# SIMPLICIAL VOLUME OF CLOSED LOCALLY SYMMETRIC SPACES OF NON-COMPACT TYPE

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Abstract. We show that closed, locally symmetric spaces of non-compact type have positive simplicial volume. This gives a positive answer to a question that was first raised by Gromov in [Gr1].

## 1. Introduction

In his paper [Gr1], Gromov introduced the notion of the simplicial volume of a connected, closed, and orientable manifold $M$. This invariant is denoted by $||\,M\,|| \in [0,\infty)$ and measures how efficiently the fundamental class of $M$ may be represented using real cycles.

In the same paper, the question was raised as to whether the simplical volume of a closed locally symmetric space of non-compact type is positive (pg. 11 in [Gr1]). Gromov cites the work of Thurston ([Th]), Inoue-Yano ([IY]), and Savage ([Sa]) in which positivity is verified for compact manifolds that are real hyperbolic, negatively curved, and locally symmetric quotients of $SL(n,\mathbb{R})/SO(n,\mathbb{R})$, respectively. Moreover, Gromov ([Gr1]) shows that characteristic classes may be represented using bounded cohomology classes implying positivity for several additional classes of locally symmetric spaces. Since then, this question has been mentioned in a variety of different sources ([Gr2], [L], [Sa], [CF1]), and has become a well-known "folk conjecture." The purpose of this paper is to answer this conjecture in the affirmative. Namely, we obtain:

**Main Theorem:** If $M^n$ is a closed locally symmetric space of non-compact type, then $||M^n|| > 0$.

The approach we use is due to Thurston [Th] and bounds the simplicial volume $||M||$ from below by the proportion between the volume of $M$ and the maximal volume of suitably defined straightened top dimensional simplices in $M$. Technically, however, we are indebted to Besson, Courtois, and Gallot for their pioneering work around the use of the barycenter method in proving the rank one minimal entropy rigidity conjecture for locally symmetric spaces [BCG] and to Connell and Farb for their subsequent development of the technique in higher rank spaces (see [CF1] for an extensive survey).

The main contribution of this paper lies in the idea of using the barycenter method in order to define the straightened simplices that are central to Thurston's argument. Roughly speaking, the barycenter method homotopes a map with negatively curved target to a $C^1$ map (often called the "natural" map) by first mapping the source space to the space of measures on the boundary at infinity of the target and then to the target by taking the center of mass of the measures. The technique, as developed by Besson, Courtois, and Gallot, then proceeds with a rather intricate argument to give a pointwise upper bound for the Jacobian of the natural map. When studying rigidity questions, one typically wants to replace this pointwise upper estimate by the best uniform estimate and determine what happens in the equality case. In this way, Besson, Courtois, and Gallot solved the minimal entropy rigidity conjecture for rank one spaces. As a major first step towards solving the minimal entropy rigidity conjecture in higher rank, Connell and Farb extended the barycenter method to higher rank spaces [CF2]. Furthermore, they established a new geometric result about locally symmetric spaces of non-compact type in order to obtain an a priori uniform bound for the Jacobian of the natural map. Here, we use Connell and Farb's extension of the technique to higher rank spaces in order to define "barycentrically" straightened simplices. The uniform upper bound for the volume of top dimensional straightened simplices required in Thurston's argument will follow directly from the detailed analysis used to obtain the uniform Jacobian bound in [CF2].

We note that a similar "barycentric" straightening was used by Kleiner in [Kl] in order to compare various notions of dimension for length spaces with curvature bounded above. We would like to thank Dick Canary, Tom Farrell, and Ralf Spatzier for their interest in our work and many helpful discussions.

## 2. Thurston's lower bound for simplicial volume

In this section, we first collect together some definitions and results about simplicial volume. We then describe how Thurston reduces the problem of establishing positivity of simplicial volume to the more geometric problem of uniformly bounding the volumes of suitably defined straightened simplices.

2.1. **Simplicial volume.** We begin with the definition of the simplicial volume and the important proportionality principle.

