ON IRREDUCIBLE, INFINITE, NON-AFFINE COXETER GROUPS

DISSERTATION

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By

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ABSTRACT

Coxeter groups arise in many parts of mathematics as groups generated by reflections. They provide an important source of examples in geometric group theory, where "virtual" properties of infinite groups, that is, properties of subgroups of finite index, are of strong interest. This dissertation focuses on virtual properties of infinite Coxeter groups.

The following results are proved: (1) The intersection of a collection of parabolic subgroups of a Coxeter group is a parabolic subgroup; (2) The center of any finite index subgroup of an irreducible, infinite, non-affine Coxeter group is trivial; (3) Any finite index subgroup of an irreducible, infinite, non-affine Coxeter group cannot be expressed as a product of two nontrivial subgroups. Then, a unique decomposition theorem for subgroups of finite index in a Coxeter group without spherical or affine factors is proved based on (2) and (3). It is also proved that the orbit of each element other than the identity under the conjugation action in an irreducible, infinite, non-affine Coxeter group is an infinite set, which implies that an irreducible, infinite Coxeter group is affine if and only if it contains an abelian subgroup of finite index.

Besides these, new proofs are given for the statement that the center of an irreducible, infinite Coxeter group is trivial, and a stronger version of this statement.

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Key Words: root system, canonical representations, irreducible Coxeter groups, parabolic subgroup, essential element, CAT(0) space, flat torus theorem, solvable subgroup theorem

To my family

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Topic 2	Coxeter Groups
Topic 3	Bifurcations of Vector Fields

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CHAPTER 1

INTRODUCTION

Geometric group theory uses topological and geometric methods to study infinite discrete groups by considering appropriate group actions on topological or metric spaces. The theory of groups of isometries on metric spaces of non-positive curvature produces abundant results and clues in this direction. Properties of isometry groups reveal some interesting structures of the spaces they act on. The Flat Torus Theorem and the Solvable Subgroup Theorem discovered by Gromoll and Wolf [14], and independently, by Lawson and Yau [19], are good illustrations of these two-sided relations.

Coxeter groups arise in many parts of mathematics as groups generated by reflections, especially, from the study of semisimple Lie theory. They provide an important source of examples in geometric group theory, where "virtual" properties of infinite groups, that is, properties of subgroups of finite index, are of great interest.

Irreducible Coxeter groups are the basic blocks of Coxeter groups since any Coxeter group can be expressed as a direct product of its irreducible components. Paris [22] proved that irreducible, infinite Coxeter groups cannot be further decomposed. So it is natural to speculate that the same holds for subgroups of finite index. But this is obviously false for irreducible, infinite, affine Coxeter groups. Here we prove that it is true for irreducible, infinite, non-affine Coxeter groups. In addition, we investigate the "virtual center", that is, the center of a finite index subgroup, of an irreducible, infinite, non-affine Coxeter group and prove that it is trivial. It turns out that these two questions are closely related, and they imply a unique decomposition theorem for subgroups of finite index in a Coxeter group without spherical or affine factors. Our method can be used to obtain another interesting property of irreducible, infinite, non-affine Coxeter groups: the orbit of each element other than the identity under the conjugation action is an infinite set.

As defined in Bourbaki [2], a *Coxeter system* (W, S) is a group W and a set S of generators such that W has a presentation of the form,

$$W = \langle\ S|(st)^{m_{st}} = 1,\ \ s, t \in S\rangle, \tag{1.1}$$

where $m_{st} = m_{ts}$ is a positive integer or ∞, and $m_{st} = 1$ if and only if $s = t$ (a relation $(st)^\infty = 1$ is interpreted as vacuous). W is called a *Coxeter group*. The cardinality $|S|$ of S is called the *rank* of W. We are mainly interested in Coxeter groups of finite rank. So, we assume $|S|$ is finite in this dissertation.

For a Coxeter system (W, S), its *Coxeter graph* is a graph with vertex set S, and with two vertices $s \neq t$ joined by an edge whenever $m_{st} \geq 3$. If $m_{st} \geq 4$, the corresponding edge is labeled by m_{st}. We say that a Coxeter group (W, S) is *irreducible* if its Coxeter graph is connected.

Associated to a Coxeter group (W, S), there is a symmetric bilinear form on a real vector space V, having a basis $\{\alpha_s | s \in S\}$ in one-to-one correspondence with S. The bilinear form $(\cdot, \cdot)$ is defined by setting

$$(\alpha_s, \alpha_t) = -\cos\frac{\pi}{m_{st}}. \tag{1.2}$$

The value on the right-hand side is interpreted to be -1 when $m_{st} = \infty$.

A well-known fact is that a Coxeter group W is finite if and only if its bilinear form is positive definite. We call W a *spherical* Coxeter group in this case.

It is stated in [2, p.137] that an irreducible, infinite Coxeter group has a trivial center, and a proof using the canonical representations of a Coxeter group, developed by J. Tits (see [2, 17]), is suggested.

If the bilinear form of an irreducible Coxeter group (W, S) is positive semi-definite but not positive definite, then $W = \mathrm{Z}^n \rtimes W_0$, where W_0 is a finite Coxeter group and $n = |S| - 1$. We call W an *irreducible, infinite, affine* Coxeter group in this situation. A natural and interesting question, which was proposed to the author by M. Davis and T. Januszkiewicz, is to determine if the center of a finite index subgroup of an *irreducible, infinite, non-affine* Coxeter group is trivial. By "*non-affine*" we mean its bilinear form is neither positive definite nor positive semi-definite. The answer is yes.

Theorem 1.1. *The center of any finite index subgroup of an irreducible, infinite, non-affine Coxeter group is trivial.*

The solution of this question was inspired by a preprint of L. Paris (an early version of [22]). In that paper, by studying the essential elements (which will be defined in Chapter 4) of a Coxeter group, Paris obtained several interesting results on irreducible Coxeter groups. One of them is that any irreducible, infinite Coxeter group cannot be written as a product of two nontrivial subgroups. This paper also brought to the author's attention D. Krammer's thesis [18].

The idea of studying essential elements (Krammer [18], Paris [22]) is important in the proof of Theorem 1.1. In addition, the author makes use of some arguments similar to those in the proofs of the Flat Torus Theorem and the Solvable Subgroup

Theorem of CAT(0) spaces. For a detailed description of CAT(0) spaces, the reader is referred to [3]. We will explain briefly in Section 2 a geometric construction associated to a Coxeter system (W, S) (see [6], [7]), which yields a PE cell complex $\Sigma = \Sigma(W, S)$ (here PE stands for "piecewise Euclidean"), now commonly called the *Davis complex*. It is proved by G. Moussong [21] that Σ is a CAT(0) space and W acts properly and cocompactly on Σ by isometries.

The proof of Theorem 1.1 also relies on the general theory of root system of a Coxeter group (see Bourbaki [2], Deodhar [11] and Krammer [18]). Deodhar [12] and M. Dyer [13] independently proved a theorem, which says that any subgroup generated by a collection of reflections in a Coxeter group is itself a Coxeter group. This theorem also plays an important role in the proof.

Using some arguments similar to the proof of Theorem 1.1, we obtain the following.

Theorem 1.2. *Any finite index subgroup of an irreducible, infinite, non-affine Coxeter group cannot be expressed as a product of two nontrivial subgroups.*

After proving this, the author discovered that in a revised version (version 2) of [22], Paris extended his discussions to include the conclusions of Theorem 1.1 and 1.2 using purely algebraic arguments. It appears that we both realized the necessity of using the reflection subgroup theorem obtained by Deodhar and Dyer to achieve this aim.

A recent work of de Cornulier and de la Harpe [5] provides a different proof of Theorem 1.2, where they mention that Theorem 1.1 can be obtained from a result in Benoist and de la Harpe [1].

Based on Theorem 1.1 and 1.2, we obtain the following result.

Theorem 1.3. *If a group W is a direct product of n irreducible, infinite, non-affine Coxeter groups, then any finite index subgroup H of W has a trivial center, and H can be expressed uniquely as a direct product of m nontrivial subgroups of H (up to the rearrangement of factors), where each factor cannot be further decomposed and $1 \leq m \leq n$.*

There are examples, which will be explained in Chapter 5, showing that the situation $m < n$ can happen.

Theorem 1.3 has an implication for the group ring $R[H]$, where H is a subgroup of finite index in a Coxeter group without spherical or affine factors and R is a commutative ring with identity. Here $R[H]$ is the free R-module generated by the elements of H. Any element of $R[H]$ is of the form $\sum_{h \in H} a(h)h$, where $a(h) \in R$ and $a(h) = 0$ for all but finitely many h. The multiplication in H extends uniquely to a R-bilinear product $R[H] \times R[H] \to R[H]$. This makes $R[H]$ a ring.

When R is the field $\mathbf{C}$ of complex numbers, more structures are of interest. Note that a Coxeter group of finite rank is a countable discrete group. Let $L^2(H)$ denote the Hilbert space of square-summable, complex-valued functions on H, i.e.,

$$L^2(H) = \{f : H \to \mathbf{C} | \ \Sigma |f(h)|^2 < \infty\}.$$

Then $L^2(H)$ is a right $\mathbf{C}[H]$-module induced by the right action of H on $L^2(H)$:

$$(f \cdot h)(h') = f(hh').$$

Indeed, $\mathbf{C}[H] \subset \mathcal{L}(L^2(H))$, the set of bounded linear operators on $L^2(H)$. The *von Neumann algebra* $\mathcal{N}(H)$ is defined to be the weak closure of the algebra of operators $\mathbf{C}[H]$ acting from the right on $L^2(H)$.

First, we have the following.

Theorem 1.4. *Given an irreducible, infinite, non-affine Coxeter group* (W, S) *and any* $w \neq 1$ *in* W*, the cardinality of the set* $\{gwg^{-1} | g \in W\}$ *is infinite.*

Remark 1.5. Ian Leary pointed out a connection between this theorem and Theorem 1.1. Indeed, Theorem 1.4 is still valid if we replace the above W by a subgroup of finite index in a Coxeter group without spherical or affine factors. Leary's comments are included at the end of Chapter 5, following our old proof of this theorem.

As a comparison, recall that an irreducible, infinite, affine Coxeter group W has a decomposition $\mathbf{Z}^n \rtimes W_0$, where W_0 is a finite Coxeter group. In this situation, the cardinality of the set $\{gwg^{-1} | g \in W\}$ is finite for $w \in \mathbf{Z}^n$. In summary, Theorem 1.1, 1.2 and 1.4 illustrate the group theoretic differences between irreducible, infinite, non-affine Coxeter groups and irreducible, infinite, affine Coxeter groups, even though the classification between "affine" and "non-affine" is based on generators and relations and the associated bilinear form. Indeed, the following corollary can be proved easily by Theorem 1.4.

Corollary 1.6. *An irreducible, infinite Coxeter group* W *is affine if and only if it contains an abelian subgroup of finite index.*

Proof. The "only if" part is obvious. For the "if" part, suppose that A is a finite index abelian subgroup of the irreducible, infinite Coxeter group W. Since the number of distinct left cosets of A in W is the same as the number of distinct right cosets of A in W, we assume that $W = \bigcup_{i=1}^{n} w_i A$, where n is a positive integer, $\{w_1, w_2, \cdots, w_n\}$ is a designated finite subset of W. Given $a \neq 1 \in A$, we have $\{waw^{-1} | w \in W\} = \{w_i a w_i^{-1} | i = 1, \cdots, n\}$. By Theorem 1.4, W cannot be non-affine. Hence, W is affine. $\square$

A different proof of this corollary can be found in [10, p.15].

With the discussion at the end of Chapter 5, the corollary for the group ring $R[H]$ is as follows, the proof of which is left to the reader.

Corollary 1.7. *Let H be a subgroup of finite index in a Coxeter group without spherical or affine factors. Then*

(1) for a commutative ring R with identity, the center of $R[H]$ is R;

(2) The center of the von Neumann algebra $\mathcal{N}(H)$ is $\mathbf{C}$.

This dissertation incorporates the author's two recent papers [23, 24] and is organized as follows. In Chapter 2, we state some basic facts about the combinatorial theory of a Coxeter group. Using these, the author gives another proof of the statement that an irreducible, infinite Coxeter group has a trivial center. This proof does not use the canonical representations of Coxeter groups and is of purely combinatorial nature. Then we describe briefly the construction of the Davis complex of a Coxeter group. The canonical representations, root systems and the Tits cone of a Coxeter group are introduced in Chapter 3, where using the root system as a tool, the author proves the following statement.

Theorem 1.8. *The intersection of a collection of parabolic subgroups of a Coxeter group is a parabolic subgroup.*

The definition of a parabolic subgroup of a Coxeter group is given in Chapter 3. Theorem 1.7 will be used when we discuss essential elements of a Coxeter group in Chapter 4.

