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Algebraic K-theory of virtually free groups

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We provide a general procedure for computing the algebraic K-theory of finitely generated virtually free groups. The procedure describes these groups in terms of the algebraic K-theory of various finite subgroups and various Farrell Nil groups. We illustrate this process by carrying out the computation for several interesting classes of examples. The first two classes serve as a check on the method and show that our algorithm recovers results that already exist in the literature. The last two classes of examples yield new computations.

1. Background material on virtually free groups

A finitely generated group Γ is *virtually free* provided Γ fits into a short exact sequence of the form

$$1 \to F_m \to \Gamma \to F \to 1,$$

where F_m is a free group on m generators, and F is a finite group. We will say that a virtually free group is *splittable* if the short exact sequence can be chosen to be split. Given a finitely generated, virtually free group Γ, we are interested in computing the groups $K_n(\mathbb{Z}\Gamma)$ for $n \in \mathbb{Z}$. In this section we provide background material on virtually free groups.

1.1. Virtually free groups, graphs of groups, and automorphisms of finite graphs

Firstly, observe that any group that acts on a simplicial tree with finite vertex stabilizers and finitely many edge orbits is automatically a virtually free group.

Solitar conjectured that the converse statement held true. This was verified by Karrass *et al*. [24] (see also [7]). In the language of Bass–Serre theory [38], this can be interpreted as saying that the following two classes of groups coincide:

(A) finitely generated virtually free groups;

(B) fundamental groups of finite graphs of groups, all of whose vertex groups are finite.

Given a finitely generated virtually free group Γ, we will denote by $\hat{\mathcal{G}}$ a corresponding graph-of-groups description. In order to simplify some of the later discussion, we now provide another characterization of virtually free groups.

Let us start by pointing out an easy method for producing virtually free groups. Given an arbitrary finite graph $\mathcal{G}$, one can consider the (finite) group of simplicial automorphisms $\mathrm{Aut}(\mathcal{G})$ of the graph. Choose $F \leqslant \mathrm{Aut}(\mathcal{G})$ to be a subgroup of the automorphism group having the property that $g \in F$ leaves an edge invariant if and only if it fixes the edge (we say such an action has *no inversions*). Note that any subgroup of $\mathrm{Aut}(\mathcal{G})$ acts without inversions on the barycentric subdivision of $\mathcal{G}$, so, at the cost of possibly subdividing all edges, one can always arrange for the group to act without inversions. If we let T denote the universal cover of the graph $\mathcal{G}$, we can now form the group Γ consisting of all lifts of elements in F to the universal cover T. There is a natural morphism $\Gamma \to F$, obtained by mapping each element in Γ to the automorphism in F which it covers. This gives a short exact sequence

$$1 \to F_m \to \Gamma \to F \to 1,$$

where $F_m = \pi_1(\mathcal{G})$, and hence Γ is virtually free.

Note that, in this situation, one can take the quotient $\hat{\mathcal{G}} := \mathcal{G}/F$, which will topologically be a graph, and label each vertex and edge in $\mathcal{G}/F$ with the appropriate stabilizer of a pre-image for the F-action on $\mathcal{G}$. This gives a graph-of-groups structure to $\hat{\mathcal{G}}$, and the fundamental group of this graph of groups is naturally isomorphic to Γ, i.e. this graph of group provides the description of Γ guaranteed by (B) above. Furthermore, the associated Bass–Serre tree can be identified with the universal cover T of the graph $\mathcal{G}$, and the natural Γ-action on T coincides with the action on T coming from Bass–Serre theory.

Conversely, given any virtually free group Γ, we have from (B) that Γ can be identified with the fundamental group of a finite graph of finite groups. Bass–Serre theory then yields a Γ-action on the corresponding Bass–Serre tree T. Now Γ contains a finite-index normal free subgroup F_m, and since this group acts *freely* on T (vertex stabilizers have finite order while F_m is torsion-free), we can define a finite graph $\mathcal{G} := T/F_m$. Observe that the quotient group $F := \Gamma/F_m$ acts on $\mathcal{G}$, so it can be identified with a subgroup of $\mathrm{Aut}(\mathcal{G})$. Moreover, if $g \in F$ leaves an edge in $\mathcal{G}$ invariant, then g actually fixes that edge (this is inherited from the corresponding property for the Γ-action on the Bass–Serre tree). So we have established that the class of finitely generated virtually free groups can additionally be identified with the class

(C) groups that arise as lifts to the universal cover of a finite group action on a finite graph, where the group acts with no edge inversions.

We will henceforth assume that the virtually free group Γ that we are interested in is given to us in the form of a finite group F acting on a finite graph $\mathcal{G}$ (with corresponding graph-of-groups description given by $\hat{\mathcal{G}} := \mathcal{G}/F$).

REMARK 1.1. We see that there is an analogy between virtually free groups and orbifold fundamental groups. The graph-of-groups description given by (B) is similar to an orbifold in that it is locally modelled by finite group actions on a 1-dimensional complex (while orbifolds are locally modelled by linear actions of finite groups on $\mathbb{R}^n$). In the theory of orbifolds, there is an important distinction between 'good orbifolds' (which are quotients of a manifold by a proper group action), and 'bad orbifolds' (which are not). In the terminolody of orbifolds, the equivalence of viewpoints (B) and (C) is merely stating that there are no analogues of 'bad orbifolds' in the context of 1-dimensional complexes.

1.2. Fixed points and splittable virtually free groups

Observe that, associated to any virtually free group, the short exact sequence $1 \to F_m \to \Gamma \to F \to 1$ gives rise to a natural homomorphism from the finite quotient F into $\mathrm{Out}(F_m)$. Algebraically, this morphism is given by taking any pre-image $\tilde{g} \in \Gamma$ of a given element $g \in F$, and sending g to the outer automorphism of F_m defined via the conjugation by $\tilde{g}$ on the normal subgroup $F_m \lhd \Gamma$. If the short exact sequence is associated to the F-action on a finite graph $\mathcal{G}$ with $\pi_1(\mathcal{G}) \cong F_m$, we can also interpret this homomorphism topologically. Indeed, the F-action on the finite graph $\mathcal{G}$ gives a natural morphism $F \to \mathrm{Out}(\pi_1(\mathcal{G})) = \mathrm{Out}(F_m)$ via the induced action on the fundamental group. Note that the morphism is into $\mathrm{Out}(\pi_1(\mathcal{G}))$ rather than $\mathrm{Aut}(\pi_1(\mathcal{G}))$ because, in general, the F-action on $\mathcal{G}$ will not have a global fixed point.

A virtually free group Γ is *splittable* provided that it is isomorphic to a group of the form $\Gamma \cong F_m \rtimes_\phi F$ for some suitable map $\phi\colon F \to \mathrm{Aut}(F_m)$. One of the advantages of the formulation (C) given in the previous section is that it makes it particularly easy to identify the splittable virtually free groups. Indeed, we have that the following two classes of groups coincide:

(A′) finitely generated splittable virtually free groups, and

(C′) groups which arise as lifts to the universal cover of a finite group action on a finite graph, *where the finite group has a global fixed point.*

To see the equivalence of these two classes, we note that if Γ is splittable, one can identify the quotient group $\Gamma/F_m \cong F$ with a finite subgroup of Γ, which we also denote by F. Under the Γ-action on *any* tree T, this finite subgroup must have a fixed point $\tilde{v}$, which in turn descends to a point $v \in \mathcal{G} = T/F_m$ that must be fixed by F. We conclude that the F-action on the corresponding finite graph $\mathcal{G}$ has a global fixed point. Conversely, if we have a finite group F-acting on a finite graph $\mathcal{G}$ with a global fixed point v, we choose any pre-image $\tilde{v}$ of v in the universal cover T. Recalling that Γ is defined to be all possible lifts of elements in F to the universal cover, the lifts which fix $\tilde{v}$ provides a subgroup $\tilde{F}$ of Γ that projects down isomorphically to F. This provides a splitting of Γ, as desired. The equivalence (A′) = (C′) will be useful in § 2.5, where it will be used to identify virtually cyclic subgroups of Γ.

2. The procedure: general case

In this section we explain how to compute the groups $K_n(\mathbb{Z}\Gamma)$ when Γ is a virtually free group. We will assume throughout that Γ is given via a graph-of-groups description $\hat{\mathcal{G}}$ (as in (B)), or via a finite group F acting on a finite graph $\mathcal{G}$ (as in (C)). Of course, the two viewpoints are closely related, as one can identify $\hat{\mathcal{G}} = \mathcal{G}/F$.

In § 2.1 we provide background information on the Farrell–Jones isomorphism conjecture (FJIC), and provide a general splitting formula (see equation (2.1)) for the groups $K_n(\mathbb{Z}\Gamma)$. In § 2.2 we analyse the first term appearing in the splitting, and explain how this can be computed from the quotient graph of groups $\hat{\mathcal{G}}$. In §§ 2.3–2.5 we analyse the remaining terms in the splitting and describe how these can be recognized in terms of the F-action on $\mathcal{G}$.