**Definition.** *Let $M$ be a topological space, $C^0(\Delta^k, M)$ be the set of singular $k$-simplices, and let $c = \sum_{i=1}^{j} r_i \cdot f_i$ with each $r_i \in \mathbb{R}$ and $f_i \in C^0(\Delta^k, M)$ be a singular real chain. The $l^1$ norm of $c$ is defined by $||c||_1 = \sum_i |r_i|$. The $l^1$ (pseudo-)norm of a real singular homology class $[\alpha] \in H_k^{sing}(M, \mathbb{R})$ is defined by $||\,[\alpha]\,||_1 = inf\{||c||_1 \,|\, \partial(c) = 0\,,\, [c] = [\alpha]\}$.*

**Definition.** *Let $M^n$ be an oriented closed connected $n$-manifold with fundamental class $[M^n]$. The simplicial volume of $M^n$ is defined as $||M^n|| = ||i([M^n])||_1$, where*

*$i : H_n(M,\mathbb{Z}) \to H_n(M,\mathbb{R})$ is the change of coefficients homomorphism, and $[M^n]$ is the fundamental class arising from the orientation of $M^n$.*

The proportionality principle ([Gr1],[Th],[St]) for simplicial volume is expressed in the following:

**Theorem 2.1.** *Let $M$ and $M'$ be two closed Riemannian manifolds with isometric universal covers. Then*

$$\frac{||M||}{Vol(M)} = \frac{||M'||}{Vol(M')}.$$

In addition, the simplicial volume is particularly well behaved with respect to products. Namely, the following holds:

**Theorem 2.2.** *For a pair of closed manifolds $M_1$, $M_2$, we have that:*

$$C \cdot ||M_1|| \cdot ||M_2|| \geq ||M_1 \times M_2|| \geq ||M_1|| \cdot ||M_2||$$

*where $C > 1$ is a constant that depends only on the dimension of $M_1 \times M_2$.*

**Theorem 2.3.** *For $n \geq 3$, the connected sums of a pair of $n$-dimensional manifolds $M_1$ and $M_2$ satisfy:*

$$||M_1 \sharp M_2|| = ||M_1|| + ||M_2||$$

The proofs of Theorem 2.2 and Theorem 2.3 may be found in [BP] and in [Gr1].

2.2. **Thurston's approach.** Our proof of the Main Theorem relies on Thurston's observation that the ratio of the Riemannian volume of a manifold $M^n$ to the maximal volume of a straightened top dimensional singular simplex in $M^n$ gives a lower bound for the simplicial volume (see [Th]). We summarize his approach in the following:

**Definition.** *Let us denote by $\tilde{M}^n$ the universal cover of $M^n$, by $\Gamma$ the fundamental group of $M^n$, and by $C^0(\Delta^k, \tilde{M}^n)$ the set of singular $k$-simplices in $\tilde{M}^n$, where $\Delta^k$ is assumed to be equipped with a fixed Riemmanian metric. Assume there is a collection of maps $st_k : C^0(\Delta^k, \tilde{M}^n) \to C^0(\Delta^k, \tilde{M}^n)$. We will say this collection of maps is a straightening provided it satisfies the following four formal properties:*

1. *the maps $st_k$ are $\Gamma$-equivariant,*
2. *the maps $st_*$ induce a chain map $st_* : C_*^{sing}(\tilde{M}^n, \mathbb{R}) \to C_*^{sing}(\tilde{M}^n, \mathbb{R})$ which is $\Gamma$-equivariantly chain homotopic to the identity,*
3. *the image of $st_n$ lies in $C^1(\Delta^n, \tilde{M}^n)$, i.e. straightened top-dimensional simplices are $C^1$,*
4. *there exists a constant $C > 0$, depending solely on $\tilde{M}^n$ and the chosen Riemannian metric on $\Delta^n$, such that for any $f \in C^0(\Delta^n, \tilde{M}^n)$, and corresponding straightened simplex $st_n(f) : \Delta^n \to \tilde{M}^n$, there is a uniform upper bound on the Jacobian of $st_n(f)$:*

$$|Jac(st_n(f))(\delta)| \leq C$$

*where $\delta \in \Delta^n$ is arbitrary, and the Jacobian is computed relative to the fixed Riemannian metric on $\Delta^n$.*

Thurston established:

**Theorem 2.4.** *If $\tilde{M}^n$ supports a straightening, then $||M^n|| > 0$.*

We now recall how Theorem 2.4 is proven. Assume that $\tilde{M}^n$ supports a straightening and note that property (1) implies that the straightening maps descend to straightening maps on the compact quotient $M^n$. Property (2) ensures that the homology of $M^n$ obtained via the complex of straightened chains coincides with the ordinary singular homology of $M^n$. Furthermore, since the straightening procedure is a projection operator on the level of chains, it is contracting in the $l^1$-norm. In particular, if $\sum r_i f_i$ is a real $n$-chain representing the fundamental class of $M^n$, then so is $\sum r_i st(f_i)$, and we have the inequality $||\sum r_i f_i||_1 \geq ||\sum r_i st(f_i)||_1$.

