

EQUIVARIANT K -HOMOLOGY FOR HYPERBOLIC REFLECTION GROUPS

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Abstract

We compute the equivariant K -homology of the classifying space for proper actions, for cocompact 3-dimensional hyperbolic reflection groups. This coincides, via the Baum-Connes Conjecture, with the topological K -theory of the reduced C^* -algebra associated to the group. We show that for any such reflection group, the associated K -theory groups are torsion-free. As a result we can promote previous rational computations to integral computations. Our proof relies on a new efficient algebraic criterion for checking torsion-freeness of K -theory groups, which could be applied to many other classes of groups.

1. Introduction

For a discrete group Γ , a general problem is to compute $K_*(C_r^*\Gamma)$, the topological K -theory of the reduced C^* -algebra of Γ . The Baum–Connes Conjecture predicts that this functor can be determined, in a homological manner, from the complex representation rings of the finite subgroups of Γ . This viewpoint led to general recipes for computing the rational topological K -theory $K_*(C_r^*\Gamma) \otimes \mathbb{Q}$ of groups, through the use of Chern characters (see for instance Lück and Oliver

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[17] and Lück [14, 16], as well as related earlier work of Adem [1]). When Γ has small homological dimension, one can sometimes even give completely explicit formulas for the rational topological K -theory, see for instance Lafont *et al.* [12] for the case where Γ is a 3-orbifold group.

On the other hand, performing integral calculations for these K -theory groups is much harder. For 2-dimensional crystallographic groups, such calculations have been done in M. Yang's thesis [26]. This was subsequently extended to the class of cocompact planar groups by Lück and Stamm [19], and to certain higher dimensional crystallographic groups by Davis and Lück [5] (see also Langer and Lück [13]). For 3-dimensional groups, Lück [15] completed this calculation for the semidirect product $\text{Hei}_3(\mathbb{Z}) \rtimes C_4$ of the 3-dimensional integral Heisenberg group with a specific action of the cyclic group C_4 . Some further computations were completed by Isely [8] for groups of the form $\mathbb{Z}^2 \rtimes \mathbb{Z}$; by Rahm [23] for the class of Bianchi groups; by Pooya and Valette [22] for solvable Baumslag–Solitar groups; and by Flores *et al.* [6] for lamplighter groups of finite groups.

Our present paper has two main goals. Our first goal is to add to the list of examples, by providing a formula for the integral K -theory groups of cocompact 3-dimensional hyperbolic reflection groups. The study of hyperbolic reflection groups has a long history, and is motivated for instance in Davis' book [4].

MAIN THEOREM. *Let Γ be a cocompact 3-dimensional hyperbolic reflection group, generated by reflections in the sides of a hyperbolic polyhedron $\mathcal{P} \subset \mathbb{H}^3$. Then*

$$K_0(C_r^*(\Gamma)) \cong \mathbb{Z}^{cf(\Gamma)} \quad \text{and} \quad K_1(C_r^*(\Gamma)) \cong \mathbb{Z}^{cf(\Gamma) - \chi(\mathcal{C})},$$

where the integers $cf(\Gamma)$, $\chi(\mathcal{C})$ can be explicitly computed from the combinatorics of the polyhedron \mathcal{P} .

Here, $cf(\Gamma)$ denotes the number of conjugacy classes of elements of finite order in Γ , and $\chi(\mathcal{C})$ denotes the Euler characteristic of the Bredon chain complex. By a celebrated result of Andreev [2], there is a simple algorithm that inputs a Coxeter group Γ , and decides whether or not there exists a hyperbolic polyhedron $P_\Gamma \subset \mathbb{H}^3$ which generates Γ . In particular, given an arbitrary Coxeter group, one can easily verify if it satisfies the hypotheses of our Main Theorem.

Note that the lack of torsion in the K -theory is not a property shared by all discrete groups acting on hyperbolic 3-space. For example, 2-torsion occurs in $K_0(C_r^*(\Gamma))$ whenever Γ is a Bianchi group containing a 2-dihedral subgroup $C_2 \times C_2$ (see [23]). In fact, the key difficulty in the proof of our Main Theorem lies in showing that these K -theory groups are torsion-free. Some previous integral computations yielded K -theory groups that are torsion-free, though in those papers the torsion-freeness was a consequence of ad-hoc computations. Our second goal is to give a general criterion which explains the lack of torsion, and can be efficiently checked. This allows a systematic, algorithmic approach to the question of whether a K -theory group is torsion-free.

Let us briefly describe the contents of the paper. In Section 2, we provide background material on hyperbolic reflection groups, topological K -theory, and the Baum–Connes Conjecture. We also introduce our main tool, the Atiyah–Hirzebruch type spectral sequence. In Section 3, we use the spectral sequence to show that the K -theory groups we are interested in coincide with the Bredon homology groups $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ and $H_1^{\text{fin}}(\Gamma; R_{\mathbb{C}})$, respectively. We also explain, using the Γ -action on \mathbb{H}^3 , why the homology group $H_1^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ is torsion-free. In contrast, showing that $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ is torsion-free is much more difficult. In Section 4, we give a geometric proof for this fact in a restricted setting. In Section 5, we give a linear algebraic proof in the general case,

inspired by the ‘representation ring splitting’ technique of [23]. In particular, we establish a novel criterion (Theorem 5.2) for verifying that $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ is torsion-free, for any collection of groups Γ with prescribed types of finite subgroups. In Sections 6.1 and 6.2, we further illustrate this criterion by applying it to the Heisenberg semidirect product group of [15] and to the crystallographic groups of [5], respectively. As the rank of the K -theory groups can be easily computed, this gives an alternate proof of the integral K -theoretic computations in [5, 15]. Finally, in Section 7, we return to our Coxeter groups, and provide an explicit formula for the rank of the Bredon homology groups (and hence for the K -groups we are interested in), in terms of the combinatorics of the polyhedron \mathcal{P} .

Our results rely on fairly standard, but rather long representation theoretic calculations which have not been included in this article, but are available for the interested reader in the two appendices of the arXived version of the present paper [10], making our results self-contained and fully verifiable.

2. Background material

2.1. K -theory and the Baum–Connes Conjecture

Associated to a discrete group Γ , one has $C_r^*\Gamma$, the *reduced C^* -algebra* of Γ . This algebra is defined to be the closure, with respect to the operator norm, of the linear span of the image of the regular representation $\lambda: \Gamma \rightarrow B(l^2(\Gamma))$ of Γ on the Hilbert space $l^2(\Gamma)$ of square-summable complex valued functions on Γ . This algebra encodes various analytic properties of the group [21].

For a C^* -algebra A , one can define the *topological K -theory* groups $K_*(A) := \pi_{*-1}(GL(A))$, which are the homotopy groups of the space $GL(A)$ of invertible matrices with entries in A . Due to Bott periodicity, there are canonical isomorphisms $K_*(A) \cong K_{*+2}(A)$, and thus it is sufficient to consider $K_0(A)$ and $K_1(A)$.

In the special case where $A = C_r^*\Gamma$, the *Baum–Connes Conjecture* predicts that there is a canonical isomorphism $K_n^\Gamma(X) \rightarrow K_n(C_r^*(\Gamma))$, where X is a model for $\underline{E}\Gamma$ (the classifying space for Γ -actions with isotropy in the family of finite subgroups), and $K_*^\Gamma(-)$ is the equivariant K -homology functor. The Baum–Connes conjecture has been verified for many classes of groups. We refer the interested reader to the monograph by Mislin and Valette [21], or the survey article by Lück and Reich [18] for more information on these topics.

2.2. Hyperbolic reflection groups

We will assume some familiarity with the geometry and topology of Coxeter groups, which the reader can obtain from Davis’ book [4]. By a d -dimensional *hyperbolic polyhedron*, we mean a bounded region of hyperbolic d -space \mathbb{H}^d delimited by a given finite number of (geodesic) hyperplanes, that is, the intersection of a collection of half-spaces associated to the hyperplanes. Let $\mathcal{P} \subset \mathbb{H}^d$ be a polyhedron such that all the interior angles between intersecting faces are of the form π/m_{ij} , where the $m_{ij} \geq 2$ are integer (although some pairs of faces may be disjoint). Let $\Gamma = \Gamma_{\mathcal{P}}$ be the associated Coxeter group, generated by reflections in the hyperplanes containing the faces of \mathcal{P} .

The Γ -space \mathbb{H}^d is then a model for $\underline{E}\Gamma$, with fundamental domain \mathcal{P} . This is a strict fundamental domain —no further points of \mathcal{P} are identified under the group action —and hence $\mathcal{P} = \Gamma \backslash \mathbb{H}^d$. Recall that Γ admits the following Coxeter presentation:

$$\Gamma = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} \rangle, \quad (2.1)$$

where n is the number of distinct hyperplanes enclosing \mathcal{P} , s_i denotes the reflection on the i th face, and $m_{ij} \geq 2$ are integers such that: $m_{ii} = 1$ for all i , and, if $i \neq j$, the corresponding faces meet with

interior angle π/m_{ij} . We will write $m_{ij} = \infty$ if the corresponding faces do not intersect. For the rest of this article, $d = 3$, and X is \mathbb{H}^3 with the Γ -action described above, with fundamental domain \mathcal{P} .

2.3. Cell structure of the orbit space

Fix an ordering of the faces of \mathcal{P} with indexing set $J = \{1, \dots, n\}$. We will write $\langle S \rangle$ for the subgroup generated by a subset $S \subset \Gamma$. At a vertex of \mathcal{P} , the concurrent faces (a minimum of three) must generate a reflection group acting on the 2-sphere, hence it must be a spherical triangle group. This forces the number of incident faces to be exactly three. The only finite Coxeter groups acting by reflections on S^2 are the triangle groups $\Delta(2, 2, m)$ for some $m \geq 2$, $\Delta(2, 3, 3)$, $\Delta(2, 3, 4)$ and $\Delta(2, 3, 5)$, where we use the notation

$$\Delta(p, q, r) = \langle s_1, s_2, s_3 \mid s_1^2, s_2^2, s_3^2, (s_1s_2)^p, (s_1s_3)^q, (s_2s_3)^r \rangle. \tag{2.2}$$

From our compact polyhedron \mathcal{P} , we obtain an induced Γ -CW-structure on $X = \mathbb{H}^3$ with:

- one orbit of 3-cells, with trivial stabilizer;
- n orbits of 2-cells (faces) with stabilizers $\langle s_i \mid s_i^2 \rangle \cong C_2$, the cycle group of order 2 ($i = 1, \dots, n$);
- one orbit of 1-cells (edges) for each unordered pair $i, j \in J$ with $m_{ij} \neq \infty$, with stabilizer a dihedral group $D_{m_{ij}}$ —this group structure can be read off straight from the Coxeter presentation $\langle s_i, s_j \mid s_i^2, s_j^2, (s_i s_j)^{m_{ij}} \rangle$;
- one orbit of 0-cells (vertices) per unordered triple $i, j, k \in J$ with $\langle s_i, s_j, s_k \rangle$ finite, with stabilizer the triangle group $\langle s_i, s_j, s_k \rangle \cong \Delta(m_{ij}, m_{ik}, m_{jk})$.

We introduce the following notation for the simplices of \mathcal{P} :

$$\begin{aligned} f_i & \quad \text{(faces),} \\ e_{ij} &= f_i \cap f_j \quad \text{(edges),} \\ v_{ijk} &= f_i \cap f_j \cap f_k = e_{ij} \cap e_{ik} \cap e_{jk} \quad \text{(vertices),} \end{aligned} \tag{2.3}$$

whenever the intersections are non-empty.