In Section 3.1, using the root system of a Coxeter group, we give a proof of the statement that for an irreducible, infinite Coxeter group (W, S) and $w \in W$, if

$wSw^{-1} = S$, then $w = 1$. Our proof appears to be different from the one suggested in Bourbaki [2] by using the Tits cone. This statement is stronger than the statement that the center of an irreducible, infinite Coxeter group is trivial.

Some important results from Krammer's thesis [18] and Paris [22] are summarized in Chapter 4, where we provide a proof of a key statement in [22] about the characterization of essential elements of a Coxeter group and we outline Krammer's proof of his result on irreducible, infinite, non-affine Coxeter groups. All these discussions are important in the proofs of Theorems 1.1-1.4, which are given in Chapter 5.

CHAPTER 2

BASIC COMBINATORIAL THEORY OF COXETER GROUPS

A Coxeter group may be characterized by some combinatorial conditions, which are stated below. For now, let W be a group generated by a subset S of *involutions* (elements of order 2). The *length* $l(w)$ or $l_S(w)$ of an element $w \in W$, with respect to S, is the smallest number d such that $w = s_1 \cdots s_d$, with all $s_i \in S$. This expression for w is called a *reduced decomposition* of w if $d = l(w)$.

Consider the following conditions.

(D) Deletion Condition. *If $w = s_1 \cdots s_d$ with $d > l(w)$, then there are indices $i < j$ such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d$. where the hats indicate deleted letters.*

(E) Exchange Condition. *Given $w \in W$, $s \in S$, and any reduced decomposition $w = s_1 \cdots s_d$ of w, either $l(sw) = d + 1$ or else there is an index i such that $w = ss_1 \cdots \hat{s}_i \cdots s_d$.*

(F) Folding Condition. *Given $w \in W$ and $s, t \in S$ such that $l(sw) = l(w) + 1$ and $l(wt) = l(w) + 1$, either $l(swt) = l(w) + 2$ or else $sw = wt$.*

The proof of the following theorem can be found in [2], [4] or [9].

Theorem 2.1. *A group W generated by a set S of involutions gives a Coxeter system (W, S) if and only if W satisfies any one of the conditions (D), (E) and (F), with the length function $l(w) = l_S(w)$ defined as above.*

Given a Coxeter system (W, S), for each subset T of S, let W_T be the subgroup generated by T. Call it a *special subgroup* of W. Then any element $w \in W$ can be expressed as $w = w_0 a$ where $a \in W_T$ and w_0 is the shortest element in the left coset wW_T. w_0 is characterized by the property $l(w_0 t) = l(w_0) + 1$ for any $t \in T$ and it is unique in wW_T. We say w_0 is $(\emptyset, T)$*-reduced* in this situation. It is clear this type of decomposition for w is unique and w_0 satisfies $l(w_0 b) = l(w_0) + l(b)$ for any $b \in W_T$. Similar discussions for right cosets give a "right-hand version" of the decomposition and the definition of $(T, \emptyset)$*-reduced* elements.

For $w \in W$, define a subset $\mathrm{In}(w)$ of S by

$$\mathrm{In}(w) = \{s \in S | l(ws) = l(w) - 1\},$$

and put

$$\mathrm{Ou}(w) = S - \mathrm{In}(w).$$

We collect some basic facts about finite special subgroups of a Coxeter group.

Lemma 2.2. *Suppose W_T is a finite subgroup, where $T \subset S$. Then there is a unique element w_T in W_T of longest length. Moreover, the following statements are true.*

(1) w_T is an involution.

(2) For any $x \in W_T$, $x = w_T$ if and only if $\mathrm{In}(x) = T$.

(3) For any $x \in W_T$, $l(w_T x) = l(w_T) - l(x)$.

This lemma is taken from exercises in Chapter 4 of [2]. The proof of this lemma and the following two lemmas can be found in Chapter 3 of Davis [9]. For readers' convenience, we include the proof of Lemma 2.4.

Lemma 2.3. ([8]) *For any $x \in W$, $W_{\mathrm{In}(x)}$ is a finite subgroup.*

Lemma 2.4. ([8]) *If W_T is a finite subgroup of W and w_T is the longest element in W_T, then for $s \in S - T$, $sw_T = w_T s$ if and only if $m_{st} = 2$ for all $t \in T$.*

Proof. If $m_{st} = 2$, then s and t commute. Hence, if $m_{st} = 2$ for all $t \in T$, then s and w_T commute.

Conversely, suppose s and w_T commute, where $s \notin T$. Then $l(w_T s) = l(w_T)+1$, so $s \in \mathrm{In}(w_T s)$. Since $w_T s = s w_T$, $T \subset \mathrm{In}(w_T s)$. Therefore, $\mathrm{In}(w_T s) = T \cup \{s\}$, $w_{T\cup\{s\}} = w_T s$. We want to show that $m_{st} = 2$ for all $t \in T$. Suppose, to the contrary, that $m_{st} > 2$, for some $t \in T$. Then $l(sts) = 3$, $l((w_T s)(sts)) = l(w_T) + 1 - 3 = l(w_T) - 2$ by Lemma 2.2. On the other hand, $l((w_T s)(sts)) = l(w_T ts) = l(w_T t) + 1 = l(w_T)$, a contradiction. Hence, the conclusion of Lemma 2.4 holds.

$\square$

Now we prove the statement.

Proposition 2.5. *The center of an irreducible, infinite Coxeter group (W, S) is trivial.*

Proof. If $w \neq 1$ is in the center of W, then $ws = sw$ for any $s \in S$. Put $S_1 = \mathrm{In}(w)$ and $S_2 = \mathrm{Ou}(w)$. Then $S_1 \neq \emptyset$. Write $w = w_0 a$ with $a \in W_{S_1}$ and w_0 being $(\emptyset, S_1)$ reduced. Notice that $l(w) - 1 = l(ws) = l(w_0 as)$ for any $s \in S_1$, it follows that $l(as) = l(a) - 1$ for all $s \in S_1$. By Lemma 2.2 and Lemma 2.3, a is the (unique) longest element in the finite subgroup W_{S_1}, and $a^2 = 1$.

Now, continue our discussion and consider the "right-hand version" of the above-mentioned decomposition of w. Since w is in the center of W, we have $w = aw_1$, where w_1 is $(S_1, \emptyset)$ reduced, a is the longest element in W_{S_1}. Hence, $w_0 = wa = aw = w_1$, $w = aw_0 = w_0 a$.

Notice that for any $t \in S_2$, $l(wt) = l(w) + 1$, it follows that $l(w_0 t) = l(w_0) + 1$, since otherwise we would have $l(wt) \leq l(a) + l(w_0 t) \leq l(w) - 1$, contradicting the definition of S_2. Therefore, w_0 is $(\emptyset, S_2)$-reduced, and hence is $(\emptyset, S)$-reduced. This implies that $w_0 = 1$ and $w = a$, i.e., $w = w_{S_1}$. So, w_{S_1} commutes with every element in $S_2 = S - S_1$. Now, Lemma 2.4 implies that $m_{st} = 2$ for any $s \in S_1$, $t \in S_2$. The irreducibility of W implies $S_2 = \emptyset$ and hence W is a finite Coxeter group, a contradiction. This finishes the proof of Proposition 2.5. □

To prove the theorems stated in the introduction, we need the fact that a Coxeter group acts properly and cocompactly on a CAT(0) space. Here we give a brief description of the Davis complex. For a more complete account of it, the reader is referred to [6, 7, 8].

Let (W, S) be a Coxeter system. We define a poset, denoted $\mathcal{S}^f(W, S)$ (or simply $\mathcal{S}^f$), by putting

$$\mathcal{S}^f = \{T | T \subset S \text{ and } W_T \text{ is finite}\}.$$

This poset is partially ordered by inclusion. It is clear that $\mathcal{S}^f - \{\emptyset\}$ is isomorphic to the poset of simplices of an abstract simplicial complex, which is denoted by $N(W, S)$ (or simply N). N is called the *nerve* of (W, S).

Theorem 2.6. (Davis, Moussong [7, 21]). *Associated to a Coxeter system* (W, S)*, there is a PE cell complex* $\Sigma(W, S)$ $(= \Sigma)$ *with the following properties.*

(1) The poset of cells in Σ is the poset of cosets

$$W\mathcal{S}^f = \coprod_{T \in \mathrm{S}^f} W/W_T.$$

(2) W acts by isometries on Σ with finite stabilizers and with compact quotient.

(3) Each cell in Σ is simple (so that the link of each vertex is a simplicial cell complex). In fact, this link is just $N(W,S)$.

(4) Σ is CAT(0).

CHAPTER 3

CANONICAL REPRESENTATIONS OF A COXETER GROUP

3.1 Canonical representations and root systems of a Coxeter group

In this section we collect a few basic facts about the canonical representations and root systems of a Coxeter group. These materials are taken from Chapter 5 of [17]. At the end of this section, we give a proof of the statement that for an irreducible, infinite Coxeter group (W, S) and $w \in W$, if $wSw^{-1} = S$, then $w = 1$. The proof is different from the one suggested in Bourbaki [2] by using the Tits cone. This statement is stronger than Proposition 2.5, which says that the center of an irreducible, infinite Coxeter group is trivial.

Recall from the introduction that for a Coxeter system (W, S), there is a symmetric bilinear form $(\cdot, \cdot)$ on a real vector space V, having a basis $\Pi = \{\alpha_s | s \in S\}$ in one-to-one correspondence with S. Now, for each $s \in S$, define a linear transformation $\sigma_s : V \to V$ by $\sigma_s\lambda = \lambda - 2(\alpha_s, \lambda)\alpha_s$. Then σ_s is a linear reflection. It has order 2 and fixes the hyperplane $H_s = \{\delta \in V | (\delta, \alpha_s) = 0\}$ pointwise, and $\sigma_s\alpha_s = -\alpha_s$. We have the following theorem (see [11, 17]).

Theorem 3.1. *There is a unique homomorphism $\sigma : W \to GL(V)$ sending s to σ_s. This homomorphism is a faithful representation of W and the group $\sigma(W)$ preserves the bilinear form. Moreover, for each pair $s, t \in S$, the order of st in W is precisely m_{st}.*

From now on, we write $w(\alpha)$ for $\sigma(w)(\alpha)$, when $\alpha \in V$ and $w \in W$.

The *root system* Φ of W, is defined to be the collection of all vectors $w(\alpha_s)$, where $w \in W$ and $s \in S$. An important fact about the root system is that any root $\alpha \in \Phi$ can be expressed as

$$\alpha = \sum_{s \in S} c_s \alpha_s,$$

where all the coefficients satisfy $c_s \geq 0$ (we call α a positive root and write $\alpha > 0$), or all the coefficients satisfy $c_s \leq 0$ (call α a negative root and write $\alpha < 0$). The α_s is called a (positive) *simple root*, for $s \in S$. Write Φ^+ and Φ^- for the respective sets of positive and negative roots. Then $\Phi^+ \bigcap \Phi^- = \emptyset$, $\Phi^+ \bigcup \Phi^- = \Phi$ and $\Phi^- = -\Phi^+$. The map from Φ to $R = \{wtw^{-1} | w \in W, t \in S\}$ (the set of reflections in W) given by $\alpha = w(\alpha_s) \mapsto wsw^{-1}$ is well-defined and restricts to a bijection from Φ^+ (Φ^-) to R, and $\sigma(wsw^{-1}) = t_\alpha$, where t_α is the linear reflection given by $t_\alpha \lambda = \lambda - 2(\alpha, \lambda)\alpha$. The following fact is important when discussing root systems.

Proposition 3.2. ([11, 17]) *Let $w \in W$, $\alpha \in \Phi^+$. Then $l(wt_\alpha) > l(w)$ if and only if $w(\alpha) > 0$.*

For a Coxeter group (W, S) and a subset I of S, the *special subgroup* W_I of W is the subgroup generated by $s \in I$. At the extremes, $W_\emptyset = \{1\}$ and $W_S = W$. When W_I is a finite subgroup, I is called a *spherical* subset of S.

With the representation $\sigma : W \to \mathrm{GL}(V)$ in mind, we define a dual representation $\sigma^* : W \to \mathrm{GL}(V^*)$ as follows (and we abuse the notations by identifying w with $\sigma(w)$ or $\sigma^*(w)$),

$$\langle w(f), \lambda \rangle = \langle f, w^{-1}(\lambda) \rangle \text{ for } w \in W, f \in V^*, \lambda \in V,$$

where V^* is the dual space of V and the natural pairing of V^* with V is denoted by $\langle f, \lambda \rangle$. For $I \subset S$, write

$$C_I = \{f \in V^* | \langle f, \alpha_s \rangle > 0 \text{ for } s \in S - I \text{ and } \langle f, \alpha_s \rangle = 0 \text{ for } s \in I\}.$$

Notice that $C_S = \{0\}$ and write $C = C_\emptyset$, $\overline{C} = \bigcup_{I \subset S} C_I$. Define U to be the union of all $w(\overline{C})$, $w \in W$. U is a cone in V^*, called the *Tits cone* of W. A *face* of U is a set of the form $w(C_I)$, $w \in W$, $I \subset S$.

