ASYMPTOTIC CONES, BI-LIPSCHITZ ULTRAFLATS, AND THE GEOMETRIC RANK OF GEODESICS.

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Abstract. Let M be a closed non-positively curved Riemannian (NPCR) manifold, $\tilde{M}$ its universal cover, and X an ultralimit of $\tilde{M}$. For $\gamma \subset \tilde{M}$ a geodesic, let γ_ω be a geodesic in X obtained as an ultralimit of γ. We show that if γ_ω is contained in a flat in X, then the original geodesic γ supports a non-trivial, normal, parallel Jacobi field. In particular, the rank of a geodesic can be detected from the ultralimit of the universal cover. We strengthen this result by allowing for bi-Lipschitz flats satisfying certain additional hypotheses.

As applications we obtain (1) constraints on the behavior of quasi-isometries between complete, simply connected, NPCR manifolds, and (2) constraints on the NPCR metrics supported by certain manifolds, and (3) a correspondence between metric splittings of complete, simply connected NPCR manifolds, and metric splittings of its asymptotic cones. Furthermore, combining our results with the Ballmann-Burns-Spatzier rigidity theorem and the classic Mostow rigidity, we also obtain (4) a new proof of Gromov's rigidity theorem for higher rank locally symmetric spaces.

1. Introduction.

Ultralimits have revealed themselves to be a particularly useful tool in geometric group theory. Indeed, a number of spectacular results have been obtained via the use of ultralimits, including:

- Gromov's polynomial growth theorem [G], [VW]
- Kleiner and Leeb's quasi-isometric rigidity theorem for lattices in higher rank semi-simple Lie groups [KlL]
- Kapovich, Kleiner, and Leeb's theorem on detecting de Rham decompositions for universal covers of Hadamard manifolds [KKL]
- Kapovich and Leeb's proof that quasi-isometries preserve the JSJ decomposition of Haken 3-manifolds [KaL]
- Drutu and Sapir's characterization of (strongly) relatively hyperbolic groups in terms of ultralimits [DS]

In the present note, we show that ultralimits of simply connected Riemannian manifolds M of non-positive sectional curvature can be used to detect the geometric rank of geodesics in M. More precisely, we establish the following:

Theorem 1.1. *Let M be a simply connected, complete, Riemannian manifold of non-positive sectional curvature, and let $Cone(M)$ be an asymptotic cone of M. For $\gamma \subset M$ an arbitrary geodesic, let $\gamma_\omega \subset Cone(M)$ be the corresponding geodesic in the asymptotic cone. If there exists a flat plane $F \subset Cone(M)$ with $\gamma_\omega \subset F$, then there exists a non-trivial parallel Jacobi field J along γ satisfying $\langle J(t), \dot{\gamma}(t)\rangle = 0$. In particular, the geodesic γ has higher rank.*

Let us briefly explain the layout of the present paper. In Section 2, we provide a quick review of the requisite notions concerning asymptotic cones, variation of arclength formulas for geodesic variations, and other background material. In Section 3, we provide conditions ensuring existence of a non-trivial, orthogonal, Jacobi field along a geodesic γ. The conditions involve existence of what we call *pointed flattening sequences* of 4-tuples for the geodesic γ. The arguments in this section are purely differential geometric in nature. In Section 4, we show that if $\gamma_\omega \subset Cone(M)$ is contained in a flat, then pointed flattening sequences of 4-tuples can be constructed along γ (completing the proof of Theorem 1.1). The arguments here rely on some elementary arguments concerning asymptotic cones and the "large-scale geometry" of the manifold M. In Section 5, we establish some improvements by allowing for $\gamma_\omega \subset Cone(M)$ to be contained in a *bi-Lipschitz flat*. The precise result is contained in:

Theorem 1.2. *Let M be a simply connected, complete, Riemannian manifold of non-positive sectional curvature, and let $Cone(M)$ be an asymptotic cone of M. For $\gamma \subset M$ an arbitrary geodesic, let $\gamma_\omega \subset Cone(M)$ be the corresponding geodesic in the asymptotic cone. Assume that:*

- *there exists $g \in Isom(M)$ which stabilizes and acts cocompactly on γ, and*
- *there exists a bi-Lipschitzly embedded flat $\phi : \mathbb{R}^2 \hookrightarrow Cone(M)$ mapping the x-axis onto γ_ω.*

Then the original geodesic γ has higher rank.

Finally, in Section 6, we apply our Theorem 1.2 to obtain various geometrical corollaries. These include:

- constraints on the possible quasi-isometries between certain non-positively curved Riemannian manifolds.
- restrictions on the possible non-positively curved Riemannian metrics that are supported by certain manifolds.
- a proof that splittings of simply connected non-positively curved Riemannian manifolds correspond exactly with metric splittings of the asymptotic cones.
- a new proof of Gromov's rigidity theorem [BGS]: a closed higher rank locally symmetric space supports a unique metric of non-positive curvature (up to homothety).

Finally, we point out that various authors have studied geometric properties encoded in the asymptotic cone of non-positively curved manifolds. Perhaps the viewpoint closest to ours is that of Kapovich-Kleiner-Leeb paper [KKL], which focus on studying the (local homological) topology of the asymptotic cone to recover geometric information on the original space.

We should also mention the recent preprint of Bestvina-Fujiwara [BeFu], which gives a bounded cohomological characterization of higher rank symmetric spaces. Although they do not specifically discuss ultralimits, their discussion of rank 1 isometries seems to bear some philosophical similarities to our work.

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2. Background Material

2.1. **Introduction to asymptotic cones.** In this section, we provide some background on ultralimits and asymptotic cones of metric spaces. Let us start with some basic reminders on ultrafilters.

Definition. *A non-principal ultrafilter on the natural numbers* $\mathbb{N}$ *is a collection* $\mathcal{U}$ *of subsets of* $\mathbb{N}$*, satisfying the following four axioms:*

1. *if* $S \in \mathcal{U}$*, and* $S' \supset S$*, then* $S' \in \mathcal{U}$*,*
2. *if* $S \subset \mathbb{N}$ *is a finite subset, then* $S \notin \mathcal{U}$*,*
3. *if* $S, S' \in \mathcal{U}$*, then* $S \cap S' \in \mathcal{U}$*,*
4. *given any finite partition* $\mathbb{N} = S_1 \cup \ldots \cup S_k$ *into pairwise disjoint sets, there is a unique* S_i *satisfying* $S_i \in \mathcal{U}$*.*

Zorn's Lemma guarantees the existence of non-principal ultrafilters. Now given a compact Hausdorff space X and a map $f : \mathbb{N} \to X$, there is a unique point $f_\omega \in X$ such that every neighborhood U of f_ω satisfies $f^{-1}(U) \in \mathcal{U}$. This point is called the $\omega-$limit of the sequence $\{f(i)\}$; we write $\omega \lim f(i) = f_\omega$. In particular, if the target space X is the compact space $[0, \infty]$, we have that f_ω is a well-defined real number (or ∞).

Definition. *Let* $(X, d, *)$ *be a pointed metric space,* $X^{\mathbb{N}}$ *the collection of* X*-valued sequences, and* $\lambda : \mathbb{N} \to (0, \infty) \subset [0, \infty]$ *a sequence of real numbers satisfying* $\lambda_\omega = \infty$*. Given any pair of points* $\{x_i\}, \{y_i\}$ *in* $X^{\mathbb{N}}$*, we define the pseudo-distance* $d_\omega(\{x_i\}, \{y_i\})$

between them to be f_ω, where $f : \mathbb{N} \to [0, \infty)$ is the function $f(k) = d(x_k, y_k)/\lambda(k)$. Observe that this pseudo-distance takes on values in $[0, \infty]$.

Next, note that $X^{\mathbb{N}}$ has a distinguished point, corresponding to the constant sequence $\{*\}$. Restricting to the subset of $X^{\mathbb{N}}$ consisting of sequences $\{x_i\}$ satisfying $d_\omega(\{x_i\}, \{*\}) < \infty$, and identifying sequences whose d_ω distance is zero, one obtains a genuine pointed metric space $(X_\omega, d_\omega, *_\omega)$, which we call an asymptotic cone of the pointed metric space $(X, d, *)$.

