for K an arbitrary locally CAT(0)-manifold, the product $M \times K$ is a locally CAT(0)-manifold which does not support any Riemannian smoothing.*

Proof. To see this, we first note that the product of the locally CAT(0)-metrics on M and K provide a locally CAT(0)-metric on $M \times K$. Now assume that $M \times K$ supported a Riemannian smoothing $f : N \to M \times K$, and let g be the associated Riemannian metric of nonpositive sectional curvature on N. Since $\pi_1(N) \cong \pi_1(M) \times \pi_1(K)$, the classical splitting theorems (see Gromoll and Wolf [GW], Lawson and Yau [LY], and Schroeder [Sc]) imply that we have a corresponding *geometric* splitting $(N, g) \cong (M', g_1) \times (K', g_2)$, having the property that:

- each factor can be identified with a totally geodesic submanifold of (N, g),
- the factors satisfy $\pi_1(M) \cong \pi_1(M')$, and $\pi_1(K) \cong \pi_1(K')$.

So we see that M' is a Riemannian manifold of nonpositive sectional curvature, of dimension ≥ 5, and satisfying $\pi_1(M) \cong \pi_1(M')$. Since the Borel conjecture is known to hold for this class of manifolds (see Farrell and Jones [FJ]), there exists a homeomorphism $M' \to M$ realizing the isomorphism of fundamental groups. This provides a Riemannian smoothing of M, giving us the desired contradiction. □

We remark that property (4) in our Main Theorem can be deduced from a virtually identical argument: instead of appealing to the Borel Conjecture to obtain a contradiction, we resort instead to property (3) in our Main Theorem.

3. SPECIAL TRIANGULATIONS OF S^3.

Recall that a simplicial complex is *flag* provided it is determined by its 1-skeleton, i.e. every k-tuple of pairwise incident vertices spans a $(k-1)$-simplex σ^{k-1} (for $k \geq 3$). A subcomplex Σ' of a simplicial complex Σ is *full* provided every simplex $\sigma \subset \Sigma$ whose vertices lie in Σ' satisfies $\sigma \subset \Sigma'$. We will say a cyclically ordered

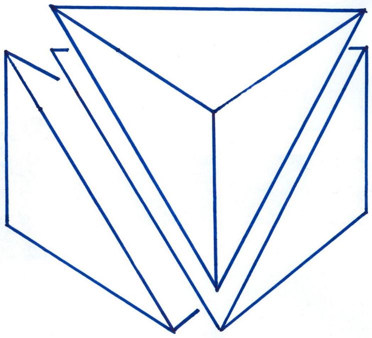

Figure 1. Basic triangulation of a triangular prism.

4-tuple of vertices (v_1, v_2, v_3, v_4) in a simplicial complex forms a *square* provided each consecutive pair of vertices determines an edge in the complex, while the pairs (v_1, v_3) and (v_2, v_4) do *not* determine an edge.

Definition 2. A flag triangulation of S^3 is said to have *isolated squares* provided no two squares in the triangulation intersect (i.e. each vertex lies in at most one square). For such a triangulation, the collection of squares form a link in S^3. We call the isotopy class of this link the *type* of the triangulation.

In this section, we establish:

Theorem 3. *Let $k \subset S^3$ be any prescribed link in the 3-sphere. Then there exists a flag triangulation of S^3, with isolated squares, and with type the given link k.*

We establish this result in several steps, gradually building up the triangulation to have the properties we desire.

Step 1: Triangulating the solid torus.

As a first step, we describe a triangulation on a solid torus $\mathbb{D}^2 \times S^1$. Recall that there is a canonical decomposition of the 3-dimensional cube $[0,1]^3 \subset \mathbb{R}^3$ into six tetrahedra. This triangulation is determined by the inequalities $0 \leq x_{\sigma(1)} \leq x_{\sigma(2)} \leq x_{\sigma(3)} \leq 1$, where σ ranges over the six possible permutations of the index set $\{1,2,3\}$. Now if we restrict to the region where $x_1 \leq x_2$, we obtain a triangulation of the triangular prism $\Delta^2 \times [0,1]$ into exactly three tetrahedra. Let us denote by F, G the two square faces of the triangular prism defined via the hyperplanes $x_1 = 0$ and $x_1 = x_2$ respectively. The triangulation of the prism cuts each of these squares into two triangles, along the diagonal originating at the origin. We call the *bottom* of the prism the triangle corresponding to the intersection with the hyperplane $x_3 = 0$, and call the *top* of the prism the triangle arising from the intersection with the hyperplane $x_3 = 1$. Figure 1 contains an illustration of this decomposition of the triangular prism (drawn to respect the orientation of the "bottom" and "top"). In the picture, the two square sides facing us are F and G respectively.

We can now take three copies of the triangular prism, and cyclically identify each F_i to the corresponding G_{i+1}. This gives a new triangulation of a triangular prism (with nine tetrahedra), with an inherited notion of "top" and "bottom". This new triangulation has the following key properties:

- there exists a unique edge e of the triangulation joining the center of the bottom triangle to the center of the top triangle,
- the center of the bottom triangle is adjacent to *every* vertex in the triangulation, and
- aside from the center of the bottom triangle, the center of the top triangle is adjacent to *no other* vertices in the bottom of the prism.

We will call a copy of this canonical triangulation of the triangular prism a *block*. Fixing an identification of $\mathbb{D}^2$ with the base of the triangular prism, we can think of a block as a triangulation of $\mathbb{D}^2 \times [0,1]$.

To obtain the desired triangulation of the solid torus $\mathbb{D}^2 \times S^1$, we "stack" four blocks together. More precisely, we take four blocks and cyclically identify the top of each block with the bottom of the next block. This gives us a triangulation of the solid torus $\mathbb{D}^2 \times S^1$ into thirty-six tetrahedra. We say blocks are *adjacent* or *opposite*, according to whether they share a vertex or not. Corresponding to the above properties for the individual blocks, this triangulation of the solid torus satisfies:

- the triangulation contains a canonical, unique square having the property that it is entirely contained within the *interior* of $\mathbb{D}^2 \times S^1$; the four vertices of this square will be called *interior vertices.*
- all the remaining vertices of the triangulation lie on the boundary of $\mathbb{D}^2 \times S^1$, and will be called *boundary vertices.*
- every tetrahedron in the triangulation contains at least one interior vertex.
- every interior vertex has the property that, if one looks at all adjacent boundary vertices, these vertices are all contained in single block (the unique block whose bottom contains the given interior vertex).