Theorem 3.3. ([17])(a) *Let $w \in W$ and $I, J \subset S$. If $w(C_I) \bigcap C_J \neq \emptyset$, then $I = J$ and $w \in W_I$, so $w(C_I) = C_I$. In particular, W_I is precisely the stabilizer in W of each point of C_I, and the $w(C_I)$, where $w \in W$, $I \subset S$, form a partition of the Tits cone U.*

(b) *$\overline{C}$ is a fundamental domain for the action of W on U: the W-orbit of each point of U meets $\overline{C}$ in exactly one point.*

(c) *The Tits cone U is convex, and every closed line segment in U meets just finitely many of the sets of the family $\{w(C_I) | I \subset S\}$.*

Both σ and σ^* are called canonical representations.

Now we use the basic facts of root systems to prove the following.

Proposition 3.4. *Let (W, S) be an irreducible, infinite Coxeter group and $w \in W$. If $wSw^{-1} = S$, then $w = 1$.*

Proof. Suppose $w \neq 1$. We use the notations introduced in Chapter 2. Let $S_1 = \mathrm{In}(w) = \{s_1, \ldots, s_k\}$, and $S_2 = \mathrm{Ou}(w) = S - \mathrm{In}(w) = \{s_{k+1}, \ldots, s_n\}$. Since W is assumed to be infinite, S_1 and S_2 are nonempty. Let the set Π of positive simple roots corresponding to S be $\{\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$. Write $w = w_0 a$, where a is the longest element in W_{S_1} and w_0 is $(\emptyset, S_1)$ reduced.

Given $j \in \{k+1, \ldots, n\}$, $a(\alpha_j) > 0$ by Proposition 3.2, and $a(\alpha_j) = \alpha_j + \sum_{l=1}^{k} c_l \alpha_l$. So $c_l \geq 0$ for $1 \leq l \leq k$. Since w_0 is $(\emptyset, S_1)$ reduced, $0 < w_0(\alpha_l) = \beta_l$, for $1 \leq l \leq k$. We claim that $\beta_l \in \Pi$. The reason is that, if for some l, $1 \leq l \leq k$, $w_0(\alpha_l) = c_1\beta_1 + c_2\beta_2 + \cdots$, for different β_1, $\beta_2 \in \Pi$, with $c_1, c_2 > 0$, we would have a contradiction. Since by Lemma 2.2, $l(asa) = l(a) - l(sa) = l(a) - l(a) + 1 = 1$, for any $s \in S_1$, so $a(\{\alpha_1, \ldots, \alpha_k\}) = \{-\alpha_1, \ldots, -\alpha_k\}$, we could have $w(-\alpha_{l'}) = w_0 a(-\alpha_{l'}) = w_0(\alpha_l) = c_1\beta_1 + c_2\beta_2 + \cdots$, for some l', $1 \leq l' \leq k$, contradicting the assumption $wSw^{-1} = S$. Thus $w_0(\{\alpha_1, \ldots, \alpha_k\}) \subset \Pi$.

Given $j \in \{k+1, \ldots, n\}$, since $0 < w(\alpha_j) = w_0 a(\alpha_j) = w_0(\alpha_j + \sum_{l=1}^{k} c_l \alpha_l)$, and $w_0(\alpha_j) \notin \mathrm{Span}(\{\beta_1, \ldots, \beta_k\})$, it follows that $w_0(\alpha_j) > 0$ for $j \geq k+1$. Here $\mathrm{Span}(X)$ means the real vector subspace spanned by the set X. Thus, $\mathrm{Ou}(w_0) = S$, $w_0 = 1$.

Now, $aSa^{-1} = S$ and $aS_1a^{-1} = S_1$, thus $aS_2a^{-1} = S_2$. Since $0 < a(\alpha_j) = \alpha_j + \sum_{l=1}^{k} c_{jl}\alpha_l$, for $k+1 \leq j \leq n$, so $c_{jl} = 0$, for $1 \leq l \leq k$. This means that a commutes with each $s \in S_2$, Then similar arguments as in the proof of Proposition 2.5 yield a contradiction to the irreducibility of (W, S). The proof is completed. $\square$

3.2 The interior U^0 of the Tits cone U

The following theorem is important for the discussions in Chapter 4.

Theorem 3.5. *The topological interior U^0 of the Tits cone U relative to V^* equals the union of all $w(C_I)$, where $w \in W$ and $I \subset S$ is spherical.*

To prove this theorem, we use the ideas suggested by Krammer [18]. But our proof of a key statement, Theorem 3.9, is different from that given in [18].

For a spherical subset $I \subset S$, write V_I =Span$\{\alpha_i | i \in I\}$ and $V_I^{\perp} = \{x \in V | (x, \alpha_i) = 0 \text{ for all } i \in I\}$. Then $V = V_I \oplus V_I^{\perp}$, and $V^* = Y_I \oplus Z_I$, where Y_I =Ann$(V_I^{\perp})$, Z_I =Ann(V_I). Define $p_I : V \to V_I$, $q_I : V^* \to Z_I$ to be the projections with respect to these decompositions.

Lemma 3.6. ([18]) *Let $I \subset S$ be spherical and $x \in V^*$. Then $q_I x = \frac{1}{|W_I|} \sum_{w \in W_I} wx$.*

Proof. Given $x \in V^*$ and $v \in V$, write $v = v_1 + v_2$, with $v_1 \in V_I$ and $v_2 \in V_I^{\perp}$. Then $q_I x(v) = x(v_2)$. Let $v_0 = \frac{1}{|W_I|} \sum_{w \in W_I} wv$. Note $t_{\alpha_i} v_0 = v_0$ for any $i \in I$, so $v_0 \in V_I^{\perp}$. It can be checked that $v - v_0 = \frac{1}{|W_I|} \sum_{w \in W_I} (v - wv)$ is a linear combination of the α_i, $i \in I$. So, for the decomposition of v, $v_2 = v_0 = \frac{1}{|W_I|} \sum_{w \in W_I} wv$. Thus $q_I x(v) = \langle x, \frac{1}{|W_I|} \sum_{w \in W_I} wv \rangle = (\frac{1}{|W_I|} \sum_{w \in W_I} wx)(v)$. This proves the desired identity. $\square$

It is clear that for any $x \in V^*$ and $v \in V$,

$$\langle x, v \rangle = \langle x, p_I v \rangle + \langle x, v - p_I v \rangle = \langle x, p_I v \rangle + \langle q_I x, v \rangle. \tag{3.1}$$

Lemma 3.7. ([18]) *Let f_s be the dual basis of α_s in V^* for any $s \in S$. Suppose that $I \subset S$ is spherical. Then*

(a) For any $s \in I$, $t \in S - I$, $\langle f_s, p_I \alpha_t \rangle \leq 0$;

(b) For any $s \in I$, $t \in S - I$, $\langle q_I f_s, \alpha_t \rangle \geq 0$.

Proof. (a) Write $p_I \alpha_t = \sum_{s \in I} b_s \alpha_s$. The b_s are determined by the equations $(\alpha_u, \alpha_t - p_I \alpha_t) = 0$, $u \in I$, i.e., $\sum_{s \in I} a_{us} b_s = a_{ut}$, $u \in I$, where $a_{ij} = (\alpha_i, \alpha_j)$, for $i, j \in S$.

Let $A_I = (a_{ij})_{(i,j)\in I\times I}$, $c = (a_{ut})_{u\in I}$ and $b = (b_s)_{s\in I}$ (both as column vectors). Then $b = A_I^{-1}c$. By Lemma 3.8 below, the entries of A_I^{-1} are non-negative. Since $t \in S - I$, the entries of c are are non-positive. Hence for any $s \in I$, $\langle f_s, p_I\alpha_t\rangle = b_s \le 0$.

Statement (b) follows from (a), the fact $\langle f_s, \alpha_t\rangle = 0$, and (3.1). □

Lemma 3.8. *Let $A = (a_{ij})$ be a positive definite real square symmetric matrix with $a_{ii} = 1$ and $a_{ij} \le 0$ $(i \ne j)$. Then the entries of A^{-1} are non-negative.*

A proof of this lemma can be found in Moussong [21].

Now we give the proof of the following.

Theorem 3.9. ([18]) *Let $I \subset S$ be spherical. Then $q_I(C) = C_I$.*

Proof. We use the notations introduced in Lemma 3.7. First we prove $q_I(C) \subset C_I$. Let $x \in C$. Note that the decomposition of α_s is $\alpha_s + 0$ for $s \in I$. So $q_Ix(\alpha_s) = x(0) = 0$ for $s \in I$. Let $t \in S - I$. Then $q_I(\alpha_t) = x(\alpha_t) - \sum\limits_{s\in I} b_s x(\alpha_s) > 0$, by Lemma 3.7. Thus $q_I(C) \subset C_I$.

Then to show that $q_I(C) \supset C_I$, given $y \in C_I$, we want to find $x = \sum\limits_{j=1}^{|S|} x_j f_j$, with $x_j > 0$, such that $q_I(x) = y$, i.e., we want to solve $x_u - \sum\limits_{i\in I} b_{ui}x_i = y(\alpha_u)$, with all $x_j > 0$, for $u \in S - I$, where $p_I\alpha_u = \sum\limits_{i\in I} b_{ui}\alpha_i$. Since $y(\alpha_u) > 0$ for $u \in S - I$. It is always possible to choose $x_i > 0$ but sufficiently small for $i \in I$ such that $x_u = y(\alpha_u) + \sum\limits_{i\in I} b_{ui}x_i > 0$ for all $u \in S - I$. This proves $q_I(C) \supset C_I$. □

Proof of Theorem 3.5. ([18]) Let $I \subset S$ be spherical. It follows from Lemma 3.6 and Theorem 3.9 that C_I is contained in $\sum\limits_{w_1\in W_I} w_1(C)$, which is an open subset of U. Hence $C_I \subset U^0$, and so is $w(C_I)$, for any $w \in W$.

Now suppose $x \in C_J \cap U^0$, where $J \subset S$. Let $B \subset U^0$ be an open neighborhood of x. We may suppose that B is symmetric in the sense that $B - x = -(B - x)$, i.e.,

$B = 2x - B$. Since $x \in \overline{C}$, there exists $y \in B \cap C$. Then $z = 2x - y \in B \subset U^0$. For $s \in J$, we have $\langle x, \alpha_s \rangle = 0$, $\langle y, \alpha_s \rangle > 0$. So $\langle z, \alpha_s \rangle < 0$. Suppose $z \in w(\overline{C})$. Then $w^{-1}\alpha_s < 0$ and hence $J \subset \{t \in S | l(w^{-1}s) < l(w^{-1})\}$, by Proposition 3.2. This implies that J is spherical by Lemma 2.3 and thus completes the proof of Theorem 3.5. □

Theorem 3.5 implies the following.

Theorem 3.10. (Tits) *Let $H \subset W$ be a finite subgroup. Then $H \subset wW_Iw^{-1}$ for some $w \in W$ and $I \subset S$ spherical.*

Proof. Choose $x \in U^0$. Let $y = \sum\limits_{h \in H} hx$. Then $y \in U^0$ and $Hy = y$. By Theorem 3.3 and 3.5, the stabilizer of y equals wW_Iw^{-1} for some $w \in W$ and $I \subset S$ spherical. It is obvious that $H \subset wW_Iw^{-1}$. □

3.3 Root system of a parabolic subgroup and the parabolic closure of a set

Given a Coxeter group (W, S), for any $w \in W$ and $I \subset S$, the subgroup wW_Iw^{-1} is called a *parabolic subgroup* of W. Krammer [18] defines the *parabolic closure* $\mathrm{Pc}(A)$ of a subset A of W to be the intersection of all parabolic subgroups containing A. It is believed that a parabolic closure is a parabolic subgroup. However, the author has not seen a proof in the literature that a parabolic closure which, by definition, is the intersection of a collection of parabolic subgroups, must be a parabolic subgroup. Perhaps the result closest to this aim is

Proposition 3.11. *The intersection of two parabolic subgroups of a Coxeter group is a parabolic subgroup.*

This result appears in geometric form in [26] and a proof using algebraic argument is given in [25]. For the reader's convenience, the author gives a proof of this proposition using the canonical representation of a Coxeter group.