2.1. The Farrell–Jones isomorphism conjecture and splitting formula for the *K*-theory

The starting point in any such computation lies with the FJIC [14]. Recall that, for a group Γ, the FJIC asserts that the natural assembly map

$$H_n^{\Gamma}(\underline{\underline{E}}\Gamma; \mathbb{K}\mathbb{Z}^{-\infty}) \to H_n^{\Gamma}(pt; \mathbb{K}\mathbb{Z}^{-\infty}) \cong K_n(\mathbb{Z}\Gamma)$$

is an isomorphism for all $n \in \mathbb{Z}$. Here, the left-hand side is a specific generalized equivariant homology theory, applied to the Γ-CW complex $\underline{\underline{E}}\Gamma$. This space is a model for the classifying space for Γ-actions with isotropy in the family $\mathcal{VC}$ of virtually cyclic subgroups of Γ. The classifying space $\underline{\underline{E}}\Gamma$ is characterized, up to Γ-equivariant homotopy equivalence, by the following two properties:

- $\underline{\underline{E}}\Gamma^H$ is contractible, for every subgroup $H \in \mathcal{VC}$;
- $\underline{\underline{E}}\Gamma^H$ is empty, for every subgroup $H \notin \mathcal{VC}$.

A recent result of Bartels *et al*. [2] establishes the FJIC for δ-hyperbolic groups. Since our virtually free group Γ contains a finite-index finitely generated free subgroup, it is δ-hyperbolic and hence satisfies FJIC. For the particular groups that we are interested in, we note that the FJIC was previously known to hold in dimensions $n \leqslant 1$ by [36] (see also the related work [15]). So, to compute $K_n(\mathbb{Z}\Gamma)$, it is sufficient to compute the equivariant generalized homology groups $H_n^{\Gamma}(\underline{\underline{E}}\Gamma; \mathbb{K}\mathbb{Z}^{-\infty})$.

Leary and Juan-Pineda [23] gave a construction for an $\underline{\underline{E}}\Gamma$ where Γ is a δ-hyperbolic group (see also [28]). Using the specific model they constructed, they showed that the corresponding homology group splits as a direct sum:

$$H_n^{\Gamma}(\underline{\underline{E}}\Gamma; \mathbb{K}\mathbb{Z}^{-\infty}) \cong H_n^{\Gamma}(\underline{E}\Gamma; \mathbb{K}\mathbb{Z}^{-\infty}) \oplus \bigoplus_{V \in \mathcal{V}} \mathrm{cok}_n(V). \tag{2.1}$$

Let us explain the various terms appearing in this splitting. The space $\underline{E}\Gamma$ refers to the classifying space for Γ-actions with isotropy in the family $\mathcal{FIN}$ of finite subgroups of Γ. For any group G, the containment $\mathcal{FIN} \subseteq \mathcal{VC}$ gives a well-defined (up to G-equivariant homotopy) map $\underline{E}G \to \underline{\underline{E}}G$. A result of [1] states that the induced map

$$H_n^{G}(\underline{E}G; \mathbb{K}\mathbb{Z}^{-\infty}) \to H_n^{G}(\underline{\underline{E}}G; \mathbb{K}\mathbb{Z}^{-\infty})$$

is split injective; we denote the cokernel by $\mathrm{cok}_n(G)$. Finally, to complete the description of the splitting, we need to describe the collection $\mathcal{V}$ of subgroups of Γ. This collection consists of one representative from each conjugacy class of maximal infinite virtually cyclic subgroup of Γ.

2.2. Equivariant homology of the Bass–Serre tree

In this section, we identify the first term in the splitting (2.1). Since Γ has a graph-of-groups description, Bass–Serre theory [38] tells us that Γ acts (without inversions) on the corresponding Bass–Serre tree T. For a group acting on a tree, every finite subgroup fixes a non-empty subtree. In particular, we see that if $H \leqslant \Gamma$ is a finite subgroup, then T^H is contractible. Conversely, if $K \leqslant \Gamma$ is any subgroup satisfying $T^K \neq \emptyset$, then K fixes a vertex in T. But the Γ-action on T has finite vertex stabilizers, which forces K to be a finite subgroup. We conclude that the Γ-action on T has the following two properties:

1. if $H \leqslant \Gamma$ is finite, then T^H is contractible;

2. if $H \leqslant \Gamma$ is infinite, then $T^H = \emptyset$.

This means that T is a model for $\underline{E}\Gamma$ and hence the first term in (2.1) is given by

$$H_n^{\Gamma}(\underline{E}\Gamma; \mathbb{KZ}^{-\infty}) = H_n^{\Gamma}(T; \mathbb{KZ}^{-\infty}).$$

We now compute the equivariant homology of the tree T, which will give us the first term in the splitting (2.1). Let us start by fixing some notation. In the graph-of-groups description $\hat{\mathcal{G}}$ for Γ, let $\boldsymbol{V}$ and $\boldsymbol{E}$ denote the vertex and edge sets, respectively. For $v \in \boldsymbol{V}$ (or $e \in \boldsymbol{E}$) denote by Γ_v (respectively, Γ_e) the finite group associated to the vertex v (or to the edge e). Recall that we can identify the graph of groups with the quotient space T/Γ. For a vertex $v \in \boldsymbol{V} \subseteq T/\Gamma$, the stabilizer $\mathrm{Stab}_{\Gamma}(\tilde{v})$ of any pre-image $\tilde{v} \in T$ of the vertex v is naturally isomorphic to the corresponding Γ_v.

To compute the equivariant homology groups $H_*^{\Gamma}(X; \mathbb{KZ}^{-\infty})$ for a given Γ-CW-complex X, we can use an Atiyah–Hirzebruch-type spectral sequence (see, for example, [33, § 8] or [8, § 4]). The E^2-terms of this spectral sequence are obtained as follows. Let $(X/\Gamma)^p$ denote the p-skeleton of the orbit space X/Γ. Then $E^2_{p,q} = H_p(X; \{K_q(\mathbb{Z}\Gamma_{\sigma^p})\})$, the pth-homology group of the chain complex

$$\cdots \to \bigoplus_{\sigma^{p+1} \in (X/\Gamma)^{p+1}} K_q(\mathbb{Z}\Gamma_{\sigma^{p+1}}) \to \bigoplus_{\sigma^p \in (X/\Gamma)^p} K_q(\mathbb{Z}\Gamma_{\sigma^p}) \to \cdots,$$

where Γ_σ denotes the stabilizer of a pre-image $\tilde{\sigma} \in X$ of $\sigma \in X/\Gamma$, and the morphisms in the chain complex are induced by the natural inclusions (up to conjugacy).

In our special case, we have that $X = T$, the Bass–Serre tree, and we only have cells in dimensions 0 and 1. So the chain complex simplifies to

$$0 \to \bigoplus_{e \in \boldsymbol{E}} K_q(\mathbb{Z}\Gamma_e) \to \bigoplus_{v \in \boldsymbol{V}} K_q(\mathbb{Z}\Gamma_v) \to 0.$$

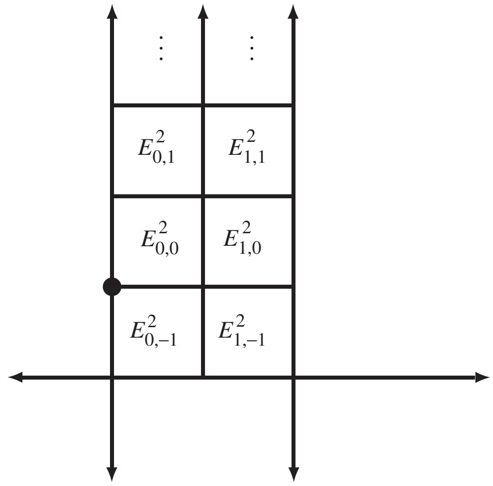

Figure 1. Potentially non-zero terms at the E^2-stage.

This immediately tells us that $E^2_{p,q} = 0$ for all $p \neq 0, 1$, and that

$$E^2_{0,q} = \operatorname{coker}\biggl(\bigoplus_{e \in \boldsymbol{E}} K_q(\mathbb{Z}\Gamma_e) \to \bigoplus_{v \in \boldsymbol{V}} K_q(\mathbb{Z}\Gamma_v)\biggr),$$

$$E^2_{1,q} = \ker\biggl(\bigoplus_{e \in \boldsymbol{E}} K_q(\mathbb{Z}\Gamma_e) \to \bigoplus_{v \in \boldsymbol{V}} K_q(\mathbb{Z}\Gamma_v)\biggr),$$

i.e. the only non-zero terms at the E^2-stage of the spectral sequence occur in the first two columns ($p = 0, 1$).

Moreover, Carter [6] has shown that, for finite groups G, we have that $K_q(\mathbb{Z}G) = 0$ for all $q \leqslant -2$ (and a well-known conjecture of Hsiang states that $|G| < \infty$ is an unnecessary hypothesis). Since, in our special case, all edge and vertex stabilizers are finite, Carter's result implies that $E^2_{p,q} = 0$ for all $q \leqslant -2$. To summarize, the only potentially non-zero terms at the E^2-stage are given in figure 1.

From the form of the differentials, we conclude that the spectral sequence collapses at the E^2-stage. The homology groups we are trying to compute are given by

$$\begin{aligned} H^\Gamma_n(\underline{E}\Gamma; \mathbb{KZ}^{-\infty}) &= H^\Gamma_n(T; \mathbb{KZ}^{-\infty}) \\ &\cong E^2_{0,n} \oplus E^2_{1,n-1} \\ &\cong \operatorname{coker}\biggl(\bigoplus_{e \in \boldsymbol{E}} K_n(\mathbb{Z}\Gamma_e) \to \bigoplus_{v \in \boldsymbol{V}} K_n(\mathbb{Z}\Gamma_v)\biggr) \\ &\quad \oplus \ker\biggl(\bigoplus_{e \in \boldsymbol{E}} K_{n-1}(\mathbb{Z}\Gamma_e) \to \bigoplus_{v \in \boldsymbol{V}} K_{n-1}(\mathbb{Z}\Gamma_v)\biggr). \end{aligned}$$

This completely reduces the computation of the first term in the splitting (2.1) to the problem of computing the algebraic K-theory of certain explicit finite groups (occurring in the graph of groups $\hat{\mathcal{G}}$ description), and understanding the morphisms induced on K-theory by the inclusion of certain subgroups.