We will usually denote an asymptotic cone by $Cone(X)$. The reader should keep in mind that the construction of $Cone(X)$ involves a number of choices (basepoints, sequence λ_i, choice of non-principal ultrafilters) and that different choices could give different (non-homeomorphic) asymptotic cones (see the papers [TV], [KSTT], [OS]).

We will require the following facts concerning asymptotic cones of non-positively curved spaces:

- if (X, d) is a CAT(0) space, then $Cone(X)$ is likewise a CAT(0) space,
- if $\phi : X \to Y$ is a (C, K)-quasi-isometric map, then ϕ induces a bi-Lipschitz map $\phi_\omega : Cone(X) \to Cone(Y)$,
- if $\gamma \subset X$ is a geodesic, then $\gamma_\omega := Cone(\gamma) \subset Cone(X)$ is a geodesic,
- if $\{a_i\}, \{b_i\} \in Cone(X)$ are an arbitrary pair of points, then the ultralimit of the geodesic segments $\overline{a_i b_i}$ gives a geodesic segment $\overline{\{a_i\}\{b_i\}}$ joining $\{a_i\}$ to $\{b_i\}$.

Concerning the second point above, we remind the reader that a (C, K)-quasi-isometric map $\phi : (X, d_X) \to (Y, d_Y)$ between metric spaces is a (not necessarily continuous) map having the property that:

$$\frac{1}{C} \cdot d_X(p, q) - K \leq d_Y(\phi(p), \phi(q)) \leq C \cdot d_X(p, q) + K.$$

We now proceed to establish two Lemmas which will be used in some of our proofs.

Lemma 2.1 (Choosing good sequences)**.** *Let X be a CAT(0) space, $Cone(X)$ an asymptotic cone of X, $\gamma \subset X$ a geodesic, and $\gamma_\omega \subset Cone(X)$ the corresponding geodesic in the asymptotic cone. Assume that $\{A, B, C, D\} \subset Cone(X)$ is a 4-tuple of points having the property that $A, B \in \gamma_\omega$ are the closest points on γ_ω to the points D, C (respectively). Let $\{C_i\}, \{D_i\} \subset X$ be two sequences representing the points $C, D \in Cone(X)$ respectively. Then*

(1) *if $A_i, B_i \in \gamma$ are the closest points to D_i, C_i (respectively), then $\{A_i\}, \{B_i\}$ represent $A, B \in Cone(X)$ respectively.*

(2) *if $\{r_i\} \subset \mathbb{R}^+$ is a sequence of real numbers satisfying $\omega \lim\{r_i/\lambda(i)\} = d_\omega(A, D)$, and $D'_i \in \overrightarrow{A_i D_i}$ satisfies $d(A_i, D'_i) = r_i$, then the sequence $\{D'_i\}$ represents $D \in Cone(X)$.*

Lemma 2.1 allows us to replace, in certain circumstances, a given sequence of 4-tuples representing $\{A, B, C, D\} \subset Cone(X)$ by a new sequence of 4-tuples that are geometrically better behaved (i.e. have better metric properties).

Proof (Lemma 2.1). To establish (1), we assume without loss of generality that the constant sequence $\{*\}$ of basepoints used to define $* \in Cone(X)$ is chosen to lie on γ. Then the triangle inequality, combined with the fact that A_i is the closest point to D_i on γ, immediately implies:

$$d(*, A_i) \leq d(A_i, D_i) + d(D_i, *) \leq 2d(D_i, *)$$

This in turns implies that $d_\omega(\{A_i\}, *) \leq 2d_\omega(\{D_i\}, *) < \infty$, i.e. $\{A_i\}$ does define a point $A_\omega \in Cone(X)$. An identical argument shows that $\{B_i\}$ defines a point $B_\omega \in Cone(X)$. Furthermore, since all the points A_i, B_i are on γ, we have that $A_\omega, B_\omega \in \gamma_\omega \subset Cone(X)$. We now claim that $A_\omega = A$ and $B_\omega = B$. To see this, we note that the sequence of geodesic segments $\{\overline{D_i A_i}\}$ gives rise to a geodesic segment $\overline{DA_\omega}$ joining $D \in Cone(X)$ to the point $A_\omega \in \gamma_\omega \subset Cone(X)$. Since each $\overline{D_i A_i}$ was a minimal length segment joining D_i to γ, the segment $\overline{DA_\omega}$ is likewise a minimal length segment joining D to γ_ω. But we know that the closest point on γ_ω to D is A (and this is the unique such point, as $Cone(X)$ is CAT(0)). We conclude that $A_\omega = A$, as desired. An identical argument applies to show $B_\omega = B$, completing the argument for (1).

To establish (2), we first note that the sequence of geodesic rays $\{\overrightarrow{A_i D_i}\}$ define some geodesic ray $\vec{\eta} \subset Cone(X)$. Furthermore, by construction, we have that $\vec{\eta}$ originates at A, and passes through D. Now again, an easy application of the triangle inequality implies that the sequence $\{D_i'\}$ represents a point $D_\omega \in Cone(X)$, which we are claiming coincides with the point D. Since each D_i' is chosen to lie on the corresponding geodesic ray $\overrightarrow{A_i D_i}$, we immediately get $D_\omega \in \vec{\eta}$. Finally, let us calculate the distance between D_ω and the point A: $d_\omega(A, D_\omega) = \omega \lim\{d(A_i, D_i')/\lambda(i)\} = \omega \lim\{r_i/\lambda(i)\} = d_\omega(A, D)$. So we see that D_ω, D are a pair of points on the geodesic ray $\vec{\eta}$, having the property that they are both at the exact same distance from the basepoint A of the geodesic ray. This immediately implies that they have to coincide, completing the argument for (2), and hence the proof of Lemma 2.1.

$\square$

Lemma 2.2 (Translations on asymptotic cone). *Let X be a geodesic space, $\gamma \subset X$ a geodesic, and $\gamma_\omega \subset Cone(X)$ the corresponding geodesic in an asymptotic cone $Cone(X)$ of X. Assume that there exists an element $g \in Isom(X)$ with the property that g leaves γ invariant, and acts cocompactly on γ. Then for any pair of points $p, q \in \gamma_\omega$, there is an isometry $\Phi : Cone(X) \to Cone(X)$ satisfying $\Phi(p) = q$.*

Proof (Lemma 2.2). Let $\{p_i\}, \{q_i\} \subset \gamma \subset X$ be sequences defining the points p, q respectively. Since g leaves γ invariant, and acts cocompactly on γ, there exists a real number $R > 0$ and a sequence of exponents $k_i \in \mathbb{Z}$ with the property that for every index i, we have $d(g^{k_i}(p_i), q_i) \leq R$.

Now observe that the sequence $\{g^{k_i}\}$ of isometries of X defines a self-map (defined componentwise) of the space $X^{\mathbb{N}}$ of sequences of points in X. Let us denote by g_ω this self-map, which we now proceed to show induces the desired isometry on $Cone(X)$. First note that it is immediate that g_ω preserves the pseudo-distance d_ω on $X^{\mathbb{N}}$, and has the property that $d_\omega(\{g^{k_i}(p_i)\}, \{q_i\}) = 0$. So to see that g_ω descends to an isometry of $Cone(X)$, all we have to establish is that for $\{x_i\}$ a sequence satisfying $d_\omega(\{x_i\}, *) < \infty$, the image sequence also satisfies $d_\omega(\{g^{k_i}(x_i)\}, *) < \infty$. But we have the series of equivalences:

$$
\begin{aligned}
d_\omega(\{x_i\}, *) < \infty \iff & d_\omega(\{x_i\}, \{p_i\}) < \infty \\
\iff & d_\omega(\{g^{k_i}(x_i)\}, \{g^{k_i}(p_i)\}) < \infty \\
\iff & d_\omega(\{g^{k_i}(x_i)\}, \{q_i\}) < \infty \\
\iff & d_\omega(\{g^{k_i}(x_i)\}, *) < \infty
\end{aligned}
$$

where the first and last equivalences come from applying the triangle inequality in the pseudo-metric space $(X^{\mathbb{N}}, d_\omega)$, and the second and third equivalences follow from our earlier comments. We conclude that the induced isometry g_ω on the pseudo-metric space $X^{\mathbb{N}}$ of sequences leaves invariant the subset of sequences at finite distance from the distinguished constant sequence, and hence descends to an isometry of $Cone(X)$. Finally, it is immediate from the definition of the isometry g_ω that it will leave γ_ω invariant, as each g^{k_i} leaves γ invariant. This concludes the proof of Lemma 2.2.

