Finite automorphisms of negatively curved Poincaré Duality groups

F. T. Farrell * & J.-F. Lafont

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Abstract

In this paper, we show that if G is a finite p-group (p prime) acting by automorphisms on a δ-hyperbolic Poincaré Duality group over $\mathbb{Z}$, then the fixed subgroup is a Poincaré Duality group over $\mathbb{Z}_p$. We also provide a family of examples to show that the fixed subgroup might not be a Poincaré Duality group over $\mathbb{Z}$. In fact, the fixed subgroups in our examples even fail to be duality groups over $\mathbb{Z}$.

1 Introduction.

The study of finite group actions on topological spaces has a long and distinguished history. A frequent theme is to try and understand the topology of the fixed point set, both in it's intrinsic form, and as a subspace of the original space. The classic work of Smith shows that for finite p-groups acting on spheres, the fixed point set has the $\mathbb{Z}_p$ cohomology of a sphere. However, there are examples of 'exotic' actions on spheres, where the fixed point set is not *homeomorphic* to a sphere (indeed, does not even have the $\mathbb{Z}$ cohomology of a sphere). In this short paper, we are interested in relating actions on a hyperbolic group with the induced action on its boundary at infinity.

We will start by relating the fixed subgroup of an automorphism with the fixed subset of the induced action on the boundary at infinity. In particular, this will allow us to use the classic theorem of Smith to prove that if one starts with a δ-hyperbolic Poincaré Duality group over $\mathbb{Z}$, and the group that is acting is a finite p-group (p a prime), then the fixed subgroup is a Poincaré Duality group over $\mathbb{Z}_p$. We will then

use the strict hyperbolization technique due to Charney and Davis [6] to construct examples of involutions of a Poincaré Duality group over $\mathbb{Z}$ whose fixed subgroup fails to be a Poincaré Duality group over $\mathbb{Z}$ (and in fact, aren't even duality groups over $\mathbb{Z}$). These examples also provide examples of 'exotic' involution on a sphere (the boundary at infinity) which can be realized geometrically (i.e. by an isometry of a $CAT(-1)$ space). They also show that, in general, one could have involutions of $CAT(-1)$ spaces having a sphere as the boundary at infinity, where the induced involution on the boundary has a fixed point set which is *not* an ANR.

Remark: This paper was motivated by the following more specific questions (each of which is still open). Let $\Gamma = \pi_1(M)$ where M is a closed negatively curved Riemannian manifold, and let $\alpha : \Gamma \to \Gamma$ be an automorphism with $\alpha^2 = Id_\Gamma$.

Question 1: Is the fixed subgroup Γ^α a Poincaré Duality group over $\mathbb{Z}$?

Question 2: Is α induced by an involution of M? That is to say, does there exist a self-homeomorphism $f : M \to M$ with $f^2 = Id_M$, and $f_\sharp = \alpha$?

Question 3: Let $\hat{\alpha} : \partial^\infty \tilde{M} \to \partial^\infty \tilde{M}$ be the induced involution of the sphere at infinity of the universal cover of M. Is the fixed point set of $\hat{\alpha}$ (when non-empty) an ANR?

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Acknowledgements.
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2 Main results.

2.1 A positive result.

Proposition 2.1. *Let Γ be a δ-hyperbolic group, $\bar{\sigma}$ an automorphism of Γ of finite order m, and $\bar{\sigma}_\infty$ the induced action of $\bar{\sigma}$ on $\partial^\infty \Gamma$. Then $(\partial^\infty \Gamma)^{\bar{\sigma}_\infty}$ is homeomorphic to $\partial^\infty(\Gamma^{\bar{\sigma}})$.*

Proof. Let Σ be a symmetric generating set for Γ, and consider the action of $\bar{\sigma}$ on Γ. Observe that if we define a new generating set $\Sigma' := \bigcup_{i=1}^m \bar{\sigma}^i(\Sigma)$, then $\bar{\sigma}$ acts by isometries on the Cayley graph $Cay(\Gamma, \Sigma')$ of Γ with respect to these generators. Indeed, we note that given any pair of elements g, h in Γ, we have that:

$$d_{\Sigma'}(g,h) = \inf\{i \mid g^{-1}h = \alpha_1 \ldots \alpha_i, \alpha_j \in \Sigma'\}$$