2.4. A spectral sequence

We ultimately want to compute the K -theory groups of the reduced C^* -algebra of Γ via the Baum–Connes conjecture. Note that the conjecture holds for these groups: Coxeter groups have the Haagerup property [3] and hence satisfy Baum–Connes [7]. Therefore, it suffices to compute the equivariant K -homology groups $K_*^\Gamma(X)$, since X is a model of $\underline{E}\Gamma$. In turn, these groups can be obtained from the Bredon homology of X , calculated via an equivariant Atiyah–Hirzebruch spectral sequence coming from the skeletal filtration of the Γ -CW-complex X (cf. [20]). The second page of this spectral sequence is given by the Bredon homology groups

$$E_{p,q}^2 = \begin{cases} H_p^{\text{fin}}(\Gamma; R_{\mathbb{C}}) & q \text{ even,} \\ 0 & q \text{ odd.} \end{cases} \tag{2.4}$$

The coefficients $R_{\mathbb{C}}$ of the Bredon homology groups are given by the complex representation ring of the cell stabilizers, which are finite subgroups. In order to simplify notation, we will often write H_p to denote $H_p^{\delta^{\text{in}}}(\Gamma; R_{\mathbb{C}})$. In our case $\dim(X) = 3$, so the Atiyah–Hirzebruch spectral sequence is particularly easy to analyze, and gives the following:

PROPOSITION 2.1 *There are short exact sequences*

$$0 \longrightarrow \text{coker}(d_{3,0}^3) \longrightarrow K_0^{\Gamma}(X) \longrightarrow H_2 \longrightarrow 0$$

and

$$0 \longrightarrow H_1 \longrightarrow K_1^{\Gamma}(X) \longrightarrow \ker(d_{3,0}^3) \longrightarrow 0,$$

where $d_{3,0}^3: E_{3,0}^3 = H_3 \longrightarrow E_{0,2}^3 = H_2$ is the differential on the E^3 -page of the Atiyah–Hirzebruch spectral sequence.

Proof. This follows at once from a result of Mislin [20, Theorem 5.29]. \square

2.5. Bredon homology

To lighten the notation, we write Γ_e for the stabilizer in Γ of the cell e . The Bredon homology groups in Equation (2.4) are defined to be the homology groups of the following chain complex:

$$\dots \longrightarrow \bigoplus_{e \in I_d} R_{\mathbb{C}}(\Gamma_e) \xrightarrow{\partial_d} \bigoplus_{e \in I_{d-1}} R_{\mathbb{C}}(\Gamma_e) \longrightarrow \dots, \quad (2.5)$$

where I_d is a set of orbit representatives of d -cells ($d \geq 0$) in X . The differentials ∂_d are defined via the geometric boundary map and induction between representation rings. More precisely, if ge' is in the boundary of e ($e \in I_d$, $e' \in I_{d-1}$, $g \in \Gamma$), then ∂ restricted to $R_{\mathbb{C}}(\Gamma_e) \rightarrow R_{\mathbb{C}}(\Gamma_{e'})$ is given by the composition

$$R_{\mathbb{C}}(\Gamma_e) \xrightarrow{\text{Ind}} R_{\mathbb{C}}(\Gamma_{ge'}) \xrightarrow{\cong} R_{\mathbb{C}}(\Gamma_{e'}),$$

where the first map is the induction homomorphism of representation rings associated to the subgroup inclusion $\Gamma_e \subset \Gamma_{ge'}$, and the second is the isomorphism induced by conjugation $\Gamma_{ge'} = g\Gamma_{e'}g^{-1}$. Finally, we add a sign depending on a chosen (and thereafter fixed) orientation on the faces of \mathcal{P} . The value $\partial_d(e)$ equals the sum of these maps over all boundary cells of e .

Since \mathcal{P} is a strict fundamental domain, we can choose the faces of \mathcal{P} as orbit representatives. With this choice of orbit representatives, the element g in the previous paragraph is always the identity. We will implicitly make this assumption from now on.

3. Analyzing the Bredon chain complex for Γ

Let $S = \{s_i: 1 \leq i \leq n\}$ be the set of Coxeter generators and $J = \{1, \dots, n\}$. Since X is 3-dimensional, the Bredon chain complex associated to X reduces to

$$0 \longrightarrow \mathcal{C}_3 \xrightarrow{\partial_3} \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \longrightarrow 0. \quad (3.1)$$

We will now analyse the differentials in the above chain complex in turn.

First, let us recall some basic concepts from representation theory [9, 24] and explain the upcoming notation. For a finite group G , the complex representation ring $R_{\mathbb{C}}(G)$ is a free abelian group with basis the set of irreducible representations of G (the ring structure is not relevant in this setting). Hence, $R_{\mathbb{C}}(G) \cong \mathbb{Z}^{c(G)}$, where we write $c(G)$ for the set of conjugacy classes in G . If H is a subgroup of G , we write $\text{Ind}_H^G: R_{\mathbb{C}}(H) \rightarrow R_{\mathbb{C}}(G)$ for the induction homomorphism and denote $\text{Ind}_H^G(\rho)$ simply by $\rho \uparrow$ when the groups are clear from the context. Similarly, we use $\text{Res}_H^G(\rho)$ or $\rho \downarrow$ for restriction of representations.

We will use the character tables of the groups involved in the Bredon chain complex (3.1), that is, the finite Coxeter subgroups of Γ up to rank three. These are based on the representation theory described in for example, [9], where all these character tables are constructed. In the character tables below, rows correspond to irreducible representations, and columns to representatives of conjugacy classes, written in term of the Coxeter generators s_1, \dots, s_n in a fixed Coxeter presentation of Γ (2.1).

The induction homomorphisms appearing in the Bredon chain complex (3.1) can be easily computed using Fröbenius reciprocity [24]: $\langle \text{Ind}_H^G(\rho), \tau \rangle = \langle \rho, \text{Res}_H^G(\tau) \rangle$, where $\langle \cdot, \cdot \rangle$ is the scalar product of representations. Computing restrictions and scalar products is straightforward, and thus Fröbenius reciprocity give $\rho \uparrow$ as a linear combination of the irreducible representation of the larger group. Note that, for consistency across subgroups, one must pay particular attention to the order of the Coxeter generators within a subgroup when computing character tables or induction homomorphisms.

Let us write e for the identity element in Γ and, when discussing a dihedral group D_m , use the hat $\hat{}$ to denote an entry which appears only when m is even.

3.1. Analysis of ∂_3

Let G be a finite group with irreducible representations ρ_1, \dots, ρ_m of degree n_1, \dots, n_m , and τ the only representation of the trivial subgroup $\{1_G\} \leq G$. Then τ induces the regular representation of G :

$$\text{Ind}_{\{1_G\}}^G(\tau) = n_1 \rho_1 + \dots + n_m \rho_m. \quad (3.2)$$

LEMMA 3.1 *Let X be a Γ -CW-complex with finite stabilizers, and $k \in \mathbb{N}$. If there is a unique orbit of k -cells and this orbit has trivial stabilizer, then $H_k = 0$, provided that $\partial_k \neq 0$.*

Proof. The Bredon module \mathcal{C}_k equals $R_{\mathbb{C}}(\langle 1 \rangle) \cong \mathbb{Z}$ with generator τ , the trivial representation. Then $\partial_k(\tau) \neq 0$ implies $\ker(\partial_k) = 0$; and therefore the corresponding homology group vanishes. \square

From the lemma, we can easily see that $H_3 = 0$ if $\partial_3 \neq 0$. Indeed, for $\partial_3 = 0$ to occur, one would need all boundary faces of \mathcal{P} to be pairwise identified. This cannot happen since \mathcal{P} is a strict fundamental domain—the group acts by reflections on the faces. The Lemma then forces $H_3 = 0$, and Proposition 2.1 gives us

Table 1. Character table of $\langle s_i \rangle \cong C_2$

C_2	e	s_i
ρ_1	1	1
ρ_2	1	-1

COROLLARY 3.2 *We have $K_1^\Gamma(X) \cong H_1$, and there is a short exact sequence*

$$0 \longrightarrow H_0 \longrightarrow K_0^\Gamma(X) \longrightarrow H_2 \longrightarrow 0.$$

3.2. Analysis of ∂_2

Let f be a face of \mathcal{P} and $e \in \partial f$ an edge. Suppose, using the notation of Equation (2.3), that $f = f_i$ and $e = e_{ij}$. Then we have a map $R_{\mathbb{C}}(\langle s_i \rangle) \rightarrow R_{\mathbb{C}}(\langle s_i, s_j \rangle)$ induced by inclusion. Recall that $\langle s_i \rangle \cong C_2$ and $\langle s_i, s_j \rangle \cong D_{m_{ij}}$. Denote the characters of these two finite groups as specified in Tables 1 and 2; and denote by a character name with the suffix ‘ \uparrow ’ the character induced in the ambient larger group.

A straightforward analysis (cf. [11, B.2] and Tables 1 and 2) shows that

$$\begin{aligned} \rho_1 \uparrow &= \chi_1 + \widehat{\chi}_4 + \sum \phi_p, \\ \rho_2 \uparrow &= \chi_2 + \widehat{\chi}_3 + \sum \phi_p, \end{aligned}$$

if $i < j$, or

$$\begin{aligned} \rho_1 \uparrow &= \chi_1 + \widehat{\chi}_3 + \sum \phi_p, \\ \rho_2 \uparrow &= \chi_2 + \widehat{\chi}_4 + \sum \phi_p, \end{aligned}$$

if $j < i$. Thus, the induction map on the representation rings is the morphism of free abelian groups $\mathbb{Z}^2 \rightarrow \mathbb{Z}^{c(D_{m_{ij}})}$ given by

$$\begin{aligned} (a, b) &\mapsto \pm (a, b, \widehat{b}, \widehat{a}, a + b, \dots, a + b) \quad \text{or} \\ (a, b) &\mapsto \pm (a, b, \widehat{a}, \widehat{b}, a + b, \dots, a + b), \end{aligned}$$

where again the hat $\widehat{}$ denotes an entry which appears only when m_{ij} is even. Using the analysis above, we can now show the following.

THEOREM 3.3 *If \mathcal{P} is compact, then $H_2 = 0$.*

From this theorem and Corollary 3.2, we immediately obtain:

COROLLARY 3.4 $K_0^\Gamma(X) = H_0$ and $K_1^\Gamma(X) = H_1$.

Table 2. Character table of $\langle s_i, s_j \rangle \cong D_m$, where $i < j$ and $0 \leq r \leq m - 1$, while $1 \leq p \leq m/2 - 1$ if m is even, and $1 \leq p \leq (m - 1)/2$ if m is odd, and where the hat $\widehat{}$ denotes a character which appears only when m is even

D_m	$(s_i s_j)^r$	$s_j (s_i s_j)^r$
χ_1	1	1
χ_2	1	-1
$\widehat{\chi_3}$	$(-1)^r$	$(-1)^r$
$\widehat{\chi_4}$	$(-1)^r$	$(-1)^{r+1}$
ϕ_p	$2 \cos\left(\frac{2\pi pr}{m}\right)$	0

Proof of Theorem 3.3. Fix an orientation on the polyhedron \mathcal{P} , and consider the induced orientations on the faces. At an edge, we have two incident faces f_i and f_j with opposite orientations. So without loss of generality we have, as a map of free abelian groups,

$$R_{\mathbb{C}}(\langle s_i \rangle) \oplus R_{\mathbb{C}}(\langle s_j \rangle) \rightarrow R_{\mathbb{C}}(\langle s_i, s_j \rangle),$$

$$(a_i, b_i \mid a_j, b_j) \mapsto (a_i - a_j, b_i - b_j, \widehat{a_i - b_j}, \widehat{b_i - a_j}, S, \dots, S), \tag{3.3}$$

where $S = a_i + b_i - a_j - b_j$, and the elements with a hat $\widehat{}$ appear only when m_{ij} is even. Note that we use vertical bars ‘|’ for clarity, to separate elements coming from different representation rings.

By the preceding analysis, $\partial_2(x) = 0$ implies that, for each $i, j \in J$ such that the corresponding faces f_i and f_j meet, we have

- (1) $a_i = a_j$ and $b_i = b_j$, and
- (2) $a_i = a_j = b_i = b_j$, if m_{ij} is even.