As a consequence, in order to show that the simplicial volume of $M^n$ is positive, it is sufficient to give a lower bound for the $l^1$-norm of a straightened chain representing the fundamental class. But now observe that, by property (3), the straightened chain is $C^1$, and hence we can compute the volume of $M^n$ by:

$$Vol(M^n) = \int_{\sum r_i st(f_i)} dV_{M^n} = \sum r_i \int_{st(f_i)} dV_{M^n},$$

where $dV_{M^n}$ is the volume form on $M^n$. On the other hand, we have the bound:

$$\sum r_i \int_{st(f_i)} dV_{M^n} \leq \sum |r_i| \int_{\Delta^n} |Jac(st(f_i))| dV_{\Delta^n},$$

where $dV_{\Delta^n}$ is the volume form for the fixed Riemannian metric on $\Delta^n$. Now by property (4) the Jacobian of straightened simplices is bounded uniformly from above, and hence we have a uniform upper bound:

$$\int_{\Delta^n} |Jac(st(f_i))| dV_{\Delta^n} \leq K$$

where $K > 0$ depends only on $M^n$ and the chosen metric on $\Delta^n$. This yields the inequality:

$$Vol(M^n) \leq K \cdot \sum |r_i|$$

which upon dividing, and passing to the infimum over all straightened chains, provides the positive lower bound $||M^n|| \geq Vol(M^n)/K > 0$.

We note that to prove Theorem 2.4, one could replace properties (3) and (4) of a straightening by the more general condition that the volume of the images of straightned top-dimensional simplices are uniformly bounded above. In fact, this more general approach was taken in the aforementioned works [Th], [IY], and [Sa]. Our proof of the main theorem involves a new straightening procedure for locally symmetric spaces of non-compact type which do not have any local $\mathbb{H}^2$ or $SL_3(\mathbb{R})/SO_3(\mathbb{R})$

direct factors. Our formulation of Theorem 2.3 isolates properties (3) and (4) because the barycenter method we will use is particularly well adapted to establishing these properties.

## 3. Straightening simplices

In this section we introduce a new *straightening procedure* (as defined in Section 2.2) that works for locally symmetric spaces of non-compact type with no local direct factors locally isometric to $\mathbb{H}^2$ or $SL(3,\mathbb{R})/SO(3,\mathbb{R})$. The straightening will be defined using the barycenter method of Besson, Courtois, and Gallot as developed in the higher rank setting by Connell and Farb. As such, we will begin with a review of Connell and Farb's work that we use. We conclude the section by defining the straightening and verifying the four formal requirements.

For more details on locally symmetric spaces of non-compact type, we refer the reader to the text [E]. Throughout, $(M^n, g_0)$ denotes a closed locally symmetric space of non-compact type. We let $(X, g)$ be the symmetric universal covering space of $M$ and fix a basepoint $p \in X$.

3.1. **The barycenter method in higher rank.** We start with the following conventions:

- Let $\Gamma = \pi_1(M^n)$, $G = \mathrm{Isom}(X)^0$, and $K = \mathrm{Stab}_p(G)$. The basepoint $p$ determines a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of $G$. Here, $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{p}$ is identified with the tangent space $T_pX$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a fixed maximal abelian subalgebra of $\mathfrak{g}$. Fix a regular vector $A \in \mathfrak{a}$ and let $b$ denote the *barycenter* of the Weyl chamber $W(A)$ determined by $A$ (see [CF2] pg. 9 for more details).
- The visual boundary and Furstenberg boundaries are denoted by $\partial X$ and by $\partial_F X \cong G/P$, where $P$ is a minimal parabolic subgroup of $G$. We will view the Furstenberg boundary as a subset of the visual boundary by identifying it with the $G$ orbit of the boundary point determined by the barycenter vector $b$.
- The $||b||$-conformal density associated to the group $\Gamma$ given by the family of Patterson-Sullivan measures is denoted by $\nu : X \to \mathcal{M}(\partial X)$, where $\mathcal{M}(\partial X)$ denotes the space of atomless probability measures on the visual boundary of $X$. We use Albuquerque's theory of these measures in higher rank ([Al]) and recall that they are $\Gamma$-equivariant and fully supported on $\partial_F X \subset \partial X$.
- $B : X \times X \times \partial X \to \mathbb{R}$ denotes the Busemann function on $X$. Specifically,

$$B(x, y, \theta) = \lim_{t\to\infty} d_X(y, l_\theta(t)) - t$$

  where $l_\theta$ is the unique geodesic ray starting at $x$ with endpoint $\theta$. It follows that the Busemann function is invariant under the diagonal action of $\Gamma$ on $X \times X \times \partial X$.