Taking a minimal such expression, and applying $\bar{\sigma}$ to it, we see that:

$$\bar{\sigma}(g)^{-1}\bar{\sigma}(h) = \bar{\sigma}(g^{-1}h) = \bar{\sigma}(\alpha_1) \ldots \bar{\sigma}(\alpha_i)$$

But by invariance of Σ' under $\bar{\sigma}$, we immediately get an expression for $\bar{\sigma}(g)^{-1}\bar{\sigma}(h)$ as a product of i elements of Σ'. This forces $d_{\Sigma'}(\bar{\sigma}(g), \bar{\sigma}(h)) \leq d_{\Sigma'}(g,h)$. But now $\bar{\sigma}$, by hypothesis, has finite order m. So by iterating our inequality we get that:

$$d_{\Sigma'}(g,h) \geq d_{\Sigma'}(\bar{\sigma}(g), \bar{\sigma}(h)) \geq \cdots \geq d_{\Sigma'}(\bar{\sigma}^m(g), \bar{\sigma}^m(h)) = d_{\Sigma'}(g,h)$$

which implies that all the inequalities are in fact equalities, and hence that $\bar{\sigma}$ does indeed act by isometries on $Cay(\Gamma, \Sigma')$. From now on, we will omit the subscript Σ' from our distance function in order to simplify notation.

Our next step is to define certain subsets of the Cayley graph $Cay(\Gamma, \Sigma')$ in terms of their behavior under $\bar{\sigma}$, and to control the distance between these subsets. We let $F_k := \{g \mid d(g, \bar{\sigma}g) \leq k\}$, and observe that $\Gamma^{\bar{\sigma}} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$. Neumann [15] has shown that, for each i, there exists a K_i such that $d(F_i, F_0) < K_i$.

Next we observe that, by Neumann [15], the subgroup $\Gamma^{\bar{\sigma}} = F_0$ is quasi-convex in Γ. In particular, $\partial^\infty F_0$ embeds in $\partial^\infty \Gamma$. Now note that, trivially, we have that $\partial^\infty(\Gamma^{\bar{\sigma}})$ is in fact a subset of $(\partial^\infty \Gamma)^{\bar{\sigma}_\infty}$. To prove equality, we need to show the reverse inclusion. So let us take a point $p \in (\partial^\infty \Gamma)^{\bar{\sigma}_\infty}$, and let $\gamma \subset Cay(\Gamma, \Sigma')$ be a geodesic ray based at the identity and with $\gamma(\infty) = p$. Now by our choice of generators, we know that $\eta := \bar{\sigma}(\gamma)$ will also be a geodesic ray (since $\bar{\sigma}$ acts isometrically on the Cayley graph), and since the point $p = \gamma(\infty)$ is fixed by $\bar{\sigma}_\infty$, we must have $d(\gamma, \eta) \leq C$ for some constant C.

Our next claim is that, for each n, the inequality $d(\gamma(n), \eta(n)) \leq 2C$ holds (and hence, as $\eta = \bar{\sigma}(\gamma)$, forces $\gamma \subset F_{2C}$). In order to see this, we consider the following construction: given an integer n, we define $f(n)$ to be an integer satisfying $d(\gamma(n), \eta(f(n))) \leq C$ (note that both $\gamma(n)$ and $\eta(f(n))$ correspond to elements in Γ). We claim that $|f(n) - n| \leq C$ for all n. By way of contradiction, assume that $f(n) - n > C$. The triangle inequality gives us:

$$f(n) = d(\eta(0), \eta(f(n))) \leq d(\eta(0), \gamma(n)) + d(\gamma(n), \eta(f(n))) \leq n + C < f(n)$$

a contradiction (recall that $\eta(0), \gamma(0)$ are both the identity element in Γ). The case $n - f(n) > C$ can be dealt with in an analogous manner.