Proof of Proposition 3.11. Given two parabolic subgroups G_1 and G_2 of W. Pick $x_i \in U$, $i = 1, 2$, such that G_i is the stabilizer of x_i. Then $G_1 \bigcap G_2$ fixes the line segment $\overline{x_1 x_2}$. By (c) of Theorem 3.3, there exist $y_1 \neq y_2$ on $\overline{x_1 x_2}$ such that they belong to the same $w(C_I)$. So y_1 and y_2 have the same stabilizer $P = wW_I w^{-1}$. Now P fixes the line segment $\overline{x_1 x_2}$ and hence $P \subset G_i$, $P \subset G_1 \bigcap G_2$. Since $G_1 \bigcap G_2$ fixes $\overline{x_1 x_2}$, the reverse inclusion is obvious. This completes the proof. □

However, the above proof does not establish the conclusion that a parabolic closure is a parabolic subgroup.

We now use the general notion of root systems to establish some technical lemmas on the root system of a parabolic subgroup and use them to prove

Theorem 3.12. *The parabolic closure of a subset of a Coxeter group is a parabolic subgroup.*

The statement of Theorem 3.12 is equivalent to that of Theorem 1.8. An alternate description of the parabolic closure is given at the end of this section.

First, we describe a lemma on the root system Φ_I of a special subgroup W_I, where $\Phi_I = \{w(\alpha_s) | w \in W_I, s \in I\}$.

Lemma 3.13. $\Phi_I = \Phi \cap \mathrm{Span}\{\alpha_s | s \in I\}$.

It is obvious that $\Phi_I \subset \Phi \bigcap \mathrm{Span}\{\alpha_s | s \in I\}$. When W is finite, arguments similar to that given on page 11 of [17] yield the reverse inclusion. In the case that W is of finite rank, the nontrivial fact that W is isomorphic to a discrete subgroup of $\mathrm{GL}(\mathbf{R}^n)$ $(n = |S|)$ implies that Φ is a discrete set of V, which makes similar arguments work. However, Lemma 3.13 holds even when $|S| = \infty$, as the following proof demonstrates. In fact, it follows from the basic properties of Coxeter groups.

Proof of Lemma 3.13. We want to prove that $\Phi \bigcap \mathrm{Span}\{\alpha_s | s \in I\} \subset \Phi_I$. Pick an arbitrary $\phi \in \Phi \bigcap \mathrm{Span}\{\alpha_s | s \in I\}$, $\phi > 0$. Write $\phi = c_1\alpha_{s_1} + \cdots + c_n\alpha_{s_n}$, where $c_i > 0$, $s_i \in I$, $i = 1, \cdots, n$, $s_i \neq s_j$ when $i \neq j$. We assume $n \geq 2$, otherwise $\phi = \alpha_{s_1} \in \Phi_I$. Now use induction on the length $l(t_\phi)$ of t_ϕ . Recall from Section 3.1 that $t_\phi(\lambda) = \lambda - 2(\phi, \lambda)\phi$.

Notice that $1 = (\phi, \phi) = \sum_{j=1}^{n} c_j(\phi, \alpha_{s_j})$, we know $(\phi, \alpha_{s_i}) > 0$ for some i. A simple calculation shows that $s_i t_\phi s_i = t_{s_i(\phi)}$ and we want to show $l(s_i t_\phi s_i) < l(t_\phi)$. First, it follows from

$$t_\phi(\alpha_{s_i}) = \alpha_{s_i} - 2(\phi, \alpha_{s_i})\phi < 0 \tag{3.2}$$

that $l(t_\phi s_i) = l(t_\phi) - 1$ by Proposition 3.2 and hence $l(s_i t_\phi) = l(t_\phi) - 1$. If $s_i t_\phi(\alpha_{s_i}) > 0$, then (3.2) implies that $t_\phi(\alpha_{s_i}) = -\alpha_{s_i}$, i.e., $\alpha_{s_i} - 2(\phi, \alpha_{s_i})\phi = -\alpha_{s_i}$; hence, $\phi = \alpha_{s_i}$, contradicting the assumption that $n \geq 2$. Therefore, $s_i t_\phi(\alpha_{s_i}) < 0$ and $l(t_{s_i(\phi)}) = l(s_i t_\phi s_i) = l(s_i t_\phi) - 1 = l(t_\phi) - 2$. The induction hypothesis now applies and the proof is completed. $\square$

Now we discuss parabolic closures. For any $K \subset S$, write Δ_K for the set $\{\alpha_s | s \in K\}$.

Lemma 3.14. *If $W_I = wW_Jw^{-1}$ for some $w \in W$, $I, J \subset S$, then $|I| = |J|$, and $w_0(\Delta_J) = \Delta_I$ for some $w_0 \in wW_J$, so $I = w_0 J w_0^{-1}$.*

This lemma is stated and proved in Section 4.5 of [9]. The proof given there is mainly combinatorial (without using the root system), although some topological considerations (of connected components separated by some "walls" of the corresponding Cayley graph) are used. Here we give another proof.

Proof of Lemma 3.14. We employ a few basic facts of Coxeter groups (see Davis [9], Chapter 4). First, if $xtx^{-1} \in W_K$, where $x \in W$, $t \in S$ and $K \subset S$, then $xtx^{-1} = w_1 s w_1^{-1}$ for some $w_1 \in W_K$ and $s \in K$; that is, if a reflection of a Coxeter group W lies in a special subgroup W_K, it is indeed a reflection in W_K (considering W_K as a Coxeter group by itself). Second, $wW_J = w_0 W_J$, where w_0 satisfies that $l(w_0 t) = l(w_0) + 1$ for any $t \in J$, i.e., w_0 is the shortest element in wW_J.

Now using the above w_0, we have $W_I = w_0 W_J w_0^{-1}$. It follows from the correspondence of root system and reflections in the Coxeter group W that $\Phi_I = w_0(\Phi_J)$. Comparing the maximal numbers of linearly independent positive roots in these sets (by the choice of w_0, $w_0(\alpha_t) > 0$, for $t \in J$), we have $|I| = |J|$. The fact $\Phi_I = w_0(\Phi_J)$ implies each element of Φ_I is a positive or negative linear combination of $w_0(\Delta_J)$, so $\Delta_I = w_0(\Delta_J)$, and hence, $I = w_0 J w_0^{-1}$. This finishes the proof of Lemma 3.14. □

Lemma 3.15. *If $W_I \not\subseteq wW_Jw^{-1}$ and $wW_Jw^{-1} \not\subseteq W_I$, then $W_I \bigcap wW_Jw^{-1} = xW_Kx^{-1}$ with $|K| < \min\{|I|, |J|\}$.*

Proof. The statement that $W_I \bigcap wW_Jw^{-1} = xW_Kx^{-1}$ for some $x \in W$ and $K \subset S$ is guaranteed by Proposition 3.11. Since $xW_Kx^{-1} = x_0 W_K x_0^{-1} \subset W_I$, where x_0 is the shortest element in xW_K, any root corresponding to a reflection in $x_0 W_K x_0^{-1}$ lies in Φ_I, that is, $x_0(\Phi_K) \subset \Phi_I$. Comparing the maximal numbers of linearly independent positive roots in these sets, we have $|K| \leq |I|$.

Notice that $x_0(\Phi) = \Phi$, it follows from Lemma 3.13 that

$$x_0(\Phi_K) = x_0(\Phi \cap \mathrm{Span}\Delta_K) = \Phi \cap \mathrm{Span}\{x_0(\Delta_K)\}.$$

If $|K| = |I|$, noticing that $x_0(\Delta_K) \subset \Phi_I$ and $\mathrm{Span}\{x_0(\Delta_K)\} \subset \mathrm{Span}\Delta_I$, we would have

$$\Phi_I = \Phi \cap \mathrm{Span}\Delta_I = \Phi \cap \mathrm{Span}\{x_0(\Delta_K)\} = x_0(\Phi_K),$$

and hence, $\Delta_I = x_0(\Delta_K)$, $W_I = x_0 W_K x_0^{-1} = x W_K x^{-1}$, contradicting the assumption of the lemma. Hence $|K| < |I|$. Similarly, $|K| < |J|$. Therefore, $|K| < \min\{|I|, |J|\}$.

□

Another description of the parabolic closure is

Theorem 3.16. *The parabolic closure* $\mathrm{Pc}(A)$ *of a subset* A *of* W *is the parabolic subgroup* wW_Jw^{-1} *containing* A*, for* $|J|$ *minimal.*

This is now obvious. The statement that the above mentioned parabolic subgroup is contained in any parabolic subgroup containing A follows from Lemma 3.15 and the fact (whose proof is essentially contained in the proof of Lemma 3.15) that if $xW_Kx^{-1} \subset W_I$, then $|K| \leq |I|$ and if xW_Kx^{-1} is a proper subgroup of W_I, then $|K| < |I|$.

CHAPTER 4

SOME THEOREMS OF KRAMMER AND PARIS

In this chapter we quote many results from Krammer [18] and prove a few of them. The discussions lead to an important conclusion in [18] on irreducible, infinite, non-affine Coxeter groups (Theorem 4.17). Following Paris [22], we make revisions of some definitions and proofs of some statements given in [18]. We will use the notations and conclusions of the canonical representations and root systems introduced in Chapter 3.

4.1 Periodic, even and odd roots

For a Coxeter group (W, S) and a root $\alpha \in \Phi$, the associated *half-space* $A = A(\alpha)$ in W is the subset $\{w \in W | w^{-1}\alpha > 0\}$. It is clear that $A(-\alpha) = W - A$. The definition of a half-space in a Coxeter group given here is the same as that given in Davis [9], where it is based on the discussion of the associated Cayley graph of a Coxeter group. We call $A(\alpha)$ and $A(-\alpha)$ the two half-spaces determined by α, or the different sides of the "hyperplane" corresponding to α.

Let u, $v \in W$ and $\alpha \in \Phi$. We say that α *separates* u and v if $u\alpha \in \Phi^{\epsilon}$ and $v\alpha \in \Phi^{-\epsilon}$, where $\epsilon \in \{+, -\}$. Let $w \in W$ and $\alpha \in \Phi$. We say that α is *w-periodic* if there is some positive integer m such that $w^m\alpha = \alpha$.

The following lemma follows from the fact that there are only a finite number of roots or hyperplanes separating 1 and w^{-1}.

Lemma 4.1. ([22]) *Let $w \in W$ and $\alpha \in \Phi$. Then exactly one of the following holds.*

(1) α is w-periodic.

(2) α is not w-periodic, and the set $\{m \in \mathbf{Z} | \alpha$ separates w^m and $w^{m+1}\}$ is finite and has even cardinality.

(3) α is not w-periodic, and the set $\{m \in \mathbf{Z} | \alpha$ separates w^m and $w^{m+1}\}$ is finite and has odd cardinality.

We say that α is *w-even* in Case 2, and *w-odd* in Case 3. The statement α is w-odd means that for large n, w^n and w^{-n} lie on the different sides of the hyperplane corresponding to α, while α is w-even means that they are on the same side for large n. Now the following lemma is obvious.

Lemma 4.2. ([22]) *Let $\alpha \in \Phi$, $w \in W$, and $p \in \mathbf{N}$, $p \geq 1$. Then*

(1) α is w-periodic if and only if α is w^p-periodic.

(2) α is w-even (resp., w-odd) if and only if α is w^p-even (resp., w-odd).

A partial ordering $\preceq$ on Φ is defined by $\alpha \preceq \beta \Leftrightarrow A(\alpha) \subset A(\beta)$ and is characterized by the following.

Proposition 4.3. ([18]) *Let α, β be two roots. Then the following hold.*

(a) If $|(\alpha, \beta)| < 1$ then all of the four intersections $A(\pm\alpha) \cap A(\pm\beta)$ are non-empty.

(b) $(\alpha, \beta) \geq 1$ if and only $\alpha \preceq \beta$ or $\beta \preceq \alpha$.

(c) If $\alpha \preceq \beta$, say (by (b)) $(\alpha, \beta) = (p + p^{-1})/2$, $p \geq 1$, then for all $x \in U$, we have

$$\langle x, \alpha \rangle \geq p\langle x, \beta \rangle, \ \langle x, \alpha \rangle \geq p^{-1}\langle x, \beta \rangle.$$

(d) $\alpha \preceq \beta$ if and only if $\langle U, \beta - \alpha \rangle \subset \mathbf{R}_{\geq 0}$.

(e) Let $X \subset \Phi$ be a root subbasis (see Section 4.4), and let $\preceq_X$ denote the ordering on the associated root system $\Phi(X)$. Then for any α, $\beta \in \Phi(X)$, we have $\alpha \preceq \beta$ if and only if $\alpha \preceq_X \beta$.

Proposition 4.3, and the fact that $\{\alpha, w\alpha, \ldots, w^m\alpha\}$ is linearly dependent for any $\alpha \in \Phi$ and $m = |S|$, imply the following.

