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Simplicial volume of closed locally symmetric spaces of non-compact type

by

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1. Introduction

In his paper [10], Gromov introduced the notion of simplicial volume of a connected, closed and orientable manifold M. This invariant is denoted by $\|M\|\in[0,\infty)$, and measures how efficiently the fundamental class of M may be represented using real cycles.

In the same paper, the question was raised as to whether the simplical volume of a closed locally symmetric space of non-compact type is positive [10, p. 11]. Gromov cites the work of Thurston [18], Inoue–Yano [13] and Savage [16], in which positivity is verified for compact manifolds that are real hyperbolic, negatively curved and locally symmetric quotients of $\mathrm{SL}(n,\mathbf{R})/\mathrm{SO}(n,\mathbf{R})$, respectively. Moreover, Gromov [10] shows that characteristic classes may be represented using bounded cohomology classes implying positivity for several additional classes of locally symmetric spaces. Since then, this question has been mentioned in a variety of different sources [8], [11], [15], [16], and has become a well-known "folk conjecture". Recently, Bucher-Karlsson found a mistake in Savage's paper (as explained in [5]). In the case of closed locally symmetric spaces covered by $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$, she also provided (in [6]) the analytic estimates required in order to salvage Savage's argument. The purpose of this paper is to answer Gromov's conjecture for all remaining closed locally symmetric spaces of non-compact type in the affirmative. Namely, we obtain the following result.

Theorem 1.1. (Main theorem) *If M^n is a closed locally symmetric space of non-compact type, then $\|M^n\|>0$.*

The approach that we use is due to Thurston [18], and bounds the simplicial volume $\|M\|$ from below by the proportion between the volume of M and the maximal volume of suitably defined straightened top-dimensional simplices in M. Technically, however, we are indebted to Besson, Courtois and Gallot for their pioneering work around the use of the barycenter method in proving the rank-one minimal entropy rigidity conjecture for locally symmetric spaces [3], and to Connell and Farb for their subsequent development of the technique in higher-rank spaces (see [8] for an extensive survey).

The main contribution of this paper lies in the idea of using the barycenter method in order to define the straightened simplices that are central to Thurston's argument. Roughly speaking, the barycenter method homotopes a map with negatively curved target to a C^1 map (often called the "natural" map), by first mapping the source space to the space of measures on the boundary at infinity of the target, and then to the target by taking the center of mass of the measures. The technique, as developed by Besson, Courtois and Gallot, then proceeds with a rather intricate argument to give a pointwise upper bound for the Jacobian of the natural map.

When studying rigidity questions, one typically wants to replace this pointwise upper estimate by the best uniform estimate and determine what happens in the equality case. In this way, Besson, Courtois and Gallot solved the minimal entropy rigidity conjecture for rank-one spaces.

As a major first step towards solving the minimal entropy rigidity conjecture in higher rank, Connell and Farb extended the barycenter method to higher-rank spaces [7]. Furthermore, they established a new geometric result about locally symmetric spaces of non-compact type with no local factors locally isometric to $\mathbf{H}^2$ or $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$, in order to obtain an "a priori" uniform bound for the Jacobian of the natural map. Here, we use Connell's and Farb's extension of the technique to higher-rank spaces in order to define "barycentrically" straightened simplices. The uniform upper bound for the volume of top-dimensional straightened simplices required in Thurston's argument will follow directly from the detailed analysis used to obtain the uniform Jacobian bound in [7]. As this approach excludes those locally symmetric spaces with local $\mathbf{H}^2$ or $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$ factors, we appeal to Thurston [18] and Bucher-Karlsson [6] to cover the remaining cases.

We note that a similar "barycentric" straightening was used by Kleiner in [14] in order to compare various notions of dimension for length spaces with curvature bounded above. We would like to thank Dick Canary, Tom Farrell and Ralf Spatzier for their interest in our work and many helpful discussions.

2. Thurston's lower bound for simplicial volume

In this section, we first collect together some definitions and results about simplicial volume. We then describe how Thurston reduces the problem of establishing positivity of simplicial volume to the more geometric problem of uniformly bounding the volumes of suitably defined straightened simplices.

2.1. Simplicial volume

We begin with the definition of the simplicial volume and the important proportionality principle.

Definition. Let M be a topological space, $C^0(\Delta^k, M)$ be the set of singular k-simplices, and let $c=\sum_{i=1}^{j} r_i f_i$ with each $r_i \in \mathbf{R}$ and $f_i \in C^0(\Delta^k, M)$ be a singular real chain. The l^1-*norm* of c is defined by $\|c\|_1=\sum_i |r_i|$. The l^1-*(pseudo)norm* of a real singular homology class $[\alpha]\in H_k^{\mathrm{sing}}(M, \mathbf{R})$ is defined by

$$\|[\alpha]\|_1 = \inf\{\|c\|_1 : \partial(c)=0 \text{ and } [c]=[\alpha]\}.$$

Definition. Let M^n be an oriented closed connected n-manifold. The *simplicial volume of* M^n is defined as $\|M^n\|=\|i([M^n])\|_1$, where $i: H_n(M, \mathbf{Z}) \to H_n(M, \mathbf{R})$ is the change of coefficients homomorphism, and $[M^n]$ is the fundamental class arising from the orientation of M^n.