- We will denote by:
$$dB_{(x,\theta)}(\cdot) : T_xX \to \mathbb{R}$$
$$DdB_{(x,\theta)}(\cdot,\cdot) : T_xX \otimes T_xX \to \mathbb{R}$$
the 1-form and 2-form obtained by differentiating the function $B(p,\cdot,\theta)$ at the point $x \in X$.
- For a measure $\mu \in \mathcal{M}(\partial X)$, let
$$g_\mu(\cdot) = \int_{\partial X} B(p,\cdot,\theta)\, d(\mu)(\theta).$$
When $g_\mu : X \to \mathbb{R}$ has a unique minimum, the *barycenter* of $\mu$, denoted by $bar(\mu) \in X$, is defined to be the unique point of $X$ where $g_\mu$ is minimized. Though the chosen basepoint $p \in X$ is used to define the function $g_\mu$, it follows easily from the properties of the Busemann function that $bar(\mu)$ (when defined) is actually independent of this choice.

Next we recall the results from [CF2] that play a central role in our straightening. As $X$ is non-positively curved, the functions $B(p,\cdot,\theta)$ appearing in the above definition of $g_\mu$ are known to be convex functions on $X$. When the measure $\mu \in \mathcal{M}(\partial X)$ is fully supported on $\partial_F X$, Connell and Farb's proof of Proposition 3.1 in [CF2] shows that $g_\mu$ is strictly convex. Furthermore, they note that when $\mu$ is a Patterson-Sullivan measure, the function $g_\mu$ is proper. We summarize this in the following:

**Lemma 3.1.** *Let $\mu \in \mathcal{M}(\partial X)$ be a finite weighted sum of Patterson-Sullivan measures on $X$. Then $bar(\mu)$ is well defined.*

The following result of Connell and Farb from Section 4 in [CF2] will be used to verify properties (3) and (4) for our barycentric straightening:

**Theorem 3.2.** *Let $M$ be a closed locally symmetric space of non-compact type with no local direct factors locally isometric to $\mathbb{H}^2$ or $SL_3(\mathbb{R})/SO_3(\mathbb{R})$, and let $X$ be its universal cover. Let $\mu \in \mathcal{M}(\partial X)$ be a probability measure fully supported on $\partial_F X$ and let $x \in X$. Consider the endomorphisms $K_x(\mu)$, $H_x(\mu)$, defined on $T_xX$ by:*
$$\langle K_x(\mu)u, u\rangle = \int_{\partial_F X} DdB_{(x,\theta)}(u,u)\, d(\mu)(\theta)$$
*and*
$$\langle H_x(\mu)u, u\rangle = \int_{\partial_F X} dB^2_{(x,\theta)}(u)\, d(\mu)(\theta).$$
*Then $\det(K_x(\mu)) > 0$ and there is a positive constant $C := C(X) > 0$ depending only on $X$ such that:*
$$J_x(\mu) := \frac{\det(H_x(\mu))^{1/2}}{\det(K_x(\mu))} \leq C.$$
*Furthermore, the constant $C$ is explicitly computable.*

3.2. **Barycentric straightening.** In this section we always assume that $M^n$ has no local direct factors locally isometric to $\mathbb{H}^2$ or $SL(3,\mathbb{R})/SO(3,\mathbb{R})$.

We first fix some notation. Recall that a singular $k$-simplex in $X$ is a continuous map $f : \Delta^k \to X$, where $\Delta^k$ is the standard Euclidean $k$-simplex realized as the convex hull of the standard unit basis vectors in $\mathbb{R}^{k+1}$. For our purpose it is more convenient to work with the spherical $k$-simplex $\Delta^k_s = \{(a_1, \ldots, a_{k+1}) |\ a_i \geq 0, \sum_{i=1}^{k+1} a_i^2 = 1\} \subset \mathbb{R}^{k+1}$, equipped with the Riemannian metric induced from $\mathbb{R}^{k+1}$. We will denote by $e_i$ $(1 \leq i \leq k+1)$ the standard basis vectors for $\mathbb{R}^{k+1}$. We associate to each singular simplex $f : \Delta^k_s \to X$ its ordered vertex set $V := \{f(e_1), \ldots, f(e_{k+1})\}$. We now define the straightening procedure we will use.

Given an ordered set $V = \{x_1, \ldots, x_{k+1}\} \subset X$, define the map

$$\widehat{V} : \Delta^k_s \to \mathcal{M}(\partial X)$$

by $\widehat{V}(\sum_{i=1}^{k+1} a_i e_i) = \sum_{i=1}^{k+1} a_i^2 \nu(x_i)$.

