Pergamon

Topology Vol. 34, No. 2, pp. 329–350, 1995

0040-9383/95 $9.50 + 0.00

0040-9383(94)00027-1

STRICT HYPERBOLIZATION

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(Received 12 September 1992; in revised form 22 March 1994)

0. INTRODUCTION

IN [12, Section 3.4] Gromov described several procedures for converting a cell complex K into a new polyhedron $\mathscr{H}(K)$ with a piecewise Euclidean metric of nonpositive curvature—$\mathscr{H}(K)$ is called a "hyperbolization" of K. Expositions of this idea are given in [9, 21, 17].

Gromov claims in [12, Section 4.3] that in two of his constructions the metric can be perturbed so that the curvature is strictly negative. A "proof" of this claim, and a similar claim of the "product with interval" construction of [9, p. 377], is provided in [21, Theorem 4.3(iv)]. However, there is a gap in the proof; in fact, as we shall see in Section 4, the claim is false.

The goal of this paper is to find a "strict" hyperbolization, i.e. a procedure in which the result has curvature bounded above by some negative number. (Here the notion of "curvature bounded from above" is defined via comparison triangles as in [1, 12, 11].) The motivation for finding a strict hyperbolization procedure is that the fundamental group of a compact, strictly negatively curved space is "word hyperbolic" in the sense of [11–13]. Hence, strict hyperbolization gives a method for producing examples of word hyperbolic groups. One application of our strict hyperbolization procedure is to show that any triangulable manifold is cobordant to a triangulable manifold of strictly negative curvature (Theorem 7.7). This strengthens Gromov's result that any triangulable manifold is cobordant to a nonpositively curved one. Other applications may be found in [9, Section 5c; 2].

In all previous hyperbolization constructions, the result is naturally a cubical cell complex where each cube is isometric to a regular Euclidean cube. The naive idea for perturbing the metric is to replace each Euclidean cube by a regular cube in hyperbolic space. We shall call this the "naive perturbation". As was pointed out in [12, p. 123], the naive perturbation may fail to have curvature bounded from above. This can be understood by considering a two-dimensional example. Suppose a surface S is tiled by Euclidean squares in such a fashion that at least four squares meet at each vertex. The sum of the angles at each vertex is then $\geq 2\pi$. It follows that S is nonpositively curved. (The "curvature at vertex" is 0 if the angle sum is equal to 2π and $-\infty$ if it is greater than 2π.) For a square in the hyperbolic plane each interior angle is $<\pi/2$; hence, if we naively perturb the metric on S, the angle sum at a vertex where only four squares meet will be $<2\pi$. In effect, the metric will have curvature $+\infty$ at such a vertex. The problem occurs whenever a link of a vertex contains a circuit with four edges. Such circuits in links of cubes cause similar problems in

†Partially supported by NSF grant DMS-8905378.

all dimensions. (The condition that no such circuit occurs is referred to as "Siebenmann's no $\square$ condition" in [12, p. 123].)

The situation can be remedied as follows. We shall prove (as Corollary 6.2) the following result.

THEOREM. *In each dimension n there is a compact, connected, orientable, hyperbolic manifold with corners X^n such that* (a) *the codimension* 1 *faces of X^n are totally geodesic and intersect orthogonally and* (b) *the poset of faces of X^n is isomorphic to the poset of faces of an n-cube $\square^n$. In particular, the link of a face in X^n is isometric to the link of a corresponding face in $\square^n$.*

Rather than use the naive perturbation, our approach is to replace each Euclidean cube in $\mathscr{H}(K)$ by an appropriate face of X^n. The result will be piecewise hyperbolic and (since links are unchanged) will have curvature bounded above by -1. This gives the following result which is proved as Theorem 7.6, below.

THEOREM. *There is a strict hyperbolization procedure which associates to a simplicial complex K a piecewise hyperbolic space $\mathscr{G}_X(K)$ of curvature ≤ -1.*

Note, however, that this is not a perturbation: we have altered the topology. Strict hyperbolization has, therefore, been accomplished in two steps: first, we use one of Gromov's techniques to construct $\mathscr{H}(K)$ and then we replace the cubes in $\mathscr{H}(K)$ by faces of X^n to obtain $\mathscr{G}_X(K)$.

1. PRELIMINARIES CONCERNING POLYHEDRA OF PIECEWISE CONSTANT CURVATURE

Constant curvature space. For $\chi \in \mathbb{R}$ let $\mathbb{M}^n_\chi$ denote the n-dimensional, complete, simply-connected Riemannian manifold of constant sectional curvature χ (called *constant curvature space*). For $\chi > 0$, $\mathbb{M}^n_\chi$ is the n-sphere of radius $1/\sqrt{\chi}$, for $\chi = 0$ it is the Euclidean n-space $\mathbb{R}^n$, and for $\chi < 0$ it is the hyperbolic n-space of curvature χ. We shall use the notations $\mathbb{S}^n$ and $\mathbb{H}^n$ for $\mathbb{M}^n_1$ and $\mathbb{M}^n_{-1}$, respectively.

Cells: In $\mathbb{M}^n_\chi$ the notion "half-space" makes sense. A *cell* (="convex polytope") in $\mathbb{M}^n_\chi$ is defined to be a nonempty compact intersection of a finite number of half-spaces and hyperplanes. If $\chi > 0$, then we also require that a cell contain no pair of antipodal points.

If C is a cell, then $\mathscr{F}(C)$ denotes its set of faces, partially ordered by inclusion. Its derived complex $\mathscr{F}(C)'$ is the poset of chains in $\mathscr{F}(C)$; as an abstract simplicial complex it can be identified with the poset of simplices in the barycentric subdivision of C. A *combinatorial equivalence* from a cell C_1 to another C_2 is an isomorphism of posets $\varphi : \mathscr{F}(C_1) \to \mathscr{F}(C_2)$. Such a φ induces a simplicial isomorphism $\varphi : \mathscr{F}(C_1)' \to \mathscr{F}(C_2)'$ and the geometric realization of φ' is a face-preserving PL-homeomorphism from C_1 to C_2. Any cell in $\mathbb{M}^n_\chi$ is combinatoricaly equivalent to one in $\mathbb{R}^n$.

A cell C is *regular* if its isometry group acts transitively on the set of top-dimensional simplices in $\mathscr{F}(C)'$. It follows that for a regular cell, its isometry group is equal to its group of combinatorial automorphisms.

Cell complexes: By a *combinatorial cell complex* (or simply a "cell complex") we shall mean a space X formed by gluing together cells via (geometric realizations of) combinatorial equivalences of their faces, together with the decomposition of X into cells. We want to allow the possibility that distinct cells intersect in a union of proper faces (rather than just one face); however, each cell should be embedded (i.e. we do not want to allow the possibility of gluing together two faces in the boundary of a single cell). We shall also always require that the decomposition into cells be locally finite. The space X is called the *underlying polyhedron* of the combinatorial cell complex.

A cell complex is *simplicial* if each cell is a simplex; it is *cubical* if each k-cell is combinatorially equivalent to a k-cube.

To define an $\mathbb{M}_\chi$*-cell complex* (or a "geometric cell complex") one requires that the space X be formed by gluing together cells in $\mathbb{M}_\chi^n$, for some n, via isometries of their faces. (For a precise definition, see [21, Definition 3.4, p. 332].)

A metric on a space Y is *intrinsic* (or "inner") if the distance between any two points y_0 and y_1 is the infimum of the lengths of paths connecting them. If, in addition, this infimum can always be realized by a path of minimal length then Y is a *geodesic space* (or a "length space"). The image of such a minimal length path is a *geodesic segment* in Y. A subspace Z of a geodesic space Y is *totally geodesic* (or "locally convex") if, locally, any geodesic segment between two points in Z lies entirely in Z.

Arc length makes sense in the underlying polyhedron X of an $\mathbb{M}_\chi$-cell complex; hence, X has a natural intrinsic metric: the distance between two points is the infimum of the lengths of paths connecting them. (This metric agrees locally with the given metric on each cell of X, so after suitable subdivision, we may assume that the intrinsic metric restricts to the given metric on each cell.) The underlying polyhedron of an $\mathbb{M}_\chi$-cell complex together with its natural metric is a *polyhedron of piecewise constant curvature* χ. We shall also say that X is *piecewise spherical, Euclidean* or *hyperbolic*, as $\chi = +1, 0$, or -1, respectively.

Definition 1.1. An $\mathbb{M}_0$-cell complex is called a *cubical Euclidean cell complex* if each k-cell is isometric to a regular Euclidean k-cube.

Remark 1.2. Since the group of combinatorial automorphisms of a regular cube is equal to its isometry group, it follows that to each combinatorial cubical cell complex there corresponds a cubical $\mathbb{M}_\chi$-cell complex (in which each cube is regular). In particular, for $\chi = 0$, we get a cubical Euclidean cell complex. Similarly, given an n-dimensional simplicial complex we can give it an $\mathbb{M}_\chi$-structure by declaring each simplex to be a regular simplex in $\mathbb{M}_\chi^n$.

Links: The *link* of a k-face of an n-cell in $\mathbb{M}_\chi^n$ is the set of unit vectors normal to the k-plane supported by the k-face and pointing into the n-cell. Such a link is naturally an $(n-k-1)$-cell in $\mathbb{S}^{n-1}$.

Given a k-cell σ in an $\mathbb{M}_\chi$-cell complex P, the links of σ in the cells containing it fit together to form an $\mathbb{M}_1$-cell complex, denoted $\mathrm{Link}(\sigma, P)$. Thus, the link of any cell has a natural piecewise spherical structure.

Definition 1.3. A regular spherical k-simplex is *all right* if the length of each edge is $\pi/2$. (This terminology is due to G. Moussong.) Thus, a k-simplex is all right if it is isometric to the simplex in $\mathbb{S}^k$ spanned by the standard basis of $\mathbb{R}^{k+1}$. An $\mathbb{M}_1$-simplicial complex is *all right* if each of its simplices are all right.

Observation 1.4. The link of a k-face of a regular n-cube is an all right $(n-k-1)$-simplex. Hence, the link of any cell of a cubical Euclidean cell complex is all right.

Curvature bounded from above: The notion of a triangle in a geodesic space Y has an obvious meaning: "edges" are geodesic segments. Using comparison triangles and the so-called "CAT-inequalities" (cf. [12, p. 106]) one can define what it means for Y to have curvature bounded above by a real number χ: given a triangle T in a sufficiently small open set of Y and a vertex x of T, the distance from x to a point on the opposite edge must be less than or equal to the corresponding distance in a comparison triangle in $\mathbb{M}^2_\chi$.

Definition 1.5. A piecewise spherical polyhedron L is *large* if there is a unique geodesic connecting any two points of distance $<\pi$. (Equivalently, L is large if it satisfies $CAT(1)$.) The *systole* of L, denoted $sys(L)$, is the infimum of the lengths of all closed geodesics in L.

These concepts are related as follows: L is large if and only if L and the link of every cell in L has systole $\geq 2\pi$ (see [5, Theorem 3.1]).

The following is a result of Gromov [12, 4.2.A, p. 120] (see Ballman's article in [11, Chapter 10] for a proof).

THEOREM 1.6 (Gromov). *An $\mathbb{M}_\chi$-cell complex has curvature $\leq \chi$ if and only if the link of each cell is large.*

Definition 1.7 (compare [4, p. 28]). An abstract simplicial complex K is a *flag complex* if anytime $K^{(1)}$ contains the complete graph on a finite set of vertices S then it actually contains the full simplex on S. (K is a flag complex if and only if K and all links of simplices in K satisfy Gromov's "no Δ condition", cf. [12, p. 122].) The simplicial complex K satisfies the *no $\square$ condition* if any circuit of length 4 in $K^{(1)}$ is the boundary of the union of two 2-simplices along a common edge.

For a proof of the following result see [12, p. 122], as well as [20].