2.3. Geodesics in the Bass–Serre tree and the family $\mathcal{V}$

Let us now turn to the second term in the splitting (2.1). Recall that this second term takes the form

$$\bigoplus_{V\in\mathcal{V}} \operatorname{cok}_n(V),$$

where $\mathcal{V}$ consists of one representative from each conjugacy class of maximal infinite virtually cyclic subgroups of Γ. Using the Γ-action on the Bass–Serre tree T, we can interpret these subgroups geometrically. Indeed, it is well known (see, for example, [26, p. 314]) that, for the groups we are considering, there is a bijective correspondence between

1. maximal infinite virtually cyclic subgroups of Γ, and
2. stabilizers of geodesics $\gamma \subseteq T$ having the property that the stabilizer

   $$|\operatorname{Stab}_\Gamma(\gamma)| = \infty,$$

   i.e. with the property that Γ acts periodically on $\gamma \subseteq T$.

Note that, under this bijective correspondence, conjugation of subgroups corresponds to geodesics in T lying in the same Γ-orbit. As such, we can think of the set $\mathcal{V}$ as consisting of one representative from each Γ-orbit of geodesic in T, with infinite stabilizer, and the corresponding subgroups V to consist of $\operatorname{Stab}_\Gamma(\gamma)$. This gives a description of the family $\mathcal{V}$ in terms of the Γ-action on T.

Now instead of parametrizing the family $\mathcal{V}$ via periodic geodesics in T, it will be slightly more convenient for our purposes to parametrize them via primitive closed geodesics in the finite graph $\mathcal{G}$. This graph was obtained as a quotient $\mathcal{G} := T/F_m$ of the tree T by the action of a finite-index normal free subgroup $F_m \triangleleft \Gamma$. Since the subgroup F_m acts freely on T, the map $T \to \mathcal{G}$ is a covering map, and the (finite) quotient group $F := \Gamma/F_m$ acts by simplicial automorphisms (without inversions) on $\mathcal{G}$. A geodesic $\gamma \subset T$ will have infinite stabilizer under the Γ-action if and only if it projects to a closed geodesic $\rho(\gamma)$ in the finite graph $\mathcal{G}$. Conversely, any primitive closed geodesic in $\mathcal{G}$ lifts to a geodesic in T whose stabilizer is infinite. Moreover, the stabilizer of two geodesics $\gamma_1, \gamma_2 \subset T$ in the tree will be conjugate if and only if the corresponding primitive closed geodesics $\rho(\gamma_1), \rho(\gamma_2)$ differ by an element in F, i.e. if there exists a $g \in F$ with $g \cdot \rho(\gamma_1) = \rho(\gamma_2)$. As such, we will henceforth parametrize the family $\mathcal{V}$ in terms of F-orbits of primitive closed geodesics in the finite graph $\mathcal{G}$.

2.4. Bass, Farrell and Waldhausen Nil groups

Having geometrically identified the family $\mathcal{V}$ appearing in the splitting (2.1), our next step is to understand the individual terms $\operatorname{cok}_n(V)$. In order to do this, we first need to recall some basic facts concerning virtually cyclic groups.

The classification of infinite virtually cyclic groups is well known (see, for example, [37]), and such groups come in two types.

1. Those that surject onto $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, which can always be written as an amalgamation $V = G_1 *_H G_2$, where G_i and H are finite and $[G_i : H] = 2$.

2. Those that do not surject onto D_∞, which can always be written as a semi-direct product $V = F \rtimes_\alpha \mathbb{Z}$ for a suitable finite group F and $\alpha \in \mathrm{Aut}(F)$.

For groups of the first type, the corresponding cokernel $\mathrm{cok}_n(V)$ is known to be equal to the Waldhausen Nil group

$$\mathrm{nil}^{\mathrm{W}}_*(\mathbb{Z}H; \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H]).$$

We refer the reader to the seminal works [40,41] for a detailed discussion of these Nil groups. For groups of the second type, the cokernel equals two copies of the Farrell Nil group $NK_*(\mathbb{Z}F, \alpha)$ (see [12] for a discussion). In the particular case where the automorphism $\alpha \in \mathrm{Aut}(F)$ is trivial, we write $NK_*(\mathbb{Z}F)$ for the Farrell Nil group, which in this case coincides with the classical Bass Nil group (see [3, § 12.6]).

Next, note that any infinite virtually cyclic group of the first type contains a canonical index-two subgroup of the second type. In this situation, the index-two subgroup contained inside the group $G_1 *_H G_2$ is automatically of the form $H \rtimes_\alpha \mathbb{Z}$, for a suitable $\alpha \in \mathrm{Aut}\, H$. Recent work [9,10] has related the Waldhausen Nil groups with the corresponding Farrell Nil groups. More precisely, it was established that

$$\mathrm{nil}^{\mathrm{W}}_*(\mathbb{Z}H; \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H]) \cong NK_*(\mathbb{Z}H, \alpha).$$

This implies that, up to possibly passing to an index-two subgroup, it is sufficient (computationally) to consider maximal infinite virtually cyclic subgroups that do not surject onto D_∞.

Now, in the geometric situation that we are considering, we can recognize maximal infinite virtually cyclic subgroups as stabilizers of geodesics in the tree T. In fact, it is easy to recognize the type of $\mathrm{Stab}_\Gamma(\gamma)$ from its action on γ. Since γ has two ends, we have a natural homomorphism $\mathrm{Stab}_\Gamma(\gamma) \to \mathbb{Z}_2$ given by the action on the ends of γ. The group $\mathrm{Stab}_\Gamma(\gamma)$ surjects onto D_∞ if and only if the above homomorphism is surjective. In that case, the canonical index-two subgroup mentioned earlier coincides with the kernel of the homomorphism $\mathrm{Stab}_\Gamma(\gamma) \twoheadrightarrow \mathbb{Z}_2$.

Denote by $\mathrm{Stab}^{\mathrm{o}}_\Gamma(\gamma)$ the subgroup (of at most index-two) of $\mathrm{Stab}_\Gamma(\gamma)$ that preserves the ends of γ. We know that each of these groups is of the form $H_\gamma \rtimes_{\phi_\gamma} \mathbb{Z}$ for a suitable finite group $H_\gamma \leqslant \Gamma$ and suitable automorphism $\phi_\gamma \in \mathrm{Aut}(H_\gamma)$. From such a description we also know that the cokernels $\mathrm{cok}_n(V)$ coincide with one or two copies of the corresponding Farrell Nil group $NK_*(\mathbb{Z}H_\gamma, \phi_\gamma)$ (depending on the index of $\mathrm{Stab}^{\mathrm{o}}_\Gamma(\gamma) \leqslant \mathrm{Stab}_\Gamma(\gamma)$). Our next goal is to see how we can recognize the subgroups $\mathrm{Stab}^{\mathrm{o}}_\Gamma(\gamma) = H_\gamma \rtimes_{\phi_\gamma} \mathbb{Z}$ from the geometry of the F-action on the finite graph $\mathcal{G}$.

2.5. Identifying the stabilizer of an oriented geodesic

Given a periodic geodesic $\gamma \subset T$ in the tree T, we now want to identify the corresponding group $\mathrm{Stab}^{\mathrm{o}}_\Gamma(\gamma)$. First note that we have a natural split short exact sequence

$$0 \to \mathrm{Fix}_\Gamma(\gamma) \to \mathrm{Stab}^{\mathrm{o}}_\Gamma(\gamma) \to \mathbb{Z} \to 0, \tag{2.2}$$

where $\mathrm{Fix}_\Gamma(\gamma)$ is the subgroup of Γ that fixes γ pointwise. But we know that $|H_\gamma| < \infty$, and hence it must fix a point in T. This forces $H_\gamma \leqslant \mathrm{Fix}_\Gamma(\gamma)$, since H_γ

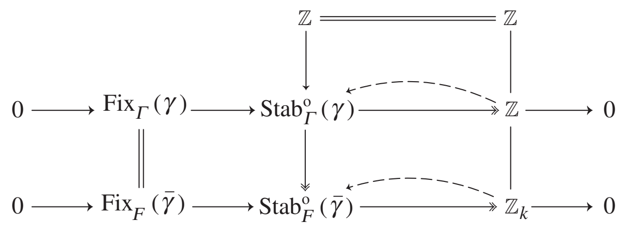

Figure 2. Relating the two short exact sequences.

fixes a point in γ and cannot reverse the ends of γ. Since H_γ is the maximal finite subgroup in $H_\gamma \rtimes_{\phi_\gamma} \mathbb{Z} = \mathrm{Stab}^{\mathrm{o}}_\Gamma(\gamma)$, this forces $H_\gamma = \mathrm{Fix}_\Gamma(\gamma)$.