We call the unique square in the interior of this triangulation of $\mathbb{D}^2 \times S^1$ the *core* of the solid torus. Observe that, out of the thirty-six tetrahedra occuring in the triangulation, exactly twenty-four of them arise as the join of a triangle in $\partial\mathbb{D}^2 \times S^1$ with an interior vertex, while the remaining twelve occur as the join of an edge in $\partial\mathbb{D}^2 \times S^1$ with an edge in the core.

Step 2: Getting squares realizing the link k.

Next, let us take the desired link k, and take pairwise disjoint regular closed neighborhoods $\hat{N}_i$ of the individual components of the link. Each of these neighborhoods is homeomorphic to a solid torus, and we denote by $N_i \subset \hat{N}_i$ the slightly smaller solid torus of radius half as large. We proceed to construct a triangulation of S^3 as follows: first, within each of the tori N_i, we use the triangulation described in Step 1, identifying the components of the link with the cores of the various triangulated solid tori. Secondly, removing the interiors of all of the $\hat{N}_i$, we obtain a compact 3-manifold M with boundary $\partial M = \coprod \partial\hat{N}_i$. Since 3-manifolds are triangularizable, we now choose an arbitrary triangulation of this 3-manifold M, obtaining a triangulation of $M \cup \coprod N_i \subset S^3$. The closure of the complementary region is a disjoint union of the sets $\hat{N}_i \setminus N_i$, each of which is topologically a fattened torus $S^1 \times S^1 \times [0,1]$. Furthermore, we are given triangulations $\mathcal{T}_0, \mathcal{T}_1$ of the two boundaries $S^1 \times S^1 \times \{0\}$, $S^1 \times S^1 \times \{1\}$ (coming from the triangulations of ∂N_i

and ∂M respectively). But any two triangulations of the 2-torus $S^1 \times S^1$ have subdivisions which are simplicially isomorphic. Letting $\mathcal{T}'$ denote such a triangulation, we assign this triangulation on the level set $S^1 \times S^1 \times \{1/2\}$.

Finally, we extend the triangulation into the two regions $S^1 \times S^1 \times [0, 1/2]$ and $S^1 \times S^1 \times [1/2, 1]$ using the following procedure. On each of these two regions, we have a triangulation $\mathcal{T}_i$ on one of the boundary components, and a subdivision $\mathcal{T}'$ of the triangulation on the other boundary component. We proceed to inductively subdivide each of the regions $\sigma \times I$, where σ ranges over the simplices of the triangulation $\mathcal{T}_i$. First of all, we add in edges $\sigma^0 \times I$ for each vertex in the triangulation $\mathcal{T}_i$. Now assuming that we have already triangulated the product $\mathcal{T}_i^{(k-1)} \times I$ of the $(k-1)$-skeleton of $\mathcal{T}_i$ with the interval, let us extend the triangulation to $\mathcal{T}_i^k \times I$. Given a k-simplex σ^k, we have that the region $\sigma^k \times I$ is topologically a closed $(k+1)$-dimensional ball, with boundary that can be identified with $(\sigma^k \times \{0\}) \coprod (\sigma^k \times \{1\}) \coprod (\partial\sigma^k \times I)$. Furthermore, the bottom level consists of a simplex (the original $\sigma^k \in \mathcal{T}_i$), the top level consists of a subdivision of the simplex (the subdivision of σ^k inside $\mathcal{T}'$), and each of the faces have already been triangulated. In other words, we see that we have a topological $\mathbb{D}^{k+1}$, along with a given triangulation of $\partial\mathbb{D}^{k+1}$. But it is now easy to extend: just cone the given triangulation on the boundary inwards. Performing this process on each of the $\sigma^k \times I$ now provides us with a triangulation of the set $\mathcal{T}_i^k \times I$. This results in a triangulation of the 3-sphere with the following two properties:

- the triangulation contains a collection of squares, whose union realize the given link k,
- for each of the squares, the union of the simplices incident the the square form a regular neighborhood $\mathbb{D}^2 \times S^1$, triangulated as in Step 1, and
- all of these regular neighborhoods are pairwise disjoint.

Step 3: Getting rid of all other squares.

At this stage, we have constructed a triangulation of S^3, which contains a collection of squares realizing the given link k. However, there are still two problematic issues: our triangulation might not be flag, and it might fail the isolated squares condition. The third step is to modify the triangulation in order to ensure these two additional conditions. To fix some notation, we will keep using N_i to denote the regular neighborhood of the squares we are interested in keeping. Recall that each of these is topologically a solid torus $\mathbb{D}^2 \times S^1$, with triangulation combinatorially isomorphic to the triangulation given in Step 1. We will first modify the given triangulation in the complement of the N_i, and subsequently change it within the regions N_i.

Let us denote by X the closure of the complement of the union of the N_i. This is topologically a 3-manifold with boundary, equipped with a triangulation (from the previous two steps). Now the standard method of obtaining a flag triangulation is to take the barycentric subdivision of a given triangulation. But unfortunately, this process creates lots of squares. Recently, Przytycki and Świątkowski [PS], building on earlier work of Dranishnikov [Dr], have found a different subdivision process that takes a 3-dimensional simplicial complex and returns a subdivision of the complex that is flag *and has no squares.* For an arbitrary simplicial complex Z, we will denote by Z^* the simplicial complex obtained by applying this procedure to Z. We modify the given triangulation of S^3 in two stages: first we modify the triangulation

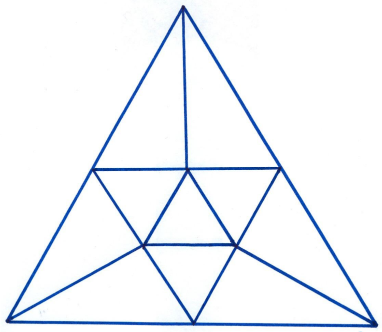

FIGURE 2. Dranishnikov subdivision of triangles.

in X, by replacing X by X^*. Next, we describe the extension of this triangulation into the various components N_i. For the original triangulation of each of the N_i, we see that the thirty-six tetrahedra are of one of two types:

(a) twenty-four of them are the join of one of the interior vertices with a triangle on ∂N_i, and
(b) twelve of them are the join of one of the four edges on the core square with an edge on ∂N_i.

