Journal of Topology and Analysis
Vol. 11, No. 1 (2019) 233–247

DOI: 10.1142/S1793525319500109

Hyperbolic groups with boundary an n-dimensional Sierpinski space

Jean-François Lafont

Department of Mathematics
Ohio State University, Columbus, OH 43210, USA
jlafont@math.ohio-state.edu

Bena Tshishiku*

Department of Mathematics
University of Chicago, Chicago, IL 60615, USA
tshishikub@gmail.com

Received 9 January 2017
Accepted 23 July 2017
Published 24 August 2017

For $n \geq 7$, we show that if G is a torsion-free hyperbolic group whose visual boundary $\partial_\infty G \simeq \mathscr{S}^{n-2}$ is an $(n-2)$-dimensional Sierpinski space, then $G = \pi_1(W)$ for some aspherical n-manifold W with non-empty boundary. Concerning the converse, we construct, for each $n \geq 4$, examples of aspherical manifolds with boundary, whose fundamental group G is hyperbolic, but with visual boundary $\partial_\infty G$ *not* homeomorphic to $\mathscr{S}^{n-2}$. Our examples even support (metric) negative curvature, and have totally geodesic boundary.

Keywords: Geometric group theory; hyperbolic groups; Sierpinski curves; aspherical manifolds; surgery theory.

AMS Subject Classification: Primary: 20F65, 20F67; Secondary: 57P10, 57R67

1. Introduction

One of the basic invariants for a hyperbolic group is its boundary at infinity, and a fundamental question is to determine what properties of the group are captured by the topology of the boundary at infinity. For example, the famous *Cannon conjecture* postulates that a hyperbolic group whose boundary at infinity is the 2-sphere S^2 must admit a properly discontinuous, isometric, cocompact action on hyperbolic 3-space $\mathbb{H}^3$.

In [20], Kapovich and Kleiner study groups whose boundary at infinity is a Sierpinski carpet — a boundary version of the Cannon conjecture. In [4], Bartels,

*Corresponding author

Lück, and Weinberger study groups whose boundary at infinity is a sphere S^n of dimension $n \geq 5$ — a high-dimensional version of the Cannon conjecture. In this paper, we consider groups whose boundary at infinity are high-dimensional Sierpinski spaces, thus lying somewhere between the work of Kapovich–Kleiner and that of Bartels–Lück–Weinberger.

The two main theorems are as follows. Let $\mathscr{S}^{n-2}$ denote an $(n-2)$-dimensional *Sierpinski space*. See Sec. 2 for the definition.

Theorem 1. *Fix $n \geq 7$ and let G be a torsion-free hyperbolic group. If the visual boundary $\partial_\infty G$ is homeomorphic to $\mathscr{S}^{n-2}$, then there exists an n-dimensional compact aspherical topological manifold W with non-empty boundary such that $\pi_1(W) \cong G$. Furthermore, W is unique up to homeomorphism.*

Note that the fundamental group π of a closed aspherical manifold M is an example of a Poincaré duality group. Whether or not all finitely presented Poincaré duality groups arise in this fashion is an open problem that goes back to Wall [28]. So the existence portion of Theorem 1 addresses a relative version of Wall realization problem for a special class of groups. On the other hand, the uniqueness portion of Theorem 1 verifies the Borel conjecture for this same class of groups.

Our second result shows that the converse of Theorem 1 is false even if one imposes additional strong constraints on the geometry of the aspherical manifold.

Theorem 2. *For each $n \geq 4$, there exists a compact aspherical manifold M^n with non-empty connected boundary $\partial M^n = N^{n-1}$ such that*:

(1) *$G = \pi_1(M)$ is hyperbolic, and $H = \pi_1(N)$ is a proper quasi-convex subgroup in G.*
(2) *$\partial_\infty(\pi_1(N))$ is homeomorphic to S^{n-2}, but*
(3) *$\partial_\infty G \cong \partial_\infty \widetilde{M}$ is* not *homeomorphic to $\mathscr{S}^{n-2}$.*

Moreover, when $n \geq 5$, the manifold M^n supports a locally CAT(-1) *metric with totally geodesic boundary.*

Remark 3. If one just wants a simple counterexample to the converse of Theorem 1, one can proceed as follows: start with a k-dimensional closed hyperbolic manifold K with fundamental group G, where $k < n$. Now embed the hyperbolic k-plane $\mathbb{H}^k$ isometrically inside $\mathbb{H}^n$. Then the G-action on the embedded $\mathbb{H}^k$ extends to an action on the r-neighborhood X of $\mathbb{H}^k$. Let $M = X/G$, and note that M is aspherical, diffeomorphic to $K \times \mathbb{D}^{n-k}$, with fundamental group G. Clearly $\partial_\infty G$ is homeomorphic to the $(k-1)$-sphere S^{k-1}, and not to Sierpinski $(n-2)$-space $\mathscr{S}^{n-2}$. Of course, in this example, $N = K \times S^{n-k-1}$, so the example fails to have property (1) from Theorem 2. Note that (1) is the group-theoretic analogue of a negatively curved manifold with totally geodesic boundary.

Another simple example is to take three copies of the torus with one boundary component and define X as the result of gluing the three boundaries together by homeomorphisms. A thickening M of X in $\mathbb{R}^3$ satisfies the conditions of Theorem 2, except that ∂M is not connected. In this case $\partial_\infty\big(\pi_1(M)\big)$ has local cut points, so it cannot be $\mathscr{S}^1$. It seems likely, and would be interesting to show, that $\partial_\infty G$ has no local cut points for the G constructed in Theorem 2.

Remark 4. In Theorem 2 one can construct, in dimensions $n \geq 5$, manifolds satisfying property (1), but failing to have (2). Start with a Davis–Januszkiewicz example of a locally CAT(-1) closed $(n-1)$-manifold N with $\partial_\infty \widetilde{N}$ not homeomorphic to S^{n-2}, chosen so that $N = \partial W^{n+1}$ for some compact manifold W^{n+1}. Then take M to be the relative hyperbolization of W, relative to N (see [16]). Properties of relative hyperbolization readily yield statement (1), while the choice of N ensures that (2) fails. It seems likely that such manifolds M would also have property (3). Indeed, one could visualize the boundary at infinity of $\widetilde{M}$ to be similar to a Sierpinski curve, but instead of having peripheral spheres (see Sec. 2), it would have peripheral subspaces which are Čech homology spheres instead of genuine spheres (since (2) fails). Such a space is probably not homeomorphic to $\mathscr{S}^{n-2}$. We point out, however, that this approach could not possibly work in dimension $n = 4$, as in this case the boundary would be a closed 3-manifold, which forces (2) to hold.

Structure of paper. In Sec. 2, we recall the definition of an n-dimensional Sierpinski space. In Secs. 3 and 4, we prove Theorems 1 and 2, respectively. In Sec. 5, we remark on a generalization of Theorem 1 to CAT(0) groups.