Proposition 4.4. ([18]) *There exists a constant $N = N(W)$ such that for all $w \in W$, the following hold. Let α be w-odd. Then*

(a) $\forall n \geq 1$, $(\alpha, w^n\alpha) > -1$,

(b) $\exists n$, $1 \leq n \leq N$ such that $(\alpha, w^n\alpha) \geq 1$.

Let α be w-even. Then

(c) $\forall n \geq 1$, $(\alpha, w^n\alpha) < 1$,

(d) $\exists n$, $1 \leq n \leq N$ such that $(\alpha, w^n\alpha) \leq -1$.

4.2 A pseudometric on U^0 and the axis $Q(w)$

We discussed the interior U^0 of the Tits cone in Section 3.2.

For any root α, let $f_\alpha : U^0 \to \{-1/2, 0, 1/2\}$ be defined by

$$f_\alpha(x) = \begin{cases} -1/2, & \langle x, \alpha \rangle < 0, \\ 0, & \langle x, \alpha \rangle = 0, \\ 1/2, & \langle x, \alpha \rangle > 0. \end{cases} \tag{4.1}$$

The pseudometrics d_α and d_Φ on U^0 are defined by

$$d_\alpha(x, y) = |f_\alpha(x) - f_\alpha(y)|, \; d_\Phi(x, y) = \sum_{\alpha \in \Phi^+} d_\alpha(x, y). \tag{4.2}$$

It is clear that, for any $x \in U^0$, the set $\{y \in U^0 | d_\Phi(x, y) = 0\}$ equals the face containing x. And, $d_\Phi(C_I, C)$ equals half of the number of reflections in W_I. The following facts hold.

Lemma 4.5. ([18])

(a) For any v, $w \in W$, $d_\Phi(vC, wC) = l(v^{-1}w)$.

(b) For any x, $y \in U^0$, $d_\Phi(x, y)$ is finite.

For any $w \in W$, analogous to the situation of a hyperbolic isometry acting on a CAT(0) space, the *axis* $Q(w)$ is defined to be

$$\{x \in U^0 | d_\Phi(x, w^n x) = |n| d_\Phi(x, wx) \text{ for all } n \in \mathbf{Z}\}.$$

For the infinite dihedral group $D_\infty = \langle s_1, s_2 \rangle$ and $w = s_1 s_2$, $Q(w)$ equals U^0(= the open half plane). For a finite Coxeter group W and $w \in W$, since $U^0 = V^*$ (which is naturally identified with V via the inner product $(\cdot, \cdot)$), $Q(w) = \{x \in V | wx = x\}$.

The non-emptiness of the axis $Q(w)$ (claimed in Theorem 4.9 below) is crucial when we discuss more properties of w-periodic roots, especially, in the proofs of some related theorems.

It is not difficult to prove that $Q(gwg^{-1}) = gQ(w)$ for any $g \in W$ and $x \in Q(w)$ is equivalent to that, for any $\alpha \in \Phi$, $n \mapsto f_\alpha(w^n x)$ is monotonic in $n \in \mathbf{Z}$. The properties of the function $f_\alpha(x)$ given in the following lemma are useful when discussing $Q(w)$.

Lemma 4.6. ([18]) *Let $w \in W$, $x \in U^0$, $\alpha \in \Phi$ be such that the function $g : \mathbf{Z} \to \{-1/2, 0, 1/2\}$, $g(n) = f_\alpha(w^n x)$ is monotonic. If g is not constant, then its image contains $\{-1/2, 1/2\}$. Furthermore, g is not constant if and only if α is w-odd.*

A root α is called w-*outward* if for all large positive integers n, $w^n \in A(-\alpha)$ and $w^{-n} \in A(\alpha)$. It is called w-*inward* if $-\alpha$ is w-outward. We sometimes omit the mention of w when we talk about periodic, outward, odd or even roots in the following paragraphs for brevity. It is clear that α is odd if and only if either α or

$-\alpha$ is outward. Denote the set of outward roots by $\mathrm{Out}(w)$. It is easy to verify that $\mathrm{Out}(w)$ is $\langle w\rangle$-invariant. Let $r(w) = \#(\langle w\rangle \backslash \mathrm{Out}(w))$.

Proposition 4.7. ([18]) *For any $x \in U^0$, we have $d_\Phi(x, wx) \geq r(w)$, with equality if and only if $x \in Q(w)$. In particular, $r(w) \leq l(w)$.*

Proof. Let $x \in U^0$ and $\alpha_1, \ldots, \alpha_r$ be outward roots in different orbits in $\mathrm{Out}(w)$ under the action of $\langle w\rangle$. Note that for any outward root α, $\sum_{n\in\mathbf{Z}} d_\alpha(w^n x, w^{n+1}x) \geq 1$. Hence,

$$d_\Phi(x, wx) = \sum_{\alpha\in\Phi^+} d_\alpha(x, wx) \geq \sum_i \sum_{n\in\mathbf{Z}} d_{\alpha_i}(w^n x, w^{n+1}x) \geq r.$$

This implies that $r(w) \leq d_\Phi(x, wx)$. Equality holds if and only if $x \in Q(w)$, by Lemma 4.6. $\square$

To any root $\alpha \in \Phi$ we associate *open* and *closed half-spaces* in U^0 by

$$H(\alpha) = \{x \in U^0 | \langle x, \alpha\rangle > 0\},$$

$$K(\alpha) = \{x \in U^0 | \langle x, \alpha\rangle \geq 0\}.$$

Let $\mu(\alpha) = \{x \in U^0 | \langle x, \alpha\rangle = 0\}$, and call it the *wall* associated to α. Then $\mu(\alpha) = K(\alpha) - H(\alpha)$.

Lemma 4.8. ([18]). *Let α be even. Then $Q(w) \subset H(\alpha)$ or $Q(w) \subset H(-\alpha)$.*

Proof. Replacing α by $-\alpha$ if necessary, we may assume that $w^n \in A = A(\alpha)$ for almost all $n \in \mathbf{Z}$. For any $x \in Q(w)$, suppose $x \in h(\overline{C})$. Since there are only a finite number of roots in Φ separating 1 and h^{-1}, $w^n h \in A$ for almost all $n \in \mathbf{Z}$. So $w^n x \in K(\alpha)$ for almost all n. It follows from Lemma 4.6 that either $\langle w\rangle x \subset H(\alpha)$ or $\langle w\rangle x \subset \mu(\alpha)$. The latter case implies that $\langle w\rangle\alpha$ is a finite set and hence α is w-periodic, a contradiction. Hence, $Q(w) \subset H(\alpha)$. $\square$

Krammer [18] shows $Q(w) \neq \emptyset$ by proving a particular subset of it is nonempty. Define $Q_2(w)$ to be the set of $x \in Q(w)$ such that for all outward α, $\langle x, \alpha \rangle \leq \langle x, w\alpha \rangle$, and for all periodic α, $\langle x, \alpha \rangle = \langle x, w\alpha \rangle$. It can be checked that $Q_2(w)$ is a convex subset of U^0. The following theorem claims the non-emptiness of the axis.

Theorem 4.9. ([18]) *There exists a constant $N = N(W)$ such that for any $w \in W$, we have $(1 + w + \cdots + w^{N-1})U^0 \subset Q_2(w)$.*

The proof of Theorem 4.9 uses the identity $\langle wy - y, \alpha \rangle = \langle w^N x - x, \alpha \rangle$, for $y = (1 + w + \cdots + w^{N-1})x$, $x \in U^0$, along with the fact (proved in [18]) that, for a Coxeter group W, there is an integer $M = M(W)$, such that for any w-periodic root α, $w^M \alpha = \alpha$. The discussion is then divided into the cases of w-periodic, w-odd and w-even roots separately. For the latter two cases, Proposition 4.4 and 4.3 apply.

4.3 Critical roots and essential elements of a Coxeter group

The following lemma follows from the discreteness of Φ in V and is useful in proving Theorem 4.11 about critical roots.

Lemma 4.10. ([18]) *Let $x \in U^0$, $N > 0$. Then there are only finitely many roots α with $0 \leq \langle x, \alpha \rangle \leq N$.*

Theorem 4.11. ([18]) *Let $\alpha \in \Phi$, and let $A \subset W$ be the corresponding half-space. Then the following are equivalent.*

(1) $\mathrm{Span}(\langle w \rangle \alpha)$ *is positive definite and $\Sigma_{n=1}^{k} w^n \alpha = 0$, where k is the smallest positive integer with $w^k \alpha = \alpha$.*

(2) $\bigcap_{n \in \mathbf{Z}} w^n A = \bigcap_{n \in \mathbf{Z}} w^n (W - A) = \emptyset$, and α is periodic.

(3) $Q(w) \subset \mu(\alpha)$, where $\mu(\alpha)$ is the wall associated to α.

(4) $Q_2(w) \subset \mu(\alpha)$.

We give the proof (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) to help understand the meaning of this theorem.

Proof. (1) $\Rightarrow$ (2). Let $V_1 = \mathrm{Span}(\langle w \rangle \alpha)$. Note $(\cdot,\cdot)|_{V_1}$ is positive definite and the set of roots is discrete, so $\langle w \rangle \alpha$ is a finite set, and hence α is w-periodic. Suppose $\bigcap_{n\in\mathbf{Z}} w^n A \neq \emptyset$. Let $w_1 \in \bigcap_{n\in\mathbf{Z}} w^n A$. Then $w_1^{-1} w^n \alpha > 0$ for any $n \in \mathbf{Z}$. So $0 = \langle w_1 x, 0 \rangle = \langle w_1 x, \sum_{n=1}^{k} w^n \alpha \rangle = \sum_{n=1}^{k} \langle x, w_1^{-1} w^n \alpha \rangle > 0$ for any $x \in C \subset U^0$, a contradiction. Hence $\bigcap_{n\in\mathbf{Z}} w^n A = \emptyset$. Similarly, $\bigcap_{n\in\mathbf{Z}} w^n (W - A) = \emptyset$.

(2) $\Rightarrow$ (3). We first prove that for a w-periodic α (with period k), $\bigcap_{n\in\mathbf{Z}} w^n A = \emptyset$ implies $Q(w) \subset K(-\alpha) = \{x \in U^0 | \langle x, \alpha \rangle \leq 0\}$. Suppose, on the contrary, there exists $x \in Q(w)$ and $\langle x, \alpha \rangle > 0$. Then $\langle w^{kl} x, \alpha \rangle = \langle x, w^{-kl} \alpha \rangle > 0$ and $f_\alpha(w^{kl} x) = 1$ for $l \in \mathbf{Z}$. Since $f_\alpha(w^n x)$ is monotonic in $n \in \mathbf{Z}$, for $x \in Q(w)$, $f_\alpha(w^n x) = 1/2$ and $\langle w^n x, \alpha \rangle > 0$ for $n \in \mathbf{Z}$. Suppose $x \in w_1(\overline{C})$. Then $w_1^{-1} w^n \alpha > 0$ for $n \in \mathbf{Z}$, and hence $w_1 \in \bigcap_{n\in\mathbf{Z}} w^n A$, a contradiction. Thus $Q(w) \subset K(-\alpha)$. Similarly, α being periodic and $\bigcap_{n\in\mathbf{Z}} w^n (W - A) = \emptyset$ imply $Q(w) \subset K(\alpha)$. So $Q(w) \subset \mu(\alpha)$. $\square$

A half-space, a root or a reflection is called *w-critical* if it satisfies the equivalent conditions of Theorem 4.11.

Proposition 4.12. ([18]) *Let $A \subset W$ be a half-space corresponding to the root α. Then the following are equivalent.*

(1) α is w-odd or w-critical.

(2) $\bigcap_{n\in\mathbf{Z}} w^n A = \bigcap_{n\in\mathbf{Z}} w^n (W - A) = \emptyset$.

As discussed in Chapter 3, the *parabolic closure* $\mathrm{Pc}(w)$ of an element $w \in W$ is defined to be the intersection of all parabolic subgroups containing w.

Theorem 4.13. ([18, p.56]) *The parabolic closure* $\mathrm{Pc}(w)$ *of* w *equals the subgroup of* W *generated by the set of reflections* $\{t_\alpha | \alpha$ *is* w*-odd or* w*-critical*$\}$.

Paris [22] defines an element $w \in W$ to be *essential* if it does not lie in any proper parabolic subgroup of W. So an element w is essential if and only if $\mathrm{Pc}(w) = W$.

Paris shows the existence of essential elements in [22].

Proposition 4.14. (Paris [22]) *Given a Coxeter group* (W, S), *where* $S = \{s_1, s_2, \cdots, s_n\}$, *then* $c = s_n \cdots s_2 s_1$ *is an essential element of* W.

The element c is called a *Coxeter element* of W. In [22] Paris attributes the following result to Krammer [18].