**Definition.** *Given a singular $k$-simplex $f \in C^0(\Delta^k_s, X)$, with vertex set $V = \{x_1, \ldots, x_{k+1}\}$, define $st_k(f) \in C^0(\Delta^k_s, X)$ by $st_k(f)(\delta) = bar(\widehat{V}(\delta))$ for $\delta \in \Delta^k_s$.*

The fact that this definition is well defined is the content of Lemma 3.1. Moreover, observe that $st_k(f)$ depends only on the vertex set $V$ of the original simplex $f$. We will therefore use the notation $st_V(\delta) := st_k(f)(\delta)$ when convenient. We now proceed to verify that this straightening procedure satisfies the four formal properties needed. For the convenience of the reader, we restate each property prior to proving it.

**Property (1):** The maps $st_k$ are $\Gamma$-equivariant.

*Proof.* Fix a point $\delta = \sum_i a_i e_i \in \Delta^k_s$. Then for any $\gamma \in \Gamma$, $st_{\gamma V}(\delta)$ is defined as the unique minimizer of the function $g(\cdot) = \int_{\partial_F X} B(p, \cdot, \theta) d(\sum_i a_i^2 \nu(\gamma x_i))(\theta)$. Since

$$\int_{\partial_F X} B(p, \cdot, \theta) d(\sum_i a_i^2 \nu(\gamma x_i))(\theta) =$$

$$\int_{\partial_F X} B(p, \cdot, \theta) d(\sum_i a_i^2 \gamma_* \nu(x_i))(\theta) =$$

$$\int_{\partial_F X} B(p, \cdot, \gamma\theta) d(\sum_i a_i^2 \nu(x_i))(\theta) =$$

$$\int_{\partial_F X} B(\gamma\gamma^{-1} p, \gamma\gamma^{-1} \cdot, \gamma\theta) d(\sum_i a_i^2 \nu(x_i))(\theta) =$$

$$\int_{\partial_F X} B(\gamma^{-1} p, \gamma^{-1} \cdot, \theta) d(\sum_i a_i^2 \nu(x_i))(\theta),$$

and since $B(\gamma^{-1}p,\cdot,\cdot)$ and $B(p,\cdot,\cdot)$ differ by a function $k(\theta)$ of $\theta$, it follows that the unique minimizer of $g(\cdot)$ is also the unique minimizer of the function:

$$h(\cdot)=\int_{\partial_F X} B(p,\gamma^{-1}\cdot,\theta)d(\sum_i a_i^2\nu(x_i))(\theta).$$

Indeed, we have that the difference of the two functions is:

$$g(\cdot)-h(\cdot)=\int_{\partial_F X} k(\theta)d(\sum_i a_i^2\nu(x_i))(\theta)$$

which is a constant function on $X$. But now $g$ is minimized at $st_{\gamma V}(\delta)$, while $h$ is minimized at $\gamma\cdot st_V(\delta)$. This gives us that $st_{\gamma V}(\delta)=\gamma\cdot st_V(\delta)$, completing the proof of Property (1).

**Property (2):** The maps $st_*$ induce a chain map $st_*: C_*^{sing}(X,\mathbb{R})\to C_*^{sing}(X,\mathbb{R})$ which is $\Gamma$-equivariantly chain homotopic to the identity.

*Proof.* The fact that $st_k$ commutes with the boundary operator follows from the fact that $st_k(f)$ depends solely on the vertices of the singular simplex $f$, along with the fact that $st_k(f)$ restricted to a face of $\Delta_s^k$ coincides with the straightening of that face.

To see that $st$ is chain homotopic to the identity, first note that the uniqueness of geodesics in $X$ gives rise to a well defined $\Gamma$-equivariant straight line homotopy between any simplex $f$ and its straightening $st(f)$. Hence there are canonically defined homotopies between simplices and their straightenings in $X$. Moreover, these homotopies when restricted to lower dimensional faces agree with the homotopies canonically defined on those faces. Appropriately ($\Gamma$-equivariantly) subdividing these homotopies defines the required chain homotopy, concluding the proof of Property (2).

**Property (3):** The image of $st_n$ lies in $C^1(\Delta_s^n,X)$, i.e. straightened top-dimensional simplices are $C^1$.