Now the subdivision X^* restricts to a subdivision on each simplex in ∂N_i, which changes the simplicial complex ∂N_i into $(\partial N_i)^*$. The effect of this subdivision on simplices in ∂N_i is to subdivide each edge in ∂N_i into two, and to replace each original triangle by the subdivision in Figure 2. We extend the subdivision $(\partial N_i)^*$ of ∂N_i to a subdivision N_i' of the original N_i in the most natural way possible:

(a) each tetrahedron in N_i that was a join of an interior vertex with a triangle $\sigma \subset \partial N_i$ gets replaced by the join of the same vertex with σ^* (i.e. we cone the subdivision of σ to the interior vertex), subdividing the original tetrahedron into ten new tetrahedra (the cone over Figure 2), and
(b) each tetrahedron that was a join of an edge on the square with an edge on ∂N_i gets replaced by two tetrahedra (i.e. the join of the internal edge with each of the two edges obtained from subdividing the boundary edge).

This changes the original triangulation on each N_i into a new triangulation N_i' with a total of 264 tetrahedrons. We will continue to use the term *block* to refer to the subcomplexes of the N_i' that are subdivisions of the original blocks in N_i. Observe that, in each of the N_i, our subdivision process did not introduce any new vertices in the interior of the N_i. As such, the core squares have been left unchanged (and we will still refer to them as the cores of the N_i').

Finally, we note that by construction the two subdivisions N_i' of N_i, and X^* of X coincide on their common subcomplex $\partial N_i = N_i \cap X$. In particular, they glue together to give a well defined triangulation Σ of S^3.

Step 4: Verifying that Σ has the desired properties.

Note that the triangulation Σ contains a copy of X^*, as well as copies of each N_i'. These partition the triangulation Σ into various pieces.

Lemma 4. *The complex X^*, the individual N_i', and the intersections $X^* \cap N_i'$, are all full subcomplexes of Σ.*

Proof. This follows easily from the following two facts:

- each of the intersections $X^* \cap N_i' = (X \cap N_i)^*$ is a full subcomplex of X^*,
- each of the intersections $X^* \cap N_i' = \partial N_i'$ is a full subcomplex of the corresponding N_i'.

The first statement is a direct consequence of [PS, Lemma 2.10], where it is shown that if U is any subcomplex of W, then U^* is a full subcomplex of W^*. The second statement is a consequence of the construction of the triangulation N_i', since by construction, each simplex of N_i' which is *not* contained in $\partial N_i'$ contains a vertex in the interior of N_i' (and hence in $N_i' - \partial N_i'$). □

Lemma 5. *The triangulation N_i' is flag.*

Proof. Given a collection of pairwise incident vertices V, there are three possibilities: V contains either two, one, or no interior vertices of N_i'. We consider each of these three cases in turn.

If V contains no interior vertices, then $V \subset \partial N_i'$, and since the latter is a full subcomplex of N_i' (see Lemma 4), V is in fact a collection of vertices in $\partial N_i'$ which are pairwise adjacent *within* $\partial N_i'$. But recall that $\partial N_i'$ is just the triangulation $(\partial N_i)^*$, hence is flag. This implies that V spans out a simplex in $\partial N_i'$.

If V contains one interior vertex v, then, by the previous argument, $V - \{v\}$ spans a simplex in $\partial N_i' = (\partial N_i)^*$ which is contained within some (maximal) 2-dimensional simplex σ in $(\partial N_i)^*$. Note that, since all vertices $V - \{v\}$ are adjacent to the interior vertex v, they must lie in the block B corresponding to v. So the 2-dimensional simplex $\sigma \subset (\partial N_i)^*$ can additionally be chosen to lie within that same block B. This means that there exists a 2-dimensional simplex $\tau \in \partial N_i$ with the property that σ is one of the 10 triangles in τ^* (see Figure 2). Finally, observe τ must lie within the block B, so the join of τ with the interior vertex v defines a tetrahedron inside the original triangulation N_i (of type (a) in the terminology of Step 3). But recall how the subdivision $(\partial N_i)^*$ of the triangulation ∂N_i was extended into N_i: for tetrahedra of type (a), the subdivision on the boundary was coned off to the interior vertex. This implies that the join of σ and the vertex v defines a tetrahedron in N_i', and as the set V is a subset of the vertex set of this tetrahedron, we deduce that V spans a simplex in N_i'.

Finally, if V contains two interior vertices v, w, let B_v, B_w denote the corresponding blocks. Since $V - \{v, w\}$ is a collection of vertices in $\partial N_i' = (\partial N_i)^*$ which are adjacent to *both* interior vertices, we see that the set $V - \{v, w\}$ must lie within $B_v \cap B_w$, which is a 1-dimensional complex homeomorphic to S^1 (subdivided into 6 consecutive edges). Since $V - \{v, w\}$ are pairwise adjacent, there is an edge σ in $B_v \cap B_w$ whose vertex set contains $V - \{v, w\}$. This edge is contained in a subdivision of an edge τ from the original triangulation ∂N_i, where τ is an edge which is common to the two blocks B_v and B_w. In particular, the join $\omega * \tau$ of τ with the edge ω in the core joining v to w defines a tetrahedron in the original triangulation N_i (of type (b) in the terminology of Step 3). Again, from the way the subdivision $(\partial N_i)^*$ was extended inwards, we recall that the tetrahedra $\omega * \tau$, being of type (b), gets replaced by two tetrahedra $\omega * \sigma$ and $\omega * \sigma'$, where $\tau^* = \sigma \cup \sigma'$. Since the join of σ and ω defines a tetrahedron in N_i', and the set V is a subset of the vertex set of this tetrahedron, we again deduce that V spans a simplex in N_i'. □

Corollary 6. *The triangulation Σ is flag.*

Proof. If all of the vertices are contained in X^*, then the claim follows immediately from the fact that X^* itself is flag (see [PS, Proposition 2.13]). So we can now assume that at least one of the vertices is contained in the interior of one of the N_i'.

Note that an interior vertex in one of the N_i' has its closed star entirely contained within the same N_i'. So we see that the tuple of pairwise adjacent vertices must be entirely contained within the same subcomplex N_i'. But by Lemma 5, we have that each of the subdivided N_i' are themselves flag, finishing the proof. $\square$

Proposition 7. *The only squares in Σ are the cores of the various N_i'.*

Proof. To see this, let us start with an arbitrary square (v_1, v_2, v_3, v_4) inside the triangulation Σ. Our goal is to show that all four vertices must be interior vertices to a single N_i', which would then force the square to be the core of the corresponding N_i'. To this end, we first note that, if the square does *not* contain any interior vertex to any of the N_i', then it is contained entirely within X^*. But from Lemma 4, the latter is a full subcomplex of Σ, and by the result of Przytycki and Świątkowski [PS, Proposition 2.13], has no squares. So we may assume that at least one of the vertices is an interior vertex to some N_i'.