Theorem 4.15. *For an irreducible, infinite Coxeter group* (W, S), *an element* $w \in W$ *is essential if and only if* W *is generated by the set* $\{t_\alpha | \alpha \in \Phi^+$ *and* α *is* w*-odd*$\}$.

This statement does not appear in Krammer's thesis [18], but it can be proved using some results that Krammer has established. A proof of Theorem 4.15 is included at the end of this section for the sake of completeness.

The next result follows from Theorem 4.15 and the discussions at the beginning of Section 4.1.

Corollary 4.16. ([22]) *Assume that* W *is an irreducible, infinite Coxeter group. Let* $w \in W$ *and* p *be a positive integer. Then* w *is essential if and only if* w^p *is essential.*

The following theorem appears in Krammer [18, p.69].

Theorem 4.17. *Assume that* W *is an irreducible, infinite, non-affine Coxeter group. Let* $w \in W$ *be an essential element. Then* $\langle w \rangle = \{w^m | m \in \mathbf{Z}\}$ *is a finite index subgroup of the centralizer* $C(w)$ *of* w *in* W.

Corollary 4.16 and Theorem 4.17 play an important role in the proofs of the theorems stated in the Introduction.

To prove Theorem 4.15, we need some more results from Krammer [18].

Lemma 4.18. ([18, p.54]) *The subgroup generated by* $\{t_\alpha | \alpha \text{ is } w\text{-critical}\}$ *is a finite subgroup of* W.

Let W_1 be the subgroup generated by $\{t_\alpha | \alpha \text{ is } w\text{-odd}\}$ and $\Phi_1 = \{w_1(\alpha) | w_1 \in W_1,\ \alpha \text{ is } w\text{-}odd\}$.

Lemma 4.19. ([18, p.56]) *Let* α *be* w*-periodic, then either* $(\alpha, \beta) = 0$ *for any* $\beta \in \Phi_1$*, or* $\alpha \in \Phi_1$.

Lemma 4.20. ([17, p.131]) *Assume that* (W, S) *is irreducible, then any proper* W*-invariant subspace is contained in the radical* $V^\perp$ *(of the bilinear form), i.e.,* $V^\perp = \{v \in V | (v, \alpha_s) = 0$ *for any* $s \in S\}$.

Proof of Theorem 4.15. Proposition 4.14 states that essential elements exist in a Coxeter group. Now, assume that (W, S) is an irreducible, infinite Coxeter group. For any essential element w of W, $\mathrm{Pc}(w) = W$. It follows from Lemma 4.18 and Theorem 4.13 that Φ_1 is nonempty. Let V_0 be the subspace spanned by those w-critical roots α satisfying the condition that $(\alpha, \beta) = 0$ for any $\beta \in \Phi_1$. Then Theorem 4.13 and Lemma 4.19 imply that V_0 is a W-invariant subspace. Obviously, $V_0 \neq V$, since otherwise, $(\beta, \beta) = 0$ for $\beta \in \Phi_1$, which is absurd. It follows from Lemma 4.20 that $V_0 \subset V^\perp$. This is impossible unless $V_0 = 0$, since any critical root $\alpha \in V_0$ satisfies $(\alpha, \alpha) = 1$. Therefore, if $w \in W$ is an essential element, $W = \mathrm{Pc}(w)$ is generated by w-odd reflections. The converse is clear from Theorem 4.13. □

4.4 Proof of Theorem 4.17 on irreducible, infinite, non-affine Coxeter groups

First we need an inequality.

Proposition 4.21. ([18]) *Let $\alpha \preceq \beta \preceq \gamma$ be roots. Let us write $2(\alpha, \beta) = p + p^{-1}$, $2(\beta, \gamma) = q + q^{-1}$, $2(\alpha, \gamma) = r + r^{-1}$, $p,\ q,\ r \geq 1$. Then $r \geq pq$.*

The proof of Proposition 4.21 is based on the characterization of the Tits cone of a hyperbolic Coxeter group and Lemma 4.22 below. To state the lemma precisely, we need a generalization of the root system we discussed in Chapter 3.

Assume that V is a vector space over $\mathbf{R}$ equipped with a symmetric bilinear form $(\cdot, \cdot)$ and assume that Π is a finite subset of V. The triple $(V, (\cdot, \cdot), \Pi)$ is called a *root basis* (see [18]) if it satisfies the following.

(1) For all $\alpha \in \Pi$, $(\alpha, \alpha) = 1$.

(2) For all different $\alpha,\ \beta \in \Pi$,

$$(\alpha, \beta) \in \{-\cos(\pi/m) | m \in \mathbf{Z},\ \text{and } m > 1\} \cup (-\infty, -1].$$

(3)There exists $x \in V^*$ such that $\langle x, \alpha \rangle > 0$ for all $\alpha \in \Pi$.

Note that Π does not have to span V, nor be linearly independent. Write

$$m_{\alpha\beta} = \begin{cases} m, & \text{if } (\alpha, \beta) = -\cos(\pi/m), \\ \infty, & \text{if } (\alpha, \beta) \leq -1. \end{cases}$$

As in Section 3.1, we define a linear reflection t_α by $t_\alpha \lambda = \lambda - 2(\alpha, \lambda)\alpha$, for $\lambda \in V$. Then the group $W = \langle t_\alpha | \alpha \in \Pi \rangle$ is isomorphic to the Coxeter group $\langle \Pi | (\alpha\beta)^{m_{\alpha\beta}} = 1, \alpha, \beta \in \Pi \rangle$. The root system Φ is defined to be $\{w(\alpha) | w \in W, \alpha \in \Pi\}$. The conclusions established in the preceding sections on the root system, the canonical representations of a Coxeter group, and the Tits cone remain valid in this general situation as the weaker assumption (3) suffices for the proofs .

A *root subbasis* of $(V, (\cdot,\cdot), \Pi)$ is a triple $(V, (\cdot,\cdot), X)$ such that $X \subset \Phi^+$ and for any different α, $\beta \in X$, we have

$$(\alpha, \beta) \in \{-\cos(\pi/m) | m \in \mathbf{Z},\text{ and } m > 1\} \cup (-\infty, -1].$$

A vector space equipped with a quadratic form is said to have signature (k, l, m) if it is isomorphic to $\mathbf{R}^{k+l+m}$ with the quadratic form

$$Q(x_1, \ldots, x_k, y_1, \ldots, y_l, z_1, \ldots, z_m) = (x_1^2 + \cdots + x_k^2) - (y_1^2 + \cdots + y_l^2).$$

Lemma 4.22. ([18]) *A root basis $(V, (\cdot,\cdot), \Pi)$ with $V = $Span$(\Pi)$ and $|\Pi| = 3$ cannot have signature either $(1, 2, 0)$ or $(1, 1, 1)$.*

Proposition 4.21 has the following corollary.

Corollary 4.23. ([18]) *Let $\alpha \preceq \beta \preceq \gamma$ be roots. Then $(\alpha, \beta) \leq (\alpha, \gamma)$, and $(\beta, \gamma) \leq (\alpha, \gamma)$.*

Lemma 4.24. ([18]) *Let α, β be w-outward. Then*

(a) $(\alpha, \beta) > -1$.

(b) The inequality, $(\alpha, w^n\beta) \geq 1$ for almost all $n \in \mathbf{N}$, implies $\alpha \preceq \beta$.

Proof. To prove part (a), suppose that $(\alpha, \beta) \leq -1$. This implies that $\alpha \preceq -\beta$ or $\beta \preceq -\alpha$ by Proposition 4.3. But $\alpha \preceq -\beta$ contradicts the assumption that both α and β are outward, which implies $w^n\alpha > 0$ and $w^n\beta > 0$ for large $n \in \mathbf{N}$. Hence, part (a) must be true. To prove (b), pick $x \in C \subset U^0$. By definition of outwardness, $\langle x, w^n\beta\rangle > 0$ for large $n \in \mathbf{N}$. Now, Lemma 4.10 implies $\lim_{n\to\infty} \langle x, w^n\beta\rangle = \infty$. Hence for large $n \in \mathbf{N}$, $\langle x, w^n\beta - \alpha\rangle > 0$. For these n, the additional assumption $(\alpha, w^n\beta) \geq 1$ implies that $\alpha \preceq w^n\beta$, by Proposition 4.3. □

Lemma 4.25. ([18]) *Let α, β be outward roots such that $(\beta, w^n\alpha)$ is unbounded ($n \in \mathbf{N}$). Then there exists $k \in \mathbf{N}$ such that $w^{-k}\alpha \preceq \beta$.*

Proof. Let $K = \{k \in \mathbf{N} | (w^{-k}\alpha, \beta) = (\alpha, w^k\beta) \geq 1\}$. Then $|K|$ must equal ∞, because if $|K| < \infty$, and the set $\{(\delta, \gamma) | \delta, \gamma \in \Phi\} \cap (-1, 1)$ is finite, it would imply that the sequence $(w^n\alpha, \beta)$ be periodic, contradicting its unboundness. Then arguments similar to those used in proving part (b) of Lemma 4.24 (using a subsequence of positive integers, if necessary) apply to show that there exists $k \in K$ such that $\alpha \preceq w^k\beta$. So, $w^{-k}\alpha \preceq \beta$. □

For the remainder of this section, we assume that (W, S) is irreducible, infinite, non-affine and pick an essential element $w \in W$. So $\mathrm{Pc}(w) = W$ and V is spanned by the outward roots, by Theorem 4.15. The following fact will be useful.

Proposition 4.26. ([18]) *Let $(V, (\cdot, \cdot), \Pi)$ be a root basis, associated with an irreducible, infinite, non-affine Coxeter system (W, S). Then by dividing out the radical $V^\perp$, one obtains again a root basis associated with (W, S).*

The new root basis is $\{\alpha' | \alpha \in \Pi\}$, where the prime denotes the map $V \to V/V^\perp$. One only needs to check the condition (3) of a root basis. To do that, one needs a result from convex set theory ([20, p. 65]) and the fact that for an irreducible Coxeter system (W, S) with root basis $(V, (\cdot, \cdot), \Pi)$, if there exsits $v = \sum_{\alpha \in \Pi} c_\alpha \alpha \neq 0$ in the radical $V^\perp$, such that $c_\alpha \geq 0$ for all $\alpha \in \Pi$, then (W, S) is affine.

Lemma 4.27. ([18])*Let α be odd. Then $(\alpha, w^n\alpha)$ is unbounded ($n \in \mathbf{N}$).*

This lemma does not hold for affine groups, since when the root basis is positive semi-definite, $|(\alpha, \beta)| \leq 1$ for any two roots α, β.

Proof of Lemma 4.27. Since W is irreducible, infinite, non-affine, we may assume that $(V, (\cdot, \cdot))$ is non-degenerate. The set $w^{\mathbf{N}}\alpha$ is a discrete subset of V and it is infinite because α is odd. Hence it is unbounded. Because the odd roots span V and $(\cdot, \cdot)$ is non-degenerate, there exists an odd root β such that $(\beta, w^n\alpha)$ is unbounded $(n \in \mathbf{N})$. We may assume that α and β are outward, if necessary, replacing them by their negatives. Then $\lim_{n\to\infty}(\beta, w^n\alpha) = \infty$. By Lemma 4.25, we have $w^{-k}\alpha \preceq \beta$ for some $k \in \mathbf{N}$. For any $N > 0$, Lemma 4.24 implies that there exists $M > 0$ such that for $n > M$, $\beta \preceq w^n\alpha$ and $(\beta, w^n\alpha) > N$. Now the fact $w^{-k}\alpha \preceq \beta \preceq w^n\alpha$ and Corollary 4.23 imply that $(\alpha, w^{n+k}\alpha) = (w^{-k}\alpha, w^n\alpha) \geq (\beta, w^n\alpha) > N$. Hence $(\alpha, w^n\alpha)$ is unbounded for $n \in \mathbf{N}$. $\square$

Definition 4.28. ([18]) Two roots α and β are said to be *equivalent*, denoted $\alpha \sim \beta$, if $(\alpha, w^n\beta)$ is unbounded, $n \in \mathbf{N}$.

Lemma 4.29. ([18]) *The relation $\sim$ among odd roots is an equivalence relation.*

Proof. Reflexivity: This property is proved in Lemma 4.27.

Symmetry: Let $\alpha \sim \beta$. We may assume α and β to be outward. For any $N \geq 1$, there exists $n > 0$ such that $(\beta, w^n\beta) > N$, by Lemma 4.27. Lemma 4.25 implies that there exists $k \in \mathbf{N}$ such that $w^{-k}\beta \preceq \alpha$. So $\beta \preceq w^k\alpha$. Note $\beta \preceq w^n\beta$, since otherwise we would have $B \subset w^{-ln}(B)$ for $l \in \mathbf{N}$ and hence $w^{ln} \in B$ for large positive integer l, contradicting outwardness of β, where $B = A(\beta)$. Applying Corollary 4.23 to the sequence $\beta \preceq w^n\beta \preceq w^{n+k}\alpha$, we have $(\beta, w^{n+k}\alpha) \geq (\beta, w^n\beta) > N$. Hence $\beta \sim \alpha$.