Passing to the finite quotient graph $\mathcal{G}$, we have that the periodic geodesic γ descends to a primitive closed geodesic $\bar\gamma$ inside $\mathcal{G}$. We can now compare the stabilizer of the geodesic $\gamma \subset T$ for the Γ-action on T with the stabilizer of the primitive closed geodesic $\bar\gamma \subset \mathcal{G}$ for the F-action on $\mathcal{G}$. There is a natural short exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Stab}^{\mathrm{o}}_\Gamma(\gamma) \to \mathrm{Stab}^{\mathrm{o}}_F(\bar\gamma) \to 0, \tag{2.3}$$

where $\mathrm{Stab}^{\mathrm{o}}_F(\bar\gamma)$ is the subgroup of F which leaves the geodesic invariant *and preserves its chosen orientation*. Analogous to the split short exact sequence for $\mathrm{Stab}^{\mathrm{o}}_\Gamma(\gamma)$, there is a split short exact sequence for $\mathrm{Stab}^{\mathrm{o}}_F(\bar\gamma)$,

$$0 \to \mathrm{Fix}_F(\bar\gamma) \to \mathrm{Stab}^{\mathrm{o}}_F(\bar\gamma) \to \mathbb{Z}_k \to 0, \tag{2.4}$$

where $\mathrm{Fix}_F(\bar\gamma)$ is the subgroup of F fixing $\bar\gamma$ pointwise, and the quotient group $\mathbb{Z}_k$ acts via (non-trivial) orientation preserving rotations on $\bar\gamma$. Let us now compare the two short exact sequences (2.2) and (2.4), via the map in equation (2.3).

The subgroup $\mathrm{Fix}_\Gamma(\gamma)$ maps into $\mathrm{Fix}_F(\bar\gamma)$ and, conversely, since $\mathrm{Fix}_F(\bar\gamma)$ has a fixed point on $\bar\gamma$, we can lift it back to a subgroup of $\mathrm{Fix}_\Gamma(\gamma)$ (see § 1.2). In particular, the map in equation (2.3) induces an isomorphism between these two groups. Similarly, the generator τ for the $\mathbb{Z}$ term in equation (2.2) acts via a minimal translation on the periodic geodesic γ, and hence descends to a minimal rotation on the closed geodesic $\bar\gamma$, i.e. maps to a generator $\bar\tau$ for the $\mathbb{Z}_k$ term in equation (2.4). The relationship between these two split short exact sequences can be summarized by the commutative diagram in figure 2.

From the identification $H_\gamma = \mathrm{Fix}_\Gamma(\gamma) \cong \mathrm{Fix}_F(\bar\gamma)$, we have that the subgroup H_γ can be identified from the F-action on $\mathcal{G}$. Since the short exact sequence (2.4) splits, we have that $\mathrm{Stab}^{\mathrm{o}}_F(\bar\gamma) \cong H_\gamma \rtimes_\alpha \mathbb{Z}_k$, where the $\mathbb{Z}_k$-action on H_γ is given by mapping the generator $\bar\tau \in \mathbb{Z}_k$ to the automorphism $\alpha \in \mathrm{Aut}(H_\gamma)$. In particular, this semi-direct product can be detected *purely at the level of the finite group F acting on the finite graph $\mathcal{G}$*. Finally, from the comparison between the short exact sequences (2.2) and (2.4), we obtain that $\mathrm{Stab}^{\mathrm{o}}_\Gamma(\gamma) \cong H_\gamma \rtimes_\alpha \mathbb{Z}$, where the $\mathbb{Z}$-action on H_γ is given by letting the generator τ act via the same automorphism $\alpha \in \mathrm{Aut}(H_\gamma)$. As a special case of this analysis, we deduce the following.

Remark 2.1. If the closed geodesic $\bar\gamma \subset \mathcal{G}$ has the property that

$$\mathrm{Stab}^{\mathrm{o}}_F(\bar\gamma) = \mathrm{Fix}_F(\bar\gamma) \times \mathbb{Z}_k,$$

and $\gamma \subset T$ is a lift of $\bar{\gamma}$, then the contribution of $V = \mathrm{Stab}^{\mathrm{o}}_{\Gamma}(\gamma) = \mathrm{Fix}_{\Gamma}(\gamma) \times \mathbb{Z}$ to the second term in the splitting (2.1) is just (one or two) copies of the Bass Nil group $NK_*(\mathrm{Fix}_F(\bar{\gamma}))$.

An important special case of remark 2.1 arises when $\mathrm{Stab}_F(\bar{\gamma}) = \mathrm{Fix}_F(\bar{\gamma})$, i.e. when the F-action on $\mathcal{G}$ has the property that the entire stabilizer of the closed geodesic $\bar{\gamma}$ fixes the geodesic pointwise. In this case, we see that the corresponding contribution to the splitting (2.1) will be a copy of the Bass Nil group associated to $\mathrm{Fix}_F(\bar{\gamma})$ (see § 3.4 for a concrete example in which this occurs). Another special case of remark 2.1 arises when $\mathrm{Fix}_F(\bar{\gamma})$ is trivial, in which case there will be no contribution to the splitting (2.1) from that geodesic (see § 3.3 for a concrete example in which this occurs).

Finally, we observe that we can also reverse the above process. Starting with a non-trivial (conjugacy class of) subgroup $H \leqslant F$ of the group F, consider $\mathcal{G}^H$ the subgraph of $\mathcal{G}$ that is pointwise fixed by H. Then our above analysis tells us that

- if $\pi_1(\mathcal{G}^H)$ is trivial, then there will be no closed geodesics pointwise fixed by H, and hence no contributions $NK_*(\mathbb{Z}H, \alpha)$ to the splitting (2.1),
- if $\pi_1(\mathcal{G}^H) \cong \mathbb{Z}$, then there is at most one closed geodesic pointwise fixed by H, and hence either zero, one or two terms of the form $NK_*(\mathbb{Z}H, \alpha)$ appearing in the splitting (2.1),
- if $\pi_1(\mathcal{G}^H)$ is free of rank $\geqslant 2$, then there are infinitely many closed geodesics pointwise fixed by H, and hence potentially infinitely many terms of the form $NK_*(\mathbb{Z}H, \alpha)$ appearing in the splitting (2.1).

In the last two items above, we say ‘potentially’ because there is always the possibility that a *larger* group pointwise fixes a given closed geodesic in $\mathcal{G}^H$. Finally, we also note that given a subgroup H, we have an induced action of the normalizer $N_F(H)$ on the subgraph $\mathcal{G}^H$ (with the normal subgroup $H \triangleleft N_F(H)$ acting trivially). Inductively, we can now consider the action of the finite group $N_F(H)/H$ on the finite graph $\mathcal{G}^H$. For instance, given an element $h \in N_F(H)/H$, we can look for closed geodesics in $\mathcal{G}^H \subset \mathcal{G}$ that are left invariant by h. If $\alpha \in \mathrm{Aut}(H)$ denotes the automorphism of H induced by conjugation by h, then each such geodesic would contribute a Farrell Nil group $NK_*(\mathbb{Z}H, \alpha)$ to the splitting. In this manner, we see that the remaining terms in the splitting (2.1) can all be identified geometrically in terms of the F-action on $\mathcal{G}$.

2.6. Summary of the procedure

This concludes our procedure for computing the groups $K_*(\mathbb{Z}\Gamma)$ for virtually free groups Γ. To summarize, given a virtually free group Γ in the form of a finite group F acting on a finite graph $\mathcal{G}$, one uses the splitting formula (2.1). In the latter formula, we have that

- the first term can be computed via the quotient graph of groups $\hat{\mathcal{G}} = \mathcal{G}/F$, and reduces to calculating the K-theory of various finite groups (arising as edge and vertex stabilizers for the F-action on $\mathcal{G}$), as well as the morphisms induced on K-theory by the inclusion of various subgroups,

- the direct sum appearing in the second term is indexed by the family of F-orbits of closed geodesics in the finite graph $\mathcal{G}$,
- each F-orbit of closed geodesics in $\mathcal{G}$ contributes either one or two copies of a Farrell Nil group to the direct sum, where the Farrell Nil group can be directly detected from the stabilizer of the closed geodesic under the F-action.

In the next section, we will carry out this procedure for some concrete examples of virtually free groups.

3. Examples

In this section, we illustrate our general method by carrying out the procedure for a few interesting classes of virtually free groups. The first two examples are already well known (and serve as simple checks that our algorithm recovers known results), while the last two examples are new.

3.1. Example: $F \times \mathbb{Z}$ with F finite

As a quick check of the method, we consider the toy example where $\Gamma = F \times \mathbb{Z}$, with F finite. This example is the fundamental group of the graph of group $\hat{\mathcal{G}}$ consisting of a single loop, with edges and vertices both labelled by F, and both edge-to-vertex morphisms given by the identity map. Taking into account the orientation on the single edge, the corresponding morphism from the K-theory of the edge group $K_n(\mathbb{Z}\Gamma_e) = K_n(\mathbb{Z}F)$ to the K-theory of the vertex group $K_n(\mathbb{Z}\Gamma_v) = K_n(\mathbb{Z}F)$ is the zero map (the inclusion from the positive endpoint of the edge, but *minus* the inclusion from the negative endpoint of the edge). We deduce that the first summand in our expression for $K_n(\mathbb{Z}[F \times \mathbb{Z}])$ is just the direct sum $K_n(\mathbb{Z}F) \oplus K_{n-1}(\mathbb{Z}F)$ (see § 2.2). For the second term in the splitting, we observe that the associated Bass–Serre tree is just a line, so has a single geodesic with stabilizer the entire group $\Gamma = F \times \mathbb{Z}$. Since the geodesic stabilizer preserves the orientation, the second term of the splitting consists of precisely two copies of the corresponding Bass Nil group $NK_n(\mathbb{Z}F)$. Our algorithm now gives us that

$$K_n(\mathbb{Z}[F \times \mathbb{Z}]) \cong K_n(\mathbb{Z}F) \oplus K_{n-1}(\mathbb{Z}F) \oplus NK_n(\mathbb{Z}F) \oplus NK_n(\mathbb{Z}F).$$

Since the integral group ring $\mathbb{Z}[F \times \mathbb{Z}]$ coincides with the ring $\mathbb{Z}F[t, t^{-1}]$ of Laurent polynomials over $\mathbb{Z}F$, the above expression agrees with the classic Bass–Heller–Swan formula [4].