□

Observe that the element $g \in Isom(X)$ used in Lemma 2.2 gives rise to a $\mathbb{Z}$-action on X leaving γ invariant. It is worth pointing out that Lemma 2.2 does **not** state that $g \in Isom(X)$ induces an $\mathbb{R}$-action on $Cone(X)$. The issue is that for each $r \in \mathbb{R}$, there is indeed a corresponding isometry of $Cone(X)$, but these will not in general vary continuously with respect to r (as can already be seen in the case $X = \mathbb{H}^2$).

2.2. **Variation of arclength formulas.** The classical variation formulas deal with the energy of curves within a variation. This is primarily due to the fact that the energy functional is "easier" to differentiate than the length functional. In the situation we are interested in, the asymptotic cones pick up (asymptotic) distances, and hence we need to actually work with variations for the arclength rather than the energy. We now proceed to remind the reader of the (perhaps less familiar) variation formulas for arclength. A proof of the present formulas can be found in Jost's book [Jo, pgs. 165-169].

Let us start out by setting up some notation. We consider *geodesic variations*, which are maps $\sigma : [0,1] \times (-\epsilon, \epsilon) \to M$ into a Riemannian manifold ($s \in [0,1]$ will be the first parameter, $t \in (-\epsilon, \epsilon)$ the second parameter), satisfying the following three properties:

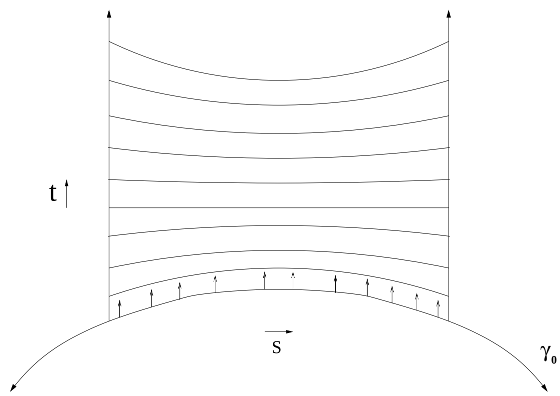

FIGURE 1. Geodesic variation.

- the curves $s \mapsto \gamma_t(s) = \sigma(s,t)$ is a geodesic for all t,
- the curves γ_t are parametrised with constant speed: $||\dot{\gamma}_t|| = L(t)$ where $L(t)$ is the length of the geodesic γ_t,
- the "lateral curves" $t \mapsto \sigma(0,t)$ and $t \mapsto \sigma(1,t)$ are geodesics.

We now denote by S, X the following vector fields:

$$S = D\sigma\left[\frac{\partial}{\partial s}\right] \qquad X = D\sigma\left[\frac{\partial}{\partial t}\right]$$

Finally, we denote by $\hat{X}$ the vector field obtained by taking the projection of X orthogonal to S.

Figure 1 provides an illustration of a geodesic variation. We have included the base geodesic (at the bottom of the picture) corresponding to $t = 0$, and have drawn the portion of σ corresponding to $t \in [0, \epsilon]$. The horizontal curves represent geodesic curves γ_t, while the two vertical curves are the "lateral curves". Along the geodesic γ, we have also illustrated a few values of the Jacobi vector field X (pointing straight up).

The variation formulas we will need are:

First variation of arclength: For $t_0 \in (-\epsilon, \epsilon)$, the first derivative of the length $L(t)$ at t_0 is given by (see [Jo, pg. 167, equation 4.1.4]):

$$\frac{dL}{dt}(t_0) = \frac{\langle S, X\rangle_{(1,t_0)} - \langle S, X\rangle_{(0,t_0)}}{L(t_0)}$$

Second variation of arclength: For $t_0 \in (-\epsilon, \epsilon)$, the second derivative of the length $L(t)$ at t_0 is given by (see [Jo, pg. 167, equation 4.1.7]):

$$\frac{d^2L}{dt^2}(t_0) = \frac{1}{L(t_0)}\Big(\int_0^1 ||\nabla_S \hat{X}||^2 - K(S \wedge \hat{X})L(t_0)^2||\hat{X}||^2 ds\Big)$$

where $K(S \wedge \hat{X})$ denotes the sectional curvature of the 2-plane spanned by S and $\hat{X}$.

Now observe that the actual arclength function L_i (and hence, its various derivatives) is in fact independent of the parametrization of the "horizontal geodesics" γ_t. Performing a change of variable, we can rewrite the second variation formula in terms of the unit speed parametrization:

$$\frac{d^2L}{dt^2}(t) = \int_0^{L(t)} ||\nabla_{\bar{S}} \hat{X}||^2 - K(\bar{S} \wedge \hat{X})||\hat{X}||^2 ds. \tag{1}$$

where now $\bar{S}$ denotes the *unit* vector in the direction of S, i.e. $\bar{S} = S/||S||$.

Notice that both X and the projection $\hat{X}$ of X orthogonal to S are Jacobi vector fields, as they arise from variations by geodesics (see Section 2.3). We point out an important consequence of the second variation formula in the context of non-positive curvature. In this setting, equation (1) immediately forces $\frac{d^2L}{dt^2}(t_0) \geq 0$ (since the expression inside the integral is ≥ 0).

2.3. **Jacobi fields, rank of geodesics, and rigidity.** For the convenience of the reader, we briefly recall some basic definitions from Riemannian geometry, referring the reader to [Jo] for more details. Given a geodesic γ in a Riemannian manifold M^n of dimension n, a vector field J along γ is said to be a *Jacobi field* if it satisfies the following second order differential equation:

$$J'' + R(J, \gamma')\gamma' \equiv 0$$

where J'' refer to the second covariant derivative of J along γ, and R denotes the curvature operator. We will require the following classical results concerning Jacobi fields:

- Jacobi fields along γ form a finite dimensional vector space (of dimension $2n$),
- a Jacobi field is uniquely determined by its value (initial conditions) at any two given points on γ,
- given a geodesic variation σ of γ as in the previous section, the "vertical vector field" X is a Jacobi field along γ,
- conversely, given a geodesic segment γ in a Riemannian manifold, and a Jacobi field J along γ, there exists a geodesic variation whose "vertical vector field" X coincides with J along γ.

Note in particular that the last two properties above tell us that Jacobi fields *exactly encode* the infinitesimal behavior of geodesic variations. A Jacobi field J that additionally satisfies $J' \equiv 0$ will be called a *parallel Jacobi fields* along γ. The *rank* of a geodesic γ is defined to be the dimension of the vector space of parallel Jacobi fields along γ. Since a concrete example of a parallel Jacobi field is given by the tangent vector field $V = \gamma'$ to the geodesic γ, we note that $rk(\gamma) \geq 1$ for every geodesic γ.

A celebrated result in the geometry of non-positively curved Riemannian manifolds is the rank rigidity theorem of Ballman-Burns-Spatzier [Ba2], [BuSp]:

Theorem 2.3 (Rank rigidity theorem)**.** *Let M be a closed non-positively curved Riemannian manifold, and $\tilde{M}$ the universal cover of M with the induced Riemannian structure. Assume that $\tilde{M}$ has* higher geometric rank*, in the sense that every geodesic $\gamma \subset \tilde{M}$ satisfies $rk(\gamma) \geq 2$. Then either:*

- *$\tilde{M}$ splits isometrically as a product of two simply connected Riemannian manifolds of non-positive curvature, or*
- *$\tilde{M}$ is an irreducible higher rank symmetric space of non-compact type.*

In Section 6, we will make extensive use of this rigidity result to obtain the various corollaries mentioned in the introduction.