Transitivity: Let $\alpha \sim \beta$ and $\beta \sim \gamma$ and suppose α, β and γ to be outward. Let $N \geq 1$. Using the method of proving Lemma 4.24, by passing to a subsequence of positive integers, if necessary, we conclude that there exist $m > n > 0$ such that $\alpha \preceq w^n\beta \preceq w^m\gamma$ and $(\alpha, w^n\beta) > N$. By Corollary 4.23, we have $(\alpha, w^m\gamma) \geq (\alpha, w^n\beta) > N$. $\square$

Theorem 4.30. ([18]) *Let α and β be odd. Then $\alpha \sim \beta$ if and only $\langle w\rangle\alpha \not\subset \{\beta\}^\perp$.*

Proof. The "only if"part is trivial. To prove the "if" part, we may assume that $c = -2(\alpha, \beta) \neq 0$, if necessary, replacing β by some $w^n\beta$. Suppose $\alpha \not\sim \beta$, then the sequence $\{(\alpha, w^n\beta) | n \in \mathbf{Z}\}$ is bounded, by the symmetry of the relation "$\sim$". Consider the root

$$\gamma = t_\beta\alpha = \alpha - 2(\alpha, \beta)\beta = \alpha + c\beta.$$

We show γ is odd. Note

$$(\gamma, w^n\gamma) = (\alpha + c\beta, w^n(\alpha + c\beta)) = (\alpha, w^n\alpha) + c^2(\beta, w^n\beta) + \text{bounded},$$

and $(\alpha, w^n\alpha)$, $(\beta, w^n\beta) > -1$ for all n by Lemma 4.24 (or Proposition 4.4). It follows from Lemma 4.27 that $\limsup_{n\to\infty}(\alpha, w^n\alpha) = \infty = \limsup_{n\to\infty}(\beta, w^n\beta)$. Hence $\limsup_{n\to\infty}(\gamma, w^n\gamma) = \infty$. Then Proposition 4.4 implies γ is odd. Note that $(\beta, w^n\gamma) = c(\beta, w^n\beta) + \text{bounded}$, we have $\beta \sim \gamma$. Similarly, $\alpha \sim \gamma$. By the transitivity of "$\sim$", $\alpha \sim \beta$, a contradiction. $\square$

Corollary 4.31. ([18]) *Any two odd roots are equivalent.*

Proof. Suppose that the set of odd roots can be written as $X \coprod Y$, X, $Y \neq \emptyset$, and $X \perp Y$. Since the set of odd roots spans V, it follows that Φ is reducible, a contradiction. Hence the set of odd roots has only one equivalence class under "$\sim$". $\square$

Corollary 4.32. ([18]) *Let α, β be outward. Then $(\alpha, w^n\beta) \to \infty$ as $n \to \infty$, and as $n \to -\infty$.*

Proof. It follows from Lemma 4.27 that there exists $m > 0$ such that $b = (\alpha, w^m\alpha) > 1$. Applying Corollary 4.23 inductively to the sequence $\alpha \preceq w^{nm}\alpha \preceq w^{(n+1)m}\alpha$, we know that $(\alpha, w^{nm}\alpha) \geq b^n \to \infty$ as $n \to \infty$. It suffices to show that for any fixed $r \in \mathbf{N}$, $(\alpha, w^{nm+r}\beta) \to \infty$ as $n \to \infty$. Note that w is essential implies $\mathrm{Pc}(w^m) = W$. By Lemma 4.2 and the definitition of outwardness, α and $w^r\beta$ are w^m-outward. The proof of Lemma 4.24 implies that $\alpha \preceq w^{km+r}\beta$ for some $k \in \mathbf{N}$. Applying Corollary 4.23 to the sequence $\alpha \preceq w^{nm}\alpha \preceq w^{(n+k)m+r}\beta$, we find $(\alpha, w^{(n+k)m+r}\beta) \geq (\alpha, w^{nm}\alpha) \geq b^n \to \infty$ as $n \to \infty$. The "$n \to -\infty$" situation is obtained by interchanging α and β. $\square$

Corollary 4.32 and Lemma 4.24 imply the following.

Corollary 4.33. ([18]) *Let α, β be outward. Then $w^{-n}\alpha \preceq \beta \preceq w^n\alpha$ for almost all $n \in \mathbf{N}$.*

Proof of Theorem 4.17. . Note that for any $g \in C(w)$ and $\alpha \in \mathrm{Out}(w)$, $g\alpha \in \mathrm{Out}(w)$, since $w^k g\alpha = gw^k\alpha$ for any $k \in \mathbf{Z}$ and there are only a finite number of roots separating 1 and g^{-1}. For outward roots α, β and $g \in C(w)$, if $\langle w\rangle g\alpha = \langle w\rangle g\beta$, then $g\alpha = w^k g\beta = gw^k\beta$ for some $k \in \mathbf{Z}$, and $\alpha = w^k\beta$. Hence $C(w)$ permutes the finite set $\langle w\rangle\backslash\mathrm{Out}(w)$. Let G be the kernel of this action. Then $[C(w) : G] < \infty$. We will show that $G = \langle w\rangle$. Let $g \in G$. Choose an outward root α. By multiplying g by a power of w, we may assume $g\alpha = \alpha$. Let β be any outward root. Then $g\beta = w^k\beta$ for some k. Note that for any $n \in \mathbf{Z}$,

$$(\alpha, w^n\beta) = (g\alpha, gw^n\beta) = (\alpha, w^n g\beta) = (\alpha, w^{n+k}\beta).$$

But Corollary 4.31 says that $\{(\alpha, w^n\beta) | n \in \mathbf{Z}\}$ is unbounded. Hence $k = 0$ and $g\beta = \beta$. Since the outward roots span V, $g = 1$. Therefore, $G = \langle w \rangle$. $\square$

The above proof in fact shows that $C(w)/\langle w \rangle$ acts faithfully on $\langle w \rangle \backslash \mathrm{Out}(w)$.

CHAPTER 5

PROOFS OF THE THEOREMS

Now we come to the proofs of Theorems 1.1-1.4 stated in the Introduction.

5.1 Proofs of Theorem 1.1 and 1.2

We need a theorem of Deodhar [11].

Theorem 5.1. *For an irreducible, infinite Coxeter group* (W, S) *and any proper subset* J *of* S, $[W : W_J]$ *is infinite.*

This theorem of Deodhar was discovered again by T. Hosaka [16], using different methods.

Lemma 5.2. *If an irreducible Coxeter group* (W, S) *contains an infinite cyclic subgroup* $\langle x \rangle$ *as a finite index subgroup, then* $W \cong D_\infty$*, the infinite dihedral group.*

Proof. The key point is that W acts on the CAT(0) space $\Sigma = \Sigma(W, S)$ properly and cocompactly. What follows is similar to the proof of the Flat Torus Theorem for CAT(0) spaces (see page 246 in [3]). Since $[W : \langle x \rangle]$ is finite, there is a positive integer k such that $\langle x^k \rangle$ is a normal subgroup. It is a finite index subgroup of W. $\mathbf{Min}(\langle x^k \rangle)$ (the set of points in Σ which are moved minimal distance by x^k) is a nonempty closed subspace of Σ isometric to one of the form $Y \times \mathbf{R}$, where Y is a closed convex subspace of Σ, $\mathbf{R}$ is the set of real numbers. (For any group Γ acting on a CAT(0) space by isometries, $\mathbf{Min}(\Gamma)$ is a certain subspace as defined on page 229 in [3].) x^k acts as a nontrivial translation on the factor $\mathbf{R}$ and acts as an identity map on Y. Because $\langle x^k \rangle \subset W$ is normal, W acts by isometries of $\mathbf{Min}(\langle x^k \rangle)$, preserving the splitting. By properties of CAT(0) spaces, the fixed point set Y_1 of the induced action of the finite group $W/\langle x^k \rangle$ on the complete CAT(0) space Y is a non-empty, closed, convex subset of Y. By construction, $Y_1 \times \mathbf{R}$ is W-invariant and the action of W on the factor Y_1 is the identity. Pick $y \in Y_1$ and consider the action of W on $L = \{y\} \times \mathbf{R}$. The restriction of each s to L is either a reflection or the identity. Since x^k is a translation on L, there are elements s_1 and s_2 in S, which act as different reflections on L. So, the order of $s_1 s_2$ is infinite. By considering the collection of right cosets $\{\langle x^k \rangle (s_1 s_2)^l | l \in \mathbf{Z}\} \subset \langle x^k \rangle \backslash W$, we know there is a positive integer d such that $(s_1 s_2)^d \in \langle x^k \rangle$. Hence, $[W : \langle s_1 s_2 \rangle]$ is finite, and therefore, so is $[W : W_{\{s_1, s_2\}}]$. This is impossible unless $W = W_{\{s_1, s_2\}} = D_\infty$, by the above theorem of Deodhar . $\square$

Proof of Theorem 1.1. Assume that (W, S) is an irreducible, infinite, non-affine Coxeter group and G is a finite index subgroup of W. We prove the center $Z(G)$ of G is trivial. The proof is divided into two cases.

Case 1. $Z(G)$ contains an element x of infinite order. By Proposition 4.14, we can pick an essential element w of W. Since the number of cosets $\{Gw^m | m \in \mathbf{Z}\}$ is finite, there is a positive integer p such that $w^p \in G$. By Corollary 4.16, w^p is essential in W. Since x is in the centralizer $C(w^p)$ of w^p, the number of cosets $\{\langle w^p \rangle x^m | m \in \mathbf{Z}\} \subset \langle w^p \rangle \backslash C(w^p)$ is finite by Theorem 4.17 so there is a positive integer q such that $x^q \in \langle w^p \rangle$. Now, x^q and hence, x is essential in W. Since G is contained in the centralizer $C(x)$ of x, it follows that the index $[G : \langle x \rangle] \leq [C(x) : \langle x \rangle]$

is finite by Theorem 4.17. Therefore, $[W : \langle x \rangle]$ is finite. Now, Lemma 5.2 implies that W is the infinite dihedral group D_∞, which is impossible since D_∞ is affine.

Case 2. $Z(G)$ is a torsion subgroup, i.e., every element in $Z(G)$ has a finite order. A preliminary result used in the proof of the Solvable Subgroup Theorem for CAT(0) spaces (see page 247 in [3]) states that *if a group* Γ *acts properly and cocompactly by isometries on a CAT(0) space, then every abelian subgroup of* Γ *is finitely generated*. It follows that $Z(G)$ is finitely generated and hence, is finite. By Theorem 3.10, the parabolic closure $\mathrm{Pc}(Z(G))$ of $Z(G)$ is a finite parabolic subgroup of W. Without loss of generality, we may assume that $\mathrm{Pc}(Z(G))$ is a finite special parabolic subgroup W_K, where $K \subset S$.

Since $Z(G)$ is normal in G, gW_Kg^{-1} is a parabolic subgroup containing $Z(G)$ for any $g \in G$. By the uniqueness of the parabolic closure (or by the discussion of the rank of the intersection of two parabolic subgroups in Section 3.3 or in [23]), we have $gW_Kg^{-1} = W_K$ and hence, $G \subset N(W_K)$ (the normalizer of W_K in W). Therefore, $[W : N(W_K)]$ is finite. This implies that the set $R_1 = \{wtw^{-1} | t \in K, w \in W\}$ is finite. Now, consider the reflection subgroup W_1 of W generated by R_1. W_1 is a Coxeter group by [12] or [13], with a set S_1 of distinguished generators, where $S_1 \subset \bigcup_{w_1 \in W_1} w_1 R_1 {w_1}^{-1}$. (It is clear that $\bigcup_{w_1 \in W_1} w_1 R_1 {w_1}^{-1} = R_1$ in the present situation.) Hence, the set of reflections in W_1, which by definition is $\{w_1 t_1 w_1^{-1} | t_1 \in S_1, w_1 \in W_1\} (\subset R_1)$, is finite. Therefore, W_1 is a finite Coxeter group. Suppose that $\mathrm{Pc}(W_1) = yW_Ly^{-1}$, where $y \in W$ and L is a proper subset of S. Because W_1 is normal in W, yW_Ly^{-1} is a (proper) normal subgroup of W. This is impossible, since any proper nontrivial special subgroup of an irreducible Coxeter group is not normal (see [17, p.

118, Exercise 2]). In the current situation, we need to replace the distinguished set S of generators by ySy^{-1}.