We now know that, if γ is an arbitrary geodesic ray originating at the identity, and having $\gamma(\infty) = p$, then $\gamma \subset F_{2C}$. However, we also have that $d(F_0, F_{2C}) \leq K$ for some constant K. In particular, we can find a geodesic ray in F_0 which has uniformly

bounded distance from γ, which forces $p \in \partial^\infty F_0 = \partial^\infty(\Gamma^{\bar{\sigma}})$, completing the proof of the proposition.

Definition 2.1. We say that a topological space X is an n-dimensional Cech cohomology sphere with R coefficients (where R is a PID) provided that $\check{\bar{H}}^k(X; R) = 0$ for all $k \neq n$, and $\check{\bar{H}}^n(X; R) = R$ ($\check{\bar{H}}^*$ refers to reduced Cech cohomology).

Definition 2.2. We say that a torsion-free group G is a *duality group of dimension n over R* (where again, R is a PID), provided that there is a right RG-module C such that one has natural isomorphisms $H^k(G; A) \cong H_{n-k}(G; C \otimes_R A)$ for all $k \in \mathbb{Z}$ and all RG-modules A (naturality is taken with respect to A, and G acts diagonally on the tensor product $C \otimes_R A$). If in addition we have that $C \cong R$, then we say that G is a *Poincaré Duality group of dimension n over R*. Finally, if G is a Poincaré Duality group of dimension n over R, and the G action on $C \cong R$ is trivial, we say that G is an *orientable Poincaré Duality group of dimension n over R*.

For background material on duality groups and Poincaré Duality groups, we refer to the lecture notes by Bieri [2]. Next, we quote the following result from Bestvina and Mess (Corollary 1.3 in their paper [1]):

Theorem 2.1 (Bestvina & Mess)**.** *Let Γ be a torsion-free δ-hyperbolic group. Then Γ is a Poincaré Duality group of dimension n over Λ if and only if $\partial^\infty\Gamma$ is an $(n-1)$-dimensional Cech cohomology sphere with Λ coefficients.*

Using their result, we obtain an immediate corollary to our previous proposition:

Corollary 2.1. *Let Γ be a torsion-free δ-hyperbolic Poincaré Duality group of dimension n over $\mathbb{Z}_p$. Let G be a finite p-group (p prime) acting by automorphisms on Γ. Then there is a $0 \leq k \leq n$ such that the subgroup Γ^G is a Poincaré Duality group of dimension k over $\mathbb{Z}_p$.*

Proof. Let us first consider the case where G is $\mathbb{Z}_p$. Then consider the induced action of G on the boundary at infinity $\partial^\infty\Gamma$. Notice that, by Bestvina and Mess' result, $\partial^\infty\Gamma$ is a compact $(n-1)$-dimensional Cech cohomology sphere with $\mathbb{Z}_p$ coefficients. So we can use a version of Smith theory (see theorem III.7.11 in Bredon [4]), to get that the fixed point set of the action on the boundary at infinity must be a $(k-1)$-dimensional Cech cohomology sphere with Z_p coefficients (for some $-1 \leq k-1 \leq n-1$). Now our Proposition 2.1 along with Bestvina and Mess' result immediately implies that the group Γ^G is a Poincaré Duality group of dimension k over $\mathbb{Z}_p$ (where $0 \leq k \leq n$).

For the more general case, we note that, since every p-group is solvable, one can find a normal subgroup $G' \leq G$. Finally one uses induction, since we have that $\Gamma^G = (\Gamma^{G'})^{G/G'}$. This gives the general case.

As was pointed out to the authors by the referee, Corollary 2.1 also follows from the result announced by Chang and Skjelbred in [5], where they explain why the fixed set of a finite p-group action on a Poincaré Duality space over $\mathbb{Z}_p$ is still a Poincaré Duality space over $\mathbb{Z}_p$.