2. *n*-Dimensional Sierpinski Space and Hyperbolic Groups

We use Cannon's definition of n-dimensional Sierpinski space [12] (Cannon uses the term Sierpinski *curve* instead of Sierpinski *space*).

Definition. Fix $n \geq 0$. Let $D_1, D_2, \ldots \subset S^{n+1}$ be a sequence of open topological balls such that

(i) $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$,
(ii) $\operatorname{diam}(D_i) \to 0$ with respect to the round metric on S^{n+1}, and
(iii) $\bigcup D_i \subset S^{n+1}$ is dense.

Then $\mathscr{S}^n := S^{n+1} \backslash \bigcup D_i$ is an *n-dimensional Sierpinski space*. The spheres $S^n \cong \partial(\overline{D_i}) \subset \mathscr{S}$ are called *peripheral spheres*.

Example. A 0-dimensional Sierpinski space $\mathscr{S}^0$ is a Cantor set, while the space $\mathscr{S}^1$ is the classical Sierpinski carpet. The Sierpinski space $\mathscr{S}^{n-2}$ arises as the visual boundary of hyperbolic groups (in the sense of Gromov [18]). For example, if W^n is a hyperbolic n-manifold with non-empty totally geodesic boundary, then $\pi_1(W)$ is a hyperbolic group whose visual boundary is a Sierpinski $(n-2)$-space. To see this, observe that the universal cover $\widetilde{W}$ can be embedded as a submanifold of hyperbolic space $\widetilde{W} \hookrightarrow \mathbb{H}^n$. Using the disk model, the visual boundary $\partial_\infty \widetilde{W}$ is a subspace of $\partial_\infty \mathbb{H}^n \cong S^{n-1}$. The boundary components of W lift to countably many disjoint geodesic hyperplanes $\mathbb{H}^{n-1} \subset \mathbb{H}^n$. Each hyperplane has as boundary

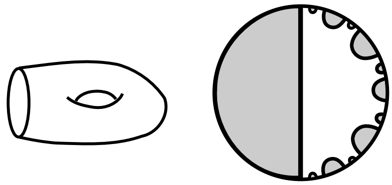

Fig. 1. A torus with one boundary component, and its universal cover inside the hyperbolic plane.

a sphere $\partial_\infty \mathbb{H}^{n-1} \cong S^{n-2}$, which bounds an open ball $\mathbb{D}^{n-1} \subset S^{n-1}$. The visual boundary of $\widetilde{W}$ is obtained by removing this countable collection of open balls, yielding a Sierpinski space $\mathscr{S}^{n-2}$.

The simplest example of this is when W is a torus with one boundary component (see Fig. 1). More examples are furnished by the following general theorem of Lafont [21].

Theorem 5. (Lafont) *Let M^n be a compact, negatively curved Riemannian manifold with non-empty totally geodesic boundary. Then $\partial_\infty \widetilde{M}$ is homeomorphic to $\mathscr{S}^{n-2}$.*

We remark that the dimension restriction in the statement of [21, Theorem 1.1] is unnecessary thanks to work of Freedman and Quinn (cf. the MathSciNet review of [26]). As a consequence of this result, the "locally CAT(−1) metric" statement in Theorem 2 cannot be replaced by "negatively curved Riemannian metric".

3. Proof of Theorem 1

Proof. We first prove the existence part of the statement, proceeding in three steps.

Step 1. (Peripheral subgroups and Poincaré duality pairs) Recall that G is a torsion-free hyperbolic group such that $\partial_\infty G \cong \mathscr{S}^{n-2}$. The stabilizer $H \leq G$ of a peripheral sphere $S^{n-2} \subset \mathscr{S}^{n-2}$ is called a *peripheral subgroup*. By the proof of Kapovich–Kleiner [20, Theorem 8(1)], there are finitely many peripheral subgroups, up to conjugacy in G. Choose representatives $H_1, \ldots, H_p$ for the conjugacy classes.

In order to show that G is the fundamental group of a manifold with boundary, we first need to establish that G has the same Poincaré duality as a manifold with boundary. To be precise, the doubling argument of Kapovich–Kleiner [20, Corollary 12] shows that $(G, \{H_i\})$ is a *group PD*(n) *pair* in the sense of Bieri–Eckmann [7]. This has the following topological consequence (see [19, Theorem 1] and [6, Sec. 6]): let (X, Y) be the CW-complex pair obtained by taking $Y = \coprod_{i=1}^p BH_i$ and defining X to be the mapping cylinder of the map $\coprod BH_i \to BG$. Then (X, Y) is a *CW-complex PD*(n) *pair* in the sense of Wall [30]. In particular, this means that there are isomorphisms $H^i(X;\mathbb{Z}) \cong H_{n-i}(X, Y;\mathbb{Z})$ and $H^{i-1}(Y;\mathbb{Z}) \cong H_{n-i}(Y;\mathbb{Z})$ induced by cap product with $[X] \in H_n(X)$ and $\partial[X] \in H_{n-1}(Y)$, respectively, and that X is a *finitely dominated* CW complex (i.e. there exists a finite CW complex L and maps $X \xrightarrow{i} L \xrightarrow{r} X$ such that $r \circ i = \mathrm{id}_X$).

Step 2. (Preparing for surgery) Let (X, Y) be the pair from Step 1. We now explain why (X, Y) is homotopy equivalent to a pair (K, N) such that

(A) K is a *finite* CW complex, and
(B) N is a manifold.

This will allow us to employ the total surgery obstruction in Step 3.

(A) Wall's finiteness obstruction $\tilde{o}(X) \in \widetilde{K}_0(X)$ vanishes if and only if X is homotopy equivalent to a finite CW complex [29]. Thus to show (A), it suffices to show $\widetilde{K}_0(X) = 0$. This is a corollary of the following powerful result (see [4, Proof of Theorem 1.2] for more information).

Theorem 6. (Bartels–Lück [2], Bartels–Lück–Reich [3]) *Let G be a torsion-free hyperbolic group G. Then*

(†) *the (non-connective) K-theory assembly map $H_i(BG; \mathbb{K}_{\mathbb{Z}}) \to K_i(\mathbb{Z}G)$ is an isomorphism for $i \leq 0$ and surjective for $i = 1$;*
(‡) *the (non-connective) L-theory assembly map $H_i(BG; {}^w\mathbb{L}_{\mathbb{Z}}^{\langle -\infty \rangle}) \to L_i^{\langle -\infty \rangle}(\mathbb{Z}G, w)$ is bijective for every $i \in \mathbb{Z}$ and every orientation homomorphism $w : G \to \{\pm 1\}$.*

The conditions (†) and (‡) are called the Farrell–Jones conjectures in K- and L-theory, respectively. Note that, since G is a torsion-free hyperbolic group, a constructive alternative is to take X a large enough Rips complex (which is automatically a finite simplicial complex). We included the non-constructive proof above, as this "obstruction" point of view will reappear in later arguments.