If all the vertices are interior to N_i', then we are done, so by way of contradiction we can also assume that the square contains a vertex which is *not* interior to N_i' (which we will call *exterior* vertices to N_i'). Now the square (v_1, v_2, v_3, v_4) contains exactly four edges, and since it contains vertices which are both interior and exterior to N_i', we must have that at least two of the four edges must connect an interior vertex to an exterior vertex (call these *intermediate* edges).

We now argue that in fact the square must contain *exactly* two intermediate edges. Indeed, if there were ≥ 3 intermediate edges, then one could find a pair of adjacent intermediate edges, which share a common exterior vertex. Up to cyclic relabeling, we may assume that v_1 is the exterior vertex. Considering the other endpoints of these two intermediate edges, we see that v_2, v_4 are interior vertices for N_i', which are both adjacent to the exterior vertex $v_1 \in \partial N_i'$. But this implies that the two blocks whose bottoms contain v_2 and v_4 cannot be opposite, so must in fact be adjacent. This forces v_2 and v_4 to be adjacent vertices in the core of N_i', contradicting the fact that (v_1, v_2, v_3, v_4) forms a square. So our hypothetical square (v_1, v_2, v_3, v_4) must have exactly two intermediate edges, leaving us with exactly two possibilities:

1. the intermediate edges are not adjacent in the square (v_1, v_2, v_3, v_4),
2. the intermediate edges are adjacent at an interior vertex of N_i', and the remaining edges are exterior.

We now explain why each of these possibilities give rise to a contradiction.

In case (1), we note that up to cyclic relabeling, we have that v_1, v_2 are adjacent vertices in the core of the N_i', while v_3, v_4 are adjacent vertices in $\partial N_i'$. We can also assume that the top of the block B_1 corresponding to v_1 attaches to the bottom of the block B_2 corresponding to v_2. Now recall that an interior vertex is *only adjacent to boundary vertices in its corresponding block.* Since v_3 is adjacent to v_2, we have that v_3 must lie in the block B_2. Similarly, the vertex v_4 being adjacent to v_1 must lie in the block B_1. Since v_3 and v_4 are adjacent, we conclude that one of these two vertices must lie in the common boundary $B_1 \cap B_2$. But such a vertex is incident to both v_1 and v_2, violating the square condition for (v_1, v_2, v_3, v_4).

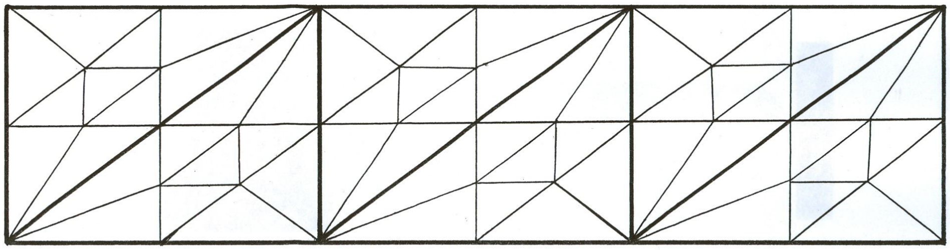

FIGURE 3. Triangulation on the boundary of a block.

It remains to rule out case (2). To this end, we may again assume that v_1 is the common interior vertex for the two intermediate edges. Now if B denotes the block corresponding to v_1, then we have that the boundary vertices v_2, v_4, both being adjacent to v_1, must actually lie in B. Moreover, for (v_1, v_2, v_3, v_4) to be a square, we must have that v_3 is **not** adjacent to v_1, and hence $v_3 \notin B$. Since v_3 is adjacent to both the vertices $v_2, v_4 \in B$, we see that the latter are either both in the top of B or both in the bottom of B, while v_3 lies in an adjacent block B'. Let us assume that the vertices lie in the top of B (the other case being completely analogous), so that we can view v_2, v_4 as lying in the *bottom* of the block B'.

We now have the following situation occurring inside the boundary of the block B': we have two vertices v_2, v_4 lying in the bottom of the block, and we have a vertex v_3 which does **not** lie in the bottom of B', but which is adjacent to both v_2 and v_4. Now recall that the triangulation of the block B' is a subdivision (given in Step 3) of a canonical triangulation of the triangular prism. This subdivision takes the boundary of the original triangulation and applies the Dranishnikov subdivision procedure to it: each edge gets subdivided into two, and each triangle gets replaced by the subdivision in Figure 2. The resulting triangulation on $S^1 \times [0,1]$ is shown in Figure 3. In the illustration, the left and right side of the rectangle have to be identified, and the "bottom" and "top" of the boundary of the block is precisely the bottom and the top of the rectangle. Note that this triangulation actually consists of six original triangles (see Step 1), each of which has been subdivided into 10 triangles as in Figure 2 (see Step 3). Finally, inspecting the triangulation in Figure 3, we observe that there are exactly six vertices which are adjacent to two distinct vertices in the bottom of the block: these are the only possibilities for v_3. But for each of these six vertices, we see that the two adjacent vertices in the bottom of the block (i.e. the corresponding v_2 and v_4) are adjacent to each other, contradicting the fact that (v_1, v_2, v_3, v_4) was a square.

Since we've ruled out all other possibilities, we see that the square cannot contain *any* intermediate edges, i.e. the four vertices of our hypothetical square (v_1, v_2, v_3, v_4) must all lie in the interior of a single N_i'. This implies that our square must coincide with the core of one of the N_i', as desired.

□

It follows from Corollary 6 that the triangulation Σ is flag, and from Proposition 7 that it has isolated squares with type given by the original link k. This completes the proof of Theorem 3.

4. Constructing the manifold.

In this section, we establish the Main Theorem. Our goal is to use some of the triangulations of S^3 constructed in the previous section to produce a 4-dimensional manifold M with the desired properties. In order to do this, we start by reviewing some properties of the Davis complex for right angled Coxeter groups.