We conclude this section by mentioning that a Poincaré Duality group over $\mathbb{Z}$ is automatically a Poincaré Duality group over $\mathbb{Z}_p$, but that the converse does not necessarily hold. In Theorem 2.2, the group $\Gamma^{\bar{\sigma}}$ will be an example of this with $p = 2$.

2.2 A family of counterexamples.

One can now ask the question of whether the previous result can be strengthened to obtain that the fixed subgroup is a Poincaré Duality group over $\mathbb{Z}$. This turns out to be false, and in this section, we will construct counterexamples. As was pointed out by the referee, similar examples were constructed by Davis & Leary [8]. Their construction used the reflection trick method (as opposed to our use of hyperbolization) and served a somewhat different purpose. We now proceed to state our main theorem.

Theorem 2.2. *Let τ be a PL involution of a sphere S^n whose fixed point set is a submanifold N^m which is not a homology sphere (with $\mathbb{Z}$ coefficients), and has dimension $m \geq 2$. Let X be the strict hyperbolization of the suspension of S^n, and σ the induced involution on X. Let Γ be the fundamental group of X, and $\bar{\sigma}$ the induced involution on Γ. Then $\bar{\sigma}$ is an involution of a (δ-hyperbolic) orientable Poincaré Duality group over $\mathbb{Z}$ whose fixed subgroup $\Gamma^{\bar{\sigma}}$ is not a duality group over $\mathbb{Z}$.*

Before starting with the proof, let us note that examples of involutions of spheres whose fixed point sets are not homology spheres do exist. In fact, Jones [11] has proved that every closed PL manifold that has the $\mathbb{Z}_2$ homology of a sphere can be realized as the fixed point set of a PL involution of some larger dimensional sphere.

For a more concrete example, we can consider Brieskorn spheres: for $n \geq 2$, define two complex functions $f_n(z_0, \ldots, z_{2n+1}) := z_0^3 + \sum_{i=1}^{2n+1} z_i^2$, and $g_n(z_0, \ldots, z_{2n}) := z_0^3 + \sum_{i=1}^{2n} z_i^2$. Using these two functions, define a pair of manifolds M_n and N_n by considering the intersection of $f_n^{-1}(0)$ and $g_n^{-1}(0)$ with a small enough ball centered at the origin in the appropriate complex vector space. It is known that M_n is PL homeomorphic to the sphere S^{4n+1}, while N_n is a $(4n-1)$-dimensional manifold that does not have the $\mathbb{Z}$-homology of a sphere (combine Lemma 8.1 with the comments on pg. 72 in Milnor [14]). Furthermore, observe that the involution $z_{2n+1} \leftrightarrow -z_{2n+1}$ on M_n has fixed point set N_n, giving us an infinite family of examples.

Proof. We start by recalling that the strict hyperbolization procedure given by Charney and Davis (section 7 in [6]) takes a simplicial complex and functorially assigns to

Figure 1: Zeeman spectral sequence for our spaces.

it a topological space (in fact, a union of compact hyperbolic manifolds with corners) that supports a metric of strict negative curvature. Let us apply this procedure to the suspension of the sphere ΣS^n (respectively ΣN^m), and call the resulting space X^{n+1} (respectively Y^{m+1}). We will omit the dimension of the spaces unless we explicitly require them for computations.

We now list out some properties of the spaces X and Y. Observe that, by a result of Illman [9], there exists a triangulation of the pair (S^n, N^m) such that the involution τ is a simplicial map. In particular, the involution on the suspension will still be simplicial, and ΣN^m is a subcomplex of ΣS^n. Functoriality of the strict hyperbolization procedure now implies that Y is a totally geodesic subspace of X, invariant under the induced involution σ on X. Since hyperbolization preserves the local structure, X will be an orientable manifold, while Y will have a pair of non-manifold points (corresponding to the two vertices of the suspension).