LEMMA 1.8. (Gromov). *Let L be an all right $\mathbb{M}_1$-simplicial complex.*

(i) *L is large if and only if it is a flag complex.*
(ii) *If L is a flag complex and, in addition, satisfies the no $\square$ condition, then $sys(K) > 2\pi$.*

COROLLARY 1.9 (Gromov). *If the link of each cell in a cubical Euclidean complex is a flag complex, then it is nonpositively curved.*

COROLLARY 1.10 (Gromov). *If the link of each cell in a combinatorial cubical cell complex X is a flag complex and satisfies the no $\square$ condition, then X can be given the structure of a cubical $\mathbb{M}_{-1}$-cell complex of curvature ≤ -1.*

The $\mathbb{M}_{-1}$-cell complex in Corollary 1.10 is constructed by the "naive perturbation" mentioned in the Introduction.

2. SOME HYPERBOLIZATION PROCEDURES

All hyperbolization constructions work roughly as follows: the hyperbolization of an n-cell is defined to be some nonpositively curved n-manifold with (totally geodesic) boundary, the hyperbolization of a cell complex K is then constructed by gluing together hyperbolized cells according to the same combinatorial pattern as are the cells of K. Since a hyperbolized cell will generally not be homeomorphic to a cell, the topology of $\mathscr{H}(K)$ is drastically altered; however, some important properties can be retained.

Each of the various hyperbolization constructions has advantages and disadvantages. We list below some properties which one might wish such a procedure $\mathscr{H}$ to have.

(1) (*Functorality*). $\mathscr{H}$ is a functor from cell complexes to nonpositively curved polyhedra in the following sense: if $i: J \to K$ is an embedding onto a subcomplex, then there is a mapping $\mathscr{H}(i): \mathscr{H}(J) \to \mathscr{H}(K)$ which is an isometric embedding onto a totally geodesic subspace.

(2) (*Preservation of local structure*). If σ^n is an n-cell in K, then $\mathscr{H}(\sigma^n)$ is an n-manifold with boundary and the link of $\mathscr{H}(\sigma)$ in $\mathscr{H}(K)$ is PL-homeomorphic to the link of σ in K.

(3) $\mathscr{H}$ (a point) = a point.

(4) (*Orientability*). For a cell σ, the manifold with boundary $\mathscr{H}(\sigma)$ is orientable.

If conditions (1) and (2) holds, then there is a continuous map $\varphi: \mathscr{H}(K) \to K$, well-defined up to homotopy, such that $\varphi(\mathscr{H}(\sigma)) \subset \sigma$ for all σ in K. Hence, we can require the following condition.

(5) (*Homological surjectivity*). The map $\varphi: \mathscr{H}(K) \to K$ induces a surjection on homology.

Condition (5) is equivalent to conditions (3) and (4). The importance of condition (3) (which in fact does not hold for all hyperbolization procedures) is that it is needed to prove Theorem 7.7 which states that every closed triangulable manifold in cobordant to a negatively curved one.

Conditions (2) and (4) imply that if K is a manifold (resp. orientable manifold) then so is $\mathscr{H}(K)$. Another condition we might require is the following.

(6) (*Covered by bundle map*). If the underlying polyhedron of K is a manifold, then $\varphi: \mathscr{H}(K)) \to K$ is covered by a map between the stable tangent bundles.

A weaker version of (6) is the following.

(6′). (*Pontryagin classes*). If K is a manifold, then the map φ pulls back the rational Pontryagin classes of K to those of $\mathscr{H}(K)$.

Following [21] we now discuss in detail the "product with interval procedure" of [9, Section 4b] and the "Moebius band procedure" of [12, Section 3.4, p. 114]. The product with interval procedure does not satisfy condition (3) above, while the Moebius band procedure does not satisfy condition (4) (orientability). There is, however, a closely related construction of Gromov [12, p. 116] which does satisfy conditions (1)–(6). It is described in detail in [9, Section 4c] and more briefly in Section 7 of this paper. On the other hand, as Gromov points out, this last procedure yields spaces whose fundamental groups obviously contain copies of $\mathbb{Z} \times \mathbb{Z}$ and hence are not word hyperbolic.

Let $\mathscr{C}_n$ denote the category of cell complexes of dimension $\leq n$. A morphism $\varphi: P \to P'$ in $\mathscr{C}_n$ is an isomorphism of P onto a subcomplex of P'. Let $\mathscr{P}\mathscr{E}_n$ denote the category of piecewise Euclidean polyhedra of dimension $\leq n$. A morphism $\theta: X \to X'$ in $\mathscr{P}\mathscr{E}_n$ is an isometry onto a totally geodesic subpolyhedron of X'.

The product with interval procedure: We shall define, for each integer $n \geq 1$, a functor $\mathscr{I}_n$ from $\mathscr{C}_n$ to $\mathscr{P}\mathscr{E}_n$. In fact, for a cell complex P, the polyhedron $\mathscr{I}_n(P)$ will have the structure of a cubical Euclidean cell complex.

If $\dim P \leq 1$, then put $\mathscr{I}_1(P) = P$ and declare each edge to be isometric to the interval $I = [-1, 1]$; $\mathscr{I}_1$ also leaves morphisms unchanged.

Let $P^{(k)}$ denote the k-skeleton of P. Suppose, inductively, that $\mathscr{I}_k$ has been defined for $k < n$. We define $\mathscr{I}_n$ as follows. If $\dim P < n$, then put $\mathscr{I}_n(P) = \mathscr{I}_{n-1}(P) \times \{-1, +1\}$. If σ^n is an n-cell, put $\mathscr{I}_n(\sigma^n) = \mathscr{I}_{n-1}(\partial\sigma^n) \times I$. Finally, if P is an arbitrary n-dimensional complex, then define $\mathscr{I}_n(P)$ to be the result of gluing a copy of $\mathscr{I}_n(\sigma^n)$ onto $\mathscr{I}_n(P^{(n-1)})$ for each n-cell σ^n

in P via the isometric embedding $\mathscr{I}_{n-1}(i)\times id:\mathscr{I}_{n-1}(\partial\sigma^n)\times\{-1,+1\}\to\mathscr{I}_{n-1}(P^{(n-1)})\times\{-1,+1\}$, where $i:\partial\sigma^n\to P^{(n-1)}$ denotes the inclusion.

By the inductive hypothesis, $\mathscr{I}_{n-1}(P^{(n-1)})$ and $\mathscr{I}_{n-1}(\partial\sigma^n)$ are both cubical Euclidean cell complexes. Since the product of a regular $(n-1)$-cube with I is a regular n-cube, $\mathscr{I}_n(\sigma^n)$ $(=\mathscr{I}_{n-1}(\partial\sigma^n)\times I)$ is a cubical Euclidean cell complex; hence, so is $\mathscr{I}_n(P)$.

If $\varphi:P\to P'$ is a morphism in $\mathscr{C}_n$ and $\dim P<n$, then $\mathscr{I}_n(\varphi)$ is defined to be $\mathscr{I}_{n-1}(\varphi)\times id_{\{-1,+1\}}$. In the general case, this gives a definition for the restriction of $\mathscr{I}_n(\varphi)$ to $P^{(n-1)}$. The definition is extended over $\mathscr{I}_n(\sigma^n)$, σ^n an n-cell of P, by the formula $\mathscr{I}_n(\varphi\,|\,\sigma^n)=\mathscr{I}_{n-1}(\varphi\,|\,\partial\sigma^n)\times id_I$. It is not difficult to show that the image of $\mathscr{I}_n(P)$ is totally geodesic in $\mathscr{I}_n(P')$.

LEMMA 2.1. *The group $(\mathbb{Z}/2)^{n-1}$ is a group of natural automorphisms of $\mathscr{I}_n$.*

Proof. For any object P in $\mathscr{C}_n$ we need to define $(n-1)$ commuting involutions $\tau_{n,1},\dots,\tau_{n,n-1}$ of $\mathscr{I}_n(P)$. The involution $\tau_{n,1}$ switches the two copies of $\mathscr{I}_{n-1}(P^{(n-1)})$ and acts on $\mathscr{I}_n(\sigma^n)$ $(=\mathscr{I}_{n-1}(\partial\sigma^n)\times I)$ as $id\times r$ where $r:I\to I$ is reflection. Suppose by induction that we have defined commuting natural involutions $\tau_{k,1},\dots,\tau_{k,k-1}$ of $\mathscr{I}_k$, for $2\le k<n$. For $i>1$ define $\tau_{n,i}$ on $\mathscr{I}_{n-1}(P^{(n-1)})\times\{-1,+1\}$ to be $\tau_{n-1,i-1}\times id_{\{-1,+1\}}$ and on $\mathscr{I}_n(\sigma^n)$ to be $\tau_{n-1,i-1}\times id_I:\mathscr{I}_{n-1}(\partial\sigma^n)\times I\to\mathscr{I}_{n-1}(\partial\sigma^n)\times I$. ∎

Remark 2.2. The orbit space $\mathscr{I}_n(P)/\tau_{n,1}$ can be identified with the subspace of $\mathscr{I}_n(P)$ consisting of $\mathscr{I}_{n-1}(P^{(n-1)})\times\{+1\}$ with a copy of $\mathscr{I}_{n-1}(\partial\sigma^n)\times[0,1]$ glued on for each n-cell σ^n. But this subspace obviously deformation retracts onto $\mathscr{I}_{n-1}(P^{(n-1)})$. Continuing in this fashion we see that $\mathscr{I}_n(P)/(\mathbb{Z}/2)^{n-1}$ can be identified with a subspace of $\mathscr{I}_n(P)$ and that this subspace deformation retracts onto $P^{(1)}$. Thus, $P^{(1)}$ is a retract of $\mathscr{I}_n(P)$.

The Moebius band procedure: Let $\mathscr{C}_n^{\text{cub}}$ be the full subcategory of $\mathscr{C}_n$ consisting of the cubical cell complexes of dimension $\le n$. By a construction very similar to the product with intervals procedure, we shall define a functor $\mathscr{M}_n:\mathscr{C}_n^{\text{cub}}\to\mathscr{PE}_n$ called the "Moebius band procedure". One advantage of this construction over the product with interval procedure is that it is independent of n, that is to say, the restriction of $\mathscr{M}_n$ to the full subcategory $\mathscr{C}_{n-1}^{\text{cub}}$ is equal to $\mathscr{M}_{n-1}$. Hence, we can safely drop the subscript and write $\mathscr{M}$ for $\mathscr{M}_n$.

The restriction of $\mathscr{M}$ to $\mathscr{C}_1^{\text{cub}}$ is equal to the restriction of $\mathscr{I}_1$, i.e. if $\dim P\le 1$, then $\mathscr{M}(P)=\mathscr{I}_1(P)=P$. Suppose, by induction, that $\mathscr{M}$ has been defined on the full subcategory $\mathscr{C}_{n-1}^{\text{cub}}$ of $\mathscr{C}_n^{\text{cub}}$. Since $\mathscr{M}$ is a functor, the group of combinatorial symmetries of an n-cube $\square^n$ acts on $\mathscr{M}(\partial\square^n)$ as a group of isometries. Let $a:\square^n\to\square^n$ denote the central symmetry (i.e. a is the antipodal map if $\square^n$ is a regular Euclidean cube centered at the origin). Define an involution τ on $\mathscr{M}(\partial\square^n)\times I$ by $\tau:(x,t)\to(\mathscr{M}(a)(x),-t)$. Put

$$\mathscr{M}(\square^n)=(\mathscr{M}(\partial\square^n)\times I)/\tau.$$

(When applied to a square this procedure yields the Moebius band; hence, the terminology.) Note that the image of $(\mathscr{M}(\partial\square^n)\times\{-1,1\})/\tau$ in $\mathscr{M}(\square^n)$ is canonically identified with $\mathscr{M}(\partial\square^n)$. Hence, for a cubical complex P of dimension n, we define $\mathscr{M}(P)$ to be the result of gluing a copy of $\mathscr{M}(\square^n)$ to $\mathscr{M}(P^{(n-1)})$ for each n-cube in P via the above identification. We may assume, inductively, that $\mathscr{M}(P^{(n-1)})$ and $\mathscr{M}(\partial\square^n)$ are cubical Euclidean complexes. Hence, $\mathscr{M}(\partial\square^n)\times I$ is naturally a cubical Euclidean complex. Since the central involution a freely permutes the faces of $\partial\square^n$, the involution τ freely permutes the cells of $\mathscr{M}(\partial\square^n)\times I$ and, hence, the cubical cell complex structure on $\mathscr{M}(\partial\square^n)\times I$ descends to one on $\mathscr{M}(\square^n)$. It follows that the cubical Euclidean complex structure on $\mathcal{M}(P^{(n-1)})$ extends across each $\mathcal{M}(\square^n)$ to give such a structure on $\mathcal{M}(P)$.