Recall that one can associate to the 1-skeleton of *any* simplicial complex L a corresponding *right angled Coxeter group* Γ_L. This group has one generator x_i of order two for each vertex v_i of the simplicial complex L, and a relation $x_ix_j = x_jx_i$ whenever the corresponding vertices v_i, v_j are adjacent in L. Let us consider the associated Davis complex $\tilde{P}_L$. This complex is obtained via the following procedure: we first consider the cubical complex $[-1,1]^{V(L)}$, that is to say, the standard cube with dimension equaling the number of vertices in the simplicial complex L. Now every face of the cube is an affine translation of $[-1,1]^S$ for some subset $S \subset V(L)$, which we call the *type* of the face. Consider the cubical subcomplex $P_L \subset [-1,1]^{V(L)}$ consisting of all faces whose type defines a simplex in L, and let $\tilde{P}_L$ to be its universal cover. Observe that the Coxeter group Γ_L acts on P_L, where each generator x_i acts by reflection on the corresponding coordinate. The kernel of the resulting morphism $\Gamma_L \to (\mathbb{Z}_2)^{|V(L)|}$ coincides with the fundamental group of P_L. There is a natural piecewise flat metric on P_L, obtained by making each k-dimensional face in the cubulation of P_L isometric to $[-1,1]^k \subset \mathbb{R}^k$. Properties of the cubical complex P_L are intimately related to properties of the simplicial complex L. For instance, we have:

(a) if L is a flag complex, then the piecewise flat metric on P_L is locally CAT(0),
(b) the links of vertices in P_L are canonically simplicially isomorphic to L,
(c) if L is the join of two subcomplexes L_1, L_2, then the space P_L splits isometrically as a product of P_{L_1} and P_{L_2},
(d) if L' is a full subcomplex of L, then the natural inclusion induces a totally geodesic embedding $P_{L'} \hookrightarrow P_L$,
(e) if the geometric realization of L is homeomorphic to an $(n-1)$-dimensional sphere, then P_L is an n-dimensional manifold,
(f) if L is a PL-triangulation of S^{n-1} then P_L is a PL-manifold, and $\partial^\infty \tilde{P}_L$ is homeomorphic to S^{n-1},
(g) if L is a *smooth* triangulation of S^{n-1}, then P_L is a smooth manifold.

These results are discussed in detail in the book [Da1].

In the previous section, we showed that given a prescribed link k in S^3, one can construct a triangulation of S^3 with isolated squares, and with type the given link. Let us apply this result in the special case where k is a nontrivial knot inside S^3. Let L denote the corresponding triangulation of S^3. Since we are in the special case of dimension = 3, the triangulation L, in addition to being flag, is automatically PL and smooth. We now consider the cubical complex $M := P_L$ associated to the corresponding right angled Coxeter group Γ_L. In view of our earlier discussion, we have the following:

Fact 1: The space M is a smooth 4-manifold (from (g) above), and the natural piecewise Euclidean metric on M induced from the cubulation is locally CAT(0) (from (a) above). Furthermore, the boundary at infinity of $\tilde{M}$ is homeomorphic to S^3 (from (f) above), and $\tilde{M}$ is diffeomorphic to $\mathbb{R}^4$.

The very last statement in **Fact 1** can be deduced from work of Stone (see [St, Theorem 1]), who showed that a metric (piecewise flat) polyhedral complex which is both CAT(0) and a PL-manifold without boundary must in fact be PL-homeomorphic to the appropriate $\mathbb{R}^n$. Since our $\tilde{M}$ satisfies these conditions, this ensures that $\tilde{M}$ is PL-homeomorphic to the standard $\mathbb{R}^4$. But in the 4-dimensional setting, there is no difference between PL and smooth, so $\tilde{M}$ is in fact diffeomorphic to $\mathbb{R}^4$.

Our goal is now to show that M has the properties postulated in our Main Theorem. Note that properties (1) and (2) are included in **Fact 1**, while property (4) can be easily deduced from property (3) (see the comment after the proof of Proposition 1). So we are left with establishing property (3): that $\pi_1(M)$ cannot be isomorphic to the fundamental group of any nonpositively curved Riemannian manifold. This last property will be established by looking at the large scale geometry of flats inside the universal cover $\tilde{M}$.

As a starting point, let us describe some flats inside $\tilde{M}$. Observe that each square inside the triangulation L is a full subcomplex isomorphic to a 4-cycle $\square$. The right angled Coxeter group associated to a 4-cycle is a direct product of two infinite dihedral groups $\Gamma_\square \cong D_\infty \times D_\infty = (\mathbb{Z}_2 * \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2)$ (see (c) above). The corresponding complex $P_\square$ is isometric to a flat torus (with cubulation given by 16 squares, obtained via the identification $S^1 \times S^1 = \square \times \square$). By considering the unique square inside the triangulation L, we obtain:

Fact 2: M contains a totally geodesic 2-dimensional flat torus T^2 (see (d) above). Furthermore, at any vertex $v \in T^2 \subset M$ of the cubulation, we have that the torus T^2 is *locally knotted* inside the ambient 4-dimensional manifold M (see (b) above), in that there is a canonical simplicial isomorphism $\big(lk_v(M), lk_v(T^2)\big) \cong (L, k)$ where k is the unique (knotted) square in the triangulation L.

Since the embedding $T^2 \hookrightarrow M$ is totally geodesic, by lifting to the universal cover, we obtain a 2-dimensional flat $F \hookrightarrow \tilde{M}$ which is locally knotted at lifts of vertices. This induces an embedding of the corresponding boundaries at infinity, giving us an embedding of $\partial^\infty F \cong S^1$ into $\partial^\infty \tilde{M} \cong S^3$. The rest of our argument will rely on the following "local-to-global" assertion:

Assertion: The embedding $\partial^\infty F \cong S^1$ into $\partial^\infty \tilde{M} \cong S^3$ defines a nontrivial knot in the boundary at infinity of $\tilde{M}$.

That is to say, the "local knottedness" of the flat propagates to "global knottedness" of its boundary at infinity. For the sake of exposition, we delay the proof of the assertion, and first show how we can use it to deduce the Main Theorem. To this end, let us assume that (M', g) is a closed manifold equipped with a Riemannian metric of nonpositive sectional curvature, and that we are given an isomorphism of fundamental groups $\phi : \Gamma = \pi_1(M) \to \pi_1(M')$. From this assumption, we want to work towards a contradiction.

The first step is to use the isomorphism of fundamental groups to obtain an equivariant homeomorphism between the corresponding boundaries at infinity. As a cautionary remark, we recall that given a pair X_1, X_2 of CAT(0)-spaces with geometric G-actions, a celebrated example of Croke and Kleiner [CK] shows that the corresponding boundaries at infinity $\partial^\infty X_1$ and $\partial^\infty X_2$ need **not** be homeomorphic. Even if the boundaries at infinity *are* homeomorphic, an example of Buyalo [Bu] shows that the homeomorphism might **not** be equivariant with respect to the G-action.