Now take a basepoint $* \in Y \subset X$, and let $\Lambda = \pi_1(Y, *)$, $\Gamma = \pi_1(X, *)$. The involution σ will give an order two automorphism $\bar{\sigma}$ of the group Γ. We note that, since Γ is the fundamental group of a closed orientable aspherical manifold, it is automatically an orientable Poincaré Duality group over $\mathbb{Z}$. Now consider the fixed subgroup $\Gamma^{\bar{\sigma}}$. In order to get information about this group, we consider a lift of the action to the universal cover $\tilde{X}$ of X.

Let $\tilde{*} \in \tilde{X}$ be a preimage of the point $*$, and let us lift the involution σ to the universal cover. Note that the fixed point set of the lifted involution is precisely the path connected lift $\tilde{Y}$ of Y that contains the point $\tilde{*}$. Furthermore, the action $\bar{\sigma}$ on Γ is compatible with the lift $\tilde{\sigma}$ of σ, in the sense that $(\bar{\sigma}(g))(\tilde{*}) = \tilde{\sigma}(g(\tilde{*}))$.

Next we note that $\Gamma^{\bar{\sigma}} = \Lambda$. Indeed Λ is automatically fixed by $\bar{\sigma}$, hence we have a containment $\Lambda \leq \Gamma^{\bar{\sigma}}$. On the other hand, for an arbitrary $g \in \Gamma^{\bar{\sigma}}$, we have that $\tilde{\sigma}(g(\tilde{*})) = (\bar{\sigma}(g))(\tilde{*}) = g(\tilde{*})$. In particular, $g(\tilde{*})$ must be fixed under $\tilde{\sigma}$, which implies $g(\tilde{*}) \in \tilde{Y}$. Since $\tilde{Y}$ is a path connected, totally geodesic subset, we can connect $\tilde{*}$ to $g(\tilde{*})$ by a path which lies entirely within $\tilde{Y}$. Looking at the projection of this path in X, we observe that it is a closed loop based at $*$, representing the element g, and lying entirely in Y. Hence $g \in \Lambda$, giving us the reverse containment. We conclude that the two groups are equal.

So in particular, Y is a topological space which happens to be a $K(\Gamma^{\bar{\sigma}}, 1)$. In particular, the group cohomology of $\Gamma^{\bar{\sigma}}$ is related to the compactly supported cohomology of $\tilde{Y}$. So we have now reduced our claim to analyzing the properties of $H^*(\tilde{Y}, \mathbb{Z})$. In order to do this, we consider the Zeeman spectral sequence; let us first

introduce some terminology. We will denote by h_p the pth local homology sheaf for Y, and by $\tilde{h}_p$ the corresponding sheaf for $\tilde{Y}$. For $x \in Y$ (respectively, in $\tilde{Y}$), we will denote by $h_p(x)$ (respectively $\tilde{h}_p(x)$) the stalk at the point x. Recall that $\tilde{Y}$ is the hyperbolization of an $(m+1)$-dimensional complex ΣN^m; we will use Y^i to denote the subspace of Y obtained from the hyperbolization of the i-skeleton of ΣN^m. Observe the following facts about the local homology sheaf:

- if $i \neq m+1$ and $x \notin Y^0$, then $h_i(x) = 0$.

- if $x \notin Y^0$, then $h_{m+1}(x) = \mathbb{Z}$.

- there exists a point $p \in Y^0$ and an integer s such that $2 \leq s < m+1$ and $h_s(p) \neq 0$.

All of the previous remarks are clear, with the possible exception of the third: let p be one of the two vertices of the suspension. Since the original link of p was *not* a homology sphere, and as hyperbolization does not change the link, there must exist an $s < m+1$ which yields the desired fact. Note that the sheafs we are considering are given by local data, so that we have that $\tilde{h}_p(\tilde{x}) = h_p(x)$, whenever $\tilde{x}$ is a lift of the point x. Hence we have that the three facts mentioned above for the stalks of the local homology sheaf h_p on Y also hold for the stalks of the local homology sheaf $\tilde{h}_p$ on $\tilde{Y}$.