As with $\mathcal{I}_n$, the restriction of $\mathcal{M}$ to $\mathscr{C}_{n-1}^{\text{cub}}$ is extended to morphisms in $\mathscr{C}_n^{\text{cub}}$ by taking the product with the identity map on the I-factor. (That this is well-defined depends on the fact that the involution a is in the center of the group of combinatorial symmetries of the n-cube.)

Both hyperbolization procedures, $\mathcal{I}_n$ and $\mathcal{M}$, described above are defined inductively by gluing together nonpositively curved polyhedra along totally geodesic subspaces. It follows from Gromov's gluing lemma (cf. [9, p. 362]) that the resulting $\mathbb{M}_0$-structures are nonpositively curved. Alternatively, this can be proved by showing that the link of each cell in $\mathcal{I}_n(P)$ or $\mathcal{M}(P)$ is a flag complex, and applying Corollary 1.9. Both of these proofs can be found in [21, Theorem 4.3]. (However, the claim there that $\mathcal{I}_n(P)$ and $\mathcal{M}(P)$ satisfy the no $\square$ condition is false, so Corollary 1.10 does not apply.)

Relative hyperbolization: There is a relative version of the product with interval construction which has been described and exploited by Hu in [16]. Suppose that K is an $\mathbb{M}_0$-cell complex, that it is a subcomplex of a (combinatorial) cell complex P, and that every cell in $P - K$ is of dimension $\leq n$. We shall define a piecewise Euclidean polyhedron $\mathcal{I}_n(P, K)$ which contains K as a totally geodesic subcomplex. To begin with, put $\mathcal{I}_1(P^{(1)} \cup K, K) = P^{(1)} \cup K$. Assuming the construction has been defined in dimension $n-1$, define $\mathcal{I}_n(P^{(n-1)} \cup K, K)$ to be $\mathcal{I}_{n-1}(P^{(n-1)} \cup K, K) \times \{-1, +1\}$; whenever σ^n is a cell which is not contained in K, define $\mathcal{I}_n(\sigma^n, K \cap \sigma^n)$ to be $\mathcal{I}_{n-1}(\partial\sigma^n, K \cap \partial\sigma^n) \times I$. Then construct $\mathcal{I}_n(P, K)$ by gluing $\mathcal{I}_n(\sigma^n, K \cap \sigma^n)$ to $\mathcal{I}_n(P^{(n-1)} \cup K, K)$ for each n-cell σ^n in P but not in K. Clearly, $K \times \{-1, +1\}^{n-1} \subset \mathcal{I}_n(P, K)$. By passing to the barycentric subdivision of P, we may assume that P and K are simplicial complexes and that K is a full subcomplex. (This means that if the vertices of σ are in K, then $\sigma \subset K$.) Since K is full, $K \cap \sigma^n$ is a face of σ^n and, hence, a totally geodesic subspace of K. Using the gluing lemma, it then follows, as before, that each copy of K is a totally geodesic subspace of $\mathcal{I}_n(P, K)$ and that if K is nonpositively curved then so is $\mathcal{I}_n(P, K)$. (It should be noted however, that the metric on K, induced by the intrinsic metric on $\mathcal{I}_n(P, K)$, while locally the same as the original metric on K, need not be globally the same.) Remark 2.2 also carries over to the relative version: $(\mathbb{Z}/2)^{n-1}$ acts on $\mathcal{I}_n(P, K)$ and $P^{(1)} \cup K$ is a retract of $\mathcal{I}_n(P, K)$ (see [16]).

Any relative hyperbolization procedure should have the following two properties: if P and K are as above then (1) a hyperbolization of P relative to K should contain K as a totally geodesic subcomplex and (2) if K is nonpositively curved, then the relative hyperbolization of P should be nonpositively curved. Some general claims concerning the existence of such procedures are made in [12, p. 117], but we only understand the construction in the above case of the product with interval procedure. (A relative version of hyperbolization is described explicitly in [9, p. 358], but the result is only a polyhedron—no metric is given. However, this version is good enough to prove that any manifold is cobordant to a nonpositively curved manifold.)

3. REFLECTION GROUPS

Coxeter groups: Given a finite set S, a *Coxeter matrix* is a function $m: S \times S \to \mathbb{Z} \cup \{\infty\}$ such that (i) $m(s, s') = m(s', s)$ for all (s, s') in $S \times S$, (ii) $m(s, s') \geq 2$ if $s \neq s'$, and (iii) $m(s, s) = 1$ for all s in S. One can associate to a Coxeter matrix m, a "Coxeter diagram", a "Coxeter group", and a "cosine matrix", as follows. The *Coxeter diagram* is the labeled graph with vertex set S; distinct vertices s and s' bound an edge if and only if $m(s, s') \geq 3$; the corresponding edge is labeled by $m(s, s')$, where, by convention, we omit the label when

$m(s, s') = 3$. The *Coxeter group* W is the group with presentation $\langle S; (ss')^{m(s,s')}\rangle$ where (s, s') ranges over $S \times S - m^{-1}(\infty)$. The *associated cosine matrix* is the function $c: S \times S \to \mathbb{R}$ defined by $c(s, s') = -\cos(\pi/m(s, s'))$. The cosine matrix defines a symmetric bilinear form on $\mathbb{R}^S$. There is a linear W-action on $\mathbb{R}^S$ preserving this form called the "canonical representation". It follows from the existence of this representation that the natural map $S \to W$ is an injection (and we henceforth identify s with its image in W), that the order of s in W is 2, and that the order of ss' in W is $m(s, s')$. The pair (W, S) is a *Coxeter system*; its *rank* is $\mathrm{Card}(S)$. The group W is finite if and only if the matrix c is positive definite; in this case the canonical representation exhibits W as a finite orthogonal reflection group on $\mathbb{R}^n$, $n = \mathrm{Card}(S)$. Conversely, if G is any finite reflection group on $\mathbb{R}^n$, then G is a Coxeter group, and if the representation is without a trivial summand, then it is equivalent to the canonical representation. The quotient space $\mathbb{R}^n/W$ (where W is finite) can be identified with one of the simplicial cones in $\mathbb{R}^n$ cut out by the hyperplanes of reflection; such a simplicial cone is called a *fundamental chamber* for W (for further details, see [3]).

The groups A_n and B_n: If an n-cell σ^n in $\mathbb{M}^n_\chi$ is regular, then its isometry group is a rank n Coxeter group $W(\sigma^n)$, the diagram of which is a straight line segment (e.g. see [8, p. 74]). An n-cell is *simple* if the link of each vertex is an $(n-1)$-simplex. Suppose σ^n is simple. Choose a simplex in the barycentric subdivision of σ^n with vertices $v_0, \ldots, v_n$, where v_i is the barycenter of a cell of dimension i. Then $S = \{s_0, \ldots, s_{n-1}\}$ is a set of fundamental generators for $W(\sigma^n)$, where s_i is the reflection across the face spanned by $\{v_0, \ldots, \hat{v}_i, \ldots, v_n\}$. It follows that if σ^n is regular and simple, then the Coxeter diagram of $(W(\sigma^n), S)$ has the form

$$\underset{s_0}{\bullet} \overset{m}{\text{——}} \bullet \text{——} \bullet \cdots \bullet \text{——} \underset{s_{n-1}}{\bullet}$$

where $m \geq 3$. For n arbitrary the only cases which occur are $m = 3$ or $m = 4$. The case $m = 3$ corresponds to the regular n-simplex Δ^n; its symmetry group is usually denoted A_n. The case $m = 4$ corresponds to the regular n-cube $\square^n$; its symmetry group is B_n. When $n = 2$ and σ^2 is a regular m-gon, we get the diagram $\bullet \overset{m}{\text{——}} \bullet$ (so that any value of $m \geq 3$ can occur). In this case, $W(\sigma^2)$ is the dihedral group of order $2m$. For $n > 2$, the only other possible value of m is 5 and it occurs only when $n = 3$ (the dodecahedron) or $n = 4$ (the "120-cell"); the corresponding Coxeter groups are denoted by H_3 and H_4, respectively. The group A_n is the symmetric group of degree $n + 1$. (Consider its action on the vertex set of Δ^n.) Thought of as a subgroup of $O(n)$, the group B_n is generated by all permutations of coordinates and sign changes; hence, as an abstract group it is the semidirect product of $(\mathbb{Z}/2)^n$ and the symmetric group of degree n. Explicitly, s_0 changes the sign of the first coordinate, and s_i, $1 \leq i \leq n-1$, switches the ith and $(i+1)$th coordinates.

Coxeter orbifolds: The quotient space of a locally smooth, proper action of a discrete group on a manifold has the structure of an orbifold, where, roughly speaking, an "orbifold" X^n is a space together with local "charts" such that each point x has a neighborhood of the form $\mathbb{R}^n/G_x$ for some finite linear group G_x. (For precise definitions see [23, Chapter 13] or [15, Section 4].) The group G_x (which is only defined up to conjugation in $GL(n, \mathbb{R})$) is called the *local group* at x. An orbifold X^n is a *reflectofold* if each local group is a finite reflection group. Since the orbit space of a finite reflection group W on $\mathbb{R}^n$ has the form $C^i \times \mathbb{R}^{n-i}$, where $\mathbb{R}^{n-i}$ is the fixed subspace and C^i is a simplicial cone in $\mathbb{R}^i$, it follows that a reflectofold X^n has a natural structure of a manifold with corners: a point $x \in X^n$ belongs to the relative interior of a codimension i stratum if and only if G_x is a Coxeter group of rank i. The only Coxeter groups of rank two are the dihedral groups: a codimension two stratum of a reflectofold is *labeled m* if the associated local group is dihedral of order $2m$.

Suppose that X^n is a reflectofold and that the codimension one strata are $F_1, \ldots, F_k$. Then X^n is a *Coxeter orbifold* if the following two conditions hold:

(a) each codimension two stratum lies in precisely two codimension one strata,
(b) for $i \neq j$, each component of $F_i \cap F_j$ has the same label.

A codimension one stratum is a *mirror*; a codimension two stratum is a *corner*. Put $S = \{s_1, \ldots, s_k\}$ where s_i corresponds to the mirror F_i. Associated to the Coxeter orbifold X^n, there is a Coxeter matrix m defined by

$$m(s_i, s_j) = \begin{cases} 1 & \text{if } i = j \\ \text{the label on } F_i \cap F_j & \text{if } i \neq j \\ \infty & \text{if } F_i \cap F_j = \emptyset. \end{cases}$$

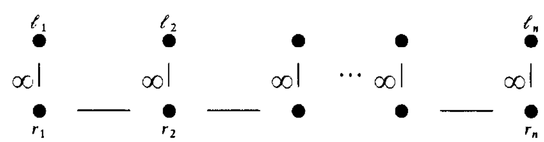
Let (W, S) denote the associated Coxeter system. There is an orbifold covering $\tilde{X}^n \to X^n$ with W as group of covering transformations. The space $\tilde{X}^n$ is a manifold; it is constructed by pasting together copies of X^n (one for each element of W) in the obvious manner (see [6, Section 13] for details). If the underlying space of X^n is simply connected, then $\tilde{X}^n$ is simply connected (cf. [6, Corollary 10.2]); hence, in this case $\tilde{X}^n$ is the universal orbifold cover of X^n.