Suppose that f_1, \dots, f_n are the faces of \mathcal{P} . Let $x = (a_1, b_1 \mid \dots \mid a_n, b_n) \in \mathcal{C}_2$ be an element in $\text{Ker}(\partial_2)$. Note that $\partial\mathcal{P}$ is connected (since P is homeomorphic to \mathbb{D}^3), so by properties (1) and (2) above, we have that $a_1 = \dots = a_n$ and $b_1 = \dots = b_n$. Since the stabilizer of a vertex is a spherical triangle group, there is an even m_{ij} , which also forces $a = b$. Therefore, we have $x = (a, a \mid \dots \mid a, a)$, so $x = \partial_3(a)$ (note that the choice of orientation above forces all signs to be positive). This yields $\text{ker}(\partial_2) \subseteq \text{im}(\partial_3)$, which gives the vanishing of the second homology group. \square

3.3. Analysis of ∂_1

A similar argument shows that H_1 is torsion-free.

THEOREM 3.5 *There is no torsion in H_1 .*

Proof. Consider the Bredon chain complex

$$\mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0.$$

To prove that $H_1 = \ker(\partial_1)/\text{im}(\partial_2)$ is torsion-free, it suffices to prove that $\mathcal{C}_1/\text{im}(\partial_2)$ is torsion-free. Let $\alpha \in \mathcal{C}_1$ and $0 \neq k \in \mathbb{Z}$ such that $k\alpha \in \text{im}(\partial_2)$. We shall prove that $\alpha \in \text{im}(\partial_2)$.

Since $k\alpha \in \text{im}(\partial_2)$, we can find $\beta \in \mathcal{C}_2$ with $\partial_2(\beta) = k\alpha$. Suppose that \mathcal{P} has n faces, and write $\beta = (a_1, b_1 | \dots | a_n, b_n) \in \mathcal{C}_2$, using vertical bars ‘|’ to separate elements coming from different representation rings. We shall see that one can find a 1-chain β' , homologous to β , and with every entry of β' a multiple of k .

At an edge e_{ij} , the differential ∂_2 is described in Equation (3.3). Since every entry of $\partial_2(\beta)$ is a multiple of k , we conclude from Equation (3.3) that for every pair of intersecting faces f_i and f_j ,

$$a_i \equiv a_j \pmod{k} \quad \text{and} \quad b_i \equiv b_j \pmod{k}.$$

Equation (3.3) also shows that $\mathbf{1}_{\partial\mathcal{P}} = (1, 1 | \dots | 1, 1)$, the formal sum over all generators of representation rings of face stabilizers of $\partial\mathcal{P} \in \mathcal{C}_2$, is in the kernel of ∂_2 . In particular, setting $\beta' = \beta - a_1 \mathbf{1}_{\partial\mathcal{P}}$, we see that $\partial_2(\beta') = \partial_2(\beta) = \alpha$ and we can assume without loss of generality that β' satisfies $a'_1 \equiv 0 \pmod{k}$.

Let us consider the coefficients for the 1-chain β' . For every face f_j intersecting f_1 , we have $a'_1 - a'_j \equiv 0 \pmod{k}$, which implies $a'_j \equiv 0 \pmod{k}$. Since $\partial\mathcal{P}$ is connected, repeating this argument we arrive at $a'_i \equiv 0 \pmod{k}$ for all i . In addition, there are even m_{ij} (the stabilizer of a vertex is a spherical triangle group), and hence (3.3) also yields $a'_i - b'_j \equiv 0 \pmod{k}$, which implies $b'_j \equiv 0 \pmod{k}$. Exactly the same argument as above then ensures that $b'_i \equiv 0 \pmod{k}$ for all i .

Since all coefficients of β' are divisible by k , we conclude that $\alpha = \partial_2(\beta'/k) \in \text{im}(\partial_2)$, as desired. \square

We note that a similar method of proof can be used, in many cases, to show that H_0 is torsion-free—though the argument becomes much more complicated. This approach is carried out in Section 4.

COROLLARY 3.6 *Let $cf(\Gamma)$ be the number of conjugacy classes of elements of finite order in Γ , and $\chi(\mathcal{C})$ the Euler characteristic of the Bredon chain complex (3.1). Then we have*

$$H_1 \cong \mathbb{Z}^{cf(\Gamma) - \chi(\mathcal{C})}.$$

Proof. The Euler characteristic of a chain complex coincides with the alternating sum of the ranks of the homology groups, giving us

$$\chi(\mathcal{C}) = \text{rank}(H_0) - \text{rank}(H_1) + \text{rank}(H_2) - \text{rank}(H_3).$$

Since $H_3 = H_2 = 0$, we have $\text{rank}(H_1) = \text{rank}(H_0) - \chi(\mathcal{C})$, and $\text{rank}(H_0) = cf(\Gamma)$ [20]. Since H_1 is torsion-free (Theorem 3.5), the result follows. \square

Note that both $cf(\Gamma)$ and $\chi(\mathcal{C})$ can be obtained directly from the geometry of the polyhedron \mathcal{P} or, equivalently, from the Coxeter integers m_{ij} . We discuss this further, and give explicit formulas, in Section 7.

REMARK 3.7 A previous article by three of the authors [12] gave an algorithm to compute the rank of the Bredon homology for groups Γ with a cocompact, 3-manifold model X for the

classifying space $\underline{E}\Gamma$. The interested reader can easily check that the computations in the present paper agree with the calculations in [12].

To complete the computation of the Bredon homology, and hence of the equivariant K -homology, all that remains is to compute the torsion subgroup of H_0 . We will show that in fact H_0 is also torsion-free.

THEOREM 3.8 *There is no torsion in H_0 .*

We postpone the proof to Section 5 below. We note the following immediate consequence of Theorem 3.8.

COROLLARY 3.9 $K_0^\Gamma(X)$ is torsion-free of rank $cf(\Gamma)$.

Our Main Theorem now follows immediately by combining Corollaries 3.4, 3.6 and 3.9. Moreover, in Section 7, we will give a formula for $cf(\Gamma)$ and $\chi(\mathcal{C})$ from the combinatorics of the polyhedron.

4. No torsion in H_0 —the geometric approach

We present a geometric proof for a restricted version of Theorem 3.8. The method of proof is similar to the proof of Theorem 3.5, but with further technical difficulties. We will show:

THEOREM 4.1 *Assume the compact polyhedron \mathcal{P} is such that all vertex stabilizers are of the form $D_m \times C_2$, where $m \geq 3$ can vary from vertex to vertex. Then there is no torsion in the 0-dimensional Bredon homology group $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$.*

First, we discuss some terminology and the over all strategy of the proof. Fix $k \geq 2$ an integer. Our over all objective is to rule out k -torsion in H_0 . Let $\beta \in \mathcal{C}_1$ (in the Bredon chain complex of Γ) such that $\partial_1(\beta)$ is the zero vector modulo k . Note that an element $\alpha \in \mathcal{C}_1$ has order k in $H_0 = \mathcal{C}_0/\text{im}(\partial_1)$ if and only if $k\alpha = \partial_1(\beta)$. Recall that

$$\mathcal{C}_1 = \bigoplus_{e_{ij} \text{ edge}} R_{\mathbb{C}}(\langle s_i, s_j \rangle),$$

is the direct sum of the representation rings (as abelian groups) of the edge stabilizers. The coefficients of x supported along a particular edge e_{ij} are by definition the projection of x to $\mathbb{Z}^{n_{ij}} \cong R_{\mathbb{C}}(\langle s_i, s_j \rangle)$, where n_{ij} is the dimension of the representation ring of the edge stabilizer. We say an element x is k -divisible along an edge e_{ij} provided the coefficients of x supported along e_{ij} are congruent to zero mod k .

By establishing k -divisibility of β along an edge e_{ij} , we mean: substituting β by a homologous element $\beta' \in \mathcal{C}_1$ (homologous means that $\partial_1(\beta) = \partial_1(\beta')$), such that β' is k -divisible along e_{ij} . When the 1-chain β is clear from the context, we will abuse terminology and simply say that the edge e is k -divisible. We sometimes refer to an edge with stabilizer D_n as an n -edge, or edge of type n .

Our goal is to replace β with a homologous chain β' for which all the edges are k -divisible, that is,

- (1) $\partial_1(\beta') = \partial_1(\beta)$, and
- (2) every coefficient in the 1-chain β' is divisible by k .

If we can do this, it follows that $\alpha = \partial_1(\beta'/k)$, and hence α is zero in H_0 .

The construction of β' is elementary, but somewhat involved. It proceeds via a series of steps, which will be described in the following subsections. Sections 4.1–4.9 contain the conceptual, geometric arguments needed for the proof. At several steps in the proof, we require some technical algebraic lemmas. For the sake of exposition, we defer these lemmas and their proofs to the very last Section 4.11.

4.1. Coloring the 1-skeleton

Recall that $\beta \in \mathcal{C}_1 = \bigoplus_{e \in \mathcal{P}^{(1)}} R_{\mathbb{C}}(\Gamma_e)$, so we can view β as a formal sum of complex representations of the stabilizers of the various edges in the 1-skeleton of \mathcal{P} . The edge stabilizers are dihedral groups. Let us 2-color the edges of the polyhedron, blue if the stabilizer is D_2 , and red if the stabilizer is D_m , where $m \geq 3$. From our constraints on the vertex groups, we see that every vertex has exactly two incident blue edges. Of course, any graph with all vertices of degree 2 decomposes as a disjoint union of cycles.

The collection of blue edges thus forms a graph, consisting of pairwise disjoint loops, separating the boundary of the polyhedron \mathcal{P} (topologically a 2-sphere) into a finite collection of regions, at least two of which must be contractible. The red edges appear in the interior of these individual regions, joining pairs of vertices on the boundary of the region. Fixing one such contractible region R_{∞} , the complement will be planar. We will henceforth fix a planar embedding of this complement. This allows us to view all the remaining regions as lying in the plane \mathbb{R}^2 .

4.2. Enumerating the regions

Our strategy for modifying β is as follows. We will work region by region. At each stage, we will modify β by only changing it on edges contained in the closure of a region. In order to do this, we need to enumerate the regions.

We have already identified the (contractible) region R_{∞} —this will be the last region dealt with. In order to decide the order in which we will deal with the remaining regions, we define a partial ordering on the set of regions. For distinct regions R, R' , we write $R < R'$ if and only if R is contained in a bounded component of $\mathbb{R}^2 \setminus R'$. This defines a partial ordering on the finite set of regions. For example, any region which is minimal with respect to this ordering must be simply connected (hence contractible). We can thus enumerate the regions R_1, R_2, \dots so that, for any $i < j$, we have $R_j < R_i$ (Fig. 1). We will now deal with the regions in the order they are enumerated. Concretely, the choice of enumeration means that by the time we get to the i th region, we have already dealt with all the regions which are ‘interior’ to R_i (that is, in the bounded components of $\mathbb{R}^2 \setminus R_i$).

4.3. Enumerating faces within a region

We now want to establish k -divisibility of the red edges inside a fixed region R . To do this, we first need to order the 2-faces inside R . Consider the graph G dual to the decomposition of R into 2-faces. This graph has one vertex for each 2-face in R , and an edge joining a pair of vertices if the corresponding 2-faces share a (necessarily red) edge. Notice that every vertex of R lies on a

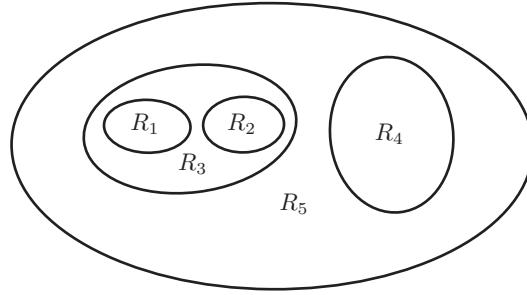


Figure 1. Example of enumeration of regions (see Section 4.2).