*Proof.* Notice that for any simplex $f\in C^0(\Delta_s^n,X)$ and any $\delta=\sum_i a_ie_i\in\Delta_s^n$, we have an implicit characterisation of the point $st_n(f)(\delta)=st_V(\delta)$ via the 1-form equation:

$$0\equiv d(g_{\widehat{V}(\delta)})_{st_V(\delta)}(\cdot)=\int_{\partial_F X} dB_{(st_V(\delta),\theta)}(\cdot)\, d(\sum_i a_i^2\nu(x_i))(\theta). \tag{1}$$

Indeed, $st_V(\delta)=bar(\sum_i a_i^2\nu(x_i))$ is defined as the unique minimum of the function

$$g_{\widehat{V}(\delta)}(\cdot)=\int_{\partial_F X} B(p,\cdot,\theta)\, d(\sum_i a_i^2\nu(x_i))(\theta),$$

yielding Equation (1) upon differentiation.

Following [BCG], we choose a frame $(e_i(x))_{i=1,\ldots,n}$ of $T_xX$ with smooth dependance on $x \in X$. Define the map $G = (G_1, \ldots, G_n) : \Delta_s^n \times X \to \mathbb{R}^n$ by

$$G_i(\delta, x) = \int_{\partial_F X} dB_{(x,\theta)}(e_i(x))\, d(\widehat{V}(\delta))(\theta).$$

Equation (1) implies that $G(\delta, st_V(\delta)) = 0$. As the Busemann functions $B(p, \cdot, \theta)$ are smooth and since $\partial_F X$ is compact, it follows that $G$ is a smooth map. To now apply the inverse function theorem, the non-degeneracy of the partial derivative of $G$ with respect to the variable $x$ must be checked. This requires that for the endomorphism $K$ defined by

$$\langle K(u), u\rangle := \int_{\partial_F X} DdB_{(x,\theta)}(u, u)\, d(\widehat{V}(\delta))(\theta),$$

the determinant be non-zero. Note however that in the notation of Theorem 3.2, the determinant of this matrix is precisely $\det(K_x(\widehat{V}(\delta)))$, and hence must be non-zero as the measure $\widehat{V}(\delta) = \sum_i a_i^2 \nu(x_i)$ has full support on the Furstenberg boundary. This completes the proof of property (3).

**Property (4):** There exists a constant $C > 0$, depending on $X$, such that for any $f \in C^0(\Delta_s^n, X)$, and corresponding straightened simplex $st_n(f) : \Delta_s^n \to X$, there is a uniform upper bound on the Jacobian of $st_n(f)$:

$$|Jac(st_n(f))(\delta)| \leq C$$

where $\delta = \sum_i a_i e_i \in \Delta_s^n$ is arbitrary, and the Jacobian is computed relative to the Riemannian metric on the spherical simplex $\Delta_s^n$ induced from $\mathbb{R}^{n+1}$.

*Proof.* Differentiating equation (1) with respect to directions in $T_\delta(\Delta_s^n)$, one obtains the equation

$$\begin{aligned} 0 \equiv D_\delta d(g_{\widehat{V}(\delta)})_{st_V(\delta)}(\cdot, \cdot) &= \sum_i 2a_i \langle \cdot, e_i\rangle_\delta \int_{\partial_F X} dB_{(st_V(\delta),\theta)}(\cdot)\, d(\nu(x_i))(\theta) \\ &\quad + \int_{\partial_F X} DdB_{(st_V(\delta),\theta)}(D(st_V)_\delta(\cdot), \cdot)\, d(\sum_i a_i^2 \nu(x_i))(\theta). \end{aligned} \tag{2}$$

defined on $T_\delta(\Delta_s^n) \otimes T_{st_V(\delta)}(X)$. Now define symmetric endomorphisms $H_\delta$ and $K_\delta$ of $T_{st_V(\delta)}(X)$ by

$$\langle H_\delta(u), u\rangle_{st_V(\delta)} = \int_{\partial_F X} dB^2_{(st_V(\delta),\theta)}(u)\, d(\sum_i a_i^2 \nu(x_i))(\theta)\text{ , and}$$

$$\langle K_\delta(u), u\rangle_{st_V(\delta)} = \int_{\partial_F X} DdB_{(st_V(\delta),\theta)}(u, u)\, d(\sum_i a_i^2 \nu(x_i))(\theta).$$