2.4. **Distorted subspaces in metric spaces.** Let (X, ρ), (Y, d) be a pair of metric spaces, and $\phi : Y \to X$ an injective map. We define the *distortion* of the map ϕ to be the supremum, over all triples of distinct points $x, y, z \in Y$, of the quantity:

$$\Big|\frac{\rho(\phi(x), \phi(y))}{\rho(\phi(y), \phi(z))} - \frac{d(x, y)}{d(y, z)}\Big|$$

We denote the distortion of ϕ by $\delta(\phi)$. Observe that the distortion $\delta(\phi)$ measures the difference between relative distances in Y, and relative distances in $\phi(Y) \subset X$.

We say that a metric space (X, ρ) contains an *undistorted copy* of a metric space (Y, d) provided there exists an injective map $\phi : (Y, d) \hookrightarrow (X, \rho)$ with $\delta(\phi) = 0$. We say that X contains *almost undistorted copies* if for any $\epsilon > 0$, one can find a map $\phi_\epsilon : (Y, d) \to (X, \rho)$ with $\delta(\phi_\epsilon) < \epsilon$. Finally, given a sequence of maps $\phi_i : Y \to X$, we say that the sequence is *undistorted in the limit*, provided we have $\lim \delta(\phi_i) = 0$.

Let $\square$ denote the 4-point metric space, consisting of the vertex set of the standard unit square in $\mathbb{R}^2$, with the induced distance, i.e. $\square$ consists of four points, with the four "side" distances equal to one, and the two "diagonal" distances equal to $\sqrt{2}$. We call pairs of points at distance one a side pair of vertices. A large part of this paper will focus on finding and using (almost) undistorted copies of $\square$ inside simply connected complete Riemannian manifolds of non-positive curvature (and inside their asymptotic cones). Given a (cyclicly ordered) 4-tuple of points $\{A, B, C, D\}$ inside a space X, we will frequently identify the 4-tuple with a copy of $\square$, with the understanding that the ordered 4-tuple of points correspond to the cyclicly ordered points in the square. We now point out an easy lemma that allows us to occasionally "ignore diagonals."

Lemma 2.4. *Let $\{A_j, B_j, C_j, D_j\}$ be a sequence of 4-tuples inside a CAT(0) space X. Assume that each of the 4-tuples satisfies the conditions:*

- *the point B_j is the closest point to C_j on the geodesic segment $\overline{A_jB_j}$,*
- *the point A_j is the closest point to D_j on the geodesic segment $\overline{A_jB_j}$,*
- *we have equality of the side lengths $d(D_j, A_j) = d(A_j, B_j) = d(B_j, C_j) = K_j$,*
- *$d(C_j, D_j) = K_j(1 + \epsilon_j)$, with $\epsilon_j \to 0$.*

Then we have that $d(A_j, C_j)/K_j \to \sqrt{2}$ and $d(B_j, D_j)/K_j \to \sqrt{2}$.

Proof. Let us temporarily ignore the indices j, and for a 4-tuple $\{A, B, C, D\}$ of points as above, we let d_1, d_2 denote the lengths of the two diagonals $\overline{AC}, \overline{BD}$. We now want to control the two ratios d_i/K in terms of ϵ, and in fact, show that the ratios tend to $\sqrt{2}$ as $\epsilon \to 0$. But this is relatively easy to do: consider a comparison triangle $\bar{A}\bar{B}\bar{C} \subset \mathbb{R}^2$ for the triangle ABC. The fact that the point B is the closest point to C on the geodesic segment $\overline{AB}$ immediately implies that, in the comparison triangle, we have $\angle \bar{B} \geq \pi/2$. This in turn forces the inequality:

$$d_1^2 = d(\bar{A}, \bar{C})^2 \geq d(\bar{A}, \bar{B})^2 + d(\bar{B}, \bar{C})^2 = 2K^2 \quad \Longrightarrow \quad d_1 \geq K\sqrt{2}$$

An identical argument establishes $d_2 \geq K\sqrt{2}$. But on the other hand, we know that CAT(0) spaces satisfy, for any 4-tuples of points $\{A, B, C, D\}$ the inequality:

$$d(A, C)^2 + d(B, D)^2 \leq d(A, B)^2 + d(B, C)^2 + d(C, D)^2 + d(D, A)^2.$$

Substituting the known quantities into our expression, we obtain:

$$2 \cdot (K\sqrt{2})^2 \leq d_1^2 + d_2^2 \leq 3 \cdot K^2 + [K(1 + \epsilon)]^2$$

Dividing out by K^2, we see that the ratios $d_1/K, d_2/K$ are a pair of real numbers $\geq \sqrt{2}$ which satisfy the inequality:

$$4 \leq (d_1/K)^2 + (d_2/K)^2 \leq 3 + (1 + \epsilon)^2.$$

Now taking the indices j back into account, it is now immediate that as $\epsilon_j \to 0$, the ratios $d_1/K_j \to \sqrt{2}$ and $d_2/K_j \to \sqrt{2}$, as desired. This concludes the proof of Lemma 2.4.

□

3. From flattening 4-tuples to parallel Jacobi fields.

In this section, we focus on establishing how certain sequences of 4-tuples of points can be used to construct parallel Jacobi fields along geodesics. More precisely, we introduce the notion of:

Definition (Good 4-tuple)**.** *Let $\tilde{M}$ be a complete, simply connected, Riemmanian manifold of non-positive sectional curvature, and let $\gamma \subset \tilde{M}$ be an arbitrary geodesic. We say that that a 4-tuple of points $\{A, B, C, D\}$ in the space $\tilde{M}$ is* good *(relative to γ) provided that $A, B \in \gamma$, $\overline{AD} \perp \gamma$, $\overline{BC} \perp \gamma$, and $d(D, A) = d(A, B) = d(B, C)$.*

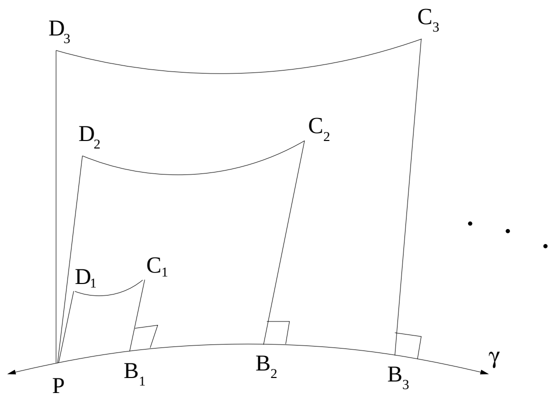

FIGURE 2. Pointed flattening sequence along a geodesic.

In effect, a good 4-tuple is a geodesic quadrilateral in $\tilde{M}$, with one side on the geodesic γ, the two adjacent sides perpendicular to γ, and with those three sides having exactly the same length.

Definition (Pointed flattening sequences)**.** *We say that γ has* pointed flattening 4-tuples *if given any point $P \in \gamma$, there exists a sequence of $\{P, B_i, C_i, D_i\}$ of 4-tuples, each of which is good (relative to γ), satisfies $\lim B_i = \gamma(\infty)$, and is undistorted in the limit.*

Figure 2 above illustrates the first three 4-tuples of a pointed flattening sequence. The sides of each quadrilateral are perpendicular to the bottom geodesic, and the length of the top edge approaches (as a ratio in the limit) the length of the remaining three edges of the quadrilaterals.

While the definition of pointed flattening sequences of 4-tuples might seem somewhat artificial, the reader will see in Sections 4 and 5 that these are relatively easy to detect from the asymptotic cone. The main goal of this section is to prove:

Theorem 3.1 (Pointed flattening sequence $\Rightarrow$ higher rank)**.** *Let $\tilde{M}$ be a complete, simply connected, Riemmanian manifold of non-positive sectional curvature, and let $\gamma \subset \tilde{M}$ be an arbitrary geodesic. If γ has pointed flattening 4-tuples, then γ supports a non-trivial, orthogonal, parallel Jacobi field. In particular, $rk(\gamma) \geq 2$.*

So let us start with some $P \in \gamma$, and let $\{P, B_i, C_i, D_i\}$ be the sequence of 4-tuples whose existence is ensured by the hypothesis that γ has pointed flattening sequences. Observe that the point $P = \gamma(r)$ divides the geodesic γ into two geodesic rays, and we denote by $\vec{\gamma}_P$ the geodesic ray obtained by restricting γ to $[r, \infty)$. Our approach will be to first construct a non-trivial, orthogonal, parallel Jacobi field along the geodesic ray $\vec{\gamma}_P$, and then let P tend to $\gamma(-\infty)$.