In conclusion, $Z(G) = \{1\}$. This finishes the proof of Theorem 1.1. $\square$

The proof of Theorem 1.2 is similar.

Proof of Theorem 1.2. Let G be a finite index subgroup of an irreducible, infinite, non-affine Coxeter group. Suppose that $G = A \times B$, where A and B are nontrivial subgroups. Pick an essential element w in W. Following the idea in the proof of Theorem 1.1, we know that there is a positive integer p such that $w^p \in G$. Now w^p is essential and $w^p = ab$ for some $a \in A$ and $b \in B$. At least one of them, say, a has an infinite order, because w^p has an infinite order, and a and b commute. Notice that since $aw^p = w^p a$, a is in the centralizer $C(w^p)$ of w^p. By considering the collection of cosets $\{\langle w^p \rangle a^m | m \in \mathbf{Z}\}$, we conclude that there is a positive integer q such that $a^q \in \langle w^p \rangle$. So, a is an essential element. Since each element of B commutes with a, $B \subset C(a)$, the centralizer of a. Since the collection of cosets $\{\langle a \rangle h | h \in B\}$ is finite and $A \cap B = \{1\}$, we know that B is finite. Without loss of generality, we can assume that $\mathrm{Pc}(B) = W_I$, a finite parabolic subgroup. Because B is normal in G, the uniqueness of parabolic closure implies that $G \subset N(W_I)$. So, $[W : N(W_I)]$ is finite. The arguments in Case 2 of the proof of Theorem 1.1 now apply to derive a contradiction. Hence, G cannot be expressed as a product of two nontrivial subgroups. The conclusion of Theorem 1.2 holds. $\square$

5.2 Proof of Theorem 1.3 and examples

Proof of Theorem 1.3. Assume that

$$W = W_1 \times W_2 \times \cdots \times W_n \text{ (internal direct product)}, \tag{5.1}$$

where each W_i is an irreducible, infinite, non-affine Coxeter group. Let H be a finite index subgroup of G. First we prove that $Z(H) = \{1\}$.

Denote by p_i the projection $p_i : W \to W_i$ of G to the ith factor W_i. Let $c = \prod_{i=1}^{n} c_i \in Z(H)$, the center of H, where $c_i \in p_i(H) \subset W_i$. Pick an arbitrary $h = \prod_{i=1}^{n} h_i \in H$ with $h_i \in p_i(H)$. Then $ch = hc$. This implies that $\prod_{i=1}^{n}(c_i h_i) = \prod_{i=1}^{n}(h_i c_i)$, so, $c_i h_i = h_i c_i$ for any i. Notice that since $h_i \in p_i(H)$ is arbitrary, $c_i \in Z(p_i(H))$. Because $W_i = p_i(W)$ and $[p_i(W) : p_i(H)] = [W : p_i^{-1}(p_i(H))] \leq [W : H] < \infty$, Theorem 1.1 implies that $c_i = 1$. Hence, $Z(H) = \{1\}$.

Now we use induction on n, the number of factors in expression (5.1), to prove that if a finite index subgroup H of W can be expressed as

$$H = H_1 \times H_2 \times \cdots \times H_m \text{ (internal direct product)}, \tag{5.2}$$

where each H_i is a nontrivial subgroup, then $m \leq n$.

The case $n = 1$ is just Theorem 1.2. Now, assume $n \geq 2$ and $m \geq 2$. Notice that $p_1(H) = p_1(H_1)p_1(H_2)\cdots p_1(H_m)$, and

$$p_1(H_i) \bigcap \{p_1(H_1)\cdots p_1(H_{i-1})p_1(H_{i+1})\cdots p_1(H_m)\} = \{1\},$$

because this intersection is contained in $Z(p_1(H))$, which is trivial by Theorem 1.1 (knowing that $p_1(H)$ is a finite index subgroup of W_1). Hence

$$p_1(H) = p_1(H_1) \times \cdots \times p_1(H_m).$$

By Theorem 1.2 only one of the factors on the right-hand side, say, $p_1(H_1)$, can be nontrivial, and all other $p_1(H_j)$ $(j \neq 1)$ are trivial. So, $p_1(H_1) = p_1(H)$. Without loss of generality, we can assume that $p_i(H_1)$ is nontrivial for $i = 1, \ldots, l$, and is trivial for $i = l+1, \ldots, m$. This implies that $p_i(H_j) = \{1\}$ for $i = 1, \ldots, l$, $j \neq 1$. Hence,

$$H_1 \subset W_1 \times \cdots \times W_l, \ H_2 \times \cdots \times H_m \subset W_{l+1} \times \cdots \times W_n.$$

Now use the induction hypothesis and the following simple fact (the proof of which is left to the reader),

Lemma 5.3. *Let G_1 and G_2 be two groups. If N_i is a subgroup of G_i, $i = 1, 2$ and $[G_1 \times G_2 : N_1 \times N_2] < \infty$, then $[G_i : N_i] < \infty$, $i = 1, 2$, and $[G_1 \times G_2 : N_1 \times N_2] = [G_1 : N_1][G_2 : N_2]$.*

We conclude that $m \leq n$.

Having this inequality in mind, we may continue to decompose some factors in expression (5.2), until each factor cannot be further decomposed. (It is a finite step procedure due to the above inequality.) From now on, when we talk about a decomposition of form (5.2), we assume that each factor cannot be further decomposed.

We prove the above decomposition is unique, up to the rearrangement of factors. Suppose that there is another decomposition

$$H = K_1 \times \cdots \times K_r, \tag{5.3}$$

where each factor K_j cannot be further decomposed. Let $q_j : H \to K_j$ be the projection of H onto its jth factor in the decomposition (5.3). We know that $K_i = q_i(H) = q_i(H_1)q_i(H_2)\cdots q_i(H_m)$. And, notice that $q_i(H_k) \bigcap \{\prod_{j \neq k} q_i(H_j)\} = \{1\}$, because this intersection is contained in the center $Z(K_i)$ of K_i, and $Z(K_i) \subset Z(H)$, while the latter is trivial by the first part of this theorem, we have the following,

$$K_i = q_i(H_1) \times \cdots \times q_i(H_m).$$

By the assumption that K_i cannot be further decomposed, $K_i = q_i(H_{\phi(i)})$ for some $\phi(i)$, and $q_i(H_j)$ is trivial for $j \neq \phi(i)$. So, ϕ defines a map from $\{1, 2, \ldots, r\}$ to $\{1, 2, \ldots, m\}$, and it is surjective because, for any $j \in \{1, 2, \ldots, m\}$, there is an $i \in \{1, 2, \ldots, r\}$ such that the restriction $q_i|_{H_j}$ is nontrivial. Hence $m \leq r$. Similar discussions also yield $r \leq m$. Therefore, $m = r$ and ϕ is a bijection. After re-indexing, we may assume $\phi = \mathrm{id}$ (the identity map). This means that $H_i \subset K_i$ and the restriction $q_i|_{H_i}$ is indeed the inclusion $H_i \hookrightarrow K_i$. Because

$$H = H_1 \times \cdots \times H_m = K_1 \times \cdots \times K_m,$$

we know that $H_i = K_i$, for $i = 1, \ldots, m$. This is the claimed unique decomposition of H and m is determined uniquely by H. The proof of Theorem 1.3 is completed. □

The situation that $m < n$ may happen. To demonstrate this, we need the following lemma.

Lemma 5.4. *Let $W = W_1 \times W_2$, where W_1 and W_2 are irreducible, infinite, non-affine Coxeter groups. If H is a finite index subgroup of W and $H = H_1 \times H_2$, where $H_i \neq \{1\}$, $i = 1, 2$, then after re-indexing, $H_i \subset W_i$.*

Proof. As in the proof of Theorem 1.3, let p_i be the projection $p_i : W \to W_i$. It follows that $p_1(H)$ is a finite index subgroup of W_1 and $p_1(H) = p_1(H_1) \times p_1(H_2)$ by repeating the arguments in the proof of Theorem 1.3. In this product decomposition, only one factor, say, $p_1(H_1)$ is nontrivial by Theorem 1.2. So, $p_1(H) = p_1(H_1)$ and

$p_1(H_2) = \{1\}$. This implies that $H_2 \subset W_2$. So, $p_2(H_2) \neq \{1\}$. By Theorem 1.2, $p_2(H_1) = \{1\}$, $H_1 \subset W_1$. $\square$

In the following two examples, let (W_i, S_i) be an irreducible, infinite, non-affine Coxeter group, $i = 1, 2$. Suppose that $W = W_1 \times W_2$ and $S = S_1 \cup S_2$.

Example 5.5. (M. Davis) Suppose that for some nontrivial finite group G, there are surjective homomorphisms $f_i : W_i \to G$ $(i = 1, 2)$. Let $H = \{(w_1, w_2) \in W | f_1(w_1) = f_2(w_2)\}$. Then $[W : H] = |G|$, $p_1(H) = W_1$ and $p_2(H) = W_2$, where p_i is the projection defined in Lemma 5.4. It follows from Lemma 5.4 that H cannot be further decomposed.

Example 5.6. Let $\phi : W \to D_1$ be the surjective homomorphism such that $\phi(s) = -1$ for any $s \in S$, where $D_1 = \{-1, 1\}$. Let $H = \ker(\phi)$. Then $[W : H] = 2$. If H had a decomposition $H = H_1 \times H_2$, $H_i \neq \{1\}$, then one of the factors, say, H_1 would be W_1 because of Lemma 5.4 and Lemma 5.3 and $[W : H] = 2$. This would imply that $\phi(s_1) = 1$ for any $s_1 \in S_1$, a contradiction. Hence, $H = \ker(\phi)$ cannot be further decomposed.

5.3 Proof of Theorem 1.4

Proof of Theorem 1.4. Let $w \neq 1$ be an element of W. We show that $|\{gwg^{-1} | g \in W\}| = \infty$. To do this, we need to prove that $[W : C(w)] = \infty$. The proof is divided into two cases.

Case 1. The order of w is finite. In this case, the parabolic closure $\mathrm{Pc}(\langle w \rangle)$ of $\langle w \rangle$ is a finite parabolic subgroup. Without loss of generality, assume that $\mathrm{Pc}(\langle w \rangle) = W_K$, where $K \subset S$. The uniqueness of the parabolic closure and the fact that $gwg^{-1} = w$

for any $g \in C(w)$ imply that $gW_Kg^{-1} = W_K$ for $g \in C(w)$. Hence, $C(w) \subset N(W_K)$. The discussion of Case 2 in the proof of Theorem 1.1 yields that $[W : N(W_K)] = \infty$ when W is an irreducible, infinite Coxeter group and W_K is a nontrivial finite special subgroup. Therefore, in this situation, $[W : C(w)] = \infty$.

Case 2. The order of w is infinite. Suppose that $[W : C(w)] < \infty$. Pick an essential element $x \in W$. Since the number of cosets $\{C(w)x^k | k \in \mathbf{Z}\}$ is finite, there is a positive integer m such that $x^m \in C(w)$. By Corollary 4.16, x^m is essential. Notice that $w \in C(x^m)$ and the number of cosets $\{\langle x^m \rangle w^l | l \in \mathbf{Z}\}$ in $C(x^m)$ is finite because of Theorem 4.17, so we conclude that there is a positive integer n such that $w^n \in \langle x^m \rangle$. Now, Corollary 4.16 implies that w^n is essential, and hence, so is w. Then, by Theorem 4.17, we have $[W : \langle w \rangle] < \infty$. It follows from Lemma 5.2 that $W = D_\infty$, contradicting the assumption that W is non-affine.

In conclusion, $[W : C(w)] = \infty$ for $w \neq 1$. The conclusion of the theorem follows immediately. $\square$

Ian Leary pointed out a connection between this theorem and Theorem 1.1. Indeed, a more general result than Theorem 1.4 can be proved easily from Theorem 1.1, without using any facts about Coxeter groups.

Proposition 5.7. *Let G be a group such that for every finite index subgroup H, the center of H is trivial. Then for every finite index subgroup H of G, every non-identity conjugacy class in H is infinite.*

Proof. For K a group and $l \in K$, let $C_K(l)$ be the centralizer of l in K:

$$C_K(l) = \{k \in K | kl = lk\}.$$

For any K, and any $l \in K$, the map $k \mapsto klk^{-1}$ induces a bijection between the set $K/C_K(l)$ of cosets of $C_K(l)$ and the set $\{klk^{-1} | k \in K\} \subset K$ of conjugates of l.

Now let H be a finite index subgroup of G, and let $h \in H$. Define C by $C = C_H(h)$. If h has only finitely many conjugates in H, then by the above remark it follows that $[H : C]$ is finite, and hence that $[G : C] = [G : H] \cdot [H : C]$ is finite. But h is in the center of C, and so $h = 1$. $\square$

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