We remark that a similar analysis can be performed in the case where $\Gamma = F \rtimes_\alpha \mathbb{Z}$, with F finite and $\alpha \in \mathrm{Aut}(F)$. In this case, the reader can verify that the result of our algorithm agrees with the computation for $K_n(\mathbb{Z}\Gamma)$ given in [13] (for $n \leqslant 1$) and extended to $n \geqslant 2$ in [17].

3.2. Example: amalgams of finite groups

Next, let us consider a group of the form $\Gamma = G_1 *_H G_2$, with H, G_1 and G_2 all finite. In this case, our algorithm tells us that the algebraic K-groups $K_n(\mathbb{Z}\Gamma)$ are given by

$$\begin{aligned}&\operatorname{coker}(K_n(\mathbb{Z}H) \to K_n(\mathbb{Z}G_1) \oplus K_n(\mathbb{Z}G_2)) \\ &\qquad \oplus \ker(K_{n-1}(\mathbb{Z}H) \to K_{n-1}(\mathbb{Z}G_1) \oplus K_{n-1}(\mathbb{Z}G_2)) \oplus \bigoplus_{V\in\mathcal{V}} \operatorname{cok}_n(V), \qquad (3.1)\end{aligned}$$

where the last term is a direct sum of various Nil groups corresponding to stabilizers of geodesics in the Bass–Serre tree.

The algebraic K-theory of an amalgamation $\Gamma = G_1 *_H G_2$ was studied in depth in [40, 41]. One of the main results was that a Mayer–Vietoris-type long exact sequence held, once a suitable 'error-term' was factored out:

$$\begin{aligned}&\cdots \to K_n(\mathbb{Z}H) \to K_n(\mathbb{Z}G_1) \oplus K_n(\mathbb{Z}G_2) \to K_n(\mathbb{Z}\Gamma)/\operatorname{NIL}_n \\ &\qquad \to K_{n-1}(\mathbb{Z}H) \to K_{n-1}(\mathbb{Z}G_1) \oplus K_{n-1}(\mathbb{Z}G_2) \to \cdots. \qquad (3.2)\end{aligned}$$

The error terms NIL_n are called the Waldhausen Nil groups associated to the amalgamation, and are direct summands in the corresponding K-groups $K_n(\mathbb{Z}\Gamma)$. It follows from [26, corollary 3] that this Waldhausen Nil group is isomorphic to the term $\bigoplus_{V\in\mathcal{V}} \operatorname{cok}_n(V)$ appearing in equation (3.1). As a result, we see that our expression for $K_n(\mathbb{Z}\Gamma)$ in equation (3.1) is compatible with the Waldhausen long exact sequence (3.2).

For a concrete example, consider the modular group $\Gamma := \operatorname{PSL}_2(\mathbb{Z})$. This group is a non-uniform lattice in $\operatorname{PSL}_2(\mathbb{R}) = \operatorname{isom}(\mathbb{H}^2)$, and hence acts isometrically on $\mathbb{H}^2$ with cofinite volume. Passing to a finite-index torsion-free subgroup, we have that the corresponding quotient of $\mathbb{H}^2$ is a non-compact surface, forcing the finite-index subgroup to be free, and hence $\operatorname{PSL}_2(\mathbb{Z})$ is virtually free. It is well known that $\operatorname{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$.

We can now apply equation (3.1) to calculate the groups $K_n(\mathbb{Z}\operatorname{PSL}_2(\mathbb{Z}))$. To identify the infinite virtually cyclic subgroups of $\operatorname{PSL}_2(\mathbb{Z})$, we note that every geodesic γ in T satisfies $\operatorname{Fix}_\Gamma(\gamma) = 1$ (since every edge stabilizer is trivial). This forces the maximal infinite virtually cyclic subgroups to be isomorphic to $\mathbb{Z}$ or D_∞. All the Nil groups associated to $\mathbb{Z}$ are known to vanish. By [9] the Nil groups associated to D_∞ are isomorphic to the Nil groups associated to $\mathbb{Z}$, and hence also vanish in all dimensions (see also [10]). We conclude that the last term in equation (3.1) vanishes (for all $n \in \mathbb{Z}$), and hence we obtain that $K_n(\mathbb{Z}[\operatorname{PSL}_2(\mathbb{Z})])$ is given by

$$\begin{aligned}&\operatorname{coker}(K_n(\mathbb{Z}) \to K_n(\mathbb{Z}[\mathbb{Z}_2]) \oplus K_n(\mathbb{Z}[\mathbb{Z}_3])) \\ &\qquad \oplus \ker(K_{n-1}(\mathbb{Z}) \to K_{n-1}(\mathbb{Z}[\mathbb{Z}_2]) \oplus K_{n-1}(\mathbb{Z}[\mathbb{Z}_3])).\end{aligned}$$

Moreover, the natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[G]$ sits as a retract of rings. Hence, the induced maps on K_q are always split injective, allowing us to identify the abelian group $K_q(\mathbb{Z})$ with a direct summand in the abelian group $K_q(\mathbb{Z}[G])$. In particular, the *second* term in the above expression automatically vanishes, giving us the more concise identification

$$K_n(\mathbb{Z}[\operatorname{PSL}_2(\mathbb{Z})]) \cong \operatorname{coker}(K_n(\mathbb{Z}) \to K_n(\mathbb{Z}[\mathbb{Z}_2]) \oplus K_n(\mathbb{Z}[\mathbb{Z}_3])). \qquad (3.3)$$

For small values of n, the above expression can be used to explicitly determine the desired K-groups. For instance, the following results are known:

- when $q \leqslant -1$, we have that $K_q(\mathbb{Z})$, $K_q(\mathbb{Z}[\mathbb{Z}_2])$ and $K_q(\mathbb{Z}[\mathbb{Z}_3])$ all vanish [5];
- when $q = 0$, we have that $K_0(\mathbb{Z}) \cong K_0(\mathbb{Z}[\mathbb{Z}_2]) \cong K_0(\mathbb{Z}[\mathbb{Z}_3]) \cong \mathbb{Z}$ and the inclusions of $\mathbb{Z}$ induce isomorphisms on K_0 (see [35, theorem 2.1]);
- when $q = 1$, we have that $K_1(\mathbb{Z}) \cong \mathbb{Z}_2$, $K_1(\mathbb{Z}[\mathbb{Z}_2]) \cong \mathbb{Z}_2^2$ and $K_1(\mathbb{Z}[\mathbb{Z}_3]) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ (these can be deduced from [19, § 5]);
- when $q = 2$, we have that $K_2(\mathbb{Z}) \cong \mathbb{Z}_2$, while [11] established that $K_2(\mathbb{Z}[\mathbb{Z}_2]) \cong (\mathbb{Z}_2)^2$ and that $K_2(\mathbb{Z}[\mathbb{Z}_3]) \cong \mathbb{Z}_2$;
- when $q > 3$, the groups $K_q(\mathbb{Z})$, $K_q(\mathbb{Z}[\mathbb{Z}_2])$ and $K_q(\mathbb{Z}[\mathbb{Z}_3])$ are known to be finitely generated abelian groups (see [32] or, more recently, [25]).

From the low-dimensional results mentioned above, we can now deduce the following results concerning $K_n(\mathbb{Z}[\mathrm{PSL}_2(\mathbb{Z})])$:

$$K_n(\mathbb{Z}\,\mathrm{PSL}_2(\mathbb{Z})) \cong \begin{cases} 0, & n \leqslant -1, \\ \mathbb{Z}, & n = 0, \\ \mathbb{Z}_2^2 \oplus \mathbb{Z}_3, & n = 1, \\ \mathbb{Z}_2^2, & n = 2, \\ \text{finitely generated}, & n \geqslant 3. \end{cases}$$

We note that this computation was essentially already done in [9, theorem 3.32].