In order to construct the desired Jacobi field along the geodesic ray $\vec{\gamma}_P$, we consider geodesic variations σ_i in the space $\tilde{M}$, each of which is constructed from the corresponding 4-tuple $\{P, B_i, C_i, D_i\}$ as follows:

- $\alpha_i : [0, T_i] \to \tilde{M}$ denotes the unit speed geodesic from P to D_i, and β_i denotes the one from B_i to C_i. We set $L_i(t) = d(\alpha_i(t), \beta_i(t))$, so in particular we have $L_i(0) = T_i$.
- σ_i is parametrised by $\{(s,t) : t \in [0, T_i],\ s \in [0, T_i]\}$.
- σ_i, when restricted to the interval $\{0\} \times [0, T_i]$, coincides with α_i, and when restricted to the interval $\{1\} \times [0, T_i]$, with β_i.
- for every $t \in [0, T_i]$, the restriction of σ_i to the interval $[0, T_i] \times \{t\}$ is the constant speed geodesic from $\alpha_i(t)$ to $\beta_i(t)$.

Note that these maps are precisely variations by geodesics of the type discussed in section 2.2. Our goal will now be to analyze properties of the functions L_i. We start with the easy:

Fact 1: For any fixed value of i, the function L_i is twice differentiable and convex.

Twice differentiability follows immediately from the formulas for the first and second variation of arclength. Convexity is immediate from the fact that $L_i''(t) \geq 0$ (see the comment immediately after equation (1)).

Fact 2: For any i and any $0 \leq x \leq t \leq T_i$, we have

$$L_i(t) = L_i(x) + (t - x)L_i'(x) + \int_x^t \int_x^y L_i''(\tau) d\tau dy$$

This is nothing but the Fundamental Theorem of Calculus applied twice.

Fact 3: For any i and any $0 \leq t \leq T_i$, we have the following expression for $L_i(t)$:

$$L_i(t) = L_i(0) + \int_0^t \int_0^y L_i''(\tau) d\tau dy \tag{2}$$

By Fact 2, it is sufficient to argue that each of the derivatives $L_i'(0)$ is equal to zero. Now recall that the maps σ_i are geodesic variations with the property that each of the "lateral curves" $\alpha_i(t) = \overline{PD_i}$ and $\beta_i(t) = \overline{B_iC_i}$ are geodesics. Furthermore, since the 4-tuple $\{P, B_i, C_i, D_i\}$ is good (relative to γ), we have that the "lateral curves" are *perpendicular* to the geodesic γ. Now applying the first variation of arclength formula (section 2.2), we immediately see that $L_i'(0) = 0$, as desired.

Note that a consequence of Fact 3 is that each L_i is monotone non-decreasing. Now recall that the variations we are considering come from a pointed flattening sequence of 4-tuples, which in particular means that the corresponding maps $\square \to \tilde{M}$ are undistorted in the limit. In our current notation, we have that $d(C_i, D_i) = L_i(T_i)$ and $d(P, B_i) = T_i$, hence we obtain:

$$\lim_{i\to\infty} \frac{L_i(T_i)}{T_i} = \lim_{i\to\infty} \frac{d(C_i, D_i)}{d(P, B_i)} = 1,$$

Furthermore, recall that $L_i(0) = d(P, B_i) = T_i$, and in particular we have $L_i(0)/T_i = 1$ (for all i). Combining this with our equation (2) in Fact 3 (applied to $t = T_i$), we see that:

$$\lim_{i\to\infty} \int_0^{T_i} \int_0^y \frac{L_i''(\tau)}{T_i}\, d\tau\, dy = 0. \tag{3}$$

The next step is to get rid of the T_i factor inside the integral.

Fact 4: We have that:

$$\lim_{i\to\infty} \int_0^{T_i/2} L_i''(\tau)\, d\tau = 0.$$

To see this, we first observe that we have the obvious series of equalities:

$$\frac{1}{2}\int_0^{T_i/2} L_i''(\tau)\, d\tau = \frac{T_i}{2}\int_0^{T_i/2} \frac{L_i''(\tau)}{T_i}\, d\tau = \int_{T_i/2}^{T_i}\int_0^{T_i/2} \frac{L_i''(\tau)}{T_i}\, d\tau\, dy$$

Now recall from Fact 1 that $L_i''(\tau) \geq 0$ (by convexity), and hence the expression inside each of the integrands above is ≥ 0. But now, by positivity of each of the functions $L_i''(\tau)/T_i$, containment of the domains of integrations yields the following inequality:

$$\int_{T_i/2}^{T_i}\int_0^{T_i/2} \frac{L_i''(\tau)}{T_i}\, d\tau\, dy \leq \int_0^{T_i}\int_0^y \frac{L_i''(\tau)}{T_i}\, d\tau dy.$$

Combining this upper estimate with equation (3) above, we immediately obtain:

$$0 \leq \lim_{i\to\infty} \int_0^{T_i/2} L_i''(\tau)\, d\tau \leq 2 \cdot \lim_{i\to\infty} \int_0^{T_i}\int_0^y \frac{L_i''(\tau)}{T_i}\, d\tau\, dy = 0$$

completing the proof of Fact 4.

Next we note that a consequence of Fact 4 is that the sequence of functions $L_i''(t)$ tends to zero for almost all $t \in [0, T_i/2]$. In particular, we can find a sequence $\{t_i\}$ satisfying the following two conditions:

(1) $t_i \in [0, T_i]$ and $\lim_{i\to\infty} t_i = 0$
(2) $\lim_{i\to\infty} L_i''(t_i) = 0$.

Let us denote by $\gamma_i : [0,1] \to \tilde{M}$ the geodesic joining $\alpha_i(t_i)$ to $\beta_i(t_i)$. Note that these geodesics are precisely the curves $\sigma_i(-, t_i) : [0, L_i(t_i)] \to \tilde{M}$, where σ_i is our sequence of variations of geodesics. We next observe that:

Fact 5: The geodesic segments γ_i tend to the geodesic ray $\vec{\gamma}_P \subset \gamma$.

To see this, we first note since the "lateral curves" for the variation σ_i are geodesics perpendicular to γ (and since we have $K \leq 0$), we have

$$d(\alpha_i(t_i), \gamma) = t_i = d(\beta_i(t_i), \gamma)$$

In particular, we see that the geodesic segments γ_i join a pair of points whose distance from γ tends to zero. Since geodesic neighborhoods of γ are convex (by the non-positive curvature hypothesis), we conclude that the distance of any point on γ_i is at most t_i away from the geodesic γ, where t_i was chosen to tend to 0. Furthermore, we clearly have that $\lim \alpha_i(t_i) = P$, and $\lim \beta_i(t_i) = \gamma(\infty)$, and hence we obtain $\lim \gamma_i = \vec{\gamma}_P$, as desired.

Now along each of the geodesic segments γ_i, we have that the corresponding geodesic variation σ_i gives rise to a Jacobi vector field X_i. We now focus our attention to this sequence of Jacobi fields.

Fact 6: The Jacobi field X_i along γ_i satisfies $||X_i(p)|| \leq 1$ for all $p \in \gamma_i$.

To see this, first observe that $X_i(0) = \alpha_i'(t_i)$ and $X_i(L_i(t_i)) = \beta_i'(t_i)$. Since α_i and β_i are unit speed parametrized, this implies that

$$||X_i(0)|| = ||X_i(L_i(t_i))|| = 1.$$

But from the non-positive curvature assumption and the Jacobi differential equation, it follows that the square-norm of a Jacobi field along a geodesic is a convex function. Since $||X_i|| = 1$ at the endpoints of the geodesic γ_i, Fact 6 follows.

Fact 7: Up to possibly passing to subsequences, the Jacobi fields X_i along γ_i converge (uniformly on compact sets), to a Jacobi field X along $\vec{\gamma}_P$.