Now the Zeeman spectral sequence (see section 2 of McCrory [13], based on previous work of Zeeman [17]) states that:

$$E_{i,j}^2 := \check{H}_c^j(\tilde{Y}; \tilde{h}_i) \overset{j}{\Longrightarrow} H_{i-j}(\tilde{Y})$$

with differentials $d_{i,j}^t : E_{i,j}^t \longrightarrow E_{i+(t-1),j+t}^t$. Observe that, by the properties listed above for the ith local homology sheaf, $E_{i,j}^2 = 0$ if $i \neq m+1$ and $j \neq 0$. So in particular, all the terms vanish except those in the 0th row and those in the $(m+1)^{st}$ column (see figure on previous page).

We now plan on working with this spectral sequence. Observe from the shape of the spectral sequence that one has isomorphisms:

$$E_{s,0}^2 \cong E_{s,0}^3 \cong \cdots \cong E_{s,0}^{m-s+2}$$

$$E_{m+1,m-s+2}^2 \cong E_{m+1,m-s+2}^3 \cong \cdots \cong E_{m+1,m-s+2}^{m-s+2}$$

and that the differential d^{m-s+2} maps $E_{s,0}^{m-s+2}$ to $E_{m+1,m-s+2}^{m-s+2}$. However, we know that $H_s(\tilde{Y}) = 0$, so the differential must be an isomorphism. This yields:

$$H_c^0(\tilde{Y}; \tilde{h}_s) = E_{0,s}^2 \cong E_{m+1,m-s+2}^2 = H_c^{m-s+2}(\tilde{Y}; \tilde{h}_{m+1})$$

(since we are dealing with complexes, Cech cohomology coincides with standard co-homology). Furthermore, $\tilde{Y}$ is simply-connected and has dimension $m+1 \geq 3$, hence $\tilde{h}_{m+1}$ is the trivial $\mathbb{Z}$ sheaf over $\tilde{Y}^{m+1} - \tilde{Y}^0$. This implies:

$$H_c^0(\tilde{Y}; \tilde{h}_s) \cong H_c^{m-s+2}(\tilde{Y}; \tilde{h}_{m+1}) = H_c^{m-s+2}(\tilde{Y}; \mathbb{Z})$$

Now focusing on the left hand term, we note that $\tilde{h}_s(q) = 0$ for all $q \notin \tilde{Y}^0$, which gives us:

$$H_c^{m-s+2}(\tilde{Y}; \mathbb{Z}) \cong H_c^0(\tilde{Y}; \tilde{h}_s) = \bigoplus_{q \in \tilde{Y}^0} \tilde{h}_s(q)$$

But now observe that if $\tilde{p}$ is a vertex in $\tilde{Y}^0$ which is a lift of p (one of the vertex points of the suspension), then $\tilde{h}_s(g \cdot \tilde{p}) = h_s(p) \neq 0$ for every element $g \in \Gamma^{\bar{\sigma}}$. Since all the points $g \cdot \tilde{p}$ lie in $\tilde{Y}^0$, and since $\Gamma^{\bar{\sigma}}$ is an infinite group, this implies that $\bigoplus_{q \in \tilde{Y}^0} \tilde{h}_s(q)$ is not finitely generated. So in particular, $H_c^{m-s+2}(\tilde{Y}; \mathbb{Z})$ is not finitely generated. Since Y is a finite complex which happens to be a $K(\Gamma^{\bar{\sigma}}, 1)$, we conclude that $H^{m-s+2}(\Gamma^{\bar{\sigma}}, \mathbb{Z}\Gamma^{\bar{\sigma}}) \cong H_c^{m-s+2}(\tilde{Y}; \mathbb{Z})$ is not finitely generated. By Bieri and Eckmann's criterion (see Bieri [2], section 9.10), this implies that $\Gamma^{\bar{\sigma}}$ cannot be a Poincaré Duality group over $\mathbb{Z}$.