The proportionality principle ([10], [17], [18]) for simplicial volume is expressed in the following result.

THEOREM 2.1. *Let M and M' be two closed Riemannian manifolds with isometric universal covers. Then*

$$\frac{\|M\|}{\mathrm{Vol}(M)} = \frac{\|M'\|}{\mathrm{Vol}(M')}.$$

In addition, the simplicial volume is particularly well behaved with respect to products and connected sums. Namely, the following results hold.

THEOREM 2.2. *For a pair of closed manifolds M_1 and M_2, we have*

$$C\|M_1\|\cdot\|M_2\| \geqslant \|M_1 \times M_2\| \geqslant \|M_1\|\cdot\|M_2\|,$$

where $C>1$ is a constant that depends only on the dimension of $M_1 \times M_2$.

THEOREM 2.3. *For $n\geqslant 3$, the connected sum $M_1\# M_2$ of a pair of n-dimensional manifolds M_1 and M_2 satisfies*

$$\|M_1\# M_2\| = \|M_1\| + \|M_2\|.$$

The proofs of Theorem 2.2 and Theorem 2.3 may be found in [2] and [10].

2.2. Thurston's approach

Our proof of the main theorem relies on Thurston's observation that the ratio between the Riemannian volume of a manifold M^n and the maximal volume of a straightened top-dimensional singular simplex in M^n gives a lower bound for the simplicial volume (see [18]). We summarize his approach in the following definition.

Definition. Let us denote by $\widetilde{M}^n$ the universal cover of M^n, by Γ the fundamental group of M^n, and by $C_*^{\rm sing}(\widetilde{M}^n, \mathbf{R})$ the real singular chain complex of $\widetilde{M}^n$. By definition, $C_k^{\rm sing}(\widetilde{M}^n, \mathbf{R})$ is the free $\mathbf{R}$-module generated by $C^0(\Delta^k, \widetilde{M}^n)$, the set of singular k-simplices in $\widetilde{M}^n$. The simplex Δ^k is assumed to be equipped with a fixed Riemmanian metric. Assume that there is a collection of maps $\mathrm{st}_k\colon C^0(\Delta^k, \widetilde{M}^n) \to C^0(\Delta^k, \widetilde{M}^n)$. We will say that this collection of maps is a *straightening* provided it satisfies the following four formal properties:

(a) the maps st_k are Γ-equivariant;

(b) the maps st_k induce a chain map $\mathrm{st}_*\colon C_*^{\rm sing}(\widetilde{M}^n, \mathbf{R}) \to C_*^{\rm sing}(\widetilde{M}^n, \mathbf{R})$ which is Γ-equivariantly chain homotopic to the identity;

(c) the image of st_n lies in $C^1(\Delta^n, \widetilde{M}^n)$, i.e. straightened top-dimensional simplices are C^1;

(d) there exists a constant $C>0$, depending solely on $\widetilde{M}^n$ and the chosen Riemannian metric on Δ^n, such that for any $f \in C^0(\Delta^n, \widetilde{M}^n)$, with corresponding straightened simplex $\mathrm{st}_n(f)\colon \Delta^n \to \widetilde{M}^n$, there is a uniform upper bound on the Jacobian of $\mathrm{st}_n(f)$:

$$|\mathrm{Jac}(\mathrm{st}_n(f))(\delta)| \leqslant C,$$

where $\delta \in \Delta^n$ is arbitrary, and the Jacobian is computed relative to the fixed Riemannian metric on Δ^n.

Thurston established the following result.

Theorem 2.4. *If $\widetilde{M}^n$ supports a straightening, then $\|M^n\|>0$.*

We now recall how Theorem 2.4 is proven. Assume that $\widetilde{M}^n$ supports a straightening and note that property (a) implies that the straightening maps descend to straightening maps on the compact quotient M^n. Property (b) ensures that the homology of M^n obtained via the complex of straightened chains coincides with the ordinary singular homology of M^n. Furthermore, since the straightening procedure is a projection operator on the level of chains, it is contracting in the l^1-norm. In particular, if $\sum_{i=1}^j r_i f_i$ is a real n-chain representing the fundamental class of M^n, then so is $\sum_{i=1}^j r_i \,\mathrm{st}(f_i)$, and we have the inequality $\|\sum_{i=1}^j r_i f_i\|_1 \geqslant \|\sum_{i=1}^j r_i \,\mathrm{st}(f_i)\|_1$.