This follows from the general fact that a Jacobi field is determined by any two of its values. Take points p_i, q_i in γ_i that converge to points $p \neq q$ of $\vec{\gamma}_P$. From Fact 6, we see that up to possibly passing to subsequences, both $X_i(p_i)$ and $X_i(q_i)$ have a limit. Moreover, Jacobi fields are solution of ordinary differential equations with smooth coefficients (in fact with the regularity of the curvature tensor) and therefore depend continuously on the initial data (the values at p_i and q_i.) It follows that X_i converge to a Jacobi field X along γ uniformly on compact sets, and in particular point-wise.

Fact 8: The sequence $\{t_i\}$ can be chosen so that the limiting vector field X is perpendicular to the geodesic ray $\vec{\gamma}_P$.

To see this, we note that for each of the variations σ_i, we have the two associated continuous vector fields S_i, X_i (see section 2.2). Furthermore, note that these two vector fields are *orthogonal* along the base geodesic γ_i. Indeed, the vector field S_i

is just γ_i', while the vector field X_i is orthogonal to γ_i at the two endpoints of the variation (recall that α_i, β_i are $\perp$ to γ_i). But from the Jacobi equation, a Jacobi field that is orthogonal to a geodesic at a pair of points is orthogonal to the geodesic at *every* point.

Next observe that the inner product between the vectors X_i and S_i varies continuously along the domain of σ_i. Since we have $\langle X_i, S_i\rangle \equiv 0$ along the geodesic γ, by choosing t_i close enough to zero, one can ensure that

$$\lim_{i\to\infty} \sup_{x\in\gamma_i} \left|\langle X_i, S_i\rangle_x\right| = 0.$$

In particular, for *any* sequence of points $\{p_i\} \subset \tilde{M}$ satisfying $p_i \in \gamma_i$ and $\lim p_i = p \in \gamma$, we have that:

$$\langle X(p), \gamma'(p)\rangle = \langle \lim_{i\to\infty} X_i(p_i), \lim_{i\to\infty} \gamma_i'(p_i)\rangle = \lim_{i\to\infty} \langle X_i(p_i), S_i(p_i)\rangle = 0$$

Applying this to the two sequences of points with distinct limits, we see that the limiting vector field X is orthogonal to $\vec{\gamma}_P$ at two distinct points, and hence is orthogonal to $\vec{\gamma}_P$ at every point. In fact, the discussion above also shows that the Jacobi field X defined on $\vec{\gamma}_P$ extends to a perpendicular Jacobi field along the entire geodesic γ.

Fact 9: The Jacobi vector field X along $\vec{\gamma}_P$ satisfies:

$$\int_{\vec{\gamma}_P} -K(X\wedge\dot{\gamma})||X||^2 + ||\nabla_{\dot{\gamma}}X||^2 ds = 0. \tag{4}$$

This follows immediately from Facts 5, 7, condition (2) in our choice of the sequence $\{t_i\}$ (see the discussion preceding Fact 5), and the second variation formula for $L_i''(t_i)$ (see section 2.2, equation (1)). Indeed, this is just an application of the Lebesgue dominated convergence theorem (the integrand is positive, bounded on compact sets, and we have point-wise convergence.)

Observe that at this point, we are almost done. Since $\tilde{M}$ has non-positive sectional curvature, we see that the expression inside the integral in equation (4) consists of a sum of two terms that are ≥ 0 (pointwise). Since the overall integral is zero, and the expression inside the integral varies continuously, this tells us that at *every* point along $\vec{\gamma}_P$, we have that:

$$-K(X\wedge\dot{\gamma})||X||^2 = 0 \quad \textbf{and} \quad ||\nabla_{\dot{\gamma}}X||^2 = 0.$$

Furthermore, at the point P we see that the vector field X is the limit of vectors of norm =1 (see Fact 6), and whose angle with γ' tends to $\pi/2$ (see Fact 8). In particular this gives:

Fact 10: The Jacobi field X is not the zero vector field, since we have $X(P) \neq 0$.

Finally, let us complete the proof of the theorem. Let $\{P_k\}$ be a sequence of points on γ, with $P_k = \gamma(t_k)$ for a strictly decreasing sequence of real numbers t_k with $\lim t_k = -\infty$ (so in particular, $\lim P_k = \gamma(-\infty)$). Let $\mathcal{J}$ denote the $(2n-2)$-dimensional vector space of orthogonal Jacobi fields along the geodesic γ. Corresponding to each P_k, we let $\mathcal{J}_k \subset \mathcal{J}$ denote the collection of all orthogonal Jacobi fields on γ having the property that they are parallel along the geodesic ray $\vec{\gamma}_{P_k}$ (with no constraints on the behavior on the rest of γ). It is obvious that each $\mathcal{J}_k$ is actually a vector subspace of $\mathcal{J}$, and our proof ensures that each $\mathcal{J}_k$ contains a non-zero vector field, and in particular, satisfies $\dim \mathcal{J}_k \geq 1$. Furthermore, whenever $k \geq k'$, we have a containment of geodesic rays $\vec{\gamma}_{P_{k'}} \subset \vec{\gamma}_{P_k}$, which immediately yields containments $\mathcal{J}_k \subset \mathcal{J}_{k'}$. Since we have a sequence of nested, non-trivial, vector subspaces of the *finite dimensional* vector space $\mathcal{J}$, we conclude that their intersection is non-zero. This implies the existence of a globally defined, non-trivial, parallel, orthogonal Jacobi field along γ, completing the proof of the theorem.

□

4. From flats in the ultralimit to flattening sequences.

In this section, we focus exclusively on finding conditions on the ultralimit $Cone(\tilde{M})$ that can be used to construct pointed flattening sequences along a geodesic γ. This entire section will be devoted to establishing the following:

Theorem 4.1 (Undistorted $\square$ in ultralimit $\Rightarrow$ Pointed flattening sequence)**.** *Let $\tilde{M}$ be a simply connected, complete, Riemmanian manifold of non-positive sectional curvature, and let $Cone(\tilde{M})$ be an asymptotic cone of $\tilde{M}$. Given a geodesic $\gamma \subset \tilde{M}$, let $\gamma_\omega \subset Cone(\tilde{M})$ be the corresponding geodesic in the ultralimit. Assume that there exists a 4-tuple of points $\{A, B, C, D\} \subset Cone(\tilde{M})$, satisfying $A, B \in \gamma_\omega$, with $* \in Int(\overline{AB})$, and so that the associated map $\square \to Cone(\tilde{M})$ is undistorted. Then the original geodesic γ has pointed flattening sequences.*

In the next section, we will establish a strengthening of this result, by considering the case where γ_ω is contained in a bi-Lipschitz flat (i.e. a bi-Lipschitz image of $\mathbb{R}^2$ equipped with the standard metric). In this more general context, and under the presence of some additional constraints we will see that γ still has pointed flattening sequences.

Before starting the proof of the theorem, let us first introduce an auxiliary notion.

Definition (Flattening sequences)**.** *We say that γ has* flattening 4-tuples *if there exists a sequence $\{A_i, B_i, C_i, D_i\}$ of 4-tuples of points each of which is good (relative to γ), has* $\lim A_i = \gamma(-\infty)$ *and* $\lim B_i = \gamma(+\infty)$*, and viewing the 4-tuples as a sequence of maps $\square \to \tilde{M}$, we also require the sequence to be undistorted in the limit.*

An illustration of a flattening sequence is provided in Figure 3. All the quadrilaterals are perpendicular along the base geodesic, have three sides of equal length, and

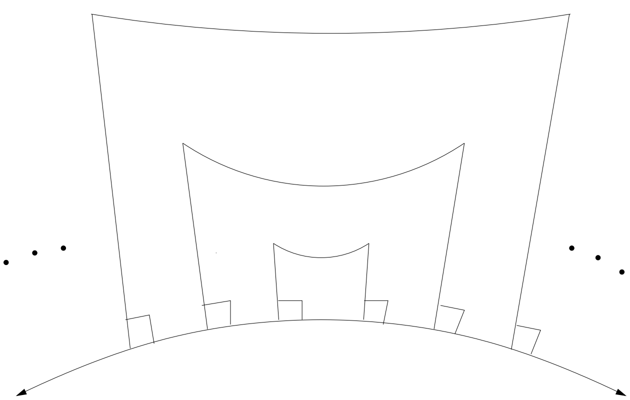

FIGURE 3. Flattening sequence along a geodesic.

have the length of the top edge tending (asymptotically, in the ratio) to the length of the remaining three edges.