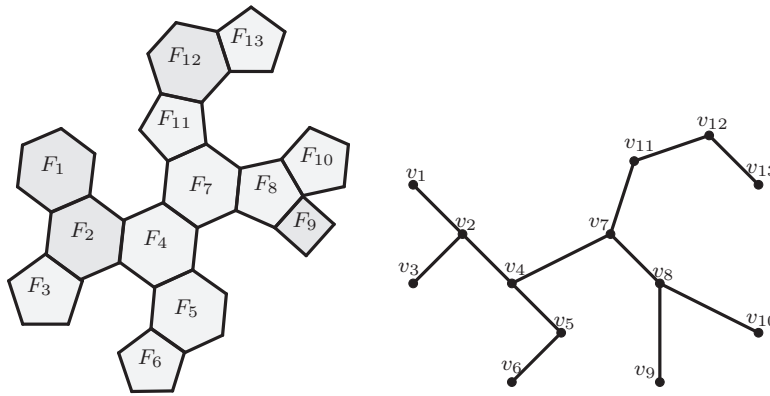


Figure 2. Contractible region, its dual tree T with an enumeration of its vertices, and the corresponding enumeration of the faces in the region (see Section 4.3).

pair of blue edges. It follows that, if we remove the blue edges from the region R , the result deformation retracts to G .

If s denotes the number of regions $R_i < R$ with $\partial R \cap \partial R_i$ non-empty, then the first homology $H_1(G) = H_1(R)$ will be free abelian of rank s . We now choose any spanning tree T for the graph G , and note that $G \setminus T$ consists of precisely s edges, each of which is dual to a red edge inside the region R . Since T is a tree, we can enumerate its vertices v_1, v_2, \dots, v_n so that

- (1) v_1 is a leaf of the tree (vertex of degree one), and
- (2) the subgraph induced by any $\{v_1, \dots, v_k\}, 1 \leq k \leq n$, is connected.

Since vertices of the dual tree correspond to 2-faces in R , this gives us an enumeration of the 2-faces F_1, F_2, \dots, F_n inside the region R . Figure 2 illustrates this process for a contractible region, while Fig. 3 gives an illustration for a non-contractible region.

4.4. Establishing k -divisibility of red edges dual to a spanning tree

Continuing to work within a fixed region R , we now explain how to establish k -divisibility of all the red edges which are dual to the edges in the spanning tree T . As explained in Section 4.3, we have an enumeration F_1, F_2, \dots of the 2-faces contained inside the region R . With our choice of

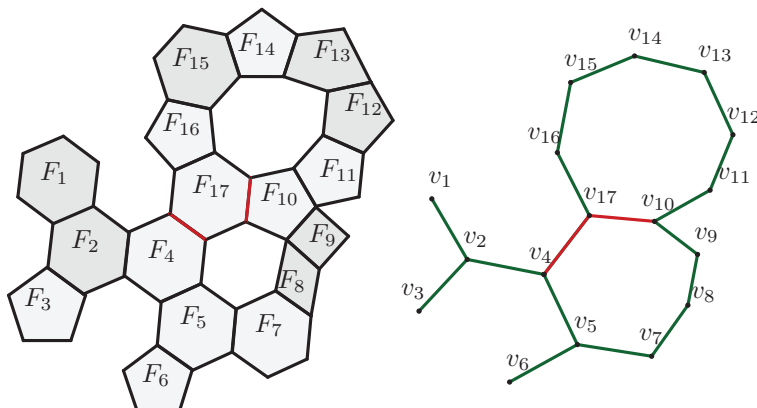


Figure 3. Non-contractible region and its dual graph G . A spanning tree T for G consisting of all edges except the red edges (v_4, v_{17}) and (v_{17}, v_{10}) . The enumeration of the vertices, and corresponding enumeration of the faces is done according to Section 4.3.

enumeration, we have guaranteed that each F_{k+1} shares precisely one distinguished red edge with $\bigcup_{i=1}^k F_i$, distinguished in the sense that this red edge is dual to the unique edge in the tree T connecting the vertex v_{k+1} to the subtree spanned by v_1, \dots, v_k .

We orient the (blue) edges along the boundary loops of R clockwise, and the red edges cutting through R in an arbitrary manner. For each 2-face F_i , we want to choose corresponding elements η_i in the representation ring $R_{\mathbb{C}}(\Gamma_{F_i}) \cong \mathbb{Z}^2$, where $\Gamma_{F_i} \cong C_2$ is the stabilizer of F_i . These η_i shall be chosen such that all red edges dual to T are k -divisible for the 1-chain $\beta + \partial_2(\sum \eta_i)$, which is clearly homologous to β .

Pick an η_1 arbitrarily. We now assume that η_1, \dots, η_k are given, and explain how to choose η_{k+1} . By our choice of enumeration of vertices, the vertex v_{k+1} is adjacent to some v_j where $j \leq k$. Dual to the two vertices v_j, v_{k+1} we have a pair F_j, F_{k+1} of 2-faces inside the region R . Dual to the edge that joins v_j to v_{k+1} is the (red) edge $F_j \cap F_{k+1}$. We see that $\partial_2(\eta_j + \eta_{k+1})$ is the only term which can change the portion of β supported on the edge $F_j \cap F_{k+1}$. Since η_j is already given, we want to choose η_{k+1} in order to ensure that the resulting 1-chain is k -divisible on the edge $F_j \cap F_{k+1}$. This will arrange key property (2) above (Section 4.3) for the (red) edge $F_j \cap F_{k+1}$.

That such an η_{k+1} can be chosen is the content of Lemma 4.6 in Section 4.11. Iterating this process, we find that the 1-chain $\beta + \partial_2(\sum \eta_i)$ is homologous to β , and that all red edges dual to edges in T are k -divisible for $\beta + \partial_2(\sum \eta_i)$.

4.5. Forming the graph B

Performing the process in Section 4.4 for each region R (including the region R_∞), we finally obtain a 1-chain homologous to our original β (which by abuse of notation we will still denote β) which is k -divisible except possibly along:

- s_i red edges inside each region R_i , where s_i denotes the number of regions entirely enclosed by the region R_i who share a boundary with R_i ;
- all the blue edges inside the 1-skeleton, which we recall decompose into a finite collection of blue loops.

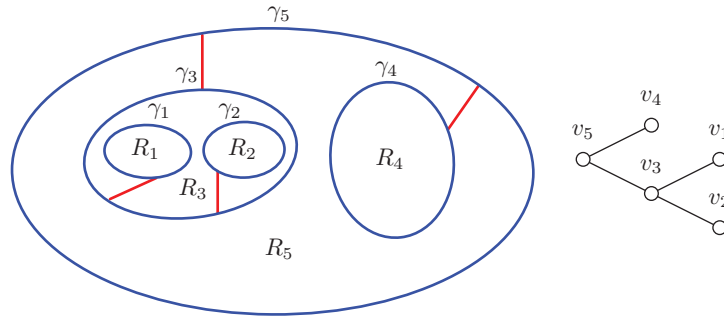


Figure 4. (Left) Non-contractible region R with blue loops and red edges. (Right) Graph B associated to R (see Section 4.5).

We now use these to form a graph B , which captures all the remaining potentially ‘bad’ edges for the chain β , that is, edges that are still not k -divisible. B is formed with one vertex for each blue loop. Note that each remaining red edge that is potentially not k -divisible joins a pair of vertices which lie on some blue loops γ_1, γ_2 (as all vertices lie on blue loops). For each such red edge, we define an edge in the graph B joining the two vertices v_i corresponding to the blue loops γ_i (Fig. 4).

LEMMA 4.2 *The graph B is a tree.*

Proof. It follows immediately from the discussion in Section 4.3 that B is connected. Thus it suffices to show B has no embedded cycles. Observe that, since blue loops separate the plane into two connected components, the corresponding vertex of B likewise partitions the graph B into two connected components (corresponding to the ‘interior’ component and the ‘exterior’ component determined by the blue loop). Thus, if there is an embedded cycle, then for each vertex v of B , it must remain within one connected component of $B \setminus \{v\}$. This means that any embedded cycle has the property that all the vertices it passes through correspond to blue components lying in the closure of a single region R (and all red edges lie inside that region R).

Now by way of contradiction, assume that e_1, \dots, e_k forms a cycle in B , cyclically joining vertices $\gamma_1, \dots, \gamma_k$, where the γ_i are blue loops inside the closure of the region R . We can concatenate the corresponding (red) edges e_i along with paths on the blue loops γ_i to obtain an embedded edge loop η contained in the closure of the region R . Now pick a pair of 2-faces F_1, F_2 inside the region R , where F_1 is contained inside the loop η , while F_2 is contained outside of the loop η , for example, pick the 2-faces on either side of the red edge e_1 . Note that, by construction, the closed loop η separates these regions from each other.

These two regions correspond to vertices v_1, v_2 in the graph G associated to the region R . Since T was a spanning tree for the graph G , it follows that we can find a sequence of edges in the tree T connecting v_1 to v_2 . This gives rise to a sequence of 2-faces connecting F_1 to F_2 , where each consecutive face share an edge distinct from any of the red edges e_i . Thus, we obtain a continuous path joining F_1 to F_2 which is completely disjoint from η , a contradiction. We conclude that B cannot contain any cycles, and hence is a tree. \square

Notice that each vertex in B corresponds to a blue loop, which lies in the closure of precisely two regions. There will thus be a unique such region which lies in the bounded component of the complement of the loop. This establishes a bijection between the regions and the vertices of B . From the enumeration of regions in Section 4.2, we can use the bijection to enumerate vertices of

B . For example, the vertex with smallest labeling will always correspond to the boundary of a contractible region. We refer the reader to Fig. 4 for an illustration of this labeling.

4.6. Coefficients along the blue loops

Our next goal is to modify β in order to make all the remaining red edges (that is, edges in the graph B) k -divisible. Note that the 1-chain β might not be an integral 1-cycle, but it is a 1-cycle mod k . We will now exploit this property to analyze the behavior of the 1-chain β along the blue loops.

PROPOSITION 4.3 *Let γ be any blue loop, oriented clockwise. Then the coefficients on the blue edges are all congruent to each other modulo k , that is, if (a, b, c, d) and (a', b', c', d') are the coefficients along any two blue edges in a given blue loop, then $(a, b, c, d) \equiv (a', b', c', d') \pmod{k}$. Moreover, along any red edge in the graph B , the coefficient is congruent to zero mod k , except possibly if the edge has an even label, in which case the coefficient is congruent to $(0, 0, \hat{z}, -\hat{z}, 0, \dots, 0)$ for some z (which may vary from edge to edge).*

Proof. To see this, we argue inductively according to the ordering of the blue connected components (see Section 4.5).

Base Case: For the initial case, consider the blue loop corresponding to vertex v_1 in the tree B . By hypothesis, this blue loop γ has a single (red) edge incident to it which is potentially not k -divisible, corresponding to the unique edge in the tree B incident to v_1 . Let w denote the single vertex on γ where that red edge is incident, allowing us to view γ as a path starting and terminating at w . Since all the remaining red edges incident to γ are k -divisible, applying Lemma 4.7 in Section 4.11 (with all the $\hat{z} \equiv 0$) for each incident k -divisible red edge shows that all the coefficients along the path γ are congruent to each other mod k . Note that, in this base case, we are always in the cases $i < j < k$ or $j < k < i$ of Lemma 4.7, according to whether the k -divisible red edge lies in the unbounded or bounded region determined by the blue loop γ . This establishes the first statement of the Proposition. To get the second statement, we apply Lemma 4.8 in Section 4.11 at the vertex w , and we are done.

Inductive Step: Now inductively, let us assume that we are focusing on the blue loop γ_i corresponding to some vertex v_i in B . We assume that all the blue loops γ_j corresponding to vertices v_j with $j < i$ already satisfy the desired property. We also assume that all red edges in the graph B connected to vertices v_j with $j < i$ have coefficients of the form described in the Proposition.

From the directed structure of the graph B , the vertex v_i has a *unique* edge e connecting to a vertex v_j with index $j > i$, and all the remaining edges in B connect to a v_j for some $j < i$.