In order to see that $\Gamma^{\bar{\sigma}}$ is not even a duality group over $\mathbb{Z}$, it is sufficient to show that the cohomological dimension of $\Gamma^{\bar{\sigma}}$ is greater than $m - s + 2$. We first note that, since $s \geq 2$, we have that $m - s + 2 \leq m$, so it is sufficient to show that $\Gamma^{\bar{\sigma}}$ has non-trivial cohomology in some dimension that is strictly greater than m. Observe that, by construction, we have that $\Gamma^{\bar{\sigma}}$ is the fundamental group of the finite aspherical $(m+1)$-dimensional space Y, which implies that the cohomological dimension of $\Gamma^{\bar{\sigma}}$ is at most $m+1$. We would be done provided we can show that the cohomological dimension of $\Gamma^{\bar{\sigma}}$ is exactly $m+1$. Looking back at the construction of Y, we observe that the submanifold N^m we started with is a $\mathbb{Z}_2$ homology sphere. Suspending the manifold, we obtain an $(m+1)$-dimensional space which is a $\mathbb{Z}_2$ homology manifold. Now Y is the hyperbolization of this space, and since the hyperbolization procedure preserves the local structure, Y is also an $(m+1)$-dimensional $\mathbb{Z}_2$ homology manifold. This implies that $H^{m+1}(\Gamma^{\bar{\sigma}}; \mathbb{Z}_2) \cong \mathbb{Z}_2 \neq 0$, which forces the cohomological dimension of $\Gamma^{\bar{\sigma}}$ to be at least $m+1$. This completes our proof.

Remark. As was pointed out to the authors by the referee, the argument in Theorem 2.2 can also be used to show that the condition that G be a p-group in Corollary 2.1 really is necessary. Namely, there are examples of a $\mathbb{Z}_6$ action on an orientable Poincaré Duality δ-hyperbolic group over $\mathbb{Z}$ whose fixed subgroup is not a Duality group over *any* PID (in which $0 \neq 1$). Indeed, note that the unit tangent bundle $S(S^{n-1})$ of an $(n-1)$-dimensional sphere can be identified with the Stiefel manifold

$V_{2,n}$ of orthonormal 2-frames in $\mathbb{R}^n$. The latter can be embedded in $\mathbb{C}^n$ via the map $f(u, v) = u + iv$ (where $u, v \in \mathbb{R}^n$ are orthonormal vectors).

Note that since u, v are orthonormal, we have that $|f(u, v)|^2 = |u|^2 + |v|^2 = 2$, and also that:

$$\sum_{j=1}^n (u_j + iv_j)^2 = \sum_{j=1}^n u_j^2 - \sum_{j=1}^n v_j^2 + 2i(\sum_{j=1}^n u_j v_v) = 0$$

This implies that $S(S^{n-1})$ is diffeomorphic to the Brieskorn variety for the polynomial $z_1^2 + \cdots z_n^2 = 0$. In particular, we see that $S(S^{n-1})$ is the fixed point set of the $\mathbb{Z}_6$ action on the Brieskorn variety for the polynomial $z_0^3 + z_1^2 + \cdots z_n^2 + z_{n+1}^2 = 0$, where the action is given by $g(z_0, z_1, \ldots, z_n, z_{n+1}) = (\theta z_0, z_1, \ldots, z_n, -z_{n+1})$, where $\theta = e^{2\pi i/3}$. Furthermore, since odd dimensional spheres have a non-zero vector field, we have that $H_*(S(S^{2n-1}); \mathbb{Z}) \cong H_*(S^{2n-1} \times S^{2n-2}; \mathbb{Z})$.

Now let N_n be the fixed point set of the above mentioned action of $\mathbb{Z}_6$ on the Brieskorn variety M_n for the polynomial $z_0^3 + z_1^2 + \cdots z_{2n}^2 + z_{2n+1}^2 = 0$. As we mentioned earlier, the Brieskorn variety M_n is PL-homeomorphic to S^{4n+1}, while by the previous paragraph, N_n is diffeomorphic to $S(S^{2n-1})$. Suspending the spaces and hyperbolizing gives us a $\mathbb{Z}_6$ action on a $CAT(-1)$ space X, where now the fixed subset Y is the hyperbolization of the suspension of $S(S^{2n-1})$.