3.3. Example: $\mathbf{GL_3(\mathbb{F}_2)}$ acting on the incidence graph of the Fano plane

For our next example, recall that the group $\mathrm{GL}_3(\mathbb{F}_2)$ is a finite simple group of order 168 which acts via linear automorphisms on the vector space $\mathbb{F}_2^3$. The corresponding projective plane $\mathbb{P}^2(\mathbb{F}_2)$ (the *Fano plane*) has 7 points and 7 lines, and inherits an action of $\mathrm{GL}_3(\mathbb{F}_2)$ by collineations. The incidence graph $\mathcal{G}$ for the Fano plane has a vertex for each point and a vertex for each line in the Fano plane (point vertices and line vertices), and an edge joining a point vertex to a line vertex if and only if the corresponding point lies on the corresponding line within the Fano plane. It follows that $\mathcal{G}$ is a bipartite 3-regular graph with exactly 14 vertices and 21 edges. The graph $\mathcal{G}$ supports an induced simplicial action of $\mathrm{GL}_3(\mathbb{F}_2)$. As discussed in § 1.1, looking at all possible lifts of the $\mathrm{GL}_3(\mathbb{F}_2)$ action to the universal cover of the graph $\mathcal{G}$ gives rise to a virtually free group Γ. Since $\mathcal{G}$ is a connected graph with 14 vertices and 21 edges, its fundamental group is free of rank 8. Hence, we obtain a short exact sequence

$$1 \to F_8 \to \Gamma \to \mathrm{GL}_3(\mathbb{F}_2) \to 1.$$

We now want to analyse the algebraic K-theory of the virtually free group Γ.

First of all, we need to look at the quotient graph of groups $\hat{\mathcal{G}} = \mathcal{G}/\,\mathrm{GL}_3(\mathbb{F}_2)$. We note that the $\mathrm{GL}_3(\mathbb{F}_2)$-action on $\mathbb{F}_2^3$ is transitive on 1-dimensional linear subspaces as well as on 2-dimensional linear subspaces. This in turn implies that the $\mathrm{GL}_3(\mathbb{F}_2)$-action on the Fano plane acts transitively on points, and transitively on lines. So at the level of the incidence graph $\mathcal{G}$, all point vertices lie in a single $\mathrm{GL}_3(\mathbb{F}_2)$-orbit, and all line vertices lie in a single $\mathrm{GL}_3(\mathbb{F}_2)$-orbit. Moreover, point vertices and line vertices clearly lie in different $\mathrm{GL}_3(\mathbb{F}_2)$-orbits, so the quotient graph of groups $\hat{\mathcal{G}}$ has precisely two vertices. Finally, we note that edges in $\mathcal{G}$ correspond to flags in the vector space $\mathbb{F}_2^3$. But the $\mathrm{GL}_3(\mathbb{F}_2)$-action is transitive on flags, so there is a single edge in $\hat{\mathcal{G}}$. This graph-of-groups description realizes Γ as an amalgamated product $\Gamma = G_1 *_H G_2$, where G_1 is the stabilizer of a 1-dimensional subspace $V_1 \subset \mathbb{F}_2^3$, G_2 is the stabilizer of a 2-dimensional subspace $V_2 \subset \mathbb{F}_2^3$ (containing V_1), and H is the common subgroup which stabilizes the pair $V_1 \subset V_2 \subset \mathbb{F}_2^3$. So provided we can identify these groups, we can apply equation (3.1) from the previous section.

More concretely, we can take V_1 to be the subspace spanned by $\langle 1,0,0\rangle$, and V_2 to be the subspace spanned by $\{\langle 1,0,0\rangle, \langle 0,1,0\rangle\}$. The subgroup G_1 will then consist of all matrices in $\mathrm{GL}_3(\mathbb{F}_2)$ whose first column is $[1,0,0]^{\mathrm{T}}$. The lower right (2×2)-minor of such a matrix must lie within $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$, while the remaining two entries in the first row can be chosen arbitrarily (so are parametrized by $\mathbb{F}_2^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$). We see that G_1 is a group of order 24, which fits into a short exact sequence

$$1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to G_1 \to S_3 \to 1.$$

In fact, looking at the matrix multiplication, we see that G_1 has a semi-direct product structure $G_1 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3$, where the action is given via the natural action of $S_3 \cong \mathrm{GL}_2(\mathbb{F}_2)$ on $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{F}_2^2$. The automorphism group of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to S_3, and it is easy to see that the $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$ action given above in fact coincides with the action of the full automorphism group $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$. This tells us that G_1 coincides with the holomorph of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is known to be isomorphic to S_4.

Note that, by projective duality in the Fano plane, we have that $G_2 \cong G_1$, but let us briefly explain how we can see this directly. The stabilizer of V_2 will consist of all matrices in $\mathrm{GL}_3(\mathbb{F}_2)$ whose first two columns come from the set $\{[1,0,0]^{\mathrm{T}}, [0,1,0]^{\mathrm{T}}, [1,1,0]^{\mathrm{T}}\}$. There are six choices for the first two columns, and the resulting top left (2×2)-minor of such a matrix defines an element of $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$. Since the first two entries in the bottom row are automatically 0, this forces the last row of the matrix to be $[0,0,1]$. So, for the last column of the matrix, there are only four possible choices, given by varying the first two entries of the column. Note that these four choices naturally correspond to a copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Taking into account the matrix multiplication again allows us to express G_2 as a semi-direct product $G_2 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3 \cong S_4$, and the obvious (matrix) symmetry provides the explicit isomorphism between G_1 and G_2.

Finally, the subgroup H stabilizes *both* V_1 and V_2. So H contains all matrices in $\mathrm{GL}_3(\mathbb{F}_2)$ whose first column is $[1,0,0]^{\mathrm{T}}$, and whose second column is chosen from the set $\{[0,1,0]^{\mathrm{T}}, [1,1,0]^{\mathrm{T}}\}$. Again, the last row of the matrix must be $[0,0,1]$, so for the last column, there are four possible choices, which are obtained by varying the first two entries. The resulting group H has order 8, and consists of all upper triangular matrices in $\mathrm{GL}_3(\mathbb{F}_2)$. This is a non-abelian group, and has exactly two elements of order 4, and hence has to be isomorphic to the dihedral group D_4 of order 8. Moreover, note that the subgroup $H \leqslant \mathrm{GL}_3(\mathbb{F}_2)$ is contained in both G_1 and G_2 (and, in fact, coincides with $G_1 \cap G_2$).

Now, from the discussion in § 2.2, we see that the first term in the splitting formula (2.1) for $K_n(\mathbb{Z}\Gamma)$ is given by

$$\mathrm{coker}(K_n(\mathbb{Z}D_4) \to 2K_n(\mathbb{Z}S_4)) \oplus \ker(K_{n-1}(\mathbb{Z}D_4) \to 2K_{n-1}(\mathbb{Z}S_4)), \tag{3.4}$$

where the morphisms are induced by the natural inclusions (described in the previous paragraphs). So, to complete the calculation, we are left with identifying the remaining terms of the splitting. To do this, we follow the procedure from §§ 2.3–2.5.

We need to consider non-trivial closed geodesics in the graph $\mathcal{G}$, and to identify their stabilizers under the $\mathrm{GL}_3(\mathbb{F}_2)$-action. We claim that the subgroup of $\mathrm{GL}_3(\mathbb{F}_2)$ that fixes pointwise a given closed geodesic γ inside $\mathcal{G}$ has to be trivial. Observe that such a geodesic γ must pass through *at least three* distinct line vertices in $\mathcal{G}$, which can be chosen to have the additional property that the corresponding three lines in the Fano plane do *not* have a common point of intersection. This is argued by contradiction: if not, then the geodesic γ passes either through exactly two line vertices, or exactly three line vertices, which are all adjacent to a single point vertex. But if we remove $\mathcal{G}$ from the graph, either

(i) all but two line vertices, or

(ii) all line vertices that are *not* adjacent to a given point vertex,

then the resulting graph falls apart into connected components that are points, along with a single tree. But there are no non-trivial closed geodesics inside a tree, which is a contradiction. Let l_1, l_2 and l_3 denote the three lines in the Fano plane corresponding to the three line vertices on γ (with $l_1 \cap l_2 \cap l_3 = \emptyset$). Now, if we have an element $g \in \mathrm{GL}_3(\mathbb{F}_2)$ that fixes the geodesic γ pointwise, then the action of this g on the Fano plane must stabilize the three lines l_i. In particular, it must fix each of the three points of intersection $P_{12} = l_1 \cap l_2$, $P_{23} = l_2 \cap l_3$ and $P_{13} = l_1 \cap l_3$. But a collineation of the Fano plane is uniquely determined by the image of any three non-collinear points. Since the three points P_{12}, P_{23} and P_{13} are non-collinear, and are fixed by g, we conclude that g must be the identity. Applying remark 2.1, we see that all the remaining terms in the splitting equation (2.1) vanish. We conclude that the algebraic K-groups $K_n(\mathbb{Z}\Gamma)$ are given by the expression in equation (3.4).