In general, for any orbifold Y, its *orbifold fundamental group*, denoted $\pi_1^{\text{orb}}(Y)$, is the group of covering transformations of the universal orbifold cover of Y (see [23, Chapter 13]). The previous paragraph is summarized by the following lemma.

LEMMA 3.1. *Suppose that X^n is a Coxeter orbifold and that its underlying space is simply connected. Then $\pi_1^{\text{orb}}(X^n)$ can be canonically identified with the associated Coxeter group W.*

The Coxeter orbifold Q^n: We shall define an orbifold which will be important in the next section. (Coincidentally, this same orbifold turns up in a different context in [7] as the quotient orbifold of an action on Tomei's manifold of isospectral tridiagonal matrices,)

The underlying space of Q^n is the cube $[0, 1]^n$. The mirrors are named as follows: the *ith-right mirror*, denoted R_i, is given by $R_i = \{x \in [0, 1]^n \mid x_i = 1\}$, the *ith-left mirror*, denoted L_i, is given by $L_i = \{x \in [0, 1]^n \mid x_i = 0\}$. The corner $R_i \cap R_{i+1}$, $1 \leq i \leq n - 1$, is labeled 3; all other corners are labeled 2. To check that this defines an orbifold structure we must check that the induced Coxeter group at each face is finite. But this is clear, for example, the local group at the rightmost vertex $R_1 \cap \cdots \cap R_n$ is A_n, at the opposite vertex $L_1 \cap \cdots \cap L_n$ it is $(A_1)^n$ $(=(\mathbb{Z}/2)^n)$. Let r_i (resp. ℓ_1) be the generator for the Coxeter group $\pi_1^{\text{orb}}(Q^n)$ corresponding to R_i (resp. L_i). The corresponding Coxeter diagram is the following:



LEMMA 3.2. *The Coxeter group $\pi_1^{\text{orb}}(Q^n)$ contains a free abelian subgroup of rank* $[(n + 1)/2]$.

Proof. The standard subgroup generated by $\{r_1, r_3, \ldots, r_{2d-1}, \ell_1, \ell_3, \ldots, \ell_{2d-1}\}$, $d = [(n + 1)/2]$, has as diagram *d disjoint copies of* $\bullet \overset{\infty}{\text{———}} \bullet$; hence, it is $(D_\infty)^d$ where D_∞ is the infinite dihedral group. ∎

COROLLARY 3.3. *For $n > 2$, the group $\pi_1^{\mathrm{orb}}(Q^n)$ is not word hyperbolic.*

Proof. Word hyperbolic groups do not contain $\mathbb{Z} \times \mathbb{Z}$ (cf. [11, Theorem 8.34, p. 156]). ∎

4. HYPERBOLIZATION OF THE BOUNDARY COMPLEX OF A REGULAR CELL

In this section we show that when the Moebius band procedure is applied to the boundary of an n-cube or when the product when interval procedure is applied to the boundary of a simple regular n-cell, the result is a finite orbifold covering of Q^{n-1} (the orbifold described in the previous section). The fundamental group of this covering is a subgroup of finite index in $\pi_1^{\mathrm{orb}}(Q^{n-1})$; hence by Corollary 3.3, it is not word hyperbolic for $n \geq 4$. For any cell complex K and any subcomplex $K' \subset K$, the hyperbolization of K' is totally geodesic in the hyperbolization of K, and hence the fundamental group of the hyperbolization of K' injects into that of K. It follows that for $n \geq 4$, the result of hyperbolizing an n-dimensional cubical or simplicial cell complex by either the Moebius band or product with interval procedure can never be given a metric of strictly negative curvature.

As in Section 3, B_n denotes the symmetry group of the n-cube $\square^n$ and $\{s_0, \dots, s_{n-1}\}$ is its set of fundamental generators. For $i \leq n$, we identify B_i with the subgroup $\langle s_0, \dots, s_{i-1} \rangle$ of B_n. Let $a_i \in B_i$ denote the central symmetry (=antipodal map of $\square^i$).

Since the Moebius band procedure $\mathcal{M}$ is a functor, the group B_n acts on $\mathcal{M}(\square^n)$, and on $\mathcal{M}(\partial\square^n)$.

PROPOSITION 4.1. *The n-dimensional orbifold $\mathcal{M}(\partial\square^{n+1})/B_{n+1}$ can be identified with Q^n and $\pi_1((\mathcal{M}(\partial\square^{n+1}))$ with the kernel of the homomorphism $\varphi_n : \pi_1^{\mathrm{orb}}(Q^n) \to B_{n+1}$ defined by*

$$\varphi_n(r_i) = s_i, \qquad \varphi_n(\ell_i) = a_i$$

where $\{r_1, \dots, r_n, \ell_1, \dots, \ell_n\}$ is the fundamental set of generators for $\pi_1^{\mathrm{orb}}(Q^n)$.

Proof. Set $\tilde{Q}^n = \mathcal{M}(\partial\square^{n+1})/B_{n+1}$ and set $\tilde{\ell}_i = \mathcal{M}(a_i)$, $\tilde{r}_i = \mathcal{M}(s_i)$, $i = 1, \dots, n$. We will show:

(1) the underlying space of $\tilde{Q}^n$ is $[0, 1]^n$;

(2) $\tilde{\ell}_i$ and $\tilde{r}_i$ act locally as reflections on $\mathcal{M}(\partial\square^{n+1})$ and the image of the fixed point set of $\tilde{\ell}_i$ (resp. $\tilde{r}_i$) in $[0, 1]^n$ is L_n (resp. R_n).

From (2), it follows that the projection $\mathcal{M}(\partial\square^{n+1}) \to \tilde{Q}^n$ is the orbifold covering corresponding to the homomorphism

$$\tilde{\varphi}_n : \pi_1^{\mathrm{orb}}(\tilde{Q}^n) \to B_{n+1}$$

defined by $\tilde{\varphi}_n(\tilde{\ell}_i) = a_i$, $\tilde{\varphi}_n(\tilde{r}_i) = s_i$. Moreover, since $\tilde{\mathcal{M}}(\partial\square^{n+1})$ is a manifold (viewed as an orbifold with all local groups trivial), $\tilde{\varphi}_n$ induces an injection on the local groups of $\tilde{Q}^n$. In particular, on a codimension two face, the local group maps isomorphically to a dihedral subgroup of B_{n+1} whose order determines the label on the face. For example, the local group on $R_i \cap R_{i+1}$ is the group generated by $\tilde{r}_i$ and $\tilde{r}_{i+1}$ which maps isomorphically onto $\langle s_i, s_{i+1} \rangle = D_6 \subset B_{n+1}$ so this face has label 3. Similarly, $R_i \cap L_j$, $i \neq j$, has local group isomorphic to $\langle s_i, a_j \rangle = \mathbb{Z}/2 \times \mathbb{Z}/2 \subset B_{n+1}$ and hence is labeled 2, and so on. Thus $\tilde{Q}^n$ can be naturally identified with Q^n and $\tilde{\varphi}_n$ with φ_n.

It remains to check (1) and (2). We do this by induction on n. For $n = 1$,

$$\mathcal{M}(\partial\Box^2) = \partial\Box^2, \qquad \mathcal{M}(a_1) = a_1 = s_0, \qquad \mathcal{M}(s_1) = s_1$$

so (1) and (2) are immediate.

Suppose $n > 1$. First note that since B_{n+1} acts transitively on top-dimensional faces of $\partial\Box^{n+1}$ and the stabilizer of such a face is B_n, the natural map

$$\mathcal{M}(\Box^n)/B_n \to \mathcal{M}(\partial\Box^{n+1})/B_{n+1} = \tilde{Q}^n$$

is a homeomorphism on underlying spaces. By definition,

$$\mathcal{M}(\Box^n) = \mathcal{M}(\partial\Box^n) \times [-1, 1]/\langle\tau\rangle$$

where $\tau = a_n \times t \in B_n \times \mathbb{Z}/2$ (and where $t:[-1, 1] \to [-1, 1]$ is defined by $t(x) = -x$). It follows that

$$\begin{aligned}\mathcal{M}(\Box^n)/B_n &= \mathcal{M}(\partial\Box^n)/B_n \times [-1, 1]/(\mathbb{Z}/2)\\ &= \tilde{Q}^{n-1} \times J\end{aligned}$$

where $J = [-1, 1]/(\mathbb{Z}/2) = [0, 1]$, viewed as an orbifold. By induction, the underlying space of $\tilde{Q}^{n-1} \times J$, and hence of $\tilde{Q}^n$, is $[0, 1]^n$.

To verify (2), we compare $\tilde{Q}^{n-1} \times J$ with $\tilde{Q}^n$ as orbifolds. The former is an orbifold with boundary (where $\partial(\tilde{Q}^{n-1} \times J) = \tilde{Q}^{n-1} \times \{1\} = R_n$ and where the local group is trivial on this boundary). On the other hand, in $\tilde{Q}^n$, R_n is a mirror. To see this, note that the element s_n of B_{n+1} takes one top-dimensional face of $\partial\Box^{n+1}$ to an adjacent one, fixing pointwise an $(n-1)$-dimensional face $\Box^{n-1}$. It follows that $\mathcal{M}(s_n)$ acting on $\mathcal{M}(\partial\Box^{n+1})$ takes one copy of $\mathcal{M}(\Box^n)$ to another, fixing their intersection $\mathcal{M}(\Box^{n-1})$. The image of $\mathcal{M}(\Box^{n-1})$ in $\tilde{Q}^n$ is precisely R_n, so we conclude that in $\tilde{Q}^n$, R_n is a mirror with corresponding reflection $\mathcal{M}(s_n)$.

For a point x in $\tilde{Q}^{n-1} \times J$ not on the boundary, the local structure for x in $\tilde{Q}^{n-1} \times J$ is the same as its local structure in $\tilde{Q}^n$. Thus by induction, for $i = 1, \ldots, n-1$, L_i and R_i are mirrors with corresponding reflections $\mathcal{M}(s_i)$ and $\mathcal{M}(a_i)$. The face L_n is also a mirror of $\tilde{Q}^{n-1} \times J$ with reflection given by $id \times t$. Since τ acts on $\mathcal{M}(\partial\Box^n) \times [-1, 1]$ via $\mathcal{M}(a_n) \times t$, it follows that on $\mathcal{M}(\Box^n) = \mathcal{M}(\partial\Box^n) \times [-1, 1]/\langle\tau\rangle$, the action of $id \times t$ is the same as the action of $\mathcal{M}(a_n)$ (which by definition is induced by the action of $\mathcal{M}(a_n) \times id$ on $\mathcal{M}(\partial\Box^n) \times [-1, 1]$). ■

In Section 2 we showed that $(A_1)^{n-1}$ is a group of natural transformations of the product with interval functor $\mathcal{I}_n$. Thus, if G is any group of symmetries of an n-dimensional cell complex K, then $G \times (A_1)^{n-1}$ acts on $\mathcal{I}_n(K)$.

By an argument entirely similar to the proof of the previous proposition we get the following.

PROPOSITION 4.2. *Let σ^{n+1} be a simple regular $(n+1)$-cell and let $W(\sigma^{n+1})$ be its symmetry group. The n-dimensional orbifold $\mathcal{I}_n(\partial\sigma^{n+1})/(W(\sigma^{n+1}) \times (A_1)^{n-1})$ can be identified with Q^n and $\pi_1(\mathcal{I}_n(\partial\sigma^{n+1}))$ with the kernel of a homomorphism $\varphi: \pi_1^{\mathrm{orb}}(Q^n) \to W(\sigma^{n+1}) \times (A_1)^{n-1}$ given by the formulas*

$$\varphi(r_i) = s_i, \quad 1 \le i \le n$$

$$\varphi(\ell_i) = t_{i-1}, \quad 2 \le i \le n$$

$$\varphi(\ell_1) = s_0$$

where $\{r_1, \ldots, r_n, \ell_1, \ldots, \ell_n\}$, $\{s_0, \ldots, s_n\}$, *and* $\{t_1, \ldots, t_{n-1}\}$ *are the fundamental generating sets for the Coxeter groups* $\pi_1^{\mathrm{orb}}(Q^n)$, $W(\sigma^{n+1})$, *and* $(A_1)^{n-1}$, *respectively.*

The following two corollaries now follow from the discussion at the beginning of this section.