As a consequence, in order to show that the simplicial volume of M^n is positive, it is sufficient to give a lower bound for the l^1-norm of a straightened chain representing the fundamental class. But now observe that, by property (c), the straightened chain is C^1, and hence we can compute the volume of M^n by

$$\mathrm{Vol}(M^n) = \int_{\sum_{i=1}^{j} r_i\,\mathrm{st}(f_i)} dV_{M^n} = \sum_{i=1}^{j} r_i \int_{\mathrm{st}(f_i)} dV_{M^n},$$

where dV_{M^n} is the volume form on M^n. On the other hand, we have the bound

$$\sum_{i=1}^{j} r_i \int_{\mathrm{st}(f_i)} dV_{M^n} \leqslant \sum_{i=1}^{j} |r_i| \int_{\Delta^n} |\mathrm{Jac}(\mathrm{st}(f_i))|\, dV_{\Delta^n},$$

where dV_{Δ^n} is the volume form for the fixed Riemannian metric on Δ^n. Now, by property (d), the Jacobian of straightened simplices is bounded uniformly from above, and hence we have a uniform upper bound

$$\int_{\Delta^n} |\mathrm{Jac}(\mathrm{st}(f_i))|\, dV_{\Delta^n} \leqslant K,$$

where $K>0$ depends only on M^n and the chosen metric on Δ^n. This yields the inequality

$$\mathrm{Vol}(M^n) \leqslant K \sum_{i=1}^{j} |r_i|,$$

which, upon dividing, and passing to the infimum over all straightened chains, provides the positive lower bound $\|M^n\| \geqslant \mathrm{Vol}(M^n)/K > 0$.

We note that, to prove Theorem 2.4, one could replace properties (c) and (d) of a straightening by the more general condition that the volume of the images of straightened top-dimensional simplices are uniformly bounded above. In fact, this more general approach was taken in the aforementioned works [13], [16] and [18]. Our proof of the main theorem involves a new straightening procedure for locally symmetric spaces of non-compact type which do not have any local $\mathbf{H}^2$ or $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$ direct factors. Our formulation of Theorem 2.3 isolates properties (c) and (d) because the barycenter method that we will use is particularly well adapted to establishing these properties.

3. Straightening simplices

In this section we introduce a new *straightening procedure* (as defined in §2.2) that works for locally symmetric spaces of non-compact type with no local direct factors locally isometric to $\mathbf{H}^2$ or $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$. The straightening will be defined using the barycenter method of Besson, Courtois and Gallot, as developed in the higher-rank setting by

Connell and Farb. As such, we will begin with a review of Connell's and Farb's work that we use. We conclude the section by defining the straightening and verifying the four formal requirements.

For more details on locally symmetric spaces of non-compact type, we refer the reader to the text [9]. Throughout, (M^n, g_0) denotes a closed locally symmetric space of non-compact type. We let (X, g) be the symmetric universal covering space of M and fix a basepoint $p\in X$.

3.1. The barycenter method in higher rank

We start with the following conventions:

- Let $\Gamma=\pi_1(M^n)$, $G=\mathrm{Isom}(X)^0$ and $K=\mathrm{Stab}_p(G)$. The basepoint p determines a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ of the Lie algebra of G. Here, $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is identified with the tangent space T_pX. Let $\mathfrak{a}\subset\mathfrak{p}$ be a fixed maximal abelian subalgebra of $\mathfrak{g}$. Fix a regular vector $A\in\mathfrak{a}$ and let b denote the *barycenter* of the Weyl chamber $W(A)$ determined by A (see [7, p. 9] for more details).
- The visual boundary and Furstenberg boundaries are denoted by ∂X and by $\partial_F X\cong G/P$, where P is a minimal parabolic subgroup of G. We will view the Furstenberg boundary as a subset of the visual boundary by identifying it with the G orbit of the boundary point determined by the barycenter vector b.
- The $\|b\|$-conformal density associated with the group Γ given by the family of Patterson–Sullivan measures is denoted by $\nu: X\to\mathcal{M}(\partial X)$, where $\mathcal{M}(\partial X)$ denotes the space of atomless probability measures on the visual boundary of X. We use Albuquerque's theory [1] of these measures in higher rank and recall that they are Γ-equivariant and fully supported on $\partial_F X\subset\partial X$.
- $B: X\times X\times\partial X\to\mathbf{R}$ denotes the Busemann function on X. Specifically,

$$B(x,y,\theta)=\lim_{t\to\infty}(d_X(y,l_\theta(t))-t),$$

where l_θ is the unique geodesic ray starting at x with endpoint θ. It follows that the Busemann function is invariant under the diagonal action of Γ on $X\times X\times\partial X$.

- We will denote by

$$dB_{(x,\theta)}(\,\cdot\,): T_xX\longrightarrow\mathbf{R}\quad\text{and}\quad DdB_{(x,\theta)}(\,\cdot\,,\cdot\,): T_xX\otimes T_xX\longrightarrow\mathbf{R}$$

the 1-form and the 2-form, respectively, obtained by differentiating the function $B(p,\cdot\,,\theta)$ at the point $x\in X$.