Next, let us provide some specific calculations for the lower algebraic K-theory of Γ. Using the pseudo-isotopy spectrum instead of the K-theory spectrum, an identical analysis yields the following analogues of formula (3.4) for the Wh, $\tilde{K}_0$, and K_{-1} functors:

$$\begin{aligned}
\mathrm{Wh}(\Gamma) &\cong \mathrm{coker}(\mathrm{Wh}(D_4) \to 2\,\mathrm{Wh}(S_4)) \oplus \ker(\tilde{K}_0(\mathbb{Z}D_4) \to 2\tilde{K}_0(\mathbb{Z}S_4)),\\
\tilde{K}_0(\mathbb{Z}\Gamma) &\cong \mathrm{coker}(\tilde{K}_0(\mathbb{Z}D_4) \to 2\tilde{K}_0(\mathbb{Z}S_4)) \oplus \ker(K_{-1}(\mathbb{Z}D_4) \to 2K_{-1}(\mathbb{Z}S_4)),\\
K_{-1}(\mathbb{Z}\Gamma) &\cong \mathrm{coker}(K_{-1}(\mathbb{Z}D_4) \to 2K_{-1}(\mathbb{Z}S_4)).
\end{aligned}$$

But it is known that the K_{-1}, $\tilde{K}_0$ and Wh all vanish for the group D_4 (see [27, §§ 3.1, 3.3 and 3.4]). Likewise, the K_{-1}, $\tilde{K}_0$ and Wh all vanish for the group S_4 (see example 3.4 for detailed references). We deduce that the K_{-1}, $\tilde{K}_0$, and Wh of

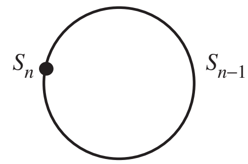

Figure 3. Graph of groups for the example in § 3.4.

the virtually free group Γ also vanish. From the known relationship between these functors and the original K-theory functors, we obtain

$$K_n(\mathbb{Z}\Gamma) \cong \begin{cases} 0, & n \leqslant -1, \\ \mathbb{Z}, & n = 0, \\ \mathbb{Z}_2 \oplus \Gamma^{ab}, & n = 1, \\ \text{finitely generated}, & n \geqslant 2. \end{cases}$$

We conclude this example by pointing out that it fits into a more general setting. Indeed, we can consider F as an arbitrary finite group of Lie type. Recall that, to any such group, there is associated a *Bruhat–Tits building*, a finite simplicial complex on which F acts. By restricting the action to the 1-skeleton of the building, we obtain an action of F by simplicial automorphisms on a graph. Using the geometry of the building, one can then proceed to analyse the algebraic K-theory of the corresponding virtually free group Γ. In the general setting, the resulting expressions are more complicated, but will always involve knowing the algebraic K-theory of various parabolic subgroups, with morphisms induced by subgroup inclusions. The contribution from the virtually cyclic subgroups will typically involve Bass Nil groups associated to the parabolic subgroups. We leave it to the interested reader to further analyse such examples.

3.4. Example: symmetric groups acting on a bouquet of circles

For our last class of examples, let us consider the group $\Gamma_n := F_n \rtimes S_n$, where the symmetric group S_n acts on the free group on n letters F_n by permuting the generators. In terms of the discussion in § 1.1, we can think of F_n as the fundamental group of the bouquet of n circles $\mathcal{G} := B_n$ and the symmetric group $S_n \leqslant \operatorname{Aut} B_n$ acting by permutation on the individual (oriented) circles in B_n. Observe that in this example, the S_n-action preserves the base point, so we really do have $S_n \to \operatorname{Aut} F_n$ (i.e. the natural map $S_n \to \operatorname{Out} F_n$ lifts to $\operatorname{Aut} F_n$).

Note that S_n acts transitively (without inversions), on the loops in B_n, so the quotient space $\hat{\mathcal{G}} = B_n/S_n$ consists of a single loop. The subgroup of S_n that leaves the base of the bouquet invariant is the full subgroup S_n, while the subgroup leaving one of the circles in B_n invariant is a copy of S_{n-1}. We conclude that the graph of finite groups description of Γ_n is a single loop, with vertex group S_n and edge group S_{n-1}. The edge-to-vertex morphisms are both given by the inclusion $S_{n-1} \hookrightarrow S_n$ of S_{n-1} as the subgroup fixing one of the n elements on which S_n acts (see figure 3).

Let T be the Bass–Serre tree of this graph of groups (which coincides with $\tilde{B}_n$, the universal cover of the bouquet of n circles). As discussed in § 2.2, the first term in the splitting (2.1) will be given by

$$H_i^{\Gamma}(T;\mathbb{KZ}^{-\infty}) \cong \operatorname{coker}(K_i(\mathbb{Z}S_{n-1}) \to K_i(\mathbb{Z}S_n)) \oplus \ker(K_{i-1}(\mathbb{Z}S_{n-1}) \to K_{i-1}(\mathbb{Z}S_n)).$$

Since both the edge-to-vertex morphisms are given by the same inclusion, taking the orientation of the edge into account, we see that the morphisms from each $K_i(\mathbb{Z}S_{n-1})$ to each $K_i(\mathbb{Z}S_n)$ is in fact the zero map. This allows the first term in the splitting to further reduce to

$$H_i^{\Gamma}(T;\mathbb{KZ}^{-\infty}) \cong K_i(\mathbb{Z}S_n) \oplus K_{i-1}(\mathbb{Z}S_{n-1}).$$

To understand the second term in the splitting (2.1), we follow the method discussed in §§ 2.3–2.5. The second term in the splitting is parametrized by S_n-orbits of closed geodesics in the bouquet B_n of n-circles. A closed geodesic $\gamma \subset B_n$ can be described by the sequence of circles through which the geodesic passes. We say that γ has *type* k (with $1 \leqslant k \leqslant n$) if it passes through precisely k of the n circles in B_n. So for a geodesic of type k, we see that $\mathrm{Fix}_{S_n}(\gamma) \cong S_{n-k}$, which can be identified with all permutations of the $n-k$ circles which γ does *not* pass through (with the convention that $S_1 \cong S_0$ are the trivial group). Consider the word in $F_k = \pi_1(B_k)$ represented by the primitive closed geodesic γ. The subgroup $\mathrm{Stab}_{S_n}(\gamma)$ will contain an element acting by a rotation on γ if and only if the corresponding (cyclic) word in F_k is invariant under some permutation of the k letters. Moreover, such a permutation gives rise to the desired element in $\mathrm{Stab}_{S_n}(\gamma)$ which acts via a rotation. We choose such a permutation τ which effects a minimal rotation on γ. The induced action of this permutation on the remaining $\mathrm{Fix}_{S_n}(\gamma) \cong S_{n-k}$ is clearly trivial, as these two permutations act on disjoint subsets. We conclude that $\mathrm{Stab}_{S_n}(\gamma) \cong S_{n-k} \times \mathbb{Z}_r$ for some r. Finally, applying remark 2.1, we conclude that each S_n-orbit of primitive closed geodesic of type k contributes either one or two copies of the Bass Nil group $NK_*(\mathbb{Z}S_{n-k})$ to the second term in the splitting formula (2.1). Now there is precisely one S_n-orbit of primitive closed geodesic of type 1, as any such geodesic can only loop around *one* of the circles (since it has type 1), and can only loop around *once* (since it is primitive). Moreover, the stabilizer of this geodesic must *preserve* the orientation, so we get a contribution of $2 \cdot NK_*(\mathbb{Z}S_{n-1})$ to the second term in the splitting (2.1). On the other hand, for $2 \leqslant k \leqslant n$, there are countably infinitely many distinct S_n-orbits of primitive closed geodesics of type k. Putting all this together, we find that the second term in the splitting (2.1) is given by the following direct sum, which we will denote by $\mathrm{NIL}_*(n)$:

$$\mathrm{NIL}_*(n) := 2 \cdot NK_*(\mathbb{Z}S_{n-1}) \oplus \bigoplus_{1 \leqslant k \leqslant n-2} \Bigl(\bigoplus_{\infty} NK_*(\mathbb{Z}S_k) \Bigr),$$

where the ∞ in the direct sum means a countable direct sum of infinitely many copies of $NK_*(\mathbb{Z}S_k)$. To conclude, the algebraic K-theory of Γ_n is given by

$$K_i(\mathbb{Z}\Gamma_n) \cong K_i(\mathbb{Z}S_n) \oplus K_{i-1}(\mathbb{Z}S_{n-1}) \oplus \mathrm{NIL}_i(n). \tag{3.5}$$

	S_1	S_2	S_3	S_4	S_5
K_1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
K_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$
K_{-1}	0	0	0	0	$\mathbb{Z}^2$

Figure 4. Non-vanishing algebraic K-groups $K_i(\mathbb{Z}S_n)$ associated to the symmetric groups S_n within the range $1 \leqslant n \leqslant 5$, $i \leqslant 1$.

Finally, let us apply the above formula to give explicit expressions for $K_i(\mathbb{Z}\Gamma_n)$ when i and n are small. Within the range $1 \leqslant n \leqslant 5$, $i \leqslant 1$, figure 4 summarizes the non-vanishing algebraic K-groups $K_i(\mathbb{Z}S_n)$ associated to the symmetric groups S_n.

To justify the entries in the chart, we first remark that Carter showed that for G, a finite group, $K_i(\mathbb{Z}G)$ vanishes for $i \leqslant -2$. This leaves the K_{-1}, K_0 and K_1 as potentially non-vanishing terms. For the symmetric groups S_n, $n \leqslant 5$, the references for the corresponding entries in the chart are as follows.

$n = 2$. The symmetric group S_2 is isomorphic to the cyclic group $\mathbb{Z}_2$ of order 2, and the lower algebraic K-theory of the latter is well known. The vanishing of K_{-1} can be found in [3, theorem 10.6]. The value of K_0 can be deduced from [35, theorem 2.1]. The value of K_1 can be found in [19, § 5].

$n = 3$. The symmetric group S_3 is isomorphic to the dihedral group D_3 of order 6, and the lower algebraic K-theory of the latter is well known. The vanishing of K_{-1} can be deduced from [6, p. 1928]. The value of K_0 can be deduced from [34, theorem 8.2]. The value of K_1 can be deduced from the main result in [29] (or see also [31, theorem 14.1]).

$n = 4$. For the symmetric group S_4, the vanishing of K_{-1} follows from a remark in [5, p. 606]. The K_0 can be deduced from [35, theorem 3.2]. The K_1 can be computed using [31, theorem 14.1].