(B) It remains to see that Y is homotopy equivalent to a closed manifold N^{n-1}. By definition Y is homotopy equivalent to $\coprod_{i=1}^{p} BH_i$. The peripheral subgroups H_i are all hyperbolic groups, and $\partial_\infty H_i$ is identified with the sphere $S^{n-2} \subset \mathscr{S}^{n-2}$ stabilized by H_i (see [20, Theorem 8]). The following result from [4, Theorem A] implies that $Y \simeq \coprod_{i=1}^{p} BH_i$ is homotopy equivalent to a manifold.

Theorem 7. (Bartels–Lück–Weinberger [4]) *Fix $n \geq 7$, and let H be a torsion-free hyperbolic group. If $\partial_\infty H \cong S^{n-2}$, then there is a closed aspherical manifold N^{n-1} such that $\pi_1(N) \cong H$.*

Step 3. (The total surgery obstruction) Let (K, N) be the pair from Step 2. The *structure set* $S_\partial^{TOP}(K)$ is defined as the set of equivalence classes of homotopy equivalences $f : (M, \partial M) \to (K, N)$ where $(M, \partial M)$ is a manifold with boundary and $f\big|_{\partial M} : \partial M \to N$ is a homeomorphism (the equivalence relation is h-cobordism rel ∂; see [25, Chap. 18]). Surgery theory provides computable obstructions to determine whether or not (K, N) is homotopy equivalent to a manifold with boundary, i.e. whether or not $S_\partial^{TOP}(K) \neq \emptyset$.

We will follow the algebraic approach detailed in Ranicki [25]. The *total surgery obstruction* $s_\partial(K)$ lives in the *structure group* $\mathbb{S}_n(K)$ and has the property that $s_\partial(K) = 0$ if and only if (K, N) is homotopy equivalent (rel boundary) to an n-manifold with boundary; see [24, Theorem 1]. The group $\mathbb{S}_n(K)$ fits into the *algebraic surgery exact sequence* [25, Definition 15.19]

$$\cdots \to H_n(K; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\pi_1(K)) \to \mathbb{S}_n(K) \to H_{n-1}(K; \mathbb{L}_\bullet) \to \cdots,$$

where A is the *assembly map* and $\mathbb{L}_\bullet$ is the 1-*connective surgery spectrum* whose 0th space is G/TOP and whose homotopy groups are $\pi_i(\mathbb{L}_\bullet) = L_i(\mathbb{Z})$ for $i \geq 1$.

To show that $S_\partial^{TOP}(K) \neq \emptyset$, we will show that $\mathbb{S}_n(K) = 0$. For this, we need to consider two other versions of the structure groups.

- The *quadratic structure groups* $\mathbb{S}_i(\mathbb{Z}, K)$ are defined in [25, Definition 14.6].
- The group $\overline{\mathbb{S}}_n(K)$ (see [25, Chap. 25]) belongs to the 4-*periodic algebraic surgery exact sequence*

$$\cdots \to H_n(K; \overline{\mathbb{L}}_\bullet) \xrightarrow{A} L_n(\pi_1(K)) \to \overline{\mathbb{S}}_n(K) \to H_{n-1}(K; \overline{\mathbb{L}}_\bullet) \to \cdots,$$

  where $\overline{\mathbb{L}}_\bullet$ is the 0-*connective surgery spectrum* whose 0th space is $L_0(\mathbb{Z}) \times G/TOP \cong \mathbb{Z} \times G/TOP$ and whose homotopy groups are $\pi_i(\overline{\mathbb{L}}_\bullet) = L_i(\mathbb{Z})$ for $i \geq 0$.

In order to show that $\mathbb{S}_n(K) = 0$, we use the following three facts.

(a) The groups $\overline{\mathbb{S}}_n(K)$ and $\mathbb{S}_n(\mathbb{Z}, K)$ are equal. This follows directly from Ranicki [25, Proposition 15.11(iii)–(iv)]. Here we have used that $\dim K \geq 6$. Note that $L_q(\mathbb{Z}) = 0$ for $q = -1$, and in Ranicki's notation $\mathbb{S}_n\langle 0\rangle(\mathbb{Z}, K) = \overline{\mathbb{S}}_n(K)$ (compare with [25, p. 289]).
(b) The quadratic structure groups $\mathbb{S}_i(\mathbb{Z}, K) \cong \mathbb{S}_i(\mathbb{Z}, BG)$ are 0 for all $i \in \mathbb{Z}$. For the proof, see [4, Proof of Theorem 1.2]. Note that this also uses Theorem 6.
(c) There is an exact sequence

$$H_n(K; L_0(\mathbb{Z})) \to \mathbb{S}_n(K) \to \overline{\mathbb{S}}_n(K).$$

See Ranicki [25, Theorem 25.3(i)].

From (a) and (b), it follows that $\overline{\mathbb{S}}_n(K) = 0$. Then, by (c), to show $\mathbb{S}_n(K) = 0$ it is suffices to show that $H_n(K; L_0(\mathbb{Z})) = H_n(K; \mathbb{Z}) = 0$. This can be seen from the long exact sequence in homology of a pair (K, N):

$$H_n(N; \mathbb{Z}) \to H_n(K; \mathbb{Z}) \to H_n(K, N; \mathbb{Z}) \xrightarrow{\partial} H_{n-1}(N; \mathbb{Z}).$$

The group $H_n(N; \mathbb{Z}) = 0$ because N is a PD$(n-1)$ complex. Also $H_n(K, N; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the fundamental class $[K]$, and $\partial[K]$ is a sum of fundamental classes of the components of N. In particular $\partial[K] \neq 0$, so $H_n(K; \mathbb{Z}) = 0$, as desired.

This concludes the proof of existence.

Uniqueness. So far we have proven the existence of a compact aspherical manifold W with $\pi_1(W) = G$. To show W is unique, we want to show that $S_\partial^{TOP}(W)$ is a singleton. By [24, Corollary 1 (rel ∂)], it suffices to show that $\mathbb{S}_{n+1}(W) = 0$. By [25, Theorem 25.3(i)], there is an exact sequence

$$0 \to \mathbb{S}_{n+1}(W) \to \overline{\mathbb{S}}_{n+1}(W) \to H_n(W;\mathbb{Z}),$$

and as noted above, $H_n(W;\mathbb{Z}) = 0$. Thus, it suffices to show that $\overline{\mathbb{S}}_{n+1}(W) = 0$. This follows because $\overline{\mathbb{S}}_{n+1}(W) = \mathbb{S}_{n+1}(\mathbb{Z}, W)$ (by the same reason as in Step 3, Fact (a) above), and $\mathbb{S}_{n+1}(\mathbb{Z}, W) = 0$ (see Step 3, Fact (b)). □

4. Proof of Theorem 2

The proof of Theorem 2 is an adaptation of [15, Sec. (5a), (5c)]. We briefly explain the relative version of [15] and the problem with extending it directly to our case.