- For a measure $\mu\in\mathcal{M}(\partial X)$, let

$$g_\mu(\,\cdot\,)=\int_{\partial X}B(p,\cdot\,,\theta)\,d(\mu)(\theta).$$

When $g_\mu: X \to \mathbf{R}$ has a unique minimum, the *barycenter* of μ, denoted by $\mathrm{bar}(\mu)$, is defined to be the unique point of X where g_μ is minimized. Though the chosen base-point $p \in X$ is used to define the function g_μ, it follows easily from the properties of the Busemann function that $\mathrm{bar}(\mu)$ (when defined) is actually independent of this choice.

Next, we recall the results from [7] that play a central role in our straightening.

As X is non-positively curved, the functions $B(p, \cdot, \theta)$ appearing in the above definition of g_μ are known to be convex functions on X. When the measure $\mu \in \mathcal{M}(\partial X)$ is fully supported on $\partial_F X$, Connell's and Farb's proof of [7, Proposition 3.1] shows that g_μ is strictly convex. Furthermore, they note that when μ is a Patterson–Sullivan measure, the function g_μ is proper. We summarize this in the following result.

LEMMA 3.1. *Let $\mu \in \mathcal{M}(\partial X)$ be a finite weighted sum of Patterson–Sullivan measures on X. Then $\mathrm{bar}(\mu)$ is well defined.*

The following result of Connell and Farb from [7, §4] will be used to verify properties (c) and (d) for our barycentric straightening.

THEOREM 3.2. *Let M be a closed locally symmetric space of non-compact type with no local direct factors locally isometric to $\mathbf{H}^2$ or $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$, and let X be its universal cover. Let $\mu \in \mathcal{M}(\partial X)$ be a probability measure fully supported on $\partial_F X$ and let $x \in X$. Consider the endomorphisms $K_x(\mu)$ and $H_x(\mu)$, defined on $T_x X$ by*

$$\langle K_x(\mu)u, u\rangle = \int_{\partial_F X} DdB_{(x,\theta)}(u,u)\, d(\mu)(\theta)$$

and

$$\langle H_x(\mu)u, u\rangle = \int_{\partial_F X} dB^2_{(x,\theta)}(u)\, d(\mu)(\theta).$$

Then $\det(K_x(\mu)) > 0$ and there is a positive constant $C := C(X) > 0$ depending only on X such that

$$J_x(\mu) := \frac{\det(H_x(\mu))^{1/2}}{\det(K_x(\mu))} \leqslant C.$$

Furthermore, the constant C is explicitly computable.

We remark that the proof of Theorem 3.2 necessarily excludes locally symmetric spaces with local direct factors locally isometric to $\mathbf{H}^2$ or $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$, because for these spaces the rank is too large relative to the dimension of the space. A cautionary example showing why this approach fails for spaces covered by $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$ appears in [8]. Our proof of the main theorem appeals to [18] and [6] to cover these cases.

3.2. Barycentric straightening

In this section we always assume that M^n has no local direct factors locally isometric to $\mathbf{H}^2$ or $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$.

We first fix some notation. Recall that a singular k-simplex in X is a continuous map $f:\Delta^k\to X$, where Δ^k is the standard Euclidean k-simplex realized as the convex hull of the standard unit basis vectors in $\mathbf{R}^{k+1}$. For our purpose it is more convenient to work with the spherical k-simplex $\Delta^k_s=\{(a_1,...,a_{k+1}):a_i\geqslant 0 \text{ and } \sum_{i=1}^{k+1}a_i^2=1\}\subset\mathbf{R}^{k+1}$, equipped with the Riemannian metric induced from $\mathbf{R}^{k+1}$. We will denote by e_i, for $1\leqslant i\leqslant k+1$, the standard basis vectors for $\mathbf{R}^{k+1}$. We associate with each singular simplex $f:\Delta^k_s\to X$ its ordered vertex set $V:=\{f(e_1),...,f(e_{k+1})\}$. Our principal contribution to solving Gromov's conjecture lies in defining the following straightening.

Given an ordered set $V=\{x_1,...,x_{k+1}\}\subset X$, define the map $\widehat{V}:\Delta^k_s\to\mathcal{M}(\partial X)$ by

$$\widehat{V}\left(\sum_{i=1}^{k+1}a_ie_i\right)=\sum_{i=1}^{k+1}a_i^2\nu(x_i).$$

Definition. Given a singular k-simplex $f\in C^0(\Delta^k_s,X)$, with corresponding vertex set $V=\{x_1,...,x_{k+1}\}$, define $\mathrm{st}_k(f)\in C^0(\Delta^k_s,X)$ by $\mathrm{st}_k(f)(\delta)=\mathrm{bar}(\widehat{V}(\delta))$ for $\delta\in\Delta^k_s$.