The proof that $\pi_1(Y)$ is not Poincaré Duality over any PID R is almost a verbatim repetition of that given for Theorem 2.2. In particular, the local homology sheaf for the space Y will have three distinct indices (namely $s = 2n - 1, 2n, 4n - 2$) for which $h_s(p) \neq 0$ (where again, p is one of the suspension points). Working through the Zeeman spectral sequence, we again find indices ($< 4n - 2$) where the cohomology of $\pi_1(Y)$ is not finitely generated. The only substantial change is in the argument showing that the cohomological dimension of $\pi_1(Y)$ over R is $4n - 2$. To do this, we merely note that the hyperbolization map $Y \to \Sigma S(S^{2n-1})$ induces a surjection on integral homology, together with the fact that $H_{4n-2}(\Sigma S(S^{2n-1}); R) \cong R$.

3   Concluding remarks.

We finish our paper with a few remarks. Firstly, we note that the results we obtain are, in some sense, dealing with exceptional automorphisms of δ-hyperbolic groups. Indeed, Levitt & Lustig [12] have shown that, in a suitable sense, 'most' automorphisms of a δ-hyperbolic group have very simple fixed point sets for their induced actions on the boundary at infinity (in fact, their fixed point sets consist of a pair of points). Also, if we start with a torsion free group, then the group of inner automorphisms will also be torsion free, hence any automorphism of finite order in some sense 'lives' in the outer automorphism group, which tends to be small.

Secondly, we should point out that, in the counterexamples we constructed, the groups Γ all have boundary at infinity which is in fact *homeomorphic* to a sphere. This follows from the fact that the link of every vertex in the space $\tilde{X}$ is PL-homeomorphic to the standard sphere, so by a result of Davis & Januszkiewicz [7], the boundary at infinity of $\tilde{X}$ is homeomorphic to a sphere.

Thirdly, we can ask related questions in a somewhat more general setting. More precisely, given an arbitrary topological space Y, we can consider the question of what type of actions can be realized *algebraically* or *geometrically*. By a geometric action, we mean one that is induced by an isometry of a δ-hyperbolic space X whose boundary is homeomorphic to Y. By an algebraic action, we mean one that is induced by an automorphism of a δ-hyperbolic group Γ whose boundary is homeomorphic to Y. Note that, at the cost of changing the set of generators for the group Γ (as in the proof of Proposition 2.1), we can always view an algebraic action as a geometric one (given by an isometry of the Cayley graph).

The fact that this question is non-trivial, even in the more general setting, can be seen by considering the situation of a Menger manifold. It is well known that there are numerous δ-hyperbolic groups whose boundary at infinity are Menger manifolds. Now a result of Iwamoto [10] states that every closed subset of a Menger manifold can be realized as the fixed point set of an involution. On the other hand, if an involution can be realized algebraically via an involution σ of a group Γ, then the fixed point set on the boundary at infinity must coincide with the boundary at infinity of the subgroup Γ^{σ}. However, the latter set cannot have any cutpoints (see Bowditch [3] and Swarup [1]). This gives a necessary condition for a closed subset of a Menger manifold to be the fixed point set of an algebraically realizable involution. What are the sufficient conditions?

Finally, we mention that these examples give involutions of a δ-hyperbolic group Γ where the fixed point set of the induced involution on the boundary at infinity is not an ANR, although $\partial^{\infty}\Gamma \cong S^n$. One could ask whether the fixed point set could display other complicated behavior. For instance, does there exist an involution of a δ-hyperbolic group Γ, with fixed subgroup Λ, with the property that $\partial^{\infty}\Gamma \cong S^n$, $\partial^{\infty}\Lambda \cong S^{n-2}$, and the embedding $S^{n-2} \cong \partial^{\infty}\Lambda \hookrightarrow \partial^{\infty}\Gamma \cong S^n$ is a locally flat, non-trivial knot?

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