COROLLARY 4.3. *Let* $\square^n$ *be an n-cube and* σ^n *any simple regular n-cell. If* $n \geq 4$, *then the fundamental groups of the manifold* $\mathcal{M}(\partial\square^n)$ *and* $\mathcal{I}_{n-1}(\partial\sigma^n)$ *are never word hyperbolic.*

COROLLARY 4.4. *Let P be a cubical complex of dimension at least* 4. *Then* $\pi_1(\mathcal{M}(P))$ *is not word hyperbolic. Similarly, if P is cubical or simplicial of dimension* ≥ 4 *and* $n \geq \sup\{4, \dim P\}$, *then* $\pi_1(\mathcal{I}_n(P))$ *is not word hyperbolic.*

5. CUTTING OPEN A MANIFOLD ALONG A SYSTEM OF SUBMANIFOLDS

Definition 5.1. Suppose that M^n is a smooth manifold and that $\mathcal{Y} = \{Y_1, \ldots, Y_k\}$ is a collection of codimension one submanifolds of M^n which intersect transversely. Then we say that $\mathcal{Y}$ is a *system of codimension one submanifolds.* The system $\mathcal{Y}$ is *two-sided* if each Y_i is two-sided in M. If a finite group G acts on M and if $\bigcup Y_i$ is a G-stable subset of M, then the system $\mathcal{Y}$ is called *G-stable.*

Example 5.2. Let T^n denote the n-torus, i.e. the n-fold Cartesian product of a circle. Let T_i denote the codimension one subtorus defined by setting the ith coordinate equal to 1. Then $\mathcal{T} = \{T_1, \ldots, T_n\}$ is a two-sided system of submanifolds of T^n; we shall call it the *standard system of subtori.*

Suppose $\mathcal{Y} = \{Y_1, \ldots, Y_k\}$ is a system of submanifolds of M. For each subset J of $\{1, \ldots, k\}$, put

$$Y_J = \bigcap_{j \in J} Y_j.$$

It is a smooth submanifold (possibly empty) of codimension $|J|$ in M (where $|J| = \mathrm{Card}(J)$). Suppose that M is oriented and that $\mathcal{Y}$ is two-sided. If we choose a "side" of each Y_i (i.e. a section of the normal S^0-bundle of Y_i in M), then there is an induced orientation on each Y_J.

LEMMA 5.3. (a) *Suppose that* $\mathcal{Y} = \{Y_1, \ldots, Y_n\}$ *is a two-sided system of codimension one submanifolds of a smooth manifold* M^n. *Then there is a smooth map* $\varphi: M^n \to T^n$ *such that* $\mathcal{Y}$ *is the transverse inverse image of the standard system of subtori in* T^n.

(b) *Suppose further that* M^n *is closed and oriented and that* $Y_1 \cap Y_2 \cdots \cap Y_n$ *is a single point* y. *For each subset* $J \subset \{1, \ldots, n\}$, *let* $\tilde{Y}_J$ *denote the component of* Y_J *which contains* y. *Then for each* J, *the map* $\varphi \mid \tilde{Y}_J: \tilde{Y}_J \to T_J$ *is degree one (for a convenient choice of sides). In particular,* φ *is degree one.*

Proof. Choose tubular neighborhoods $\lambda_i: Y_i \times \mathbb{R} \to M^n$. We may assume that λ_i takes $(Y_i \cap Y_j) \times \mathbb{R}$ into Y_j for $j \neq i$. The Pontryagin–Thom construction applied to the framed submanifold Y_i yields a map $\varphi_i: M^n \to S^1$ with $\varphi_i^{-1}(1) = Y_i$. Then $\varphi = (\varphi_1, \ldots, \varphi_n): M^n \to T^n$ is the map described in (a). To prove (b) we suppose that $Y_1 \cap \cdots \cap Y_n = \{y\}$. Then $e = (1, \ldots, 1)$ is a regular value of φ and $\varphi^{-1}(e)$ is a single point, namely y; hence $\deg(\varphi) = \pm 1$. If we arrange our orientation conventions properly, then the degree is $+1$. Similar remarks apply to each $\varphi \mid \tilde{Y}_J: \tilde{Y}_J \to T_J$. ■

Suppose that M^n is a smooth manifold and that Y^{n-1} is a smooth submanifold of codimension one. To *cut M open along Y* one removes Y and replaces it by the normal S^0-bundle of Y in M. The result is denoted by $M \odot Y$. It has the natural structure of a smooth manifold with boundary; its interior is $M - Y$; its boundary is the normal S^0-bundle of Y. There is a natural projection $\pi: M \odot Y \to M$ which is a homeomorphism on $M - Y$ and the projection map of the S^0-bundle on the boundary. If we wish to iterate this process, then we should allow M^n to be a smooth manifold with corners and Y^{n-1} a smooth submanifold with corners in the sense that it intersects each codimension i stratum of M in a codimension i stratum of Y. If this is the case, then $M \odot Y$ will again be a smooth manifold with corners.

If $\mathcal{Y} = \{Y_1, \ldots, Y_k\}$ is a system of codimension one submanifolds of M, then one can iterate the above cutting-open construction to obtain a smooth manifold with corners

$$M \odot \mathcal{Y} = M \odot Y_1 \odot Y_2 \odot \cdots \odot Y_k$$

together with a projection map $\pi: M \odot \mathcal{Y} \to M$. If y is a generic point of Y_J (i.e. if y does not belong to Y_i for $i \notin J$) then $|\pi^{-1}(y)| = 2^{|J|}$.

Example 5.4. If we cut open the n-torus along the standard system of subtori we obtain the n-cube $\square^n$; moreover, $\pi: \square^n \to T^n$ is the standard quotient map.

Remark 5.5. If a finite group G acts smoothly on M and stabilizes $\mathcal{Y}$, then there is an induced G-action on the cut-open object $M \odot \mathcal{Y}$.

We suppose for the remainder of this section that $\mathcal{Y} = \{Y_1, \ldots, Y_n\}$ is a two-sided system of codimension one submanifolds in a closed manifold M^n. The manifold with corners $M \odot \mathcal{Y}$ then has a "face structure" which is combinatorially that of an n-cube. We make this precise below.

For each subset J of $\{1, \ldots, n\}$ there is a two-sided system $\mathcal{Y}_J$ of submanifolds in Y_J defined by

$$\mathcal{Y}_J = \{Y_J \cap Y_i\}_{i \notin J}.$$

(When J is a singleton, say $\{j\}$, we shall write $\mathcal{Y}_j$ instead of $\mathcal{Y}_{\{j\}}$.)

For each i, $i = 1, \ldots, n$, choose a section s_i of the normal S^0-bundle of Y_i in M. Denote the corresponding copy of $Y_i \odot \mathcal{Y}_i$ in $\pi^{-1}(Y_i)$ by $\partial_{(i,+1)}(M \odot \mathcal{Y})$; the other copy is denoted by $\partial_{(i,-1)}(\mathcal{M} \odot \mathcal{Y})$. For an arbitrary subset J of $\{1, \ldots, n\}$, $\pi^{-1}(Y_J)$ consists of $2^{|J|}$ copies of $Y_J \odot \mathcal{Y}_J$ indexed by functions $\varepsilon: J \to \{\pm 1\}$. Such a copy is denoted $\partial_{(J,\varepsilon)}(\mathcal{M} \odot \mathcal{Y})$ and defined by

$$\partial_{(J,\varepsilon)}(\mathcal{M} \odot \mathcal{Y}) = \bigcap_{j \in J} \partial_{(j,\varepsilon(j))}(\mathcal{M} \odot \mathcal{Y}).$$

It is called a *face* of $\mathcal{M} \odot \mathcal{Y}$ of codimension $|J|$. Here we allow $J = \emptyset$, so that $\mathcal{M} \odot \mathcal{Y}$ is the codimension 0 face of itself. A face can be empty and it need not be connected; however, a face of codimension ℓ is a union of codimension ℓ strata in the manifold with corners structure on $\mathcal{M} \odot \mathcal{Y}$.

In the next lemma we record the fact that the faces of $M \odot \mathcal{Y}$ have the same index set as do the faces of $\square^n$.

LEMMA 5.6. *The faces of $M \odot \mathcal{Y}$ are indexed by pairs (J, ε) where J is a subset of $\{1, \ldots, n\}$ and $\varepsilon: J \to \{\pm 1\}$ is a function. Hence, the poset of nonempty faces of $M \odot \mathcal{Y}$ can be identified with a subset of the poset of faces of the n-cube $\square^n$.*

LEMMA 5.7. *Suppose that $Y_1 \cap \cdots \cap Y_n$ is a single point y. For each i, let $I(i) = \{1, \dots, n\} - \{i\}$ and let S_i denote the component of the one-dimensional intersection $Y_{I(i)}$ which contains y.*

(1) *$\bigcup S_i$ is a bouquet of circles and $\pi^{-1}(\bigcup S_i)$ is isomorphic to the 1-skeleton of $\square^n$.*

(2) *For any face $F = \partial_{(J,\varepsilon)}(M \odot \mathscr{Y})$, all 0-dimensional faces of F lie in a single component $\tilde{F}$. In particular, $\pi^{-1}(y)$ is contained in a single component of $M \odot \mathscr{Y}$.*

Proof. Since M is closed, S_i is a closed 1-manifold; hence, a circle. Since $S_i \cap S_j = \{y\}$ if $i \neq j$, it is clear that $\bigcup S_i$ is a bouquet of circles. Call the points of $\pi^{-1}(y)$ *vertices.* They are indexed by functions $\varepsilon: \{1, \dots, n\} \to \{\pm 1\}$. Let v_ε denote the vertex corresponding to ε. Let ε_i be the function defined by $\varepsilon_i(j) = (-1)^{\delta_{ij}}$. A component of $\pi^{-1}(S_i)$ is an edge connecting two vertices and for any ε there is such an edge connecting v_ε and $v_{\varepsilon_i \varepsilon}$. This proves (1). Statement (2) follows easily. ■

LEMMA 5.8. *Suppose that $Y_1 \cap \cdots \cap Y_n$ is a single point y. Suppose further that M is connected and that there are nontrivial commuting involutions $r_1, \dots, r_n$ on M (generating an action of $G = (\mathbb{Z}/2)^n$) such that Y_i is contained in the fixed point set R_i of r_i and such that G stabilizes $\mathscr{Y}$. Then $M \odot \mathscr{Y}$ is connected.*

Proof. Let $U = M - \bigcup Y_i$ denote the interior of $M \odot \mathscr{Y}$. We must show that U is connected. Put $W = M - \bigcup R_i$. Since W is open and dense in U, $\pi_0(W) \to \pi_0(U)$ is onto. Let $p: M \to M/G$ be the projection. Since M is connected, so is M/G. Since $p(R_i)$ cannot disconnect M/G locally, $p(W)$ is connected. Choose a vector in $T_y M$ not tangent to any R_i and use it to push y into W. Call the resulting point w. Since G acts freely on W, $p \mid W: W \to p(W)$ is a 2^n-sheeted cover and each component of W contains at least one point in Gw (the G-orbit of w). Hence, each component of U contains at least one point in Gw. By Lemma 5.7(2), $\pi^{-1}(y)$ is contained in a single component of $M \odot \mathscr{Y}$; hence, Gw is contained in a single component of U. Therefore, U (and consequently $M \odot \mathscr{Y}$) is connected. ■

In Lemma 5.3 we produced a map $\varphi: M^n \to T^n$. This map is compatible with the cutting-open operation; hence, we get a map from $M \odot \mathscr{Y}$ to $\square^n$. This gives us the following lemma, the proof of which is left to the reader.