We now begin the proof of theorem 4.1, by establishing:

Step 1: The geodesic γ has a flattening sequence.

Proof (Step 1). In the ultralimit $Cone(\tilde{M})$, let us pick out points $\{A, B, C, D\}$ to be the vertices of an undistorted square, with the property that $\overline{AB} \subset \gamma_\omega$, and $* \in Int(\overline{AB})$. We now intend to show that a suitable approximating sequence of 4-tuples in $\tilde{M}$ will give us the desired flattening sequence.

Let us start out by picking an arbitrary pair of approximating sequences $\{C_i'\}$ and $\{D_i'\}$ for the points $C, D \in Cone(\tilde{M})$. Now observe that corresponding to the geodesic $\gamma \subset \tilde{M}$, we have a well-defined projection map $\rho : \tilde{M} \to \gamma$, where $\rho(x)$ is defined to be the unique point on γ closest to the point x. We now define the sequence $\{A_i\}$ (respectively $\{B_i\}$) by setting $A_i := \rho(D_i')$ (respectively $B_i := \rho(C_i')$). Note that each of the 4-tuples of points $\{A_i, B_i, C_i', D_i'\}$ clearly satisfies the first three properties of being good for the geodesic γ (the points A_i, B_i are on γ, and the sides $\overline{A_iD_i'}$ and $\overline{B_iC_i'}$ are $\perp$ to γ). To ensure the last condition, we pick points $C_i \in \overrightarrow{B_iC_i'}$, $D_i \in \overrightarrow{A_iD_i'}$ satisfying $d(D_i, A_i) = d(A_i, B_i) = d(B_i, C_i)$. Note that this construction is exactly the sort considered in Lemma 2.1. In particular, statement (1) in Lemma 2.1 tells us that $\{A_i\} = A \in Cone(\tilde{M})$ and $\{B_i\} = B \in Cone(\tilde{M})$, while statement (2) in Lemma 2.1 ensures that $\{C_i\} = C \in Cone(\tilde{M})$ and $\{D_i\} = D \in Cone(\tilde{M})$.

Up to this point, we have constructed a sequence $\{A_i, B_i, C_i, D_i\}$ of 4-tuples, each of which is *good* for the geodesic γ, and additionally having the property that the sequence (ultra)-converges to the 4-tuple $\{A, B, C, D\} \subset Cone(\tilde{M})$. To conclude, we simply need to ensure that our sequence also satisfies the two conditions for a flattening sequence, namely, we need (1) that $\lim A_i = \gamma(-\infty)$ and $\lim B_i = \gamma(+\infty)$, and (2) that $\lim_{i\to\infty}\{d(C_i, D_i)/d(A_i, B_i)\} = 1$. We will ensure that these additional conditions are satisfied by passing to suitable subsequences of our original sequence. Let us explain the argument for (2); the argument for (1) is analogous (after a possible permutation of labels on the 4-tuples of points).

Given any $k \in \mathbb{N}$, the fact that $\omega \lim\{d(C_i, D_i)/\lambda(i)\} = d_\omega(C, D)$ implies that the set of indices i for which the following relation holds

$$\Big|\frac{d(C_i, D_i)}{\lambda(i)} - d_\omega(C, D)\Big| < \frac{1}{k} \tag{5}$$

forms a subset $I_k \in \mathcal{U}$. Similarly, the fact that $\omega \lim\{d(A_i, B_i)/\lambda(i)\} = d_\omega(A, B)$ implies that the set of indices i for which the following relation holds

$$\Big|\frac{d(A_i, B_i)}{\lambda(i)} - d_\omega(A, B)\Big| < \frac{1}{k} \tag{6}$$

likewise forms a subset $I'_k \in \mathcal{U}$. Since the ultrafilter $\mathcal{U}$ is closed under intersections, we conclude that $I_k \cap I'_k \in \mathcal{U}$. But every element in $\mathcal{U}$ is an infinite subset of $\mathbb{N}$, and in particular, non-empty. Hence there is an index i_k for which both equations (5) and (6) hold. Now consider the subsequence $\{A_{i_k}, B_{i_k}, C_{i_k}, D_{i_k}\}$ of 4-tuples in $\tilde{M}$, and observe that from (5) and (6), we have that the k^{th} 4-tuple satisfies:

$$\frac{d_\omega(C, D) - 1/k}{d_\omega(A, B) + 1/k} \leq \frac{d(C_{i_k}, D_{i_k})}{d(A_{i_k}, B_{i_k})} \leq \frac{d_\omega(C, D) + 1/k}{d_\omega(A, B) - 1/k}$$

Now it is immediate that for this subsequence, we obtain:

$$\lim_{k\to\infty} \frac{d(C_{i_k}, D_{i_k})}{d(A_{i_k}, B_{i_k})} = \frac{d_\omega(C, D)}{d_\omega(A, B)} = 1$$

where the last equality comes from the fact that the quadrilateral $\{A, B, C, D\} \subset Cone(\tilde{M})$ is an undistorted copy of $\square$. But this is precisely the desired property (2).

So we now know that the geodesic γ we were interested in has a flattening sequence. The second step in the proof of theorem 4.1 lies in *improving* the choice of our subsequence, obtaining:

Step 2: The geodesic γ has *pointed* flattening sequences.

Proof (Step 2). In the proof of Step 1, we started out by constructing a sequence of 4-tuples $\{A_i, B_i, C_i, D_i\}$, each of which was *good* for γ. The flattening sequence along γ was then obtained by picking a suitable subsequence of this sequence of 4-tuples. We now proceed to explain how, by being a bit more careful with our choice of subsequence, we can construct pointed flattening sequences.

To this end, let $\{A_i, B_i, C_i, D_i\}$ be the sequence of good 4-tuples for γ obtained in Step 1, and let $P \in \gamma$ be an arbitrary chosen point. Since $\omega \lim\{A_i\} = A$, $\omega \lim\{B_i\} = B$, $\omega \lim\{P\} = *$, and we have a containment $* \in Int(\overline{AB}) \subset Cone(\tilde{M})$, we immediately see that the set of indices $J_1 \subset \mathbb{N}$ for which the corresponding 4-tuples has the property that $P \in Int(\overline{A_iB_i})$ consists of a set in our ultrafilter: $J_1 \in \mathcal{U}$.

Now consider the nearest point projection $\rho : \tilde{M} \to \gamma$, and let us first consider indices $i \in J_1$. Since each of the 4-tuples in our sequence is good, we clearly have $\rho(D_i) = A_i$ and $\rho(C_i) = B_i$. Observe that P disconnects the geodesic γ into two components (since $i \in J$), and the points A_i, B_i lie in distinct components of $\gamma - \{P\}$. Since $\rho(\overline{D_iC_i})$ gives a path in γ joining A_i to B_i, we conclude that there must exist a point $E_i \in \overline{D_iC_i}$ satisfying $\rho(E_i) = P$. Observe that this immediately implies that $\overline{PE_i} \perp \gamma'$. For the remaining indices $i \notin J_1$, we set $E_i = C_i$. In particular, we now have a sequence of points $\{E_i\}$, with $E_i \in \overline{D_iC_i}$.

Now it is easy to verify that the sequence $\{E_i\}$ defines a point $E \in Cone(\tilde{M})$, and since each $E_i \in \overline{D_iC_i}$, we have that $E \in \overline{DC}$. Furthermore, the fact that $\rho(E_i) = P$ for a collection of indices $i \in J_1$ contained in our ultrafilter $\mathcal{U}$ implies that $\rho_\omega(E) = \omega \lim\{P\} = * \in Cone(\tilde{M})$, where $\rho_\omega : Cone(\tilde{M}) \to \gamma_\omega$ is the projection map from $Cone(\tilde{M})$ to γ_ω.