The paper [15] uses hyperbolization to construct a closed, locally CAT(-1) manifold M^n with the unusual property that $\partial_\infty \widetilde{M}$ is *not* homomorphic to S^{n-1}.

To show this, they establish that $\partial_\infty \widetilde{M} - \{\gamma_+, \gamma_-\}$ is not simply connected, where γ_+, γ_- are the endpoints of a geodesic $\gamma : (-\infty, \infty) \to \widetilde{M}$ whose link is a homology sphere H with $\pi_1(H) \neq 1$. In order to find nontrivial elements of $\pi_1\big(\partial_\infty \widetilde{M} - \{\gamma_+, \gamma_-\}\big)$, [15] studies metric spheres $S_p(r)$ centered at $p = \gamma(0)$. When $s > r$, there are natural *geodesic contraction maps* $\rho_r^s : S_p(s) \to S_p(r)$, which allow one to relate the topology of small spheres to the topology of $\partial_\infty \widetilde{M} = \varprojlim \{S_p(r)\}_{r>0}$. The central property of the maps ρ_r^s that makes the comparison work is that they are *cell-like*. We refer the reader to [17] for information concerning cell-like sets and maps.

Following [15], we will construct a triangulated, locally CAT(-1) manifold M with totally geodesic boundary ∂M whose universal cover $\widetilde{M}$ contains a geodesic $\gamma : (-\infty, \infty) \to \widetilde{M}$ whose link is a homology sphere H with $\pi_1(H) \neq 1$. As above, we wish to show $\pi_1(\partial_\infty \widetilde{M} - \{\gamma_+, \gamma_-\}) \neq 1$ (Lemma 8 below then implies that $\partial_\infty \widetilde{M}$ is *not* homeomorphic to $\mathscr{S}^{n-2}$). In this case $\widetilde{M}$ is a manifold with boundary, and the maps $\rho_r^s : S_p(s) \to S_p(r)$ are not surjective for $s \ll r$. This prevents us from proceeding directly as in [15]. To bypass this issue, we "cap off" the boundary components of $\widetilde{M}$ to obtain a CAT(-1) manifold $\widehat{M} \supset \widetilde{M}$ to which the arguments of [15] apply; in particular, $\pi_1(\partial_\infty \widehat{M} - \{\gamma_+, \gamma_-\}) \neq 1$. At this point it will be clear from the capping procedure (see specifically Lemma 9 below) that $\pi_1(\partial_\infty \widetilde{M} - \{\gamma_+, \gamma_-\}) \neq 1$.

For the proof of Theorem 2, we need the following elementary fact.

Lemma 8. *For $n \geq 2$, the n-dimensional Sierpinski space $\mathscr{S}^n$ is simply-connected. Moreover, if $F \subset \mathscr{S}^n$ is any finite collection of points in $\mathscr{S}^n$, then $\mathscr{S}^n \backslash F$ is still simply-connected.*

Proof. Model $\mathscr{S}^n$ as the complement, in the standard sphere S^{n+1}, of the interiors of a dense collection of pairwise disjoint round disks D_i whose radii r_i tend to zero. If γ is a curve in $\mathscr{S}^n \subset S^{n+1}$, we can find a bounding disk $\phi : \mathbb{D}^2 \to S^{n+1}$. Inductively define $\phi_k : \mathbb{D}^2 \to S^{n+1}$ to have image disjoint from the interiors of $D_1, \ldots, D_k$, as follows. First perturb ϕ to be transverse to D_1. Then $\phi^{-1}(\partial D_1)$ is a finite collection of curves in $\mathbb{D}^2$, and each of these curves maps to a curve η_j on $\partial D_1 \simeq S^n$. Since $n \geq 2$, we can redefine ϕ on the interior of these finitely many curves in $\mathbb{D}^2$ by sending each of these to a bounding disk in ∂D_1 for the corresponding η_j. In this way we obtain a map $\phi_1 : \mathbb{D}^2 \to S^{n+1}$ whose image is disjoint from $\mathrm{int}(D_1)$. Since the D_i are disjoint, we may continue inductively, replacing ϕ_1 by a map ϕ_2 whose image is disjoint from the interior of $D_1 \cup D_2$, and so on. Since the diameter of D_i shrinks to zero, the maps ϕ_k converge to a map $\phi_\infty : \mathbb{D}^2 \to S^{n+1}$ whose boundary coincides with γ, and whose image is disjoint from the interiors of all the D_i, i.e. the image of ϕ_∞ lies in $\mathscr{S}^n$. A similar argument works even after removing finitely many points in $\mathscr{S}^n$. □

Proof of Theorem 2. We proceed in several steps.

Step 1. (Construction) We construct M using the *strict hyperbolization* construction of Charney–Davis [13]. For simplicity we will focus primarily on the case $n \geq 5$. The case $n = 4$ will be explained at the end of Step 2.

The case $n \geq 5$ is modeled on [15, Sec. (5c)]. Fix a smooth n-manifold X with non-empty connected boundary Y, equipped with a PL-triangulation. Choose a smooth homology sphere H^{n-2} with nontrivial fundamental group, take a PL-triangulation of H, and consider the double suspension $\Sigma^2 H \cong S^n$, with the obvious induced (no longer PL) triangulation. Take the triangulated connect sum $X \sharp \Sigma^2 H$, obtained by using the interior of a pair of n-simplices in the triangulated X, $\Sigma^2 H$ to take the connect sum (and chosen so that simplex in X does not intersect the boundary of X). Note that, topologically $X \sharp \Sigma^2 H$ is homeomorphic to X, but now has a triangulation that fails to be PL — there is precisely one 4-cycle in the 1-skeleton of the triangulation whose link is H (instead of S^{n-2}). Finally, we let $M^n = h(X \sharp \Sigma^2 H)$, an n-manifold with boundary $N^{n-1} = h(Y)$, and set $G = \pi_1(M)$.

Properties of hyperbolization implies statement (1) in our theorem, while statement (2) follows from the fact that the triangulation of Y is PL (applying Davis–Januszkiewicz [15, Theorem (3b.2)]). The rest of our proof thus focuses on establishing statement (3) in the theorem — that $\partial_\infty G$ is not homeomorphic to $\mathscr{S}^{n-2}$.

Step 2. (Capping procedure) To show that $\partial_\infty G \neq \mathscr{S}^{n-2}$, first identify $\partial_\infty G \cong \partial_\infty \widetilde{M}$. We use Lemma 8 and show that $\pi_1(\partial_\infty \widetilde{M} \backslash F) \neq 1$, where $F = \{\gamma_+, \gamma_-\}$ consists of two points.