In his thesis [H], Hruska introduced CAT(0)-spaces with *isolated flats*. Subsequent work of Hruska and Kleiner [HK] established the following two foundational results for CAT(0)-spaces with isolated flats:

(1) for a pair X_1, X_2 of CAT(0)-spaces with geometric G-actions, if X_1 has isolated flats, then so does X_2 (see [HK, Corollary 4.1.3]), and there is a G-equivariant homeomorphism between $\partial^\infty X_1$ and $\partial^\infty X_2$ (see [HK, Theorem 4.1.8]).
(2) for a group G acting geometrically on a CAT(0)-space X, we have that X has the isolated flats property if and only if G is a relatively hyperbolic group with respect to a collection of virtually abelian subgroups of rank ≥ 2 (see [HK, Theorem 1.2.1]).

As such, if we could establish that our *group* Γ is a relatively hyperbolic group with respect to a collection of virtually abelian subgroups of rank ≥ 2, then result (2) above would ensure that our CAT(0)-manifold $\tilde{M}$ has the isolated flats property. Result (1) above would then give the desired Γ-equivariant homeomorphism between $\partial^\infty \tilde{M}$ and $\partial^\infty \tilde{M}'$. So our next goal is to establish:

Fact 3: The group $\Gamma = \pi_1(M)$ is hyperbolic relative to the collection of all virtually abelian subgroups of Γ of rank ≥ 2.

The notion of a group G being relatively hyperbolic with respect to a collection $\mathcal{A}$ of subgroups of G was originally suggested by Gromov [Gr], whose approach was later formalized by Bowditch [Bo]. Alternate formulations appear in Farb's thesis [Fa], in work of Druţu and Sapir [DrSa], and in the memoir of Osin [Os]. We refer the reader to the original sources for a detailed definition as well as basic properties of such groups. For our purposes, we merely need to know that the property of a group G being hyperbolic relative to a collection of virtually abelian subgroups of rank ≥ 2 is inherited by finite index subgroups of G. In particular, to show the desired property for Γ, we see that it is sufficient to establish that our original Coxeter group Γ_L is relatively hyperbolic with respect to higher rank virtually abelian subgroups (since $\Gamma \leq \Gamma_L$ is of finite index).

Caprace [Ca, Cor. D (ii)] recently provided a criterion for deciding whether a Coxeter group is hyperbolic relative to the collection of its higher rank virtually abelian subgroups. In the right-angled case the condition is that the flag complex L which defines Γ_L contains no full subcomplex isomorphic to the suspension ΣK of a subcomplex K with 3 vertices which is either

(a) the disjoint union of 3 points, or
(b) the disjoint union of an edge and 1 point.

In both cases ΣK does not have isolated squares. Since the Coxeter group Γ_L with which we are working is associated to a triangulation L of S^3 with isolated squares, we conclude that Γ_L is relatively hyperbolic with respect to the collection of all virtually abelian subgroups of rank ≥ 2. Hence, **Fact 3**.

Applying Hruska and Kleiner's results from [HK], we conclude that the original $\tilde{M}$ is a CAT(0)-space with the isolated flats property, and that there exists a Γ-equivariant homeomorphism from $\partial^\infty \tilde{M}$ to $\partial^\infty \tilde{M}'$. The nontrivial knot $\partial^\infty F \cong S^1$ inside $\partial^\infty \tilde{M} \cong S^3$ appearing in the **Assertion** can be identified with the limit set of the corresponding subgroup $\pi_1(T^2) \cong \mathbb{Z}^2 \leq \Gamma = \pi_1(M)$. Since we have an equivariant homeomorphism between the boundaries at infinity of $\tilde{M}$ and $\tilde{M}'$, this immediately yields:

Fact 4: The boundary at infinity $\partial^\infty \tilde{M}'$ is homeomorphic to S^3, and the limit set of the canonical $\mathbb{Z}^2$-subgroup in $\Gamma \cong \pi_1(M')$ defines a nontrivial knot $S^1 \hookrightarrow \partial^\infty \tilde{M}' \cong S^3$.

On the other hand, the flat torus theorem implies that there exists a $\mathbb{Z}^2$-periodic flat $F' \hookrightarrow \tilde{M}'$, with the property that $\partial^\infty F'$ coincides with the limit set of the $\mathbb{Z}^2$. In particular, $\partial^\infty F'$ defines a nontrivial knot inside $\partial^\infty M'$. But taking any point $p \in F'$, we note that geodesic retraction provides a homeomorphism $\rho : \partial^\infty \tilde{M}' \to T_p\tilde{M}'$. This homeomorphism takes the knotted subset $\partial^\infty F'$ lying inside $S^3 \cong \partial^\infty M'$ to the *unknotted* subset T_pF' lying inside $S^3 \cong T_p\tilde{M}'$. This contradiction allows us to conclude that no such Riemannian manifold (M', g) can exist.

So in order to complete the proof of the Main Theorem, we are left with establishing the **Assertion**. We note that a similar result was shown in the setting of CAT(-1)-manifolds by Farrell and Lafont [FL], the proof of which extends almost verbatim to yield the **Assertion**. For the convenience of the reader, we provide a (slightly different) self-contained argument for the **Assertion**.

The basic idea is as follows: picking a vertex $v \in F$, we have a geodesic retraction map $\rho : \partial^\infty \tilde{M} \to lk_v(\tilde{M})$. Under this map, we see that $\partial^\infty F$ maps to the link $lk_v(F)$ inside $lk_v(\tilde{M})$. But recall from **Fact 2** that the torus is locally knotted in $\tilde{M}$, i.e. the pair $\big(lk_v(\tilde{M}), lk_v(F)\big)$ is simplicially isomorphic to (S^3, k), where S^3 is the 3-sphere equipped with the triangulation L, and k is the knot in S^3 given by the unique square in the triangulation L. Now the retraction map ρ is *not* a homeomorphism, but is nevertheless "close enough" to a homeomorphism for us to use it to compare the pair $\big(\partial^\infty \tilde{M}, \partial^\infty F\big)$ with the knotted pair $\big(lk_v(\tilde{M}), lk_v(F)\big) \cong (S^3, k)$. More precisely, for any given subset $Z \subset lk_v(\tilde{M}) \cong S^3$ we denote by Z_∞ the corresponding pre-image $Z_\infty := \rho^{-1}(Z)$ inside $\partial^\infty \tilde{M}$. Then we have:

Fact 5: [FL, Proposition 2, pg. 627] For any open set $U \subset lk_v(\tilde{M})$, the map $\rho : U_\infty \to U$ is a proper homotopy equivalence. Moreover, the map ρ is a *near-homeomorphism*, i.e. can be approximated arbitrarily closely by homeomorphisms.