LEMMA 5.9. (a) *There is a smooth face-preserving map $f: M \odot \mathscr{Y} \to \square^n$ (in fact, unique up to homotopy through such maps).*

(b) *Suppose that M is closed and oriented and that $Y_1 \cap \cdots \cap Y_n$ is a single point y. Let F be a k-dimensional face of $M \odot \mathscr{Y}$ and $\tilde{F}$ the distinguished component of F containing y (as in Lemma 5.7(2)); then $f \mid \tilde{F}: \tilde{F} \to \square^k$ is degree one (and the other components are mapped by degree zero maps).*

If M has a Riemannian metric, then there is an induced Riemannian metric on $M \odot \mathscr{Y}$. In the next lemma we record two elementary observations.

LEMMA 5.10. *Suppose M is a Riemannian manifold and that $\mathscr{Y}$ is a system of codimension one submanifolds.*

(1) *If each Y_i is totally geodesic in M, then each face of $M \odot \mathscr{Y}$ is totally geodesic.*

(2) *If G is a finite group of isometries of M stabilizing $\mathscr{Y}$, then G acts isometrically on $M \odot \mathscr{Y}$.*

6. A HYPERBOLIC MANIFOLD WITH A SYSTEM OF TOTALLY GEODESIC SUBMANIFOLDS

Recall that B_n denotes the symmetry group of the n-cube. It has a *standard* orthogonal action on $\mathbb{R}^n$ generated by all permutations of coordinates and sign changes. The sign changes are generated by involutions r_i, $i = 1, \ldots, n$, where r_i denotes the linear reflection across the hyperplane $x_i = 0$.

Our goal in this section is to prove the following theorem.

THEOREM 6.1. *For each $n > 0$, there is a closed, connected hyperbolic n-manifold M^n, a system $\mathcal{Y} = \{Y_1, \ldots, Y_n\}$ of closed, connected submanifolds of codimension one in M^n, and an isometric action of B_n on M^n, stabilizing $\mathcal{Y}$, such that the following properties hold:*

(1) *Y_i is a component of the fixed point set of r_i on M^n.*
(2) *Each Y_i is totally geodesic in M^n.*
(3) *The Y_i's intersect orthogonally.*
(4) *$Y_1 \cap \cdots \cap Y_n$ is a single point y.*
(5) *B_n fixes y and the representation of B_n on $T_y M^n$ is equivalent to the standard representation.*
(6) *M^n, as well as each Y_i, is orientable.*

If we put

$$X^n = M^n \odot \mathcal{Y}$$

then the lemmas of Section 5 gives us the following corollary.

COROLLARY 6.2. *For each $n > 0$, there is a compact, connected, orientable hyperbolic n-manifold with corners X^n together with an action of B_n on X^n by isometries so that the following properties hold.*

(1) *The poset of faces of X^n is B_n-equivariantly isomorphic to the poset of faces of $\square^n$.*
(2) *Each face of X^n is totally geodesic.*
(3) *The faces of X^n intersect orthogonally.*
(4) *Each 0-dimensional face is a single point (i.e. X^n has precisely 2^n vertices).*
(5) *The map $f: X^n \to \square^n$ of Lemma 5.9 is degree one as is its restriction to each face of X^n.*

Remark. In this corollary the meaning of the word "face" is as in the previous section: a k-dimensional face of X^n is a union of k-dimensional strata. In particular, *a face need not be connected.* Statement (4) asserts that each 0-dimensional face is connected; however, in general we do not know if it is possible to find such X^n with all faces connected.

Proof of Corollary 6.2. It follows from Lemma 5.8 that X^n is connected. It is compact and orientable since M^n is. The B_n-action on M^n lifts to one on X^n (cf. Remark 5.5). Statements (2)–(4) follow, respectively, from parts (2)–(4) of Theorem 6.1. Statement (1) follows from (4) and Lemma 5.6. Statement (5) is just Lemma 5.9(b). ■

Remark. In dimension 3, it follows from the work of W. Thurston (his geometrization theorem for orbifolds) that there are lots of examples of such X^3.

All known methods for constructing closed hyperbolic manifolds in arbitrary dimensions involve the arithmetic of quadratic forms, e.g. see [14, 18, 19]. Our construction of M^n involves such a standard procedure.

Let $K = \mathbb{Q}(\sqrt{d})$ be a totally real quadratic extension of the rationals and let A denote the ring of algebraic integers in K. Denote the automorphism of K induced by $\sqrt{d} \to -\sqrt{d}$ by $\alpha \to \bar{\alpha}$. Choose ε in A so that $\varepsilon > 0$ and $\bar{\varepsilon} < 0$. Define a symmetric bilinear form on A^{n+1} by

$$\varphi(e_i, e_j) = \begin{cases} \delta_{ij}; & (i,j) \neq (0,0) \\ -\varepsilon; & (i,j) = (0,0) \end{cases}$$

where $e_0, e_1, \ldots, e_n$ is a basis for A^{n+1}. The isometry group of φ will be denoted by $O(\varphi)$.

The form φ induces a symmetric bilinear form $\varphi_{\mathbb{R}}$ on $A^{n+1} \otimes \mathbb{R} = \mathbb{R}^{n+1}$ of signature $(n, 1)$. The associated quadratic form is $q(x_0, x_1, \ldots, x_n) = -\varepsilon(x_0)^2 + (x_1)^2 + \cdots + (x_n)^2$. The hypersurface S in $\mathbb{R}^{n+1}$ defined by $q(x) = -1$ is a two-sheeted hyperboloid. The positive sheet defined by $x_0 > 0$ is hyperbolic n-space $\mathbb{H}^n$. The form $\varphi_{\mathbb{R}}$ induces a Riemannian metric on $\mathbb{H}^n$ of constant sectional curvature -1.

The isometry group of $\varphi_{\mathbb{R}}$ can be identified with the Lie group $O(n, 1)$. There are index two subgroups

$$SO(n, 1) = \{g \in O(n, 1) \mid \det(g) = 1\}$$

and

$$O_0(n, 1) = \{g \in O(n, 1) \mid \varphi_{\mathbb{R}}(ge_0, e_0) < 0\}.$$

The group $O(n, 1)$ acts on the hypersurface S and $O_0(n, 1)$ is the subgroup which preserves the sheets. In fact, $O_0(n, 1)$ is the full group of isometries of the Riemannian manifold $\mathbb{H}^n$. (Since the central element -1 of $O(n, 1)$ reverses the sheets we can also identify the isometry group of $\mathbb{H}^n$ with $PO(n, 1) = O(n, 1)/\{\pm 1\}$.) The identity component of $SO(n, 1)$ is $SO_0(n, 1) = SO(n, 1) \cap O_0(n, 1)$.

The group $O(\varphi)$ is naturally a subgroup of $O(n, 1)$. It follows from the assumption that $\bar{\varepsilon} < 0$ that $O(\varphi)$ is discrete and cocompact in $O(n, 1)$ (e.g. see [14, 2.2 and 2.3]). Thus, $O(\varphi)/\{\pm 1\}$ is a discrete group of isometries of $\mathbb{H}^n$ with compact orbit space.

We can identify B_n as a subgroup of $O(\varphi)$ generated by all permutations of $e_1, \ldots, e_n$ and reflections r_i, $i = 1, \ldots, n$, where r_i is the reflection which sends e_i to $-e_i$ and fixes the orthogonal complement to e_i. Thus, B_n fixes the vector e_0.

For $i = 1, \ldots, n$, let P_i denote the intersection of the orthogonal complement of e_i in $\mathbb{R}^{n+1}$ with $\mathbb{H}^n$, that is, P_i is the hyperplane in $\mathbb{H}^n$ defined by

$$P_i = \{x \in \mathbb{H}^n \mid x_i = 0\}.$$

Then $P_1 \cap \cdots \cap P_n$ is a single point p, where $p = (1/\sqrt{\varepsilon}, 0, \ldots, 0)$. The group B_n fixes p and acts on $T_p\mathbb{H}^n$ via the standard representation.

We suppose from now on that Γ is some torsion-free normal subgroup of $O(\varphi)$. We put

$$M^n = \mathbb{H}^n/\Gamma.$$

Then M^n is a closed, connected hyperbolic manifold. Let $\pi: \mathbb{H}^n \to M^n$ be the covering projection and put

$$Y_i = \pi(P_i)$$

$$y = \pi(p).$$

Since B_n normalizes Γ, it acts via isometries on M^n.

In the following lemma we prove a large part of Theorem 6.1. (Only statements (4) and (6) of Theorem 6.1 are missing; (4) is replaced by the weaker (4)′ in the lemma.)

LEMMA 6.3. *Let Γ be any torsion-free subgroup of $O(\varphi)$. Let $M^n = \mathbb{H}^n/\Gamma$ and $\mathscr{Y} = \{Y_1, \ldots, Y_n\}$, where the Y_i are as above. Then $\mathscr{Y}$ is a system of closed, connected submanifolds of codimension one in M^n such that the following properties hold*:

(1) *Y_i is a component of the fixed point set of r_i on M^n.*

(2) *Each Y_i is totally geodesic in M^n.*

(3) *The Y_i's intersect orthogonally.*

(4)′ *$Y_1 \cap \cdots \cap Y_n$ is a finite set which contains the point y.*

(5) *B_n fixes y and the representation of B_n on $T_y M^n$ is equivalent to the standard representation.*

Proof. Each Y_i, being the image of a connected space, is connected, and Y_i is clearly contained in the fixed set of r_i. Since the fixed set of any smooth involution is a submanifold, it follows that each Y_i is an embedded submanifold and a component of the fixed set of r_i. Thus, $\mathscr{Y}$ is a system of closed, connected, codimension one submanifolds. Since r_i is an isometry, (2) holds; since the r_i commute, (3) holds. Statements (4)′ and (5) are obvious. ∎

Hence, it remains to verify properties (4) and (6) of Theorem 6.1. With regard to (6), the following result is immediate from the fact that $SO_0(n, 1)$ is connected.

LEMMA 6.4. *Suppose that Γ is as in Lemma 6.3 and that $\Gamma \subset SO_0(n, 1)$. Then M^n, as well as each Y_i, is orientable.*

We now focus on property (4) of Theorem 6.1. Suppose that $z \in Y_1 \cap \cdots \cap Y_n$ and that $\tilde{z}$ is a lift of z to $\mathbb{H}^n$. We seek a condition on Γ which will insure that $z = y$, i.e. that $\tilde{z} = \gamma p$ for some $\gamma \in \Gamma$. Let γ_i in Γ be such that $\gamma_i P_i$ is the component of $\pi^{-1}(Y_i)$ which contains $\tilde{z}$. Since the vector e_i is a unit normal to P_i, $\gamma_i e_i$ is a unit normal to $\gamma_i P_i$. Since the $\gamma_i P_i$ intersect orthogonally at $\tilde{z}$, we have that

$$\varphi(\gamma_i e_i, \gamma_j e_j) = \delta_{ij}, \quad 1 \le i, j \le n.$$

Hence, $(\gamma_1 e_1, \ldots, \gamma_n e_n)$ is an orthogonal basis for a sublattice L of A^{n+1}. Moreover, L is equivalent to $\langle 1 \rangle \perp \cdots \perp \langle 1 \rangle$ where $\langle a \rangle$ denotes the one-dimensional lattice generated by a basis vector of norm a and where $\perp$ denotes orthogonal direct sum. It follows that A^{n+1} is the orthogonal direct sum of L and its orthogonal complement $L^\perp$ and that $L^\perp \cong \langle -\varepsilon \rangle$. Hence, there is a vector in A^{n+1} orthogonal to L and of norm $-\varepsilon$. The only possibilities are $\pm e$, where $e = \sqrt{\varepsilon}\tilde{z}$. Let $[e, \gamma_1 e_1, \ldots, \gamma_n e_n]$ denote the $(n+1)$ by $(n+1)$ matrix with column vectors $e, \gamma_1 e_1, \ldots, \gamma_n e_n$. Since $\{e, \gamma_1 e_1, \ldots, \gamma_n e_n\}$ is an orthogonal basis for A^{n+1} with respect to φ, the matrix $[e, \gamma_1 e_1, \ldots, \gamma_n e_n]$ lies in $O(\varphi)$. Since it maps e_0 to e, it takes $p = (1/\sqrt{\varepsilon})e_0$ to $\tilde{z} = (1/\sqrt{\varepsilon})e$.