Let $\{v_j\}_{j=1}^n$ be an orthonormal eigenbasis of $T_{st_V(\delta)}(X)$ for $H_\delta$. At points $\delta \in \Delta_s^n$ where the Jacobian of $st_V$ is nonzero, let $\{\tilde{u}_j\}$ be the basis of $T_\delta(\Delta_s^n)$ obtained by pulling back the $\{v_j\}$ basis by $K_\delta \circ D(st_V)_\delta$, and $\{u_j\}$ be the orthonormal basis of $T_\delta(\Delta_s^n)$ obtained from the $\{\tilde{u}_j\}$ basis by applying the Gram-Schmidt algorithm. We now have the sequence of equations (which we will justify in the next paragraph):

$$\det(K_\delta) \cdot |Jac(st_V)(\delta)| = |\det(K_\delta \circ D(st_V)_\delta)| \tag{3}$$

$$= \prod_{j=1}^n |\langle K_\delta \circ D(st_V)_\delta(u_j), v_j\rangle_{st_V(\delta)}| \tag{4}$$

$$= \prod_{j=1}^n \Big| \sum_{i=1}^{n+1} \langle u_j, e_i\rangle_\delta \cdot 2a_i \int_{\partial_F X} dB_{(st_V(\delta),\theta)}(v_j)\, d(\nu(x_i))(\theta) \Big| \tag{5}$$

$$\leq \prod_{j=1}^n \Big[ \sum_{i=1}^{n+1} \langle u_j, e_i\rangle_\delta^2 \Big]^{1/2} \Big[ \sum_{i=1}^{n+1} 4a_i^2 \Big( \int_{\partial_F X} dB_{(st_V(\delta),\theta)}(v_j)\, d(\nu(x_i))(\theta) \Big)^2 \Big]^{1/2} \tag{6}$$

$$\leq 2^n \prod_{j=1}^n \Big[ \sum_{i=1}^{n+1} a_i^2 \int_{\partial_F X} dB^2_{(st_V(\delta),\theta)}(v_j)\, d(\nu(x_i))(\theta) \Big]^{1/2} \tag{7}$$

$$= 2^n \prod_{j=1}^n \langle H_\delta(v_j), v_j\rangle_{st_V(\delta)}^{1/2} = 2^n \det(H_\delta)^{1/2}, \tag{8}$$

We now justify each step in the previous list of equations. Equation (3) follows from the definition of the Jacobian, along with the fact that $\det(AB) = \det(A) \cdot \det(B)$. Equation (4) follows from the fact that, with respect to the $\{u_j\}$ and $\{v_j\}$ bases, $K_\delta \circ D(st_V)_\delta$ is upper triangular, and hence the determinant is the product of the diagonal entries. Equation (5) follows from equations (4) and (2). Inequalities (6) and (7) follow from the Cauchy-Schwartz inequality applied in $\mathbb{R}^{n+1}$ and the spaces $L^2(\partial_F X, \nu(x_i))$, respectively, along with the fact that the $u_j$ are unit vectors in $T_\delta(\Delta_s^n) \subset T_\delta(\mathbb{R}^{n+1})$. The two equalities in (8) follow from the definition of $H_\delta$, and the fact that the $\{v_j\}_{j=1}^n$ is an orthonormal eigenbasis for $H_\delta$.

Upon dividing, we now obtain the inequality:

$$|Jac(st_V)(\delta)| \leq 2^n \frac{\det(H_\delta)^{1/2}}{\det(K_\delta)}$$

But now note that, in the notation of Theorem 3.2, the expression $\det(H_\delta)^{1/2}/\det(K_\delta)$ is exactly $J_{st_V(\delta)}(\sum a_i^2 \nu(x_i))$. Since the measure $\sum a_i^2 \nu(x_i)$ has full support in the

Furstenberg boundary, Theorem 3.2 now yields a uniform constant $C'$, depending solely on $X$, with the property that:

$$|Jac(st_V)(\delta)| \leq 2^n J_{st_V(\delta)}(\sum a_i^2 \nu(x_i)) \leq 2^n C' =: C$$

This completes the proof of Property (4).

## 4. Proof of the Main Theorem

We now prove the **Main Theorem**. In view of the barycentric straightening construction in section 3.2 and Thurston's Theorem 2.4, we've established the following:

**Theorem 4.1.** *Let $M^n$ be a closed locally symmetric space of non-compact type with no local direct factors locally isometric to $\mathbb{H}^2$ or $SL(3,\mathbb{R})/SO(3,\mathbb{R})$. Then $||M^n|| > 0$.*

**Theorem 4.2** (**Main Theorem**). *If $M^n$ is a closed locally symmetric space of non-compact type, then $||M^n|| > 0$.*