The fact that this definition is well posed is the content of Lemma 3.1. Moreover, observe that $\mathrm{st}_k(f)$ depends only on the vertex set V of the original simplex f. We will therefore use the notation $\mathrm{st}_V(\delta):=\mathrm{st}_k(f)(\delta)$ when convenient. We now proceed to verify that this straightening procedure satisfies the four formal properties needed. For the convenience of the reader, we restate each property prior to proving it.

Property (a). The maps st_k are Γ-equivariant.

Proof. Fix a point $\delta=\sum_{i=1}^{k+1}a_ie_i\in\Delta^k_s$. Then, for any $\gamma\in\Gamma$, $\mathrm{st}_{\gamma V}(\delta)$ is defined as the unique minimizer of the function $g(\,\cdot\,)=\int_{\partial_F X}B(p,\cdot\,,\theta)\,d\big(\sum_{i=1}^{k+1}a_i^2\nu(\gamma x_i)\big)(\theta)$. Since

$$\begin{aligned}\int_{\partial_F X}B(p,\cdot\,,\theta)\,d\left(\sum_{i=1}^{k+1}a_i^2\nu(\gamma x_i)\right)(\theta)&=\int_{\partial_F X}B(p,\cdot\,,\theta)\,d\left(\sum_{i=1}^{k+1}a_i^2\gamma_*\nu(x_i)\right)(\theta)\\&=\int_{\partial_F X}B(p,\cdot\,,\gamma\theta)\,d\left(\sum_{i=1}^{k+1}a_i^2\nu(x_i)\right)(\theta)\\&=\int_{\partial_F X}B(\gamma\gamma^{-1}p,\gamma\gamma^{-1}\cdot\,,\gamma\theta)\,d\left(\sum_{i=1}^{k+1}a_i^2\nu(x_i)\right)(\theta)\\&=\int_{\partial_F X}B(\gamma^{-1}p,\gamma^{-1}\cdot\,,\theta)\,d\left(\sum_{i=1}^{k+1}a_i^2\nu(x_i)\right)(\theta),\end{aligned}$$

and since $B(\gamma^{-1}p,\cdot,\cdot)$ and $B(p,\cdot,\cdot)$ differ by a function $k(\theta)$ of θ, it follows that the unique minimizer of $g(\cdot)$ is also the unique minimizer of the function

$$h(\cdot)=\int_{\partial_F X} B(p,\gamma^{-1}\cdot,\theta)\,d\biggl(\sum_{i=1}^{k+1} a_i^2\nu(x_i)\biggr)(\theta).$$

Indeed, we have that the difference of the two functions is

$$g(\cdot)-h(\cdot)=\int_{\partial_F X} k(\theta)\,d\biggl(\sum_{i=1}^{k+1} a_i^2\nu(x_i)\biggr)(\theta),$$

which is a constant function on X. But now g is minimized at $\mathrm{st}_{\gamma V}(\delta)$, while h is minimized at $\gamma\cdot st_V(\delta)$. This gives us that $\mathrm{st}_{\gamma V}(\delta)=\gamma\cdot st_V(\delta)$. □

Property (b). The maps st_k induce a chain map $\mathrm{st}_*: C_*^{\mathrm{sing}}(X,\mathbf{R})\to C_*^{\mathrm{sing}}(X,\mathbf{R})$ which is Γ-equivariantly chain homotopic to the identity.

Proof. The fact that st_k commutes with the boundary operator follows from the fact that $\mathrm{st}_k(f)$ depends solely on the vertices of the singular k-simplex f, along with the fact that $\mathrm{st}_k(f)$ restricted to a face of Δ_s^k coincides with the straightening of that face.

To see that st_* is chain homotopic to the identity, first note that the uniqueness of geodesics in X gives rise to a well-defined Γ-equivariant straight-line homotopy between any k-simplex f and its straightening $\mathrm{st}_k(f)$. Hence, there are canonically defined homotopies between simplices and their straightenings in X. Moreover, these homotopies when restricted to lower-dimensional faces agree with the homotopies canonically defined on those faces. Appropriately (Γ-equivariantly) subdividing these homotopies defines the required chain homotopy. □

Property (c). The image of st_n lies in $C^1(\Delta_s^n, X)$, i.e. straightened top-dimensional simplices are C^1.

Proof. Notice that for any simplex $f\in C^0(\Delta_s^n,X)$ and any $\delta=\sum_{i=1}^{k+1} a_i e_i\in\Delta_s^n$, we have an implicit characterisation of the point $\mathrm{st}_n(f)(\delta)=\mathrm{st}_V(\delta)$ via the 1-form equation

$$0\equiv d(g_{\widehat{V}(\delta)})_{\mathrm{st}_V(\delta)}(\cdot)=\int_{\partial_F X} dB_{(\mathrm{st}_V(\delta),\theta)}(\cdot)\,d\biggl(\sum_{i=1}^{k+1} a_i^2\nu(x_i)\biggr)(\theta). \tag{1}$$

Indeed, $\mathrm{st}_V(\delta)=\mathrm{bar}\bigl(\sum_{i=1}^{k+1} a_i^2\nu(x_i)\bigr)$ is defined as the unique minimum of the function

$$g_{\widehat{V}(\delta)}(\cdot)=\int_{\partial_F X} B(p,\cdot,\theta)\,d\biggl(\sum_{i=1}^{k+1} a_i^2\nu(x_i)\biggr)(\theta),$$

yielding equation (1) upon differentiation.