$n = 5$. For the symmetric group S_5, the K_{-1} can be computed using Carter's formula [5,6] (see below). The K_0 can be deduced from [35, theorem 3.2], and the K_1 can be computed using [31, theorem 14.1].

Concerning the $\mathrm{NIL}_*(n)$ term, we recall that [18] ensures that the Bass Nil groups $NK_1(\mathbb{Z}G)$, $NK_0(\mathbb{Z}G)$ *vanish* for groups of square-free order. Since the symmetric groups S_1, S_2 and S_3 are of square-free order, we obtain that the summands $\mathrm{NIL}_0(n)$, $\mathrm{NIL}_1(n)$ vanish for $n \leqslant 4$, and that $\mathrm{NIL}_0(5) \cong 2 \cdot NK_0(\mathbb{Z}S_4)$, $\mathrm{NIL}_1(5) \cong 2 \cdot NK_1(\mathbb{Z}S_4)$.

Substituting these various expressions into equation (3.5), we deduce the following explicit results for the lower K-groups of $\varGamma_n$, $n \leqslant 5$:

$$K_{-1}(\mathbb{Z}\varGamma_n) \cong \begin{cases} 0, & n \leqslant 4, \\ \mathbb{Z}^2, & n = 5, \end{cases}$$

$$K_0(\mathbb{Z}\varGamma_n) \cong \begin{cases} \mathbb{Z}, & n \leqslant 4, \\ \mathbb{Z} \oplus \mathbb{Z}_2 \oplus 2 \cdot NK_0(\mathbb{Z}S_4), & n = 5, \end{cases}$$

$$K_1(\mathbb{Z}\varGamma_n) \cong \begin{cases} \mathbb{Z}_2, & n = 1, \\ \mathbb{Z} \oplus \mathbb{Z}_2^2, & 2 \leqslant n \leqslant 4, \\ \mathbb{Z} \oplus \mathbb{Z}_2^2 \oplus 2 \cdot NK_1(\mathbb{Z}S_4), & n = 5. \end{cases}$$

Finally, let us comment on the case where $n \geqslant 6$. In this setting, the K_1 is still well understood. In fact, we have that $K_1(\mathbb{Z}S_n) \cong \mathbb{Z}_2^2$ for all $n \geqslant 2$ (this can be easily deduced from [31, theorem 14.1]). Likewise, the K_{-1} is also well understood. Carter [6, p. 1928] ensures that $K_{-1}(\mathbb{Z}S_n)$ is always torsion-free. Moreover, for any finite group G, Carter [5] provides the following formula for the rank of the K_{-1}:

$$rk(K_{-1}(\mathbb{Z}G)) = 1 - r_{\mathbb{Q}} + \sum_{p||G|} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}).$$

In this formula, the summation is over all primes p dividing $|G|$, $\mathbb{Q}_p$ denotes the field of p-adics, $\mathbb{F}_p$ denotes the finite field with p elements, and r_F denotes the number of irreducible FG-modules.

In the special case of a symmetric group S_n, the representation theory has been extensively studied (see, for example, [21]). The number of irreducible FS_n-modules over any field F of characteristic zero is given by $P(n)$, the number of partitions of the integer n. Over the finite fields $\mathbb{F}_p$, the number of irreducible $\mathbb{F}_pS_n$-modules is given by $P_p(n)$, the number of p-regular partitions of n, where a partition is p-regular provided it does *not* have p parts of equal size [20]. For a concrete example, when $n = 5$, we have that $P(5) = 7$, $P_2(5) = 3$, $P_3(5) = 5$ and $P_5(5) = 6$, giving us that $K_{-1}(\mathbb{Z}S_5) \cong \mathbb{Z}^2$, as reported in our chart. Since we know that $K_{-1}(\mathbb{Z}\varGamma_n) \cong K_{-1}(\mathbb{Z}S_n)$, this gives us a simple number-theoretic formula for $K_{-1}(\mathbb{Z}\varGamma_n)$, for all n.

In contrast to the K_1 and K_{-1} functors, the K_0 for the symmetric groups S_n are difficult to compute, and do not seem known for $n \geqslant 6$. Unfortunately, the K_0 of symmetric groups appear as summands in the expression for both the groups $K_0(\mathbb{Z}\varGamma_n)$ and $K_1(\mathbb{Z}\varGamma_n)$, making these two groups equally difficult to compute.

Concerning the Bass Nil groups that appear in equation (3.1), the situation is even more complicated. Other than vanishing results, very few Bass Nil groups are explicitly known. In fact, the only non-trivial Bass Nil groups known to the authors are the NK_0 and NK_1 for the groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$ and D_4 (and, for the latter, only NK_0 is known [42]). For NK_2, some non-vanishing results have been obtained. For instance, [22] has shown that $NK_2(\mathbb{Z}[\mathbb{Z}_2])$ is infinitely generated. Since this group is a summand inside $\mathrm{NIL}_2(n)$ whenever $n \geqslant 3$, this tells us that $K_2(\mathbb{Z}\varGamma_n)$ is infinitely generated for $n \geqslant 3$. When $i \geqslant 3$, virtually nothing is known about the NK_i.

4. Concluding remarks

Our procedure for computing the algebraic K-theory of virtually free groups relies on the assumption that the group is given either in the form of a finite graph of finite groups (as in (B) from § 1.1), or in the form of a finite group acting on a finite graph (as in (C) from § 1.1). This assumption is somewhat unsatisfying, which motivates the following problem.

PROBLEM 4.1. Given a finite presentation of a group Γ and a collection of elements that generate a finite-index free subgroup, can one find an efficient way to construct an associated finite graph of finite groups? Or equivalently, can one find a simple way to find an associated finite group action on a finite graph?

We mentioned in § 1.1 that the equivalence (A) = (B) was the content of Solitar's conjecture, resolved affirmatively in [24]. In fact, the proof of the Solitar conjecture (A) $\subseteq$ (B) makes use of Stallings' characterization of finitely generated groups with infinitely many ends [39]. While there are now many different proofs of this celebrated result, all of the proofs we have seen are existential in nature rather than algorithmic.

In a different vein, recall from § 2.4 that each primitive closed geodesic γ in the finite graph $\mathcal{G}$ yields a contribution to the algebraic K-theory of the corresponding virtually free group Γ (via one of the summands in the last term of the splitting (2.1)). This contribution consists of *either one or two copies* of a certain Farrell Nil group. Whether one or two copies arise depends on whether or not all the elements in F stabilizing γ preserve the orientation of γ. If we are only interested in computing the algebraic K-groups up to isomorphism, we can pose the following problem.

PROBLEM 4.2. Given a finite group G, and an automorphism $\alpha \in \mathrm{Aut}(G)$, do the associated Farrell Nil groups $NK_*(\mathbb{Z}G, \alpha)$ have the property that there is an isomorphism of groups $NK_*(\mathbb{Z}G, \alpha) \cong NK_*(\mathbb{Z}G, \alpha) \oplus NK_*(\mathbb{Z}G, \alpha)$?

An affirmative answer to this question would imply that, in our method for computing the algebraic K-groups, we would no longer need to worry about keeping track of orientations on geodesics. It would also allow us to give simple *upper bounds* on the algebraic K-groups of virtually free groups, as one could bound the last term in the splitting formula (2.1) by the direct sum of finitely many Nil groups (for instance, taking the Nil groups for all finite subgroups of F). Finally, it would imply that if a finite group H embeds in some Farrell Nil group $NK_*(\mathbb{Z}G, \alpha)$, then the countably infinite sum $\bigoplus_\infty H$ also has to embed in the same Nil group. This would tell us, for instance, that groups such as $\mathbb{Z}_4 \oplus \bigoplus_\infty \mathbb{Z}_2$ *cannot* occur as Nil groups. More generally, given an upper bound on the exponent of $NK_i(\mathbb{Z}G)$, one could deduce that there are only *finitely many* possibilities for the isomorphism type of $NK_i(\mathbb{Z}G)$ (independent of i and G).

The results in this paper focus on groups G which fit into a short exact sequence $1 \to K \to G \to Q \to 1$, where K is a finitely generated free group, and Q is a finite group. If, instead, we allow more general groups for the kernel and the quotient, we are led to the following problem.

PROBLEM 4.3. Assume that G fits into a short exact sequence $1 \to K \to G \to Q \to 1$.

- If K and Q satisfy the FJIC, does it follow that G satisfies the FJIC?
- If finite-dimensional models are given for $\underline{\underline{E}}K, \underline{\underline{E}}Q$, when can one construct a finite-dimensional model for $\underline{\underline{E}}G$?

Perhaps the special case of most interest is the case when Q is a virtually cyclic group, since [14, Appendix] suggests that this should be the key to understanding the situation for more general Q (though in that appendix they work with the stronger *fibred* FJIC). Concerning the construction of $\underline{\underline{E}}G$, we mention [16], which provides some partial answers for the special case where $Q = \mathbb{Z}$.

Also of interest is the case where $Q = \mathrm{Aut}(K)$, and $G = \mathrm{Hol}(K)$ is the holomorph of K, i.e. the universal split extension of K. For example, this was completely worked out in [30] for the case when $G = \mathrm{Hol}(F_2)$.

A third case of interest is where K is poly-free (rather than free), while Q is finite. The corresponding class of extensions includes the various braid groups on surfaces [15].

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