*Proof.* Observe that by the proportionality principle in Theorem 2.1, in order to show that $||M|| > 0$, it is sufficient to show that $||M'|| > 0$ for some locally symmetric space of non-compact type whose universal cover is $X$. Let $G$ denote the identity component of $\mathrm{Isom}(X)$ and $G = G_1 \times \cdots \times G_k$ be the product decomposition of $G$ into simple Lie groups corresponding to the product decomposition of $X$ into irreducible symmetric spaces. By a result of Borel [Bo], there are cocompact lattices $\Gamma_i \subset G_i$ for each $i \in \{1, \ldots, k\}$. Take $M'$ to be the product locally symmetric space $M_1 \times \cdots \times M_k$ obtained from the product lattice $\Gamma_1 \times \cdots \times \Gamma_k$. From Theorem 2.2, the inequality $||M_1 \times \cdots \times M_k|| \geq \prod_{i=1}^k ||M_i||$ holds. Hence, if one has the Main Theorem for *irreducible* locally symmetric spaces of non-compact type, one obtains the Main Theorem for *all* locally symmetric spaces of non-compact type.

Next we observe that for closed real hyperbolic surfaces, positivity of the simplicial volume follows from [Th]. Furthermore, for closed irreducible locally symmetric spaces with symmetric covering $SL_3(\mathbb{R})/SO_3(\mathbb{R})$, positivity of the simplicial volume follows from [Sa]. Therefore, in view of Theorem 4.1, positivity holds for *all* irreducible locally symmetric spaces of non-compact type, concluding the proof.

## 5. Concluding remarks

Let $\mathcal{M}$ be the smallest class of manifolds that (1) contains all closed locally symmetric spaces of non-compact type, (2) is closed under connected sums with arbitrary closed manifolds of dimension $\geq 3$, (3) is closed under products, and (4) is closed under fiber extensions by surfaces of genus $\geq 2$ (i.e. if $M \in \mathcal{M}$, and $M'$ fibers over $M$ with fiber a surface $S_g$ of genus $\geq 2$, then $M' \in \mathcal{M}$).

In view of our Main Theorem, Theorems 2.2 and 2.3, along with a result of Hoster and Kotschick [HK], we obtain the following:

**Corollary 5.1.** *For every manifold $M \in \mathcal{M}$, $||M|| > 0$.*

The *minimal entropy* of a smooth manifold $M$ is defined to be the infimum of the topological entropies of the geodesic flow over all complete Riemannian metrics of unit volume on $M$. There is the following inequality between simplicial volume and the minimal entropy $h$ (see pg. 37 in [Gr1]):

$$C \cdot ||M|| \leq h(M)^n$$

where $C$ is a uniform constant, depending only on the dimension $n$ of $M$. We therefore obtain the following:

**Corollary 5.2.** *Every manifold $M \in \mathcal{M}$ has positive minimal entropy.*

We say that $M$ *collapses* provided that there exists a sequence of Riemannian metrics $g_i$ on $M$, satisfying $|\mathrm{K}(g_i)| \leq 1$, and having the property that at every point $p \in M$, the injectivity radius with respect to the metric $g_i$ is $< 1/i$. Gromov showed that manifolds with positive simplicial volume do *not* collapse (pgs. 67-68 in [Gr1]), giving the following:

**Corollary 5.3.** *Manifolds $M \in \mathcal{M}$ do not collapse.*

Bounded cohomology $\hat{H}^*(M^n)$ is defined in Gromov [Gr1], where it is shown (pgs. 16-17) that $M^n$ has positive simplicial volume if and only if the map induced by inclusion of chains $i^n : \hat{H}^n(M^n) \to H^n_{sing}(M^n, \mathbb{R})$ is non-zero. This gives the following:

**Corollary 5.4.** *Each manifold $M^n \in \mathcal{M}$ has non-vanishing n-dimensional bounded cohomology.*

We conclude by pointing out some questions.

**Conjecture:** Let $M^n$ be a closed Riemannian manifold, whose sectional curvatures are $\leq 0$, and whose Ricci curvatures are $< 0$. Then $||M^n|| > 0$.

This conjecture was attributed to Gromov in [Sa]. It seems plausible that a similar approach could be used to verify this conjecture. The main difficulty lies in obtaining formal property (4) for the analogous straightening procedure when the space $M^n$ is locally irreducible and is *not* a locally symmetric space. We can also ask the:

**Question:** For a given closed locally symmetric space of non-compact type $M^n$, can one estimate or compute the value of the proportionality constant $||M^n||/\mathrm{Vol}(M^n)$ in terms of the symmetric covering of $M^n$?

One application of positivity of simplicial volume is the non-vanishing of the top-dimensional bounded cohomology. We have the natural:

**Question:** What is the dimension of $\hat{H}^n(M^n)$ for a closed locally symmetric quotient of an irreducible higher rank locally symmetric space of non-compact type? In particular, is it finite dimensional or infinite dimensional?

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