$\widetilde{M}$ is a non-compact CAT(-1) manifold with non-empty boundary, each component of which is isometric to $\widetilde{h(Y)}$. To understand $\partial_\infty \widetilde{M}$, we first define an isometric embedding $\widetilde{M} \hookrightarrow \widehat{M}$ into a CAT(-1) space without boundary. It will be easier to analyze $\widehat{M}$, which is obtained from $\widetilde{M}$ by gluing a certain space Z to each component of $\partial \widetilde{M}$. Next we define Z and describe its key features.

Let DX be the double of X across Y, with the induced triangulation. We apply a strict hyperbolization of Charney–Davis [13] to obtain a closed n-manifold $h(DX)$ equipped with a locally CAT(-1) metric. The universal cover $\widetilde{h(DX)}$ has boundary at infinity homeomorphic to S^{n-1} (see [15, Theorem (3b.2)]). Take any lift $\widetilde{h(Y)}$ of the separating codimension one submanifold $h(Y) \subset h(DX)$. Then $\widetilde{h(Y)}$ separates $\widetilde{h(DX)}$ into two (isometric) convex subsets. Denote by Z the closure of one of these convex subsets. Then Z is a non-compact locally CAT(-1) n-manifold with totally geodesic boundary $\widetilde{h(Y)}$.

Lemma 9. *The boundary at infinity $\partial_\infty Z$ of Z is homeomorphic to $\mathbb{D}^{n-1}$. The inclusion $\widetilde{h(Y)} = \partial Z$ induces, at the boundary at infinity, an identification $\partial_\infty \widetilde{h(Y)} = S^{n-2} = \partial(\mathbb{D}^{n-1})$.*

Let us momentarily assume Lemma 9 and finish the proof. Form the CAT(-1) space $\widehat{M}$ by gluing a copy of Z to each component of $\partial\widetilde{M}$, by isometrically identifying the copy of $\widetilde{h(Y)}$ inside Z with the boundary component. We have an isometric embedding $\widetilde{M} \hookrightarrow \widehat{M}$, inducing an embedding $\partial_\infty \widetilde{M} \hookrightarrow \partial_\infty \widehat{M}$. Let γ be a lift, in $\widetilde{M} \subset \widehat{M}$ of the singular geodesic in M, i.e. the geodesic whose link is the homology sphere H. The argument in [15, Proof of Theorem 5c.1(iv), p. 385] applies verbatim to show that $\partial_\infty \widehat{M} - \{\gamma_+, \gamma_-\}$ is not simply-connected. If η denotes a homotopically nontrivial loop in $\partial_\infty \widehat{M} - \{\gamma_+, \gamma_-\}$, then Lemma 9 allows us to use the same argument as in Lemma 8 to homotope η into the subset $\partial_\infty \widetilde{M} = \partial_\infty G$. We conclude that $\partial_\infty G - \{\gamma_+, \gamma_-\}$ fails to be simply connected. From Lemma 8, we conclude that $\partial_\infty G$ is not homeomorphic to $\mathscr{S}^{n-2}$.

The $n = 4$ case proceeds similarly, but is modeled instead on [15, Sec. (5a)]. Briefly, one lets X be a 4-dimensional simplicial complex whose geometric realization is a homology manifold with non-empty boundary Y, and which contains a singular point in the interior of X (whose link is, for example, the Poincaré homology 3-sphere H). One then looks at the universal cover of the hyperbolization $W = h(X)$. We can "cap off" the boundary components of $\widetilde{W}$ as in the last paragraph to obtain $\widehat{W}$. Then the arguments in [15, Sec. 3d] show that the fundamental group at infinity $\pi_1^\infty(\widehat{W})$ is nontrivial. It follows that $\pi_1(\partial_\infty \widehat{W})$ is also nontrivial by [14, Theorem 4.1]. Again, using Lemma 9, we can push a homotopically nontrivial loop in $\partial_\infty \widehat{W}$ into the subset $\partial_\infty \widetilde{W} = \partial_\infty G$. From Lemma 8, we conclude that $\partial_\infty G$ is not homeomorphic to $\mathscr{S}^2$. Finally, even though W is not a manifold, it is homotopy equivalent to a manifold: just remove a small neighborhood of the singular cone point, and replace it by a contractible manifold which bounds H. The resulting 4-manifold M has the desired properties.

Step 3. (Reducing Lemma 9) To complete the proof of the theorem, we are left with verifying Lemma 9. This is again a minor adaptation of the arguments in [15, Secs. 3b, 3c]. Choose a basepoint $x \in \partial Z$, and consider the closed metric

r-balls $\overline{B}_Z(r)$, $\overline{B}_{\partial Z}(r)$ in the spaces Z, ∂Z, centered at x, as well as the metric r-spheres $S_Z(r)$ and $S_{\partial Z}(r)$. The proof of Lemma 9 will rely on the following.

Claim 1. *For all r, the metric spheres $S_Z(r)$ are manifolds with boundary $S_{\partial Z}(r)$.*

Claim 2. *For points $p \in S_{\partial Z}(r)$, the complement $Lk(p) \backslash B_{Lk(p)}(v; \pi)$ of the metric ball of radius π, centered at $v \in \partial(Lk(p))$ in the link of p, is a cell-like set.*

It is easy to conclude from these two claim. If one takes a small enough r, then clearly $S_Z(r)$ is homeomorphic to a disk $\mathbb{D}^{n-1}$. In view of Claim 2 and the discussion in [15, p. 372], there is an $\epsilon > 0$ such that each of the geodesic contraction maps $\rho_r^s : S_Z(s) \to S_Z(r)$ is a cell-like map when $r < s < r + \epsilon$. So by Claim 1, the maps ρ_r^s are cell-like maps between manifolds with boundaries. From the work of Siebenmann [27], Quinn [23], and Armentrout [1] it follows that each ρ_r^s is a *near-homeomorphism* (i.e. can be approximated arbitrarily closely by homeomorphisms), and hence, that all the $S_Z(r)$ are homeomorphic to a disk $\mathbb{D}^{n-1}$, with boundary $\partial S_Z(r) = S_{\partial Z}(r)$.

Since we can identify $\partial_\infty Z$ with the inverse limit $\varprojlim \{S_Z(r)\}_{r>0}$, where the bonding maps are given by the near-homeomorphisms ρ_r^s (where $0 < r < s$), the main result of Brown [10] implies that $\partial_\infty Z$ is also homeomorphic to the closed disk $\mathbb{D}^{n-1}$. This confirms the first statement in Lemma 9. For the second statement, we note that $\widetilde{h(Y)} = \partial Z$ is a totally geodesic subspace of Z, and hence we have an embedding $S^{n-2} = \partial_\infty \widetilde{h(Y)} \hookrightarrow \partial_\infty Z = \mathbb{D}^{n-1}$. Since ∂Z fails to (coarsely) separate Z, an elementary argument gives that the image of $\partial_\infty(\partial Z) = S^{n-1}$ also fails to separate $\partial_\infty Z = \mathbb{D}^{n-1}$, and hence coincides with the set $\partial \mathbb{D}^{n-1}$. This gives the second statement in Lemma 9. We have thus reduced the proof of Lemma 9 (and hence also of the theorem) to checking Claims 1 and 2 — which are the last two steps of the proof.