By the inductive hypothesis, this tells us that all but one of these red edges have coefficients congruent to $(0, 0, \hat{z}_j, -\hat{z}_j, 0, \dots, 0)$ for some z_j (which might depend on the edge). Again, viewing γ_i as a path starting and terminating at the same vertex w (where e is incident to γ), we may apply Lemma 4.7 and conclude that γ_i has coefficients along all edges that are congruent to each other. Applying Lemma 4.8 at the vertex w shows that the coefficients along the edge e must also be of the form $(0, 0, \hat{z}, -\hat{z}, 0, \dots, 0)$ for some z . This completes the inductive step and the proof of the Proposition. \square

4.7. Equivalence classes of red edges

Now consider a red edge which is potentially not k -divisible, corresponding to an edge in the graph B joining vertices v_i to v_j . The red edge thus joins the blue loop γ_i to the blue loop γ_j . From

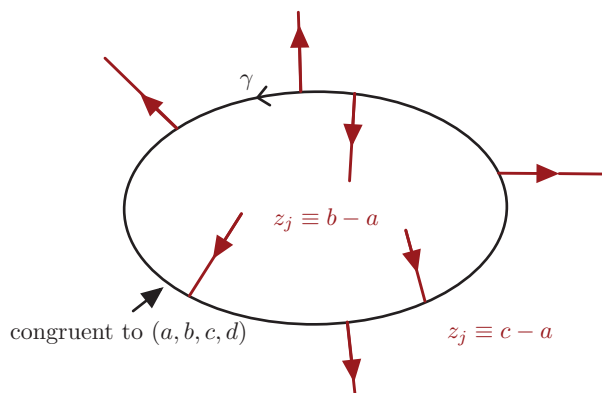


Figure 5. ‘Bad’ red edges incident to blue loop γ . All interior edges have coefficients satisfying one congruence, while all exterior edges have coefficients satisfying a different congruence (Section 4.7).

Proposition 4.3, we see that the coefficient along the red edge must be congruent to $(0, 0, \hat{z}, -\hat{z}, 0, \dots, 0)$ for some z . In particular, there is a single residue class that determines the coefficients along the red edge (modulo k).

Next let us momentarily focus on a blue loop γ , and assume the coefficients along the edges of γ are all congruent to (a, b, c, d) modulo k , as ensured by Proposition 4.3 (Fig. 5). We define an equivalence relation on all the red edges with even label incident to γ , by defining the two equivalence classes:

- (a) the incident red edges that lie in the bounded region corresponding to γ , and
- (b) those that lie in the unbounded region.

It follows from Lemma 4.7 (Section 4.11) that all edges in the equivalence class (a) have corresponding coefficients congruent to $(0, 0, \hat{z}_j, -\hat{z}_j, 0, \dots, 0)$ where each $z_j \equiv b - a$, while all edges in equivalence class (b) have corresponding coefficients congruent to $(0, 0, \hat{z}_j, -\hat{z}_j, 0, \dots, 0)$ where each $z_j \equiv c - a$ (Fig. 5).

This equivalence relation is defined locally, and can be extended over all blue loops in the 1-skeleton, resulting in an equivalence relation on the collection of all red edges with even label. Observe that, by construction, each equivalence class has the property that there is a single corresponding residue class $z \pmod k$, with the property that all the edges within that equivalence class have coefficient congruent to $(0, 0, \hat{z}, -\hat{z}, 0, \dots, 0)$, that is, the z is the same for the entire equivalence class.

COROLLARY 4.4 *The edges in the graph B that are not k -divisible are a finite union of equivalence classes for this relation.*

Proof. Let e be an edge in B , and assume that e is equivalent to an edge e' which is not an edge in B . Since all edges that are not in B are k -divisible, it follows that the coefficient on e' is congruent to zero mod k . Thus, the value of z for the equivalence class \mathcal{E} containing e' is $z = 0$. Since $e \in \mathcal{E}$, this forces e to be k -divisible, a contradiction. \square

4.8. Establishing k -divisibility of the remaining red edges

Observe that the edges in each equivalence class form a connected subgraph, and hence a subtree (see Lemma 4.2), of the graph B . This collection of subtrees partitions the graph B . Any vertex of B is incident to at most two such subtrees—the incident red edges lying ‘inside’ and ‘outside’ the corresponding blue loop.

We now proceed to establish k -divisibility of the remaining red edges for our chain. Fix an equivalence class \mathcal{E} of red edges, and associate to it a 1-chain $\alpha_{\mathcal{E}}$ whose coefficients are given as follows:

- (1) if a blue loop γ has an incident red edge $e \in \mathcal{E}$, and e lies in the *bounded* region of γ , then assign $(0, 1, 0, 0)$ to each blue edge on γ ;
- (2) if a blue loop γ has an incident red edge $e \in \mathcal{E}$, and e lies in the *unbounded* region of γ , then assign either $(0, 0, 1, 0)$ to each blue edge on γ , or $(0, 0, -1, 0)$ to each blue edge on γ ;
- (3) along the red edges in the equivalence class (recall that all these edges have even labels), assign $\pm(0, 0, \hat{1}, -\hat{1}, 0, \dots, 0)$, with sign chosen to ensure that the local 1-cycle condition holds at both endpoints (see Lemma 4.7).

Notice that one can choose the signs in (2), (3) coherently because the equivalence class defines a subtree of the tree B —and thus there are no cycles (these could have potentially forced the sign along an edge to be both positive and negative). Another key feature of the 1-chains $\alpha_{\mathcal{E}}$ is that they are linearly independent. More precisely, two distinct equivalence classes $\mathcal{E}, \mathcal{E}'$ have associated 1-chains $\alpha_{\mathcal{E}}$ and $\alpha_{\mathcal{E}'}$ whose supports are disjoint, except possibly along a single blue loop γ . In the case where the supports overlap along γ , adding multiples of $\alpha_{\mathcal{E}}$ does not affect the z -value along the class \mathcal{E}' (and vice versa).

It is now immediate from the equality case of Lemma 4.7 in Section 4.11 that the 1-chain $\alpha_{\mathcal{E}}$ is in fact an integral 1-cycle. Subtracting multiples of $\alpha_{\mathcal{E}}$ from our given chain β , we may thus obtain a homologous 1-chain for which all the red edges in \mathcal{E} are now k -divisible. Repeating this for each of the equivalence classes, we have now obtained a homologous 1-chain (still denoted β) for which *all the red edges are k -divisible*.

4.9. Establishing k -divisibility of the remaining blue edges

We now have obtained a 1-chain with prescribed differential, whose coefficients along all red edges are k -divisible. It remains to establish k -divisibility of the blue loops for the 1-chain.

If γ is one of the blue loops, then since all incident red edges are k -divisible, we see that all the edges on γ have coefficients which are congruent to either (a, b, c, d) , (a, b, a, b) , (a, a, c, c) or (a, a, a, a) , according to the equivalence classes that are incident to γ (see also Proposition 4.3).

Let us discuss, as an example, the case (a, b, a, b) . Note that this case occurs if the only incident red edges to γ with even label lie in the unbounded region determined by γ . Consider the pair of integral 1-cycles α_{13}, α_{24} supported on γ , obtained by assigning to each edge on γ the coefficient $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$, respectively. From the equality case of Lemma 4.7, we see that α_{13}, α_{24} are in fact 1-cycles. By adding multiples of α_{13}, α_{24} , we can now arrange for the coefficients along the blue loop γ to all be k -divisible. The three other cases can be dealt with similarly; we leave the details to the reader.

4.10. Completing the proof

Performing this process described in Section 4.9 for all the blue components, we finally obtain the desired 1-chain β' . Since β' now satisfies properties (1) and (2) mentioned at the beginning of the proof, we conclude that the given hypothetical torsion class $\alpha \in H_0$ was in fact the zero class. This completes the proof of Theorem 4.1.

REMARK 4.5 It is not obvious how to adapt the strategy in the geometric proof above to the case when other vertex stabilizer types are allowed. In the case of vertex stabilizers of the type $\Delta(2, 3, 3)$, $\Delta(2, 3, 4)$ and $\Delta(2, 3, 5)$, most of the arguments can be adapted. The main difficulty lies in the arguments of Section 4.4, which rely heavily on Lemma 4.7 in Section 4.11. Unfortunately, the analog of that Lemma does not seem to hold when one allows these other types of vertices as endpoints of the edge. For vertices with stabilizer $\Delta(2, 2, 2)$, additional difficulties arise, notably in Sections 4.7, 4.8 and 4.9.

4.11. Local analysis

The geometric proof that $H_0^{\text{in}}(\Gamma; R_{\mathbb{C}})$ is torsion free, Theorem 4.1, relies on a detailed local analysis of the induction homomorphism at the vertices of the polyhedron \mathcal{P} . We state and prove the results needed here. Although rather technical, they are all, unless an explicit proof is given, straightforward consequences of the induction homomorphisms (cf. [11, Appendix B]). Let us introduce some notation. Throughout this section, α will denote an integral 1-chain that is also a 1-cycle (mod k), that is, $\partial_1(\alpha) \equiv 0$. Our goal is to understand how this condition constrains the coefficients of α .

At every vertex $v = v_{ijl}$, there are three incident edges $e_1 = e_{ij}$, $e_2 = e_{il}$ and $e_3 = e_{jl}$, and let x_1 , x_2 and x_3 be α projected along those edges, written as column vectors. Write A_1 for the integer matrix representing the induction homomorphism from e_1 to v , that is, $R_{\mathbb{C}}(\langle s_i, s_j \rangle) \rightarrow R_{\mathbb{C}}(\langle s_i, s_j, s_l \rangle)$ (as always, we will implicitly identify representation rings with free abelian groups, via the bases explicitly described in [11, Appendix A]), and define A_2 and A_3 analogously for e_2 and e_3 . So each of these matrices is a submatrix of ∂_1 in matrix form. Then the value of $\partial_1(x)$ at the vertex $v = v_{ijl}$ (that is, projected to $R_{\mathbb{C}}(\langle s_i, s_j, s_l \rangle)$) is given by the matrix product $(\pm A_1 \mid \pm A_2 \mid \pm A_3) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, with signs depending on edge orientations. The product above is zero modulo k , by the hypothesis on α being a 1-cycle mod k . We can reduce modulo k all the entries and, abusing notation, still call the resulting matrices and vectors A_1, A_2, A_3, x_1, x_2 and x_3 . Furthermore, for simplicity, let us redefine A_1 as $-A_1$ etc. as needed to take account of the chosen orientations. Then we have that the column vector representing α locally at v (that is, along the incident edges) is in the kernel of the matrix representing ∂_1 locally at v , that is, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \ker(A_1|A_2|A_3)$. One important consequence is that we can perform row operations on the matrix $A = (A_1|A_2|A_3)$ without changing its kernel and, in particular, we may row reduce A , for instance into its Hermite normal form, to simplify calculations. (Obviously, row reduction must be performed modulo k , that is, in $C_k \setminus \text{cong} \cong /k\mathbb{Z}$.) Another consequence is that, to study the consequences of establishing k -divisibility of an edge, we only need to remove the corresponding matrix block and vector. For example, if k -divisibility of e_1 has been established for α , that is, if x_1 is zero modulo k , then the equation above is equivalent to $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \in \ker(A_2|A_3)$, and we can now row reduce this matrix to help us calculate its kernel, if needed.

Recall that, throughout Section 4, we are only interested in the case where all vertices have stabilizers of the form $\Delta(2, 2, m)$, with $m > 2$ (see statement of Theorem 4.1). So let us now focus on that case.