Following [3], we choose a frame $(e_i(x))_{i=1}^n$ of T_xX with smooth dependence on $x\in X$. Define the map $G=(G_1, ..., G_n)\colon \Delta_s^n\times X\to\mathbf{R}^n$ by

$$G_i(\delta,x)=\int_{\partial_F X} dB_{(x,\theta)}(e_i(x))\,d(\widehat{V}(\delta))(\theta).$$

Equation (1) implies that $G(\delta,\mathrm{st}_V(\delta))=0$. As the Busemann functions $B(p,\cdot\,,\theta)$ are smooth and since $\partial_F X$ is compact, it follows that G is a smooth map. To now apply the inverse function theorem, the non-degeneracy of the partial derivative of G with respect to the variable x must be checked. This requires that for the endomorphism K defined by

$$\langle K(u),u\rangle:=\int_{\partial_F X} DdB_{(x,\theta)}(u,u)\,d(\widehat{V}(\delta))(\theta),$$

the determinant be non-zero. Note however that in the notation of Theorem 3.2, the determinant of this matrix is precisely $\det(K_x(\widehat{V}(\delta)))$, and hence must be non-zero as the measure $\widehat{V}(\delta)=\sum_{i=1}^{k+1}a_i^2\nu(x_i)$ has full support on the Furstenberg boundary. □

Property (d). There exists a constant $C>0$, depending on X, such that for any $f\in C^0(\Delta_s^n,X)$, with corresponding straightened simplex $\mathrm{st}_n(f)\colon\Delta_s^n\to X$, there is a uniform upper bound on the Jacobian of $\mathrm{st}_n(f)$:

$$|\mathrm{Jac}(\mathrm{st}_n(f))(\delta)|\leqslant C,$$

where $\delta=\sum_{i=1}^{k+1}a_ie_i\in\Delta_s^n$ is arbitrary, and the Jacobian is computed relative to the Riemannian metric on the spherical simplex Δ_s^n induced from $\mathbf{R}^{n+1}$.

Proof. Differentiating equation (1) with respect to directions in $T_\delta(\Delta_s^n)$, one obtains the equation

$$\begin{aligned}0\equiv D_\delta d(g_{\widehat{V}(\delta)})_{\mathrm{st}_V(\delta)}(\,\cdot\,,\cdot\,)&=\sum_{i=1}^{k+1}2a_i\langle\,\cdot\,,e_i\rangle_\delta\int_{\partial_F X}dB_{(\mathrm{st}_V(\delta),\theta)}(\,\cdot\,)\,d(\nu(x_i))(\theta)\\&\quad+\int_{\partial_F X}DdB_{(\mathrm{st}_V(\delta),\theta)}(D(\mathrm{st}_V)_\delta(\,\cdot\,),\cdot\,)\,d\bigg(\sum_{i=1}^{k+1}a_i^2\nu(x_i)\bigg)(\theta)\end{aligned}\tag{2}$$

defined on $T_\delta(\Delta_s^n)\otimes T_{\mathrm{st}_V(\delta)}(X)$. Now define symmetric endomorphisms H_δ and K_δ of $T_{\mathrm{st}_V(\delta)}(X)$ by

$$\langle H_\delta(u),u\rangle_{\mathrm{st}_V(\delta)}=\int_{\partial_F X}dB^2_{(\mathrm{st}_V(\delta),\theta)}(u)\,d\bigg(\sum_{i=1}^{k+1}a_i^2\nu(x_i)\bigg)(\theta),$$

$$\langle K_\delta(u),u\rangle_{\mathrm{st}_V(\delta)}=\int_{\partial_F X}DdB_{(\mathrm{st}_V(\delta),\theta)}(u,u)\,d\bigg(\sum_{i=1}^{k+1}a_i^2\nu(x_i)\bigg)(\theta).$$