This is shown by identifying U_∞ with the inverse limit of the sets $\{U_r\}_{r\in\mathbb{R}^+}$, where each U_r is the pre-image of U under the geodesic projection from the sphere $S_v(r)$ of radius r centered at v to the link at v. For $r > s$, the bonding maps $\rho_{r,s} : U_r \to U_s$ are given by geodesic retraction, and the canonical map $\rho_{\infty,s}$ from $U_\infty = \varprojlim\{U_r\}$ to each individual U_s coincides with the geodesic retraction map.

Since the link $lk_v(\tilde{M})$ can be identified with $S_\epsilon(r)$, a small enough ϵ-sphere centered at v, the map ρ can be identified with the canonical map $\rho_{\infty,\epsilon}$ from $U_\infty = \varprojlim\{U_r\}$ to the corresponding $U_\epsilon = U$. Now by results of Davis and Januszkiewicz [DJ, Section 3] each of the bonding maps $\rho_{r,s}$ are cell-like maps, i.e. point pre-images have the shape of a point (see Dydak and Segal [DySe] for background on shape theory). Since the shape functor commutes with inverse limits, and since $\rho = \rho_{\infty,\epsilon}$, we see that ρ is also a cell-like map. A result of Edwards [Ed, Section 4] now implies that ρ is a proper homotopy equivalence, while work of Armentrout [Ar] ensures that ρ is a near homeomorphism.

Now to show that $\partial^\infty F$ defines a nontrivial knot in $\partial^\infty \tilde{M}$, we need to establish that the complement $\partial^\infty \tilde{M} - \partial^\infty F$ cannot be homeomorphic to $S^1 \times \mathbb{R}^2$. This will follow if we can show that $\pi_1\big(\partial^\infty \tilde{M} - \partial^\infty F\big)$ is a non-abelian group. To do this, let us decompose $\partial^\infty \tilde{M} - \partial^\infty F$ into a union of a suitable pair of open sets. We start by decomposing $lk_v(\tilde{M})$, and will then use the map ρ to "lift" this decomposition to $\partial^\infty \tilde{M}$. Let $lk_v(F) \subset N_1 \subset N_2 \subset lk_v(\tilde{M})$ be nested open regular neighborhoods of the knot $k = lk_v(F)$ inside $S^3 \cong lk_v(\tilde{M})$. Define open sets in $lk_v(\tilde{M})$ by setting $U_2 := N_2$, and $U_1 := lk_v(\tilde{M}) - \bar{N}_1$, where $\bar{N}_1$ denotes the closure of N_1. Note that we have homeomorphisms $U_2 \cong S_1 \times \mathbb{D}^2$ and $U_1 \cap U_2 \cong N_2 - \bar{N}_1 \cong S^1 \times S^1 \times \mathbb{R}$, while U_1 is homeomorphic to the complement of the nontrivial knot $k \subset S^3$. So at the level of π_1, we have that (a) $\pi_1(U_1 \cap U_2) \cong \mathbb{Z} \oplus \mathbb{Z}$, and (b) $\pi_1(U_1)$ is a non-abelian group. The latter fact follows from work of Papakyriakopoulos [Pa], who showed that π_1 of the complement of a nontrivial knot cannot be isomorphic to $\mathbb{Z}$. But by Alexander duality such a group must have abelianization isomorphic to $\mathbb{Z}$, hence cannot be abelian.

Now corresponding to this decomposition of $lk_v(\tilde{M})$, we have an associated open decomposition of $\partial^\infty \tilde{M}$ in terms of the corresponding $(U_1)_\infty$, $(U_2)_\infty$. We now define an open decomposition of $\partial^\infty \tilde{M} - \partial^\infty F$ by setting $U := (U_1)_\infty$ and $V := (U_2)_\infty - \partial^\infty F$. The intersection satisfies $U \cap V = (U_1 \cap U_2)_\infty$. Applying **Fact 5** to the discussion in the previous paragraph, we obtain that (a) $\pi_1(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z}$, and (b) $\pi_1(U)$ is non-abelian. From Seifert-Van Kampen, we have:

$$\pi_1\big(\partial^\infty \tilde{M} - \partial^\infty F\big) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

So to see that $\pi_1\big(\partial^\infty \tilde{M} - \partial^\infty F\big)$ is non-abelian, it suffices to show that the non-abelian group $\pi_1(U)$ injects into the amalgamation. But this will follow from:

Fact 6: The map $i_* : \pi_1(U \cap V) \to \pi_1(V)$ induced by inclusion is injective.

To establish **Fact 6**, we first choose a suitable basis for $\pi_1(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z}$. Recall that the map ρ gives a proper homotopy equivalence between $U \cap V = (U_1 \cap U_2)_\infty$ and the space $U_1 \cap U_2 = N_2 - \bar{N}_1$, where $N_1 \subset N_2$ are nested open regular neighborhoods of the knot k. Since $U_2 = N_2$ can be identified with $S^1 \times \mathbb{D}^2$, where $S^1 \times \{0\}$ corresponds to the knot k, we choose the generators for $\pi_1(N_2 - \bar{N}_1) \cong \mathbb{Z} \oplus \mathbb{Z}$ to have the following two properties:

- **(A)** the generator $\langle 1, 0\rangle$ maps to a generator represented by $[S^1 \times \{0\}] \in \pi_1(N_2) \cong \mathbb{Z}$ under the obvious inclusion, and

(B) the generator $\langle 0,1\rangle$ is chosen so that a representative curve exists which, under the natural inclusion into $N_2 \cong S^1 \times \mathbb{D}^2$, projects to a generator for $\pi_1(\mathbb{D}^2 - \{0\}) \cong \mathbb{Z}$ in the $\mathbb{D}^2$-factor, and is null-homotopic in N_2.

We choose the generators of $\pi_1(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z}$ to map to the above two generators of $\pi_1(U_1 \cap U_2)$ under the homotopy equivalence ρ.