For a $\Delta(2, 2, m)$, $m > 2$, vertex, recall that we assume $j < l$ and we have (cf. [11, Appendix B]) induction matrices $M_{2,m}^{(1)<}$ (if $i < j$) or $M_{2,m}^{(1)>}$ (if $i > j$), $M_{2,m}^{(2)<}$ (if $i < l$) or $M_{2,m}^{(2)>}$ (if $i > l$), and $M_{m,m}$ where

$$M_{2,m}^{(1)<} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \widehat{0} & \widehat{1} & \widehat{0} & \widehat{0} \\ \widehat{1} & \widehat{0} & \widehat{0} & \widehat{0} \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \widehat{0} & \widehat{0} & \widehat{0} & \widehat{1} \\ \widehat{0} & \widehat{0} & \widehat{1} & \widehat{0} \\ 0 & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{2,m}^{(2)<} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \widehat{1} & \widehat{0} & \widehat{0} & \widehat{0} \\ \widehat{0} & \widehat{1} & \widehat{0} & \widehat{0} \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \widehat{0} & \widehat{0} & \widehat{1} & \widehat{0} \\ \widehat{0} & \widehat{0} & \widehat{0} & \widehat{1} \\ 0 & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{m,m} = \begin{pmatrix} \text{Id}_{c(D_m)} \\ \text{Id}_{c(D_m)} \end{pmatrix},$$

and $M_{2,m}^{(1)>}$, respectively $M_{2,m}^{(2)>}$, equals $M_{2,m}^{(1)<}$, respectively, $M_{2,m}^{(2)<}$, with the 2nd and 3rd columns interchanged.

In the following lemmas, recall that α is an integral 1-chain which is also a 1-cycle mod k . Moreover, in Lemmas 4.7 and 4.8, let $\widehat{}$ denote coefficients that only appear when m is even, and recall the standard labeling of faces: the m -edge lies between the faces labeled F_j and F_l , and the labeling always satisfies (without loss of generality) that $j < l$. Finally, recall that we refer to an edge with stabilizer D_m as an m -edge, or edge of type m .

LEMMA 4.6 *Let α be an integral 1-chain, which we assume is also a 1-cycle mod k . Let F_1, F_2 be a pair of adjacent 2-faces, sharing a common edge e , with endpoint v whose stabilizer is of the form $\Delta(2, 2, m)$, with $m > 2$. Assume that we are given an $\eta_1 = (n_1, m_1)$ in the representation ring $R_{\mathbb{C}}(C_2)$ associated to the stabilizer of the 2-face F_1 . Then there exists a choice of $\eta_2 = (n_2, m_2)$ in the representation ring $R_{\mathbb{C}}(C_2)$ associated to the stabilizer of the 2-face F_2 , with the property that $\alpha + \partial_2(\eta_1 + \eta_2)$ has coefficient along e congruent to zero mod k (that is, the edge e is now k -divisible).*

Proof. The edge e has stabilizer D_m , with $m > 2$. We will assume the orientations along the edges and faces are as given in Fig. 6. Assume the coefficients of α supported on the edge e are given by $(a, b, \widehat{c}, \widehat{d}, r_1, \dots, r_s)$. We choose $\eta_2 := (a + n_1, r_1 + m_1 - a)$. A straightforward computation using the induction formulas shows that, with this choice of η_2 , the coefficient of $\alpha' := \alpha + \partial_2(\eta_1 + \eta_2)$ along the edge e is of the form $z := (0, b', \widehat{c}', \widehat{d}', 0, r'_2, \dots, r'_s)$. That is to say, we chose η_2 in order to force $a' = r'_1 = 0$.

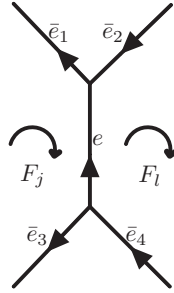


Figure 6. Local picture near the edge e (Lemma 4.6).

We are left with checking that the remaining coefficients of α' are all congruent to zero mod k . To see this, we use the fact that α' is also a 1-cycle mod k . From the labelings of the faces around the vertex v , and the order in which we label faces (one region at a time), we see that we are in one of the two cases $i < j < l$ or $j < l < i$. Let us consider the case $i < j < l$, and assume that the coefficients along the 2-edges incident to v are given by $x := (x_1, x_2, x_3, x_4)$ and $y := (y_1, y_2, y_3, y_4)$. As α' is a 1-cycle mod k , we have

$$(M_{2,m}^{(1)<} \mid -M_{2,m}^{(2)<} \mid -M_{m,m}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \mathbf{0}.$$

Note that the last s rows of the matrix are identical, giving rise to s identical relations $x_2 + x_4 - y_2 - y_4 + r'_t \equiv 0$ (for $1 \leq t \leq s$). Since $r'_1 = 0$, these equations immediately imply that all the remaining $r'_t \equiv 0$.

Let us now assume that m is odd. The first and the third row of the matrix give rise to equations $x_1 - y_1 \equiv 0$ and $x_1 + x_3 - y_1 - y_3 \equiv 0$, forcing $x_3 - y_3 \equiv 0$. Using the second row, we get $x_3 - y_3 + b' \equiv 0$, which immediately gives $b' \equiv 0$. This completes the proof when $i < j < l$ and m is odd. The case where m is even is analogous—one just uses the equations obtained from the first five rows of the matrix to conclude that a' , b' , c' and d' are all congruent to zero mod k .

Finally, if $j < l < i$, then one proceeds in a completely similar manner, but using the block matrix $(M_{2,m}^{(1)>} \mid -M_{2,m}^{(2)>} \mid -M_{m,m})$ instead. It is again straightforward to work through the equations—we leave the details to the reader. \square

LEMMA 4.7 *Consider a vertex of type $\Delta(2, 2, m)$, $m > 2$, with the incident 2-edges oriented compatibly. If the coefficients of α along the m -edge are congruent to $(0, 0, \hat{z}, -\hat{z}, 0, \dots, 0)$ for some z , then the coefficients (a, b, c, d) and (a', b', c', d') along the pair of 2-edges satisfy the following congruences:*

- (i) if $i < j < l$, then $(a, b, c, d) \equiv (a', b', c', d')$ and $\hat{z} \equiv b - a \equiv d - c$;
- (ii) if $j < i < l$, then $(a, b, c, d) \equiv (a', c', b', d')$ and $\hat{z} \equiv c - a \equiv d - b$;
- (iii) if $j < l < i$, then $(a, b, c, d) \equiv (a', b', c', d')$ and $\hat{z} \equiv c - a \equiv d - b$;

and we have oriented the m -edge so that the vertex is its source. (With the opposite orientation, simply replace \hat{z} by $-\hat{z}$.) Moreover, the same statement holds if one changes all congruences to equalities.

Proof. Let $x_1 = (a, b, c, d)$, $x_2 = (a', b', c', d')$ and $x_3 = (0, 0, \hat{z}, -\hat{z}, 0, \dots, 0)$ be the coefficients of α along the edges incident to the vertex. Consider the case $i < j < l$ first. Since α is a 1-cycle mod k ,

$$\left(M_{2,m}^{(1)<} \mid -M_{2,m}^{(2)<} \mid -M_{m,m} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \mathbf{0},$$

which gives $a - a' \equiv 0$, $b - b' \equiv 0$, $b - a' - \hat{z} \equiv 0$, $a - b' + \hat{z} \equiv 0$, $a + b - a' - b' \equiv 0$, $c - c' \equiv 0$, $d - d' \equiv 0$, $d - c' - \hat{z} \equiv 0$, $c - d' + \hat{z} \equiv 0$ and $c + d - c' - d' \equiv 0$, from which the result follows. The other two cases, $j < i < l$ and $j < l < i$, are analogous but for the block matrix $(M_{2,m}^{(1)>} \mid -M_{2,m}^{(2)<} \mid -M_{m,m})$, respectively, $(M_{2,m}^{(1)>} \mid -M_{2,m}^{(2)>} \mid -M_{m,m})$. The former gives the same congruences but with b and c interchanged, and the latter with b and c , and b' and c' , interchanged. For the opposite orientation of the m -edge, replace $-M_{m,m}$ by $M_{m,m}$ in the calculation above. \square

LEMMA 4.8 Consider a vertex of type $\Delta(2, 2, m)$, $m > 2$, with the incident 2-edges oriented compatibly. Assume the coefficients along the 2-edges are both congruent to (a, b, c, d) , that the m -edge is oriented compatibly with the first 2-edge, and that the faces are labeled so that $i < j < l$ or $j < l < i$ (so we are excluding the case $j < i < l$). Then the m -edge coefficients are congruent to $(0, 0, \hat{z}, -\hat{z}, 0, \dots, 0)$, where

- (i) if $i < j < l$, then $\hat{z} \equiv a - b$;
- (ii) if $j < l < i$, then $\hat{z} \equiv a - c$.

(If we reverse the orientation on the m -edge, the congruencies above hold with \hat{z} replaced by $-\hat{z}$.) In particular, if m is odd, the m -edge is automatically k -divisible.

Proof. We are assuming $x_1 \equiv x_2 \equiv (a, b, c, d)$, and that $x_3 = (x, y, \hat{z}, \hat{t}, r_1, \dots, r_s)$ are the coefficients of α along the edges incident to the vertex. Consider the case $i < j < l$ first. Since α is a 1-cycle mod k ,

$$\left(M_{2,m}^{(1)<} \mid -M_{2,m}^{(2)<} \mid -M_{m,m} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \mathbf{0},$$

which gives $x \equiv a - a \equiv 0$, $y \equiv b - b \equiv 0$, $\hat{c} \equiv a - b$, $\hat{d} \equiv b - a$, while all the remaining equations are of the form $r_t \equiv (a + b) - (a + b) \equiv 0$ (for $1 \leq t \leq s$). The claim follows. The case $j < l < i$ is completely analogous, but uses instead the block matrix $(M_{2,m}^{(1)>} \mid -M_{2,m}^{(2)>} \mid -M_{m,m})$. The details are left to the reader. \square

It is perhaps worth noting that the analog of Lemma 4.8 is *false* if the faces are enumerated to satisfy $j < i < l$. In particular, the corresponding block matrix $(M_{2,m}^{(1)>} \mid -M_{2,m}^{(2)<} \mid -M_{m,m})$ leads to, for example, $y \equiv b - c$, which is not necessarily zero.

5. No torsion in H_0 —the linear algebra approach

In this section, we give a proof of Theorem 3.8 inspired by the representation ring splitting technique of [23]. We do this by establishing a criterion for $H_0^{\delta_{\text{in}}}(\Gamma; R_{\mathbb{C}})$ to be torsion-free. Our criterion is efficient to check, and only requires elementary linear algebra. Furthermore, we will see it is satisfied for Γ a 3-dimensional hyperbolic Coxeter group.

The verification of our criterion relies on simultaneous base transformations of the representation rings, bringing the induction homomorphisms into the desired form. For the 3-dimensional hyperbolic Coxeter groups, these transformations are carried out in Appendices A and B in [11]. In Section 6, we will see that this condition is also satisfied for several additional classes of groups that had previously been considered by other authors.

DEFINITION 5.1 The *vertex block* of a given vertex v in a Bredon chain complex differential matrix ∂_1 consists of all the blocks of ∂_1 that are representing maps induced (on complex representation rings from $\Gamma_e \rightarrow \Gamma_v$) by edges e incident to v .

We represent elements in the Bredon chain complex as column vectors. So the matrix D for the differential ∂_1 is a $\text{rank}_{\mathbb{Z}}\mathcal{C}_0 \times \text{rank}_{\mathbb{Z}}\mathcal{C}_1$ matrix, acting by left multiplication on a column vector in \mathcal{C}_1 . For a vertex v , denote by n_0 the rank of $R_{\mathbb{C}}(\Gamma_v)$, and by n_1, n_2, n_3 , the ranks of the representation rings corresponding to the three edges e_1, e_2 and e_3 incident to v . Then the vertex block for v is a submatrix of D of size $n_0 \times (n_1 + n_2 + n_3)$. Since vertex blocks have been constructed to contain all entries from incident edges, we note that the rest of the entries in their rows are zero.

THEOREM 5.2 *If there exists a base transformation such that all minors in all vertex blocks are in the set $\{-1, 0, 1\}$, then $H_0^{\delta_{\text{in}}}(\Gamma; R_{\mathbb{C}})$ is torsion-free.*

Proof of Theorem 5.2. We start by recalling a general result on Smith Normal Forms, already observed by Smith [25]. Denote by $d_i(A)$ the i th determinant divisor $d_i(A)$, defined to be the greatest common divisor of all $i \times i$ minors of a matrix A when $i \geq 1$, and to be $d_0(A) := 1$ when $i = 0$. Then the elementary divisors of the matrix A , up to multiplication by a unit, coincide with the ratios $\alpha_i = \frac{d_i(A)}{d_{i-1}(A)}$.