LEMMA 6.5. *Let Γ be a normal, torsion-free subgroup of $O(\varphi)$. With notation as above, the following statements are equivalent*:

(i) $Y_1 \cap \cdots \cap Y_n = \{y\}$.

(ii) *Given any n-tuple $(\gamma_1, \ldots, \gamma_n)$ in $\Gamma \times \cdots \times \Gamma$ such that $\gamma_1 P_1 \cap \cdots \cap \gamma_n P_n$ is nonempty, there is an element γ in Γ such that $\gamma_1 P_1 \cap \cdots \cap \gamma_n P_n = \gamma(P_1 \cap \cdots \cap P_n)$.*

(iii) *Given any n-tuple $(\gamma_1, \ldots, \gamma_n)$ in $\Gamma \times \cdots \times \Gamma$ such that $\varphi(\gamma_i e_i, \gamma_j e_j) = \delta_{ij}$, $1 \le i, j \le n$, one of the two matrices $[\pm e, \gamma_1 e_1, \ldots, \gamma_n e_n]$ lies in $B_n\Gamma$.*

The lemma is obvious. Perhaps the only point which needs comment is the appearance of the group B_n in statement (iii). The reason is that in passing from (ii) to (iii) there is no reason to choose e_i as the unit normal to P_i rather than $-e_i$. Similarly there is no preferred ordering for $\{\gamma_1 e_1, \dots, \gamma_n e_n\}$. These sign changes and permutations of coordinates are accounted for by B_n.

We remark that if there is an element γ in Γ such that $[\pm e, \gamma_1 e_i, \dots, \gamma_n e_n] = b\gamma$, for some $b \in B_n$, then γ is unique. For if $b_1 \gamma_1 = b_2 \gamma_2$, then $\gamma_1 \gamma_2^{-1} \in B_n$. Since Γ is torsion-free, it follows that $\gamma_1 \gamma_2^{-1} = 1$.

It turns out that Lemmas 6.4 and 6.5 can be used to complete the proof of Theorem 6.1 if we choose Γ to be an appropriate congruence subgroup of $O(\varphi)$.

Let $\mathfrak{p}$ be a prime ideal in A. Then $A/\mathfrak{p}$ is a finite field and the form φ induces a nonsingular bilinear form $\varphi_\mathfrak{p}$ on $(A/\mathfrak{p})^{n+1}$. Its isometry group $O(\varphi_\mathfrak{p})$ is finite. The kernel of the natural projection $\lambda: O(\varphi) \to O(\varphi_\mathfrak{p})$ is denoted by $\Gamma(\mathfrak{p})$ and called a *congruence subgroup* of $O(\varphi)$. It is, of course, normal and of finite index in $O(\varphi)$. Moreover, if $|\mathfrak{p}|$ is sufficiently large, then $\Gamma(\mathfrak{p})$ is torsion-free. More generally, if $\mathscr{I}$ is any nonzero ideal in A, then

$$\Gamma(\mathscr{I}) = \{\gamma \in O(\varphi) \mid \gamma \equiv 1 \pmod{\mathscr{I}}\}$$

is also called a congruence subgroup.

LEMMA 6.6. *Let Γ be any torsion-free congruence subgroup of $O(\varphi)$. Then $Y_1 \cap \cdots \cap Y_n$ is a single point.*

Proof. Suppose $\Gamma = \Gamma(\mathscr{I})$. Using Lemma 6.5(iii) we consider an n-tuple $(\gamma_1, \dots, \gamma_n)$ where $\gamma_i \in \Gamma$ and $\varphi(\gamma_i e_i, \gamma_j e_j) = \delta_{ij}$. Since $\gamma_i \in \Gamma$, $\gamma_i e_i \equiv e_i \pmod{\mathscr{I}}$. Hence, the vector e, which generates the orthogonal complement to the lattice spanned by $\gamma_1 e_1, \dots, \gamma_n e_n$, satisfies $e \equiv \pm e_0 \pmod{\mathscr{I}}$. Therefore, either the matrix $\gamma_+ = [e, \gamma_1 e_1, \dots, \gamma_n e_n]$ or $\gamma_- = [-e, \gamma_1 e_1, \dots, \gamma_n e_n]$ lies in Γ. ■

To complete the proof of Theorem 6.1 we need to show that we can find a congruence subgroup which is contained in $SO_0(n, 1)$. Any $g \in O(n, 1)$ has $\det(g) = \pm 1$. If $\gamma \in \Gamma(\mathfrak{p})$, then $\det(\lambda(\gamma)) = \det(1) = 1 \in A/\mathfrak{p}$. Hence, $\det(\gamma) \equiv 1 \pmod{\mathfrak{p}}$. If $2 \notin \mathfrak{p}$, then $+1$ and -1 are not congruent modulo $\mathfrak{p}$ and so this forces γ to lie in $SO(n, 1)$. The problem of insuring that a congruence subgroup is contained in the identity component of $SO(n, 1)$ is subtler, but it has been solved in [19, Proposition 4.1, p. 120].

LEMMA 6.7 (Millson–Raganuthan). *There is an ideal $\mathscr{I}$ in A, which is a product of (finitely many) suitably chosen relatively prime ideals, such that the congruence subgroup $\Gamma(\mathscr{I})$ is contained in $SO_0(n, 1)$.*

For the reader's convenience we sketch a proof, which was explained to us by G. Prasad.

First we need to recall the notion of the "spinor norm". Suppose that φ is an m-dimensional quadratic form over a field K, and denote the associated symmetric bilinear form $K^m \times K^m \to K$ by $(v, w) \to v \cdot w$. If $v \in K^m$ is any vector of nonzero norm (i.e. if $v \cdot v \neq 0$), then *orthogonal reflection with respect* to v is the isometry r_v of φ defined by

$$r_v(x) = x - [2(x \cdot v)/(v \cdot v)]v.$$

According to [10] any isometry $g \in O(\varphi; K)$ can be written as a product of orthogonal reflections $g = r_{v_1} \cdots r_{v_s}$. The *spinor norm* $\rho_K(g)$ is defined to be the image of the product $\varphi(v_1) \cdots \varphi(v_s)$ in $K^*/(K^*)^2$. It turns out that $\rho_K: SO(\varphi; K) \to K^*/(K^*)^2$ is a well-defined homomorphism. We note that the abelian group $K^*/(K^*)^2$ is 2-torsion.

For example, if $K = \mathbb{R}$, then $(\mathbb{R}^*)^2$ consists of the positive reals and, hence $\mathbb{R}^*/(\mathbb{R}^*)^2$ is cyclic of order two. The kernel of $\rho_{\mathbb{R}}: SO(n, 1) \to \mathbb{Z}/2$ is obviously $SO_0(n, 1)$.

Proof of Lemma 6.7. Let $K = \mathbb{Q}(\sqrt{d})$ and let φ be the quadratic form on A^{n+1} defined after Corollary 6.2. We shall show that there is an ideal $\mathscr{I}$ such that $\Gamma(\mathscr{I})$ is contained in the kernel of $\rho_K: SO(\varphi) \to K^*/(K^*)^2$. Since the kernel of ρ_K is contained in the kernel of $\rho_{\mathbb{R}}$ this will complete the proof. The key observation is that the group $SO(\varphi)$ is finitely generated. (The reason is that $\mathbb{H}^2/SO(\varphi)$ is compact.) Hence, ρ_K takes $SO(\varphi)$ onto a finitely generated subgroup J of $K^*/(K^*)^2$ ($J \cong (\mathbb{Z}/2)^m$ for some m).

Since $SO(\varphi)$ is finitely generated, we can find a finite set of vectors V in K^{n+1} so that every element of some generating set for $SO(\varphi)$ can be written as a product of orthogonal reflections with respect to vectors in V. Clearing denominators, we can assume $V \subset A^{n+1}$. For a given $v \in V$, $\varphi(v)$ is a unit except at finitely many places, i.e. $|\varphi(v)|_{\mathfrak{p}} = 1$ except for finitely many places $\mathfrak{p}$. Hence, we can find a finite set of places S so that φ is nonsingular modulo primes outside of S and so that $|\varphi(v)|_{\mathfrak{p}} = 1$ for all $v \in V$ and $\mathfrak{p} \notin S$. For a prime ideal $\mathfrak{p}$ of A, let $k_{\mathfrak{p}}$ denote the residue field $A_{(\mathfrak{p})}/\mathfrak{p}A_{(\mathfrak{p})}$. For primes not in S, the spinor norm descends to $k_{\mathfrak{p}}$. (In other words, if $\varphi_{\mathfrak{p}}$ denotes the quadratic form on $k_{\mathfrak{p}}^{n+1}$induced by φ and if, for $g \in SO(\varphi)$, $g_{\mathfrak{p}}$ denotes the element of $SO(\varphi_{\mathfrak{p}})$ obtained from g by reduction mod $\mathfrak{p}$, then $\rho_{k_{\mathfrak{p}}}(g_{\mathfrak{p}})$ equals the image of $\rho_K(g)$ in $k_{\mathfrak{p}}^*$ read modulo $(k_{\mathfrak{p}}^*)^2$.)

By the Čebotarev density theorem, given any $x \in A^* - (A^*)^2$, x is not a square modulo $\mathfrak{p}$ for half the primes outside of S. Hence, choosing a representative $x \in A^*$ for each generator of J, we can pick a prime ideal $\mathfrak{p}_x$ such that x is not a square in $k_{\mathfrak{p}_x}$. Taking $\mathscr{I}$ to be the product of the $\mathfrak{p}_x$ we will then have that $\Gamma(\mathscr{I}) \subset \ker \rho_K$. ∎

7. CONVERTING A CUBICAL EUCLIDEAN CELL COMPLEX INTO A PIECEWISE HYPERBOLIC POLYHEDRON

Recall from Section 2 that $\mathscr{PE}_n$ is the category of piecewise Euclidean polyhedra of dimension $\le n$. Let $\mathscr{PH}_n$ denote the category of piecewise hyperbolic polyhedra of dimension $\le n$. Morphisms in both categories are isometries onto totally geodesic subcomplexes.

Let X^n be the hyperbolic n-manifold with corners constructed in Corollary 6.2.

PROPOSITION 7.1. *Suppose that K is a cubical Euclidean cell complex of dimension $\le n$ (cf. Definition 1.1). Then there is a piecewise hyperbolic polyhedron K_X and a map $q: K_X \to K$ such that the following properties hold*:

(1) *For each k-cell $\square^k$ in K, $q^{-1}(\square^k)$ is isometric to a k-dimensional face of X^n. Furthermore, if J is any subcomplex of K, then $q^{-1}(J)$ is isometric to J_X.*

(2) *The directions normal to $q^{-1}(\square^k)$ naturally form a piecewise spherical polyhedron, denoted Link $(q^{-1}(\square^k), K_X)$, and this polyhedron is isometric to Link$(\square^k, K)$.*

(3) *The construction of K_X from K defines a functor from $\mathscr{PE}_n$ to $\mathscr{PH}_n$. In particular, it takes totally geodesic subcomplexes to totally geodesic subcomplexes.*

(4) *The map q induces a surjection on homology.*

(5) *If K has curvature ≤ 0, then K_X has curvature ≤ -1.*

Proof. The polyhedron K can be constructed as a quotient space of a disjoint union $\hat{K}$ of copies of standard Euclidean cubes. Such a standard cube may be viewed as a face of a fixed n-cube $\square^n$. The equivalence relation on $\hat{K}$ is defined by identifying various faces via isometries viewed as elements of B_n. By Corollary 6.2(1), the poset of faces of X^n is

B_n-equivariantly isomorphic to the poset of faces of $\square^n$. To construct K_X one replaces each k-cube in $\hat{K}$ by the corresponding k-face of X^n. Call the resulting disjoint union $\hat{K}_X$. To each isometry in the equivalence relation on $\hat{K}$ the associated element of B_n gives an isometry between the corresponding faces of X^n. By definition K_X is then the quotient space of $\hat{K}_X$ by the resulting equivalence relation. The map $q: K_X \to K$ is then defined on each face of K_X via the map f of Corollary 6.2(5). Statement (1) is now obvious. Since $f: X^n \to \square^n$ is degree one on each face, statement (4) follows as in [9, Section 1d, p. 354]. Since the faces of X^n intersect orthogonally, the link of a face of X^n in X^n is isometric to the link of a face of $\square^n$ in $\square^n$. Statement (2) follows. Statements (3) and (5) follow from (2) and Theorem 1.6. ■

Definition 7.2. Suppose that K is a cubical cell complex of dimension $\leq n$. A *projection to* $\square^n$ is a cellular map $p: K \to \square^n$ such that the restriction of p to any cell is a combinatorial isomorphism.