Step 4. (Proof of Claim 1) We first argue that the ball $B_Z(r)$ of radius r is a manifold with boundary. It is clear that points $p \in \mathrm{Int}(\widetilde{M})$ at distance $< r$ from the basepoint have manifold neighborhoods. It is also immediate that points $p \in \partial\widetilde{M}$ at distance $< r$ from the basepoint have neighborhoods homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}_+$. Points at distance $= r$ from the basepoint are either in $\mathrm{Int}(\widetilde{M})$ or on $\partial\widetilde{M}$.

For points p in $\mathrm{Int}(\widetilde{M})$, the argument in [15, p. 372] shows that p has a neighborhood homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}_+$. So the only possible points to worry about are points at distance $= r$, and lying on the subset $\partial\widetilde{M}$. But for such a point p, a similar argument works as well. Let v be the point in $\mathrm{Lk}(p)$ pointing from p to the basepoint x, and consider the closed ball $\overline{B}_{\mathrm{Lk}(p)}(v; \pi/2)$ in the link of p, centered at v, of radius $\pi/2$. For any vector $w \in \overline{B}_{\mathrm{Lk}(p)}(v; \pi/2)$, one can look at the geodesic γ_w emanating from p, in the direction w (γ_w is well-defined close to p). If the direction w is at distance $< \pi/2$ from v, then for a small interval of time $[0, s(w)]$, the geodesic γ_w lies entirely in $B_Z(r)$, with $\gamma_w(s(w)) \in S_Z(r) \cup B_{\partial Z}(r)$. Note that s varies continuously and $s(w) \to 0$ as $w \to S_{\mathrm{Lk}(p)}(v; \pi/2)$. It follows that p has a neighborhood homeomorphic to the set $\hat{X}$ constructed as follows: take the product

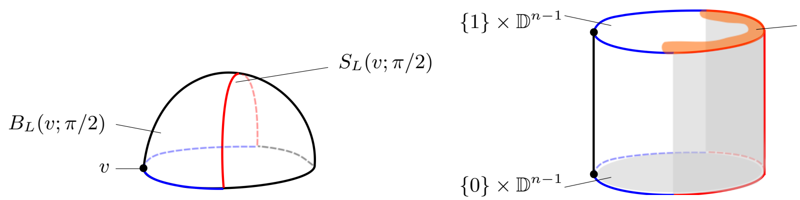

Fig. 2. Left: The link $L=\mathrm{Lk}(p)$. Right: The space $I\times\overline{B}_{\mathrm{Lk}(p)}(v;\pi/2)$, which is identified with a neighborhood $\hat{X}$ of p after quotienting by the gray region.

$I\times\overline{B}_{\mathrm{Lk}(p)}(v;\pi/2)$, collapse the fibers over the subset $S_{\mathrm{Lk}(p)}(v;\pi/2)$ to 0, and then collapse the subset $\{0\}\times\overline{B}_{\mathrm{Lk}(p)}(v;\pi/2)$ to a single point (which is identified with p), see Fig. 2. By an inductive argument (note that $\dim(\mathrm{Lk}(p))=\dim(\widetilde{M})-1$) one can assume that $\overline{B}_{\mathrm{Lk}(p)}(v;\pi/2)$ is homeomorphic to a disk $\mathbb{D}^{n-1}$, with the subset $S_{\mathrm{Lk}(p)}(v;\pi/2)$ corresponding to an embedded $\mathbb{D}^{n-2}$ inside $\partial\mathbb{D}^{n-1}\cong S^{n-2}$. Following the construction of $\hat{X}$ given above, we see that $\hat{X}$ is homeomorphic to $\mathbb{D}^n$, with the point corresponding to p lying on $\partial\mathbb{D}^n$. This shows that $B_Z(r)$ is indeed a manifold with boundary, and that the boundary of $B_Z(r)$ naturally decomposes as the union of $S_Z(r)\cup B_{\partial Z}(r)$, where the union is over the common subset $S_{\partial Z}(r)$.

Finally, we check that $S_Z(r)$ is an $(n-1)$-manifold with boundary. For points $p\in S_Z(r)$ lying in $\mathrm{Int}(\widetilde{M})$, it follows easily from [15, p. 372] that these points have neighborhoods homeomorphic to $\mathbb{D}^{n-1}$ with p lying as an interior point. In the case where $p\in S_Z(r)$ lies on $\partial\widetilde{M}$, we look at the neighborhood $\hat{X}$ of p constructed above. Within $\hat{X}$, the subset corresponding to $S_Z(r)$ consists of (the image of) a small neighborhood U of $\{1\}\times S_{\mathrm{Lk}(p)}(v;\pi/2)\cong\mathbb{D}^{n-2}$ inside the slice $\{1\}\times\overline{B}_{\mathrm{Lk}(p)}(v;\pi/2)\cong\mathbb{D}^{n-1}$. Note that the $(n-2)$-disk $S_{\mathrm{Lk}(p)}(v;\pi/2)$ lies in the boundary sphere of the $(n-1)$-disk $\overline{B}_{\mathrm{Lk}(p)}(v;\pi/2)$ (by induction). The image of U thus gives a copy of $\mathbb{D}^{n-1}$, with p lying in the boundary of $\mathbb{D}^{n-1}$. Moreover, the subset of U corresponding to $S_{\partial Z}(r)$ is just a neighborhood of p inside the boundary sphere of $\mathbb{D}^{n-1}$, i.e. is homeomorphic to $\mathbb{D}^{n-2}$. This completes the argument for Claim 1.

Step 5. (Proof of Claim 2) We want to show that the complement $\mathrm{Lk}(p)\setminus B_{\mathrm{Lk}(p)}(v;\pi)$ is cell-like. The set $\mathrm{Lk}(p)$ is homeomorphic to a disk $\mathbb{D}^{n-1}$, so we can think of the set we are interested in as lying within the double $D(\mathrm{Lk}(p))\cong S^{n-1}$.

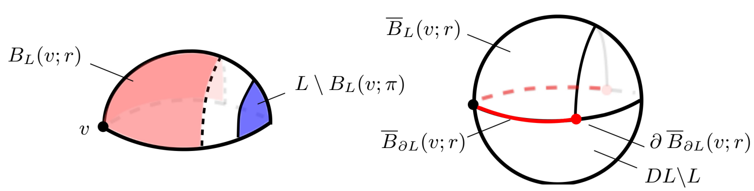

Fig. 3. The link $L=\mathrm{Lk}(p)$ and its double DL.