Let us use the notation

$$\text{pre-rank}(\partial_1) := \text{rank}_{\mathbb{Z}}\mathcal{C}_1 - \text{rank}_{\mathbb{Z}}\ker \partial_1,$$

where \mathcal{C}_1 is the module of 1-chains in the Bredon chain complex (Equation (3.1)). Observe that if A is any $i \times i$ submatrix of D of non-zero determinant, and $i < \text{pre-rank}(\partial_1)$, then A can be expanded to some $(i + 1) \times (i + 1)$ submatrix of non-zero determinant.

Now $H_0^{\delta_{\text{in}}}(\Gamma; R_{\mathbb{C}})$ is torsion-free if and only if $\alpha_i = \pm 1$ for all $1 \leq i \leq \text{pre-rank}(\partial_1)$. From the discussion above, it is sufficient to find, for each $1 \leq i \leq \text{pre-rank}(\partial_1)$, an $i \times i$ minor in the Bredon chain complex differential matrix ∂_1 with determinant ± 1 . We produce such a minor by induction on i .

Base Case: For $i = 1$, we observe there are vertices with adjacent edges, hence there are non-zero vertex blocks. As by assumption all the entries in the vertex blocks are in the set $\{-1, 0, 1\}$, there exists an entry of value ± 1 .

Inductive step: Let $2 \leq i \leq \text{pre-rank}(\partial_1)$, and assume we already have an $(i-1) \times (i-1)$ minor of ∂_1 of value ± 1 , corresponding to a submatrix B . We want to find an $i \times i$ minor of ∂_1 of value ± 1 .

Given any vertex block V , choose a maximal square submatrix B° of B which is disjoint from the rows and columns of V . At the two extremes, this submatrix could be empty (if B is contained in V) or could coincide with B (if B is completely disjoint from V). Note that after possibly permuting rows, we get a square sub-block M of V such that the submatrix B takes the form

$$B = \det \begin{pmatrix} M & 0 \\ * & B^\circ \end{pmatrix}.$$

Then in particular $\pm 1 = \det(B) = \det(M) \cdot \det(B^\circ)$, which forces $\det(B^\circ) = \pm 1$. One can then consider extending B to an $i \times i$ block B' by choosing a submatrix M' inside V of size $i - \text{size}(B^\circ)$. Such an extension might not be possible, but when it is, the resulting block B' takes the form (possibly after permuting rows)

$$B' = \det \begin{pmatrix} M' & 0 \\ * & B^\circ \end{pmatrix}.$$

Consider the collection of all $i \times i$ blocks obtained in this manner, and note that for any such block, we have $\det(B') = \det(M') \cdot \det(B^\circ)$.

Since $i \leq \text{pre-rank}(\partial_1)$, there exists a vertex block V for which this construction yields an $i \times i$ block with $\det(B') \neq 0$. We have that M' is a minor in the vertex block V , so by hypothesis $\det(M') \in \{-1, 0, 1\}$. Since $\det(B') \neq 0$, we conclude that $\det(M') = \pm 1$. And as we already noted above, the submatrix B° of B' satisfies $\det(B^\circ) \in \{-1, 1\}$. This implies our submatrix B' satisfies $\det(B') = \det(M') \cdot \det(B^\circ) = \pm 1$, which completes the inductive step and hence the proof of the theorem. \square

Our proof of Theorem 3.8 now reduces to verifying the hypotheses of Theorem 5.2, when Γ is a 3-dimensional hyperbolic reflection group. We will rely on the simultaneous base transformations that can be found in [11, Appendix A].

PROPOSITION 5.3 *For a system of finite subgroups of types $A_5 \times C_2$, S_4 , $S_4 \times C_2$, $\Delta(2, 2, 2) = (C_2)^3$ and $\Delta(2, 2, m) = C_2 \times D_m$ for $m \geq 3$ as vertex stabilizers, with their three 2-generator Coxeter subgroups as adjacent edge stabilizers, there is a simultaneous base transformation such that all vertex blocks have all their minors contained in the set $\{-1, 0, 1\}$.*

Proof. We apply the base transformation specified in [11, Appendix A]. Then we have that all of the induced maps have all of their entries in the set $\{-1, 0, 1\}$. Next, for each vertex stabilizer type, we assemble the vertex blocks from the three vertex-edge-adjacency induced maps.

Let us provide full details for the case of vertex stabilizer $\Delta(2, 2, m) = C_2 \times D_m$ for $m \geq 3$. The vertex block of a stabilizer of type $C_2 \times D_m$ for $m \geq 3$ odd consists of (cf. [11, Appendix B])

$$\text{two blocks } \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and one block } \pm \begin{pmatrix} \text{identity matrix of size } \frac{m+3}{2} \\ 0 \end{pmatrix}.$$

Note that all the columns in this matrix have a very special form: all but one of the entries are zero, and the single non-zero entry is ± 1 . An easy induction shows that when checking whether the minors all take value in the set $\{0, \pm 1\}$, such columns can always be discarded (and likewise for rows). This fact is very useful for reducing the size of the matrices to check. For the matrix above, this fact immediately lets us conclude that all minors are in $\{0, \pm 1\}$.

For $m \geq 6$ even, but not a power of 2, we have the following vertex block, where each matrix block is specified up to orientation sign (we make this assumption from now on),

$D_m \times C_2$	$D_m \hookrightarrow D_m \times C_2$	$D_2 \hookrightarrow D_m \times C_2$	$D_2 \hookrightarrow D_m \times C_2$
$\rho_1 \otimes \chi_1 \downarrow$	1 0 0 0 0 0 0	1 0 0 0	1 0 0 0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	0 1 0 1 0 0 0	0 0 0 1	0 0 0 1
$\rho_1 \otimes (\chi_3 - \chi_2) \downarrow$	0 0 1 -1 0 0 0	0 0 0 0	0 0 0 0
$\rho_1 \otimes (\chi_4 - \chi_1) \downarrow$	0 0 0 1 0 0 0	0 0 0 0	0 0 0 0
$\rho_1 \otimes (\phi_1 - \chi_3 - \chi_1) \downarrow$	0 0 0 1 1 0 0	0 0 0 0	0 0 0 0
\vdots	\vdots	\vdots	\vdots
$\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$	0 0 0 0 0 0 0	\vdots	\vdots
$\rho_1 \otimes (\phi_{\frac{m}{2}-1} - \phi_{\frac{m}{2}-2}) \downarrow$	0 0 0 0 0 0 1	0 0 0 0	0 0 0 0
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$		0 1 0 0	0 1 0 0
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$		0 0 1 0	0 0 1 0
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_2) \downarrow$		0 0 0 0	0 0 0 0
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1) \downarrow$	0	0 0 0 0	0 0 0 0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$		0 0 0 0	0 0 0 0
\vdots		\vdots	\vdots
$\rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow$		\vdots	\vdots
$\rho_1 \otimes (\phi_{\frac{m}{2}-1} - \phi_{\frac{m}{2}-2}) \downarrow$		0 0 0 0	0 0 0 0

Again, we can discard all the rows and columns which have at most one entry ± 1 (and all other entries zero). This reduces the above vertex block to the much smaller matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & \pm 1 & \pm 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

for which we can easily check that all minors lie in $\{0, \pm 1\}$.

For $m \geq 4$ a power of 2, we have the following vertex block:

$D_m \times C_2$	$D_m \hookrightarrow D_m \times C_2$	$D_2 \hookrightarrow D_m \times C_2$	$D_2 \hookrightarrow D_m \times C_2$
$\rho_1 \otimes \chi_1 \downarrow$	1 0 0 0 0 0 0	1 0 0 0	1 0 0 0
$\rho_1 \otimes (\chi_2 - \chi_1) \downarrow$	0 1 0 1 0 0 0	0 0 0 1	0 0 0 1
$\rho_1 \otimes (\chi_3 - \chi_1) \downarrow$	0 1 1 0 0 0 0	0 0 0 0	0 0 0 0
$\rho_1 \otimes (\chi_4 - \chi_2) \downarrow$	0 -1 0 0 0 0 0	0 0 0 0	0 0 0 0
$\rho_1 \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$	0 0 1 0 1 0 0	0 0 0 0	0 0 0 0
$\vdots \quad \rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow \quad \vdots$	0 0 0 0 0 \ddots 0	$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$(\phi_{\frac{m}{2}-1} - \phi_{\frac{m}{2}-2}) \downarrow$	0 0 0 0 0 0 1	0 0 0 0	0 0 0 0
$(\rho_2 - \rho_1) \otimes \chi_1 \downarrow$		0 1 0 0	0 1 0 0
$(\rho_2 - \rho_1) \otimes (\chi_2 - \chi_1) \downarrow$		0 0 1 0	0 0 1 0
$(\rho_2 - \rho_1) \otimes (\chi_3 - \chi_1) \downarrow$	0	0 0 0 0	0 0 0 0
$(\rho_2 - \rho_1) \otimes (\chi_4 + \chi_3 - \chi_2 - \chi_1) \downarrow$		0 0 0 0	0 0 0 0
$(\rho_2 - \rho_1) \otimes (\phi_1 - \chi_2 - \chi_1) \downarrow$		0 0 0 0	0 0 0 0
$\vdots \quad \rho_1 \otimes (\phi_p - \phi_{p-1}) \downarrow \quad \vdots$		$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$\rho_1 \otimes (\phi_{\frac{m}{2}-1} - \phi_{\frac{m}{2}-2}) \downarrow$		0 0 0 0	0 0 0 0

Again, we can discard the rows and columns which have at most one entry ± 1 (and all other entries zero). This reduces the above vertex block to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \pm 1 & 0 & \pm 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & \pm 1 & 0 & \pm 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for which we can easily check that it has all its minors in $\{0, \pm 1\}$. This completes the verification of the vertex block condition in the case of vertices with stabilizer $\Delta(2, 2, m) = C_2 \times D_m$ for $m \geq 3$.

For the finitely many remaining stabilizer types, we can proceed case-by-case: we input each vertex block into a computer routine which computes all of its minors. Such a routine is straightforward to implement and takes approximately two seconds per vertex block on a standard computer. The authors' implementation is available at <http://math.uni.lu/rahm/vertexBlocks/>. Note that for the groups under consideration, the matrix rank of the vertex block is at most 7, so the 8×8 minors are all zero, and it is enough to compute the $n \times n$ minors for $n \leq 7$. This computer check verifies the minor condition for the vertex blocks associated to all remaining vertex stabilizers, and completes the proof of the theorem. \square

COROLLARY 5.4 *For any Coxeter group Γ having a system of finite subgroups of types $\Delta(2, 2, 2) = (C_2)^3$, $\Delta(2, 2, m) = C_2 \times D_m$ for $m \geq 3$, S_4 , $S_4 \times C_2$ or $A_5 \times C_2$ as vertex stabilizers, we have that the Bredon homology group $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ is torsion-free.*

REMARK 5.5

- (a) When trying to extend the proof of Theorem 5.2 to $H_n^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ for $n > 0$, one should take into account the natural map $H_n^{\text{fin}}(\Gamma; R_{\mathbb{C}}) \rightarrow H_n(\underline{\mathbf{B}}\Gamma; \mathbb{Z})$ described by Mislin [21], which is an isomorphism for $n > \dim \underline{\mathbf{E}}\Gamma^{\text{sing}} + 1$, where $\underline{\mathbf{E}}\Gamma^{\text{sing}}$ consists of the non-trivially stabilized

points in $\underline{E}\Gamma$. Hence, such an extension of the theorem can only be useful when $n \leq \dim \underline{E}\Gamma^{\text{sing}} + 1$.

- (b) Note that the search for suitable base transformations for a given group Γ (as described in [11, Appendix A] in our case) can be quite laborious. If the reader wants to apply Theorem 5.2 for a given group Γ , it is prudent to first construct the vertex blocks without any base transformation and compute their elementary divisors. If there exists a suitable simultaneous base transformation which satisfies the hypotheses of Theorem 5.2, then those elementary divisors must be in the set $\{-1, 0, 1\}$.