If K admits a projection to $\square^n$, then the piecewise hyperbolic polyhedron K_X of the previous proposition can be obtained as a fiber product:

$$\begin{array}{ccc} K_X & \to & X^n \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ K & \xrightarrow{p} & \square^n \end{array}$$

That is to say, K_X can be identified with the subspace of $K \times X^n$ consisting of all (k, x) such that $p(k) = f(x)$.

PROPOSITION 7.3. *Suppose that K is a cubical cell complex homeomorphic to a smooth or PL manifold (i.e. K is a smooth or PL "cubization" of a manifold). Suppose further that K admits a projection $p: K \to \square^n$. Then*:

(1) *K_X is embedded in $K \times X^n$ with trivial normal bundle.*
(2) *The rational Pontryagin classes of K_X are the pullbacks (via q) of those of K.*

Proof. The proof of (1) is similar to that of Propositions (1f.3) and (1f.5) in [9, p. 357]. (These propositions are the PL and smooth cases, respectively.) For example, in the smooth case the argument goes as follows. Viewing $\square^n$ as $[0, 1]^n \subset \mathbb{R}^n$, the space K_X is the inverse image of $0 \in \mathbb{R}^n$ under the map $\psi: K \times X^n \to \mathbb{R}^n$ defined by $(k, x) \to p(k) - f(x)$. Assuming, as we may, that f is transverse to each face of $\square^n$, we have that 0 is a regular value of ψ; statement (1) follows. Statement (2) follows from the fact that the rational Pontryagin classes of any hyperbolic manifold (e.g. M^n or X^n) are trivial together with the Whitney product formula. ■

QUESTION 7.4. *Can we find a stably parallelizable X^n in each dimension n? (Sullivan [22, p. 553] has shown that any hyperbolic manifold is finitely covered by a stably parallelizable manifold.)*

We can apply the above construction to the result of a (nonstrict) hyperbolization procedure. For our purposes the most useful procedure is the second construction of Gromov [12, p. 116] which is described in detail in [9, Section 4c]. For the reader's convenience we shall briefly recall this construction below.

Gromov's second construction: As in Section 2, let $\mathscr{C}_n$ be the category of cell complexes of dimension $\leq n$ and let $\mathscr{C}_n^{\text{simp}}$ be the full subcategory of $\mathscr{C}_n$ consisting of simplicial complexes.

We shall define a hyperbolization functor $\mathcal{G}_n$ from $\mathcal{C}_n^{\text{simp}}$ to $\mathcal{PE}_n$ satisfying conditions (1)–(6) of Section 2. We first define the hyperbolization of an n-simplex $\mathcal{G}_n(\Delta^n)$ and a map $\mathcal{G}_n(\Delta^n) \to \Delta^n$. One then defines $\mathcal{G}_n(K)$ as the fiber product

$$\begin{array}{ccc} \mathcal{G}_n(K) & \to & \mathcal{G}_n(\Delta^n) \\ {\scriptstyle q}\downarrow & & \downarrow \\ K' & \underset{p}{\to} & \Delta^n \end{array}$$

where K' denotes the barycentric subdivision of K and p the natural projection. The definition of the hyperbolized simplex $\mathcal{G}(\Delta^n)$ is by induction on n. Put $\mathcal{G}(\Delta^1) = \Delta^1 = [0, 1]$. Assuming $\mathcal{G}_{n-1}(\partial\Delta^n)$ has been defined, we form $\mathcal{G}_n(\Delta^n)$ by taking a reflection on $\partial\Delta^n$ and then gluing the ends of $\mathcal{G}_{n-1}(\partial\Delta^n) \times [-1, 1]$ together along a "half-space" for the induced reflection on $\mathcal{G}_{n-1}(\partial\Delta^n)$. Alternatively, $\mathcal{G}_n(\Delta^n)$ can be viewed as $\mathcal{G}_{n-1}(\partial\Delta^n) \times S^1$ cut open along the half-space. Each hyperbolized simplex $\mathcal{G}_n(\Delta^n)$ has a natural structure of a cubical Euclidean cell complex and this induces such a structure on $\mathcal{G}_n(K)$. (We assume, by induction, that $\mathcal{G}_{n-1}(\partial\Delta^{n-1})$ has a cubical structure. Then $\mathcal{G}_n(\Delta^n)$ inherits a cubical structure from $\mathcal{G}_{n-1}(\partial\Delta^{n-1}) \times [-1, 1]$ where $[-1, 1]$ is subdivided into two unit intervals.)

LEMMA 7.5. *There is a projection p_n from $\mathcal{G}_n(K)$ to $\square^n$.*

Proof. It suffices to define $p_n : \mathcal{G}_n(\Delta^n) \to \square^n$ and then compose with the canonical map $\mathcal{G}_n(K) \to \mathcal{G}_n(\Delta^n)$. Identify S^1 with $[-1, 1]$ with endpoints identified and let $\lambda : S^1 \to [0, 1]$ be the map induced by $t \to |t|$. We may inductively assume that $p_{n-1} : \mathcal{G}_{n-1}(\partial\Delta^n) \to \square^{n-1}$ is defined. The p_n is defined to be the map induced by $p_{n-1} \times \lambda : G_{n-1}(\partial\Delta^n) \times S^1 \to \square^{n-1} \times [0, 1] = \square^n$. ■

Strict hyperbolization: Combining the two constructions above, we define a functor $\mathcal{G}_X$ from $\mathcal{C}_n^{\text{simp}}$ to $\mathcal{PH}_n$ by

$$\mathcal{G}_X(K) = (\mathcal{G}_n(K))_X.$$

The next theorem follows immediately from Propositions 7.1 and 7.3.

THEOREM 7.6. *There is a strict hyperbolization functor $\mathcal{G}_X$ from $\mathcal{C}_n^{\text{simp}}$ to $\mathcal{PH}_n$ satisfying conditions* (1)–(5) *and* (6)′ *of Section* 2.

If Question 7.4 can be answered affirmatively then condition (6) of Section 2 also holds.

THEOREM 7.7. *Any triangulable manifold is cobordant to an triangulable manifold of strictly negative curvature.*

Proof. The proof is the same as in [21, Section 4.3]. Let K be a triangulation of the given PL-manifold. Let $\tilde{K}$ denote $K \times [0, 1]$ with the cone on K glued on to $K \times 0$. Applying the functor $\mathcal{G}_X$, there is a unique vertex $v_0 = \mathcal{G}_X(x_0)$ of $\mathcal{G}_X(\tilde{K})$ corresponding to the cone point x_0 of $\tilde{K}$. The link of v_0 in $\mathcal{G}_X(\tilde{K})$ is PL-homeomorphic to the link of x_0 in $\tilde{K}$, namely $K \times 0$. Thus, removing a neighborhood of v_0 gives the desired cobordism from $K \times 0$ to $\mathcal{G}_X(K \times 1)$. ■

Acknowledgements—We would like to thank John Hsia and Gopal Prasad for some helpful conversations concerning the quadratic forms which are used in Section 6, and Lowell Jones for some illuminating remarks concerning the relative hyperbolization procedure of Section 2.

REFERENCES

1. A. D. ALEKSANDROV, V. N. BERESTOVSKII and I. G. NIKOLAEV: Generalized Riemannian spaces, *Russian Math. Surveys* **41** (1986), 1–54.
2. M. BESTVINA and G. MESS: The boundary of negatively curved groups, *J. Amer. Math. Soc.* **4** (1991), 469–481.
3. N. BOURBAKI: *Groupes et Algèbres de Lie*, Chapitres 4–6, Masson, Paris (1981).
4. K. S. BROWN: *Buildings*, Springer, New York (1989).
5. R. CHARNEY and M. W. DAVIS: Singular metrics of nonpositive curvature on branched covers of Riemannian manifolds, *Amer. J. Math.* **115** (1993), 929–1009.
6. M. W. DAVIS: Groups generated by reflections and aspherical manifolds not covered by Euclidean space, *Ann. Math.* (2) **117** (1983), 293–325.
7. M. W. DAVIS: Some aspherical manifolds, *Duke Math. J.* **55** (1987), 105–139.
8. M. W. DAVIS: Regular convex cell complexes, in *Geometry and topology: manifolds, varieties, and knots*, C. McCrory and T. Shifrin, Eds, Marcel Dekker, New York (1987), pp. 53–88.
9. M. W. DAVIS and T. JANUSZKIEWICZ: Hyperbolization of polyhedra, *J. Differential Geometry* **34** (1991), 347–388.
10. J. DIEUDONNÉ: *La Géométrie des Groupes Classiques*, Springer, Berlin (1963).
11. E. GHYS and P. de la HARPE (Eds): *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Progress in Math. **83**, Birkhaüser, Boston (1990).
12. M. GROMOV: Hyperbolic groups, in *Essays in group theory*, S. M. Gersten, Ed., M.S.R.I. Publ. **8**, Springer, New York (1987), pp. 75–264.
13. M. GROMOV: Asymptotic invariants of infinite groups, in *Geometric group theory, Vol. 2*, G. Niblo and M. Roller, Eds, LMS LNS **182**, Cambridge Univ. Press, Cambridge (1993).
14. M. GROMOV and I. PIATETSKI-SHAPIRO: Non-arithmetic groups in Lobachevsky spaces, *Publ. I.H.E.S.* **66** (1988), 93–103.
15. A. HAEFLIGER: Groupoides d'holonomie et classifiants, in *Structure Transverse des Feuilletages*, Toulouse (1982), *Astérisque* **116** (1984), 70–97.
16. B. HU: Retractions of closed manifolds with nonpositive curvatures, to appear in Proceedings of the Ohio State University Conference on Geometric Group theory, de Gruyter.
17. T. JANUSZKIEWICZ: Hyperbolizations, in *Group theory from a geometrical viewpoint*, E. Ghys, A. Haefliger and A. Verjovsky, Eds, World Scientific, Singapore (1991), pp. 464–490.
18. J. J. MILLSON: On the first Betti number of a constant negatively curved manifold, *Ann. Math.* **104** (1976), 235–247.
19. J. J. MILLSON and M. S. RAGHUNATHAN: Geometric construction of cohomology for arithmetic groups I, *Proc. Indian Acad. Sci. (Math. Sci.)* **90** (1981), 103–123.
20. G. MOUSSONG: Hyperbolic Coxeter groups, Ph.D. thesis, The Ohio State University (1988).
21. F. PAULIN: Constructions of hyperbolic groups via hyperbolization of polyhedra, in *Group theory from a geometrical viewpoint*, E. Ghys, A. Haefliger and A. Verjovsky, Eds, World Scientific, Singapore (1991), pp. 313–372.
22. D. SULLIVAN: Hyperbolic geometry and homeomorphisms, in *Geometric topology*, J. C. Cantrell, Ed., Academic Press, New York (1979), pp. 543–555.
23. W. THURSTON: The geometry and topology of 3-manifolds, reproduced Lecture Notes, Princeton University (1979).

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