To verify that $i_* : \pi_1(U \cap V) \to \pi_1(V)$ is injective, we first argue that an element $\langle a, b\rangle \in \ker(i_*)$ must satisfy $a = 0$. Consider the commutative diagram:

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(U \cap V) \arrow[r, "i_*"] \arrow[d, "\rho_*"] & \pi_1(V) \arrow[r] & \pi_1\big((N_2)_\infty\big) \cong \mathbb{Z} \arrow[d, "\rho_*"] \\
\mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(U_1 \cap U_2) \arrow[rr] & & \pi_1(N_2) \cong \mathbb{Z}
\end{tikzcd}$$

where all horizontal arrows are induced by the obvious inclusions, and the two vertical arrows are the isomorphisms induced by the geodesic retraction maps. By the choice of the basis on $\pi_1(U \cap V)$, we have that $\rho_*(\langle a, b\rangle) = \langle a, b\rangle \in \pi_1(U_1 \cap U_2)$, which by property **(A)** maps to $a \in \mathbb{Z} \cong \pi_1(N_2)$. From the commutativity of the diagram, we conclude that if $\langle a, b\rangle \in \ker(i_*)$, then $a = 0$. Our next goal is to show that $b = 0$.

Given a pair η_1, η_2 of disjoint oriented curves in $S^1 \times \mathbb{D}^2$, with η_1 null-homotopic, there is a well-defined linking number $L(\eta_1, \eta_2)$. For smooth curves this is obtained by looking at the oriented intersection number of η_2 with a smooth bounding disk for the curve η_1, and for continuous curves one uses an approximation by smooth curves. This linking number has the property that if $\eta_1 \sim \eta_1'$ (respectively $\eta_2 \sim \eta_2'$) are two curves homotopic to each other *in the complement of* η_2 (respectively η_1), then $L(\eta_1, \eta_2') = L(\eta_1, \eta_2) = L(\eta_1', \eta_2)$.

Now from the choice of basis on $\pi_1(U \cap V)$, along with property **(B)**, we can choose a representative curve γ for the element $\langle 0, b\rangle \in \ker(i_*) \subset \pi_1(U \cap V)$ with the property that the image curve $\rho(\gamma) \subset U_1 \cap U_2 \subset N_2 \cong S^1 \times \mathbb{D}^2$ projects to b times a generator for $\pi_1(\mathbb{D}^2 - \{0\})$. One can easily check that this forces $L\big(\rho(\gamma), S^1 \times \{0\}\big) = \pm b$. Applying **Fact 5**, we can find a homeomorphism $\rho' : (N_2)_\infty \to N_2$ which is ϵ-close to the map ρ. In view of the discussion above, and recalling that the curve $S^1 \times \{0\}$ corresponds to $lk_v(F) = \rho(\partial^\infty F)$, this gives us that:

$$\begin{aligned}
\pm b &= L\big(\rho(\gamma), S^1 \times \{0\}\big) = L\big(\rho(\gamma), \rho(\partial^\infty F)\big) \\
&= L\big(\rho'(\gamma), \rho(\partial^\infty F)\big) = L\big(\rho'(\gamma), \rho'(\partial^\infty F)\big) = L\big(\gamma, \partial^\infty F\big)
\end{aligned}$$

where for the last equality, we use the fact that ρ' is a homeomorphism, and hence preserves the linking number. But since $\langle 0, b\rangle \in \ker(i_*)$, we also have that γ bounds a disk in $V = (N_2)_\infty - \partial^\infty F$, which implies that $L\big(\gamma, \partial^\infty F\big) = 0$. This now forces $b = 0$, completing the proof of **Fact 6**.

Since the non-abelian group $\pi_1(U)$ injects into $\pi_1\big(\partial^\infty \tilde{M} - \partial^\infty F\big)$, we obtain that $\partial^\infty F \cong S^1$ defines a nontrivial knot in $\partial^\infty \tilde{M} \cong S^3$, establishing the **Assertion**, and finishing off the proof of the Main Theorem.

5. Concluding remarks.

Finally, we point out a few interesting questions that come up naturally from this work. As discussed in Section 2.2, locally CAT(0)-manifolds whose universal covers are **not** diffeomorphic to $\mathbb{R}^n$ cannot support a Riemannian smoothing. In dimensions $n \neq 4$, there is no difference between "homeomorphic to $\mathbb{R}^n$" and "diffeomorphic to $\mathbb{R}^n$". In contrast, it is known that $\mathbb{R}^4$ supports many distinct smooth structures (in fact, continuum many). Moreover, the method used to construct the Davis examples of closed aspherical manifolds whose universal covers are not homeomorphic to $\mathbb{R}^n$ requires $n \geq 5$. So one can ask:

Question: Can one find locally CAT(0) closed 4-manifolds M^4 with the property that their universal covers $\tilde{M}^4$ are

(1) not homeomorphic to $\mathbb{R}^4$?
(2) homeomorphic, but not diffeomorphic to $\mathbb{R}^4$?

Paul Thurston [Th] proved that $\tilde{M}^4$ must be homeomorphic to $\mathbb{R}^4$ if it has at least one "tame" point. We remark that the result of Stone [St] tells us that there is no hope of constructing such examples via piecewise flat metric complexes (for their universal covers would then have to be diffeomorphic to the standard $\mathbb{R}^4$). Moreover, if one asks instead for *aspherical* closed 4-manifolds, we remark that Davis [Da2] has constructed examples where the universal cover is *not* homeomorphic to $\mathbb{R}^4$ (but it is unknown whether those examples support a locally CAT(0)-metric).

Now concerning the dimension restriction in our construction, we note that this was due to the need for finding triangulations of spheres with the property that the associated Davis complex had the isolated flats condition (in order to obtain a well-defined boundary at infinity). The "isolated squares" condition we introduced was designed to ensure that Caprace's criterion was fulfilled. Attempting to generalize this construction to higher dimensions, the difficulty we run into is that, by work of Januszkiewicz and Świątkowski [JS, Section 2.2] (see also the discussion in [PS, Appendix]), there is no higher-dimensional analogue of the Dranishnikov-Przytycki-Świątkowski procedure for modifying triangulations in order to get rid of squares.

Finally, we remark that our construction relies on the presence of flats with specific large scale behavior in order to obstruct Riemannian smoothings. As such, our methods require the presence of zero curvature. If one desires examples which are *strictly negatively curved*, we are brought to the following:

Question: Can one construct examples of smooth, locally CAT(-1)-manifolds M^n with the property that $\partial^\infty \tilde{M}$ is homeomorphic to S^{n-1}, but which do *not* support any Riemannian metric of nonpositive sectional curvature?

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