Given an $r \in (0, \pi)$, consider the subset $U_r \subset D(\mathrm{Lk}(p)) \cong S^{n-1}$ defined to be the union of $D(\mathrm{Lk}(p))\backslash\mathrm{Lk}(p)$ and the set $B_{\mathrm{Lk}(p)}(v; r)$. See Fig. 3. We will show each such U_r is homeomorphic to $\mathbb{R}^{n-1}$. Then by a result of Brown [11] it follows that the union $U_\infty := \bigcup_{r\in(0,\pi)} U_r \subset D(\mathrm{Lk}(p)) \cong S^{n-1}$ is also homeomorphic to $\mathbb{R}^{n-1}$. But if a subset of S^{n-1} is homeomorphic to $\mathbb{R}^{n-1}$, its complement is automatically cell-like [17, p. 114]. Since the complement of U_∞ coincides with $\mathrm{Lk}(p)\backslash B_{\mathrm{Lk}(p)}(v; \pi)$, this would establish Claim 2.

To see that each U_r is homeomorphic to $\mathbb{R}^{n-1}$, we consider their closures $\overline{U}_r$. We have that $U_r = \mathrm{Int}(\overline{U}_r)$, and that $\overline{U}_r$ can be written as the union of a copy of $\mathrm{Lk}(p)$ along with $\overline{B}_{\mathrm{Lk}(p)}(v; r)$, where the union is taken over the common subset $\overline{B}_{\partial\mathrm{Lk}(p)}(v; r)$. Let us analyze the two pieces in this decomposition.

On one of the sides, the subset $\mathrm{Lk}(p)$ is homeomorphic to $\mathbb{D}^{n-1}$, and the common subset $\overline{B}_{\partial\mathrm{Lk}(p)}(v; r)$ is homeomorphic to an embedded $(n-2)$-disk $\mathbb{D}^{n-2}$ inside the boundary sphere $\partial\mathrm{Lk}(p) \cong S^{n-2}$. Note that, by varying the parameter r, we see that

$$S^{n-3} \simeq \partial\overline{B}_{\partial\mathrm{Lk}(p)}(v; r) \subset \partial\mathrm{Lk}(p) \simeq S^{n-2}$$

is bicollared. On the other side, the subset $\overline{B}_{\mathrm{Lk}(p)}(v; r)$ is also homeomorphic to $\mathbb{D}^{n-1}$, and the gluing disk $\mathbb{D}^{n-2} \cong \overline{B}_{\partial\mathrm{Lk}(p)}(v; r)$ inside the boundary sphere $S^{n-2} \cong \partial\overline{B}_{\mathrm{Lk}(p)}(v; r)$ also has as complement a disk (by the argument in Claim 1). Thus, we see that $\overline{U}_r$ is obtained by gluing together two closed $(n-1)$-disks, by identifying together two copies of an $(n-2)$-disk, where each copy is nicely embedded in the respective boundary spheres $S^{n-2} \cong \mathbb{D}^{n-1}$. It follows that $\overline{U}_r$ is also homeomorphic to $\mathbb{D}^{n-1}$. This completes the proof of Claim 2 and the proof of the theorem. □

Remark 10. Let us make a small comment on approximating cell-like maps by homeomorphisms, in the case of manifolds with boundary. The attentive reader will probably notice that, in Siebenmann's work [27], there are two cases that require special care. In the 5-dimensional case, he requires the restriction of the map to the boundary to be a homeomorphism (rather than just a cell-like map). This is due to the fact that, at the time [27] was written, it was unclear whether or not cell-like maps of (closed) 4-manifolds could be approximated by homeomorphisms — hence the need of a stronger hypothesis on the boundary map. In view of Quinn's subsequent proof of the 4-dimensional case [23], this stronger hypothesis is no longer needed in the 5-dimensional boundary case. Note that, in our context, the bonding maps, when restricted to the boundary, are always cell-like (but are not homeomorphisms).

The other special case has to do with 3-dimensions. Here there is an added hypothesis that every point pre-image has a neighborhood N which isn't just contractible, but in addition is prime (i.e. if $N = M_1 \# M_2$, then one of the M_i is a standard 3-sphere). The only way this could fail is if one of the M_i were instead a homotopy 3-sphere — but by Perelman's resolution of the Poincaré Conjecture, such a manifold is automatically S^3. So again, in the 3-dimensional case, this additional hypothesis is now unnecessary.

5. Remarks on CAT(0) Groups

In this section we remark on generalizing the main result from hyperbolic groups to CAT(0) groups. A proper geodesic space X is called CAT(0) if geodesic triangles in X are at least as thin as triangles in Euclidean space [8]. A group G is called CAT(0) if there exists a CAT(0) space X on which G acts *geometrically* (that is, isometrically, properly, and compactly).

A CAT(0) space X has a visual boundary $\partial_\infty X$, and if G acts geometrically on X, then G acts on $\partial_\infty X$ by homeomorphisms. In this case $\partial_\infty X$ is called *a* boundary of G. With this terminology we have the following theorem.

Theorem 11. *Let G be a CAT(0) group for which S^{n-1} is a boundary. If $n \geq 6$, then there exists a closed n-dimensional aspherical manifold W such that $\pi_1(W) \simeq G$.*

The proof is almost identical to the proof of Theorem 7 in [4]. We give a short explanation for how to extend that argument to the CAT(0) case.

Proof of Theorem 11. By assumption G acts geometrically on an X with $\partial_\infty X = S^{n-1}$. Denote $\overline{X} = X \cup \partial_\infty X$. We proceed in three steps.

Step 1. BG is homotopy equivalent to a closed aspherical homology n-manifold W such that W has the disjoint disk property. To show this, it suffices to show that G is a PD(n) group and to note that CAT(0) groups satisfy the Farrell–Jones conjectures in K- and L-theory. For then we may use [4, Theorem 1.2], which says that for such a group, BG is homotopy equivalent to a closed aspherical homology n-manifold M with the disjoint disk property.

We explain why G is PD(n) group. First, we know G is of type FP once we know that there exists a finite CW complex $K \simeq BG$ (for then the cellular chain complex of the universal cover $\widetilde{K}$ is a finite length resolution of $\mathbb{Z}$ by finitely generated free G modules). A finite CW complex $K \simeq BG$ for a group G that acts geometrically on a proper CAT(0) space is shown to exist by Lück [22]. Now G is a $PD(n)$ group because

$$H^i(G;\mathbb{Z}G) \cong H^i_c(X) \cong \widetilde{H}^{i-1}(\partial_\infty X) = \widetilde{H}^{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{else.} \end{cases}$$

The first two isomorphisms are described by Bestvina [5]. That this implies G is a PD(n) group is explained in [9, VIII.10.1].