Let $\{v_j\}_{j=1}^n$ be an orthonormal eigenbasis of $T_{\mathrm{st}_V(\delta)}(X)$ for H_δ. At points $\delta\in\Delta_s^n$ where the Jacobian of st_V is non-zero, let $\{\tilde{u}_j\}_{j=1}^n$ be the basis of $T_\delta(\Delta_s^n)$ obtained by pulling back the $\{v_j\}_{j=1}^n$ basis by $K_\delta\circ D(\mathrm{st}_V)_\delta$, and $\{u_j\}_{j=1}^n$ be the orthonormal basis of $T_\delta(\Delta_s^n)$ obtained from the $\{\tilde{u}_j\}_{j=1}^n$ basis by applying the Gram–Schmidt algorithm. We now have (each step will be justified in the next paragraph)

$$
\begin{aligned}
&\det(K_\delta)|\mathrm{Jac}(\mathrm{st}_V)(\delta)| \\
&\qquad = |\det(K_\delta\circ D(\mathrm{st}_V)_\delta)| &(3)\\
&\qquad = \prod_{j=1}^n |\langle K_\delta\circ D(\mathrm{st}_V)_\delta(u_j), v_j\rangle_{\mathrm{st}_V(\delta)}| &(4)\\
&\qquad = \prod_{j=1}^n \left|\sum_{i=1}^{n+1}\langle u_j, e_i\rangle_\delta\cdot 2a_i\int_{\partial_F X} dB_{(\mathrm{st}_V(\delta),\theta)}(v_j)\, d(\nu(x_i))(\theta)\right| &(5)\\
&\qquad \leqslant \prod_{j=1}^n \left[\sum_{i=1}^{n+1}\langle u_j, e_i\rangle_\delta^2\right]^{1/2}\left[\sum_{i=1}^{n+1}4a_i^2\left(\int_{\partial_F X} dB_{(\mathrm{st}_V(\delta),\theta)}(v_j)\, d(\nu(x_i))(\theta)\right)^2\right]^{1/2} &(6)\\
&\qquad \leqslant 2^n\prod_{j=1}^n \left[\sum_{i=1}^{n+1}a_i^2\int_{\partial_F X} dB^2_{(\mathrm{st}_V(\delta),\theta)}(v_j)\, d(\nu(x_i))(\theta)\right]^{1/2} &(7)\\
&\qquad = 2^n\prod_{j=1}^n \langle H_\delta(v_j), v_j\rangle_{\mathrm{st}_V(\delta)}^{1/2} &(8)\\
&\qquad = 2^n\det(H_\delta)^{1/2}. &(9)
\end{aligned}
$$

Equality (3) follows from the definition of the Jacobian, along with the fact that $\det(AB)=\det(A)\cdot\det(B)$. Equality (4) follows from the fact that, with respect to the $\{u_j\}_{j=1}^n$ and $\{v_j\}_{j=1}^n$ bases, $K_\delta\circ D(\mathrm{st}_V)_\delta$ is upper triangular, and hence the determinant is the product of the diagonal entries. Equality (5) follows from equalities (4) and (2). Inequalities (6) and (7) follow from the Cauchy–Schwartz inequality applied in $\mathbf{R}^{n+1}$ and the spaces $L^2(\partial_F X, \nu(x_i))$, respectively, along with the fact that the u_j's are unit vectors in $T_\delta(\Delta_s^n)\subset T_\delta(\mathbf{R}^{n+1})$. The two equalities in (8) and (9) follow from the definition of H_δ and from the fact that $\{v_j\}_{j=1}^n$ is an orthonormal eigenbasis for H_δ.

Upon dividing, we now obtain the inequality

$$|\mathrm{Jac}(\mathrm{st}_V)(\delta)|\leqslant 2^n\frac{\det(H_\delta)^{1/2}}{\det(K_\delta)}.$$

But now note that, in the notation of Theorem 3.2, the expression $\det(H_\delta)^{1/2}/\det(K_\delta)$ is exactly $J_{\mathrm{st}_V(\delta)}\big(\sum_{i=1}^{k+1}a_i^2\nu(x_i)\big)$. Since the measure $\sum_{i=1}^{k+1}a_i^2\nu(x_i)$ has full support in the Furstenberg boundary, Theorem 3.2 now yields a uniform constant C', depending solely on X, with the property that

$$|\mathrm{Jac}(\mathrm{st}_V)(\delta)| \leqslant 2^n J_{\mathrm{st}_V(\delta)}\left(\sum_{i=1}^{k+1} a_i^2 \nu(x_i)\right) \leqslant 2^n C' =: C.$$

This completes the proof of property (d). □

4. Proof of the main theorem

In view of the barycentric straightening construction in §3.2 and Thurston's Theorem 2.4, we have established the following result.

THEOREM 4.1. *Let M^n be a closed locally symmetric space of non-compact type with no local direct factors locally isometric to* $\mathbf{H}^2$ *or* $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$. *Then* $\|M^n\|>0$.

We can now prove the main theorem.

Proof of Theorem 1.1. Observe that by the proportionality principle in Theorem 2.1, in order to show that $\|M\|>0$, it is sufficient to show that $\|M'\|>0$ for some locally symmetric space M' of non-compact type whose universal cover is X. Let G denote the identity component of Isom(X) and $G=G_1\times\ldots\times G_k$ be the product decomposition of G into simple Lie groups corresponding to the product decomposition of X into irreducible symmetric spaces. By a result of Borel [4], there are cocompact lattices $\Gamma_i\subset G_i$ for each $i\in\{1,\ldots,k\}$. Take M' to be the product locally symmetric space $M_1\times\ldots\times M_k$ obtained from the product lattice $\Gamma_1\times\ldots\times\Gamma_k$. From Theorem 2.2, the inequality $\|M_1\times\ldots\times M_k\|\geqslant\prod_{i=1}^k\|M_i\|$ holds. Hence, if the main theorem holds for *irreducible* locally symmetric spaces of non-compact type, then it holds for *all* locally symmetric spaces of non-compact type.