6. Further examples with torsion-free $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$

In this section, we briefly steer away from Coxeter groups, and instead give some further examples illustrating our criterion for the Bredon homology group $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ to be torsion-free.

6.1. The Heisenberg semidirect product group

Let us show that $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ is torsion-free for Γ the Heisenberg semidirect product group of Lück's paper. In Tables 3 and 4, we transform the character tables of all the non-trivial finite subgroups of the Heisenberg semidirect product group, as identified by Lück [15].

In Tables 5, 6 and 7, we compute all possible induction homomorphisms $R_{\mathbb{C}}(H) \rightarrow R_{\mathbb{C}}(G)$ appearing in any possible Bredon chain complex.

Obviously, any concatenation of copies of the three matrices given in Tables 5, 6 and 7 yields a matrix with all of its minors contained in the set $\{-1, 0, 1\}$. For the inclusions into cyclic groups of order 2, an analogous (and even simpler) procedure works. Hence by Theorem 5.2, $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ is torsion-free for Γ the Heisenberg semidirect product group of Lück's article [15].

6.2. Crystallographic groups

Davis and Lück [5] consider the semidirect product of \mathbb{Z}^n with the cyclic p -group C_p , where the action of C_p on \mathbb{Z}^n is given by an integral representation, which is assumed to act freely on the complement of zero. The action of this semidirect product group Γ on $\underline{E}\Gamma \cong \mathbb{R}^n$ is crystallographic, with \mathbb{Z}^n acting by lattice translations, and C_p acting with a single fixed point. In particular, all cell stabilizers are trivial except for one orbit of vertices of stabilizer type C_p . So all maps in the Bredon chain complex are induced by the trivial representation, and we can easily apply Theorem 5.2 to see that $H_0^{\text{fin}}(\Gamma; R_{\mathbb{C}})$ is torsion-free for Γ .

7. $cf(\Gamma)$ and $\chi(\mathcal{C})$ from the geometry of \mathcal{P}

Let Γ be the reflection group of the compact 3-dimensional hyperbolic polyhedron \mathcal{P} . In this final section, we compute the number of conjugacy classes of elements of finite order of Γ , $cf(\Gamma)$, and the Euler characteristic of the Bredon chain complex (2.5), $\chi(\mathcal{C})$, from the geometry of the polyhedron \mathcal{P} . This gives us explicit combinatorial formulas for the Bredon homology and equivariant K -theory groups computed in our Main Theorem.

Table 3. Character table of the cyclic group C_2 of order 2, with generator s , and base transformation (Section 6.1)

$$\left(\begin{array}{c|cc} C_2 & e & s \\ \hline \rho_1 & 1 & 1 \\ \rho_2 & 1 & -1 \end{array} \right) \mapsto \left(\begin{array}{c|cc} C_2 & e & s \\ \hline \rho_1 + \rho_2 & 2 & 0 \\ \rho_2 & 1 & -1 \end{array} \right)$$

Table 4. Character table of the cyclic group C_4 of order 4, with generator s , and base transformation (Section 6.1). We let $i^2 = -1$

$$\left(\begin{array}{c|cccc} C_4 & e & s & s^2 & s^3 \\ \hline \rho_1 & 1 & 1 & 1 & 1 \\ \rho_2 & 1 & -1 & 1 & -1 \\ \rho_3 & 1 & i & -1 & -i \\ \rho_4 & 1 & -i & -1 & i \end{array} \right) \mapsto \left(\begin{array}{c|cccc} C_4 & e & s & s^2 & s^3 \\ \hline \rho_1 & 1 & 1 & 1 & 1 \\ \rho_2 - \rho_1 & 0 & -2 & 0 & -2 \\ \rho_3 - \rho_1 & 0 & i - 1 & -2 & -i - 1 \\ \rho_4 - \rho_3 & 0 & -2i & 0 & 2i \end{array} \right)$$

Table 5. The only non-trivial inclusion $C_2 \hookrightarrow C_4$ of a cyclic group of order 2 into a cyclic group of order 4: $s \mapsto s^2$, in terms of scalar products of characters

$C_2 \hookrightarrow C_4$	e	s^2	$(\cdot \rho_1 + \rho_2)$	$(\cdot \rho_2)$
$\rho_1 \downarrow$	1	1	1	0
$(\rho_2 - \rho_1) \downarrow$	0	0	0	0
$(\rho_3 - \rho_1) \downarrow$	0	-2	0	1
$(\rho_4 + \rho_3 - \rho_2 - \rho_1) \downarrow$	0	0	0	0

Table 6. The trivial inclusion $C_2 \hookrightarrow C_4$ of a cyclic group of order 2 into a cyclic group of order 4: $s \mapsto e$, in terms of scalar products of characters

$C_2 \hookrightarrow C_4$	e	e	$(\cdot \rho_1 + \rho_2)$	$(\cdot \rho_2)$
$\rho_1 \downarrow$	1	1	1	0
$(\rho_2 - \rho_1) \downarrow$	0	0	0	0
$(\rho_3 - \rho_1) \downarrow$	0	0	0	0
$(\rho_4 + \rho_3 - \rho_2 - \rho_1) \downarrow$	0	0	0	0

Table 7. The only inclusion $\{1\} \hookrightarrow C_4$ of the trivial group into a cyclic group of order 4, in terms of scalar products of characters.

$\{1\} \hookrightarrow C_4$	e	$(\cdot \tau)$
$\rho_1 \downarrow$	1	1
$(\rho_2 - \rho_1) \downarrow$	0	0
$(\rho_3 - \rho_1) \downarrow$	0	0
$(\rho_4 + \rho_3 - \rho_2 - \rho_1) \downarrow$	0	0

7.1. Conjugacy classes of elements of finite order

We now give an algorithm to calculate $cf(\Gamma)$, the number of conjugacy classes of elements of finite order in the Coxeter group Γ . We know that each element of finite order can be conjugated to one which stabilizes one of the k -dimensional faces of the polyhedron, for some $k \in \{0, 1, 2\}$. Of course, the only element which stabilizes *all* faces is the identity element. Let us set that aside, and consider the non-identity elements, to which we associate the integer k . We now count the elements according to the integer k , in descending order.

Case $k = 2$: These are the conjugacy classes represented by the canonical generators of the Coxeter group Γ . The number of these is given by the total number $|\mathcal{P}^{(2)}|$ of facets of the polyhedron \mathcal{P} .

Case $k = 1$: These elements are edge stabilizers which are not conjugate to the stabilizer of a face. We first note that there are some possible conjugacies between edge stabilizers. Geometrically, these occur when there is a geodesic $\gamma \subset \mathbb{H}^3$ whose projection onto the fundamental domain \mathcal{P} covers multiple edges inside the 1-skeleton $\mathcal{P}^{(1)}$. A detailed analysis of when this can happen is given in [10]. Following the description in that paper, we decompose the 1-skeleton into equivalence classes of edges, where two edges are equivalent if there exists a geodesic whose projection passes through both edges. Denote by $[\mathcal{P}^{(1)}]$ the set of equivalence classes of edges, and note that each equivalence class $[e]$ has a well-defined group associated to it, which is just the dihedral group Γ_e stabilizing a representative edge. We can thus count the conjugacy classes in the corresponding dihedral group, and discard the three conjugacy classes already accounted for (the conjugacy class of the two canonical generators counted in case $k = 2$, as well as the identity). Thus, the contribution from finite elements of this type is given by

$$\sum_{[e] \in [\mathcal{P}^{(1)}]} (c(\Gamma_e) - 3).$$

(Recall that $c(D_m)$, the number of conjugacy classes in a dihedral group of order $2m$, is $m/2 + 3$ if m even, and $(m - 1)/2 + 2$ if m is odd.)

Case $k = 0$: Finally, we consider the contribution from the elements in the vertex stabilizers *which have not already been counted*. That is to say, for each vertex $v \in \mathcal{P}^{(0)}$, we count the conjugacy classes of elements in the corresponding 3-generated spherical triangle group, which cannot be conjugated into one of the canonical 2-generated special subgroups. This number, $\bar{c}(\Gamma_v)$,

depends only on the isomorphism type of the spherical triangle group Γ_v , and is given in Table 8. The contribution from these types of finite elements is thus

$$\sum_{v \in \mathcal{P}^{(0)}} \bar{c}(\Gamma_v).$$

Combining all these, we obtain the desired combinatorial formula for the number of conjugacy classes of elements of finite order inside the group Γ :

$$cf(\Gamma) = 1 + |\mathcal{P}^{(2)}| + \sum_{[e] \in [\mathcal{P}^{(1)}]} (c(\Gamma_e) - 3) + \sum_{v \in \mathcal{P}^{(0)}} \bar{c}(\Gamma_v).$$

7.2. Euler characteristic

The Euler characteristic of the Bredon chain complex can be easily calculated from the stabilizers of the various faces of the polyhedron \mathcal{P} , according to the formula:

$$\chi(\mathcal{C}) = \sum_{f \in \mathcal{P}} (-1)^{\dim(f)} \dim(R_{\mathbb{C}}(\Gamma_f)).$$

Depending on the dimension of the faces, we know exactly what the dimension of the complex representation ring is (the number of conjugacy classes in the stabilizer):

- for the 3-dimensional face (the interior), the stabilizer is trivial, so there is a 1-dimensional complex representation ring;
- for the 2-dimensional faces, the stabilizer is C_2 , and there is a 2-dimensional complex representation ring;
- for the 1-dimensional faces e , the stabilizers are dihedral groups, and there is a $c(\Gamma_e)$ -dimensional complex representation ring;
- for the 0-dimensional faces v , the stabilizers are spherical triangle groups, and there is a $c(\Gamma_v)$ -dimensional complex representation ring.

Putting these together, we obtain

$$\chi(\mathcal{C}) = -1 + 2|\mathcal{P}^{(2)}| - \sum_{e \in \mathcal{P}^{(1)}} c(\Gamma_e) + \sum_{v \in \mathcal{P}^{(0)}} c(\Gamma_v).$$

Table 8. Number of conjugacy classes in spherical triangle groups. The left column is the total number, and the right column the number of those not conjugated into one of the three canonical 2-generated special subgroups

Γ_v	$c(\Gamma_v)$	$\bar{c}(\Gamma_v)$
$\Delta(2, 2, m)$	$2c(D_m)$	$c(D_m) - 3$
$\Delta(2, 3, 3)$	5	1
$\Delta(2, 3, 4)$	10	3
$\Delta(2, 3, 5)$	10	5

Finally, we obtain the desired explicit version of the Main Theorem, expressing the K -theory groups in terms of the geometry of the polyhedron \mathcal{P} .

MAIN THEOREM (EXPLICIT) Let Γ be a cocompact 3-dimensional hyperbolic reflection group, generated by reflections in the side of a hyperbolic polyhedron $\mathcal{P} \subset \mathbb{H}^3$. Then $K_0(C_r^*(\Gamma))$ is a torsion-free abelian group of rank

$$cf(\Gamma) = 1 + |\mathcal{P}^{(2)}| + \sum_{[e] \in [\mathcal{P}^{(1)}]} (c(\Gamma_e) - 3) + \sum_{v \in \mathcal{P}^{(0)}} \bar{c}(\Gamma_v),$$

and $K_1(C_r^*(\Gamma))$ is a torsion-free abelian group of rank

$$cf(\Gamma) - \chi(\mathcal{C}) = 2 - |\mathcal{P}^{(2)}| + \sum_{[e] \in [\mathcal{P}^{(1)}]} (c(\Gamma_e) - 3) + \sum_{e \in \mathcal{P}^{(1)}} c(\Gamma_e) - \sum_{v \in \mathcal{P}^{(0)}} (c(\Gamma_v) - \bar{c}(\Gamma_v)),$$

where the possible values for the $c(\Gamma_v)$ and $\bar{c}(\Gamma_v)$ are listed in Table 8.

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