Step 2. The universal cover $\widetilde{W}$ can be compactified $N = \widetilde{W} \cup \partial_\infty X$ such that N is a homology manifold with boundary. To show that N is a homology manifold with boundary, it suffices to show that N is a finite-dimensional locally compact ANR and $\partial_\infty X$ is a Z-set in N (see [4, Proposition 2.5]). The pair $(\overline{X}, \partial_\infty X)$ is a Z-structure on G by Bestvina [5, Example 1.2(ii)]. Furthermore, by [5, Lemma 1.4] for any other finite model K for BG, there is a natural Z-structure on $(\overline{K}, \partial_\infty X)$, where $\overline{K} = K \cup \partial_\infty X$. Thus $(N, \partial_\infty X)$ admits a Z-set structure; in particular, N is a Euclidean retract, finite dimensional, and S^{n-1} is a Z-set inside N.

Step 3. $\widetilde{W}$ (and hence also W) is a manifold. This part of the argument is identical to that given in [4, Theorem A]. Quinn's invariant allows one to recognize manifolds among homology manifolds with the disjoint disk property. By the local nature of Quinn's invariant, if $(B, \partial B)$ is a homology manifold with boundary and ∂B is a manifold, then $\mathrm{int}(B)$ is a manifold. □

In light of this result and Theorem 1 above, it is natural to ask the following question.

Question. Let G be a CAT(0) group which admits $\mathscr{S}^{n-2}$ as a boundary. Is G the fundamental group of an n-dimensional aspherical manifold withboundary?

Examples of G satisfying the hypothesis of this question are given by Ruane [26]: every nonuniform lattice $\Gamma \leq \mathrm{SO}(n, 1)$ is an example. For these examples, an aspherical manifold with boundary can be obtained by "truncating the cusps" of $\mathbb{H}^n/\Gamma$.

There are some basic problems with answering this question with the techniques of this paper. For example, it is not obvious that peripheral subgroups of a CAT(0) group with Sierpinski space boundary are CAT(0), or that the double of a CAT(0) group along peripheral subgroups is CAT(0).

Acknowledgments

The authors wish to thank S. Weinberger and S. Ferry for useful conversations about surgery theory, and M. Davis for helpful discussions on approximating cellular maps. The authors thank B. Farb for comments on a draft of this paper. We would also like to thank I. Belegradek, D. Canary and G. Walsh for the remarks on Theorem 2. The first author was partially supported by the NSF, Under Grants DMS-1207782, DMS-1510640, and the second author was partially supported by the NSF Grant DGE-1144082.

References

1. S. Armentrout, *Cellular Decompositions of* 3-*Manifolds that Yield* 3-*Manifolds* (Amer. Math. Soc., 1971).
2. A. Bartels and W. Lück, The Borel conjecture for hyperbolic and CAT(0)-groups, *Ann. of Math. (2)* **175** (2012) 631–689.
3. A. Bartels, W. Lück and H. Reich, The K-theoretic Farrell–Jones conjecture for hyperbolic groups, *Invent. Math.* **172** (2008) 29–70.
4. A. Bartels, W. Lück and S. Weinberger, On hyperbolic groups with spheres as boundary, *J. Differential Geom.* **86** (2010) 1–16.
5. M. Bestvina, Local homology properties of boundaries of groups, *Michigan Math. J.* **43** (1996) 123–139.
6. R. Bieri and B. Eckmann, Groups with homological duality generalizing Poincaré duality, *Invent. Math.* **20** (1973) 103–124.

7. R. Bieri and B. Eckmann, Relative homology and Poincaré duality for group pairs, *J. Pure Appl. Algebra* **13** (1978) 277–319.
8. M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften, Vol. 319 Fundamental Principles of Mathematical Sciences (Springer-Verlag, 1999).
9. K. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics, Vol. 87 (Springer-Verlag, 1982).
10. M. Brown, Some applications of an approximation theorem for inverse limits, *Proc. Amer. Math. Soc.* **11** (1960) 478–483.
11. M. Brown, The monotone union of open n-cells is an open n-cell, *Proc. Amer. Math. Soc.* **12** (1961) 812–814.
12. J. W. Cannon, A positional characterization of the $(n-1)$-dimensional Sierpiński curve in $S^n (n \neq 4)$, *Fund. Math.* **79** (1973) 107–112.
13. R. Charney and M. Davis, Strict hyperbolization, *Topology* **34** (1995) 329–350.
14. G. Conner and H. Fischer, The fundamental group of a visual boundary versus the fundamental group at infinity, *Topology Appl.* **129** (2003) 73–78.
15. M. Davis and T. Januszkiewicz, Hyperbolization of polyhedra, *J. Differential Geom.* **34** (1991) 347–388.
16. M. Davis, T. Januszkiewicz and S. Weinberger, Relative hyperbolization and aspherical bordisms: An addendum to "hyperbolization of polyhedra", *J. Diff. Geom.* **3** (2001) 535–541.
17. R. Edwards, The topology of manifolds and cell-like maps, in *Proc. the International Congress of Mathematicians* Helsinki, 1978 (Acad. Sci. Fennica, 1980), pp. 111–127.
18. M. Gromov, Hyperbolic groups, in *Essays in Group Theory*, Math. Sci. Res. Inst. Publ. Vol. 8 (Springer, 1987), pp. 75–263.
19. F. E. A. Johnson and C. T. C. Wall, On groups satisfying Poincaré duality, *Ann. of Math. (2)* **96** (1972) 592–598.
20. M. Kapovich and B. Kleiner, Hyperbolic groups with low-dimensional boundary, *Ann. Sci. École Norm. Sup. (4)* **33** (2000) 647–669.
21. J. Lafont, A boundary version of Cartan–Hadamard and applications to rigidity, *J. Topol. Anal.* **1** (2009) 431–459.
22. W. Lück, On the classifying space of the family of virtually cyclic subgroups for CAT(0)-groups, *Münster J. Math.* **2** (2009) 201–214.
23. F. Quinn, Ends of maps. III. Dimensions 4 and 5, *J. Differential Geom.* **17** (1982) 503–521.
24. A. Ranicki, The total surgery obstruction, in *Algebraic Topology*, Aarhus 1978, Proc. Sympos., Univ. Aarhus, Aarhus, 1978, Lecture Notes in Math., Vol. 763 (Springer, 1979), pp. 275–316.
25. A. Ranicki, *Algebraic L-Theory and Topological Manifolds*, Cambridge Tracts in Mathematics, Vol. 102 (Cambridge Univ. Press, 1992).
26. K. Ruane, CAT(0) boundaries of truncated hyperbolic space, *Topol. Proc.* **29** (2005) 317–331.
27. L. C. Siebenmann, Approximating cellular maps by homeomorphisms, *Topology* **11** (1972) 271–294.
28. (Ed.) C. T. C. Wall, List of problems, in *Homological Group Theory*, Proc. Sympos., Durham, 1977, of London Math. Soc. Lecture Note Ser., Vol. 36 (Cambridge Univ. Press, 1979), pp. 369–394.
29. C. T. C. Wall, An obstruction to finiteness of CW-complexes, *Bull. Amer. Math. Soc.* **70** (1964) 269–270.
30. C. T. C. Wall, Poincaré complexes. I, *Ann. of Math. (2)* **86** (1967) 213–245.