Next we observe that for closed real hyperbolic surfaces, positivity of the simplicial volume follows from [18]. Furthermore, for closed irreducible locally symmetric spaces with symmetric covering $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$, positivity of the simplicial volume follows from [6]. Therefore, in view of Theorem 4.1, positivity holds for *all* irreducible locally symmetric spaces of non-compact type, concluding the proof. □

5. Concluding remarks

Let $\mathcal{M}$ be the smallest class of manifolds that (i) contains all closed locally symmetric spaces of non-compact type, (ii) is closed under connected sums with arbitrary closed manifolds of dimension $\geqslant 3$, (iii) is closed under products, and (iv) is closed under fiber extensions by surfaces of genus $\geqslant 2$ (i.e. if $M \in \mathcal{M}$, and M' fibers over M with a surface S_g of genus $\geqslant 2$ as fiber, then $M' \in \mathcal{M}$).

In view of Theorems 1.1, 2.2 and 2.3, along with a result of Hoster and Kotschick [12], we obtain the following corollary.

COROLLARY 5.1. *For every manifold* $M \in \mathcal{M}$, $\|M\| > 0$.

The *minimal entropy* $h(M)$ of a smooth manifold M, is defined to be the infimum of the topological entropies of the geodesic flow over all complete Riemannian metrics of unit volume on M. The *minimal volume* $\mathrm{MinVol}(M)$ of a smooth manifold M is defined to be the infimum of the volume over all complete Riemannian metrics with sectional curvatures bounded between -1 and 1. The following inequalities were established in [10, p. 37]:

$$C \cdot \|M\| \leqslant h(M)^n,$$
$$C' \cdot \|M\| \leqslant \mathrm{MinVol}(M),$$

where C and C' are uniform constants, depending only on the dimension n of M. We therefore obtain the following result.

COROLLARY 5.2. *Every manifold* $M \in \mathcal{M}$ *has positive minimal entropy and positive minimal volume.*

We say that M *collapses* provided that there exists a sequence of Riemannian metrics g_i on M, satisfying $|\mathrm{K}(g_i)| \leqslant 1$, and having the property that at every point $p \in M$, the injectivity radius with respect to the metric g_i is less than $1/i$. Gromov showed that manifolds with positive simplicial volume *do not* collapse [10, pp. 67–68], giving the following result.

COROLLARY 5.3. *Manifolds* $M \in \mathcal{M}$ *do not collapse.*

The simplicial volume also provides control on the possible degree of a map into the given space (see [16, §8]). This yields the following result.

COROLLARY 5.4. *Let* $M \in \mathcal{M}$, *and assume that* $f: N \to M$ *is a continuous map from a manifold* N. *Then we have*

$$\deg(f) \leqslant \frac{\|N\|}{\|M\|}.$$

Bounded cohomology $\widehat{H}^*(M^n)$ was defined by Gromov in [10], where it is shown that M^n has positive simplicial volume if and only if the map induced by inclusion of chains $i^n: \widehat{H}^n(M^n) \to H^n_{\mathrm{sing}}(M^n, \mathbf{R})$ is non-zero [10, pp. 16–17]. This has the following consequence.

Corollary 5.5. *Each manifold $M^n \in \mathcal{M}$ has non-vanishing n-dimensional bounded cohomology.*

We conclude by pointing out some questions.

Conjecture. Let M^n be a closed Riemannian manifold, whose sectional curvatures are $\leqslant 0$, and whose Ricci curvatures are negative. Then $\|M^n\| > 0$.

This conjecture was attributed to Gromov in [16]. It seems plausible that a similar approach could be used to verify this conjecture. The main difficulty lies in obtaining the formal property (d) for the analogous straightening procedure when the space M^n is locally irreducible and is *not* a locally symmetric space. Namely, the estimates from [7] rely heavily on the fact that locally symmetric spaces come from algebraic groups. In particular, the various inequalities and minimization problems that arise in the general case may no longer be rephrased as problems about Lie groups. We can also ask the following question.

Question. For a given closed locally symmetric space M^n of non-compact type, can one compute the value of the proportionality constant $\|M^n\|/\mathrm{Vol}(M^n)$ in terms of the symmetric covering of M^n?

One application of positivity of simplicial volume is the non-vanishing of the top-dimensional bounded cohomology. We have the following natural question.

Question. What is the dimension of $\widehat{H}^n(M^n)$ for a closed locally symmetric quotient of an irreducible higher-rank locally symmetric space of non-compact type? In particular, is it finite-dimensional or infinite-dimensional?

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