Observe that the 4-tuple of points $\{A, B, C, D\} \subset Cone(\tilde{M})$, corresponding to an undistorted $\square$, satisfies the equality:

$$1 = \frac{d_\omega(A,C)^2 + d_\omega(B,D)^2 - d_\omega(C,D)^2 - d_\omega(A,B)^2}{2 \cdot d_\omega(A,D) \cdot d_\omega(B,C)} \tag{7}$$

But inside a geodesic space, a 4-tuple of (distinct, non-colinear) points satisfies the equality in equation (7) if and only if the 4-tuple of points are the vertices of a flat parallelogram (see Berg-Nikolaev [BeNi, Theorem 15]). Applying this to the given 4-tuple of points in $Cone(\tilde{M})$, we see that there exists an isometric embedding $\mathcal{P} \hookrightarrow Cone(\tilde{M})$ from a square $\mathcal{P} \subset \mathbb{R}^2$, with the property that the vertices map precisely to the points $\{A, B, C, D\}$. But now we immediately see that the point $E \in \overline{CD}$ must be the point satisfying $d_\omega(E, C) = d_\omega(P, B)$, and in particular, that the 4-tuple $\{P, B, C, E\}$ are the vertices of a flat rectangle in $Cone(\tilde{M})$, with $d_\omega(C, E) = d_\omega(P, B) < d_\omega(P, E) = d_\omega(B, C)$.

So we can find a collection of indices $J_2 \in \mathcal{U}$ with the property that for all $i \in J_2$, $d(P, B_i) < d(P, E_i)$ and $d(P, B_i) < d(B_i, C_i)$. For each of the indices $i \in J_2$, we can now choose points $F_i \in \overline{PE_i}$, $G_i \in \overline{B_iC_i}$ to satisfy $d(P, F_i) = d(P, B_i) = d(B_i, G_i)$. For indices $i \notin J_2$, we set $F_i = E_i$, $G_i = C_i$. It is again easy to verity that the sequences $\{F_i\}, \{G_i\}$ define points $F, G \in Cone(\tilde{M})$. Furthermore one can verify that the 4-tuple $\{P, B, G, F\}$ define an undistorted $\square$ in $Cone(\tilde{M})$.

Figure 4 contains an illustration of where the various points E_i, F_i, G_i are chosen from the original 4-tuple $\{A_i, B_i, C_i, D_i\}$ (for indices $i \in J_1 \cap J_2$). We observe that for our sequence of 4-tuples we now have:

- the sequence of 4-tuples $\{P, B_i, G_i, F_i\}$ ultra-converges to the vertices of an undistorted square in $Cone(\tilde{M})$, and

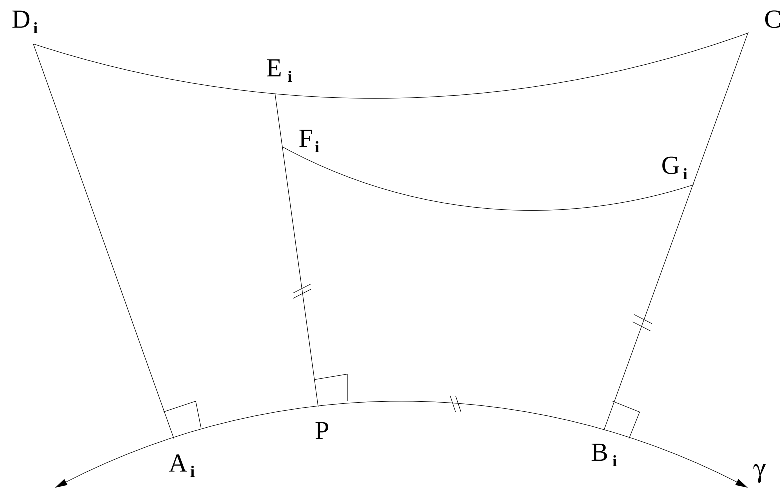

FIGURE 4. Constructing pointed flattening sequences.

- for each index $i \in J_1 \cap J_2$, the corresponding 4-tuple $\{P, B_i, G_i, F_i\}$ is good.

We now want to pick a subsequence of 4-tuples, within the index set $J_1 \cap J_2$, which satisfies the additional property that $\lim_{k\to\infty}\{d(F_{i_k}, G_{i_k})/d(P, B_{i_k})\} = 1$. So define, for $k \in \mathbb{N}$, the sets $I_k, I'_k \in \mathcal{U}$ to be the set of indices satisfying the obvious analogues of equations (5) and (6) from Step 1. Then we see that, from the closure of ultrafilters under finite intersections, we have for each $k \in \mathbb{N}$ the set $I_k \cap I'_k \cap J_1 \cap J_2$ lies in our ultrafilter $\mathcal{U}$, and hence is non-empty. In particular, we can select indices $i_k \in I_k \cap I'_k \cap J_1 \cap J_2$, and the argument given at the end of Step 1 extends verbatim to see that the subsequence $\{P, B_{i_k}, G_{i_k}, F_{i_k}\}$ satisfies the desired additional property. This completes the proof of Step 2, and hence of Theorem 4.1.

□

Finally, we conclude this section by pointing out that if the geodesic $\gamma_\omega \subset Cone(\tilde{M})$ is contained in a flat, then the combination of Theorem 4.1 and Theorem 3.1 immediately tells us that γ must be of higher rank. In particular, this completes the proof of Theorem 1.1.

5. FROM BI-LIPSCHITZ FLATS TO FLATTENING SEQUENCES.

In the previous section, we saw that we can use the presence of flats in the asymptotic cone $Cone(\tilde{M})$ to construct flattening sequences (which in turn could be used to construct pointed flattening sequences). For our applications, it will be important for us to be able to use *bi-Lipschitz flats* instead of genuine flats. The reason for this is that bi-Lipschitz flats in $Cone(M)$ appear naturally as ultralimits of quasi-flats in the original M.

To establish the result, we will first show (Lemma 5.1) that we can use suitable sequences of maps $\square \to Cone(\tilde{M})$ that are undistorted in the limit to construct flattening sequences along γ. We will then show (Theorem 5.2) that in the presence of certain bi-Lipschitz flats, one can construct the desired sequence of maps $\square \to Cone(\tilde{M})$ that are undistorted in the limit.

Lemma 5.1 (Almost undistorted metric squares $\Rightarrow$ flattening sequence)**.** *Let $\tilde{M}$ be a complete, simply-connected, Riemannian manifold of non-positive sectional curvature, and $Cone(\tilde{M})$ an asymptotic cone of $\tilde{M}$. For $\gamma \subset \tilde{M}$ a geodesic, let $\gamma_\omega \subset Cone(\tilde{M})$ be the corresponding geodesic in the asymptotic cone of $\tilde{M}$. Assume that for each $\epsilon > 0$, one can find an ϵ-undistorted copy $\{A, B, C, D\}$ of $\square$ in $Cone(\tilde{M})$ satisfying the properties:*

- *$A, B \in \gamma_\omega$ with $* \in Int(\overline{AB})$,*
- *A, B are the closest points to D, C (respectively) on the geodesic γ_ω.*

Then γ has a flattening sequence of 4-tuples.

Proof. We want to build a sequence of good 4-tuples in $\tilde{M}$ which are undistorted in the limit. We will explain how to find, for a given $\epsilon > 0$, a good 4-tuple in $\tilde{M}$ with distortion $< \epsilon$. Choosing such 4-tuples for a sequence of error terms $\epsilon_k \to 0$ will yield the desired flattening sequence.

From the hypotheses in our Lemma, we can find a 4-tuple $\{A, B, C, D\} \subset Cone(\tilde{M})$ satisfying $* \in Int(\overline{AB}) \subset \gamma_\omega$, with distortion $< \epsilon/3$. Now this 4-tuple in the asymptotic cone corresponds to a sequence $\{A_i, B_i, C_i, D_i\} \subset \tilde{M}$ of 4-tuples satisfying $\{A_i\} = A, \{B_i\} = B, \{C_i\} = C, \{D_i\} = D$. Applying Lemma 2.1, we can replace this sequence of 4-tuples by another sequence $\{A'_i, B'_i, C'_i, D'_i\} \subset \tilde{M}$ satisfying the additional property that: A'_i, B'_i are the closest points on γ to the points D'_i, C'_i respectively.

Since the distortion of $\{A, B, C, D\} \subset Cone(\tilde{M})$ is $< \epsilon/3$, we have that for some index i, the distortion of $\{A'_i, B'_i, C'_i, D'_i\} \subset \tilde{M}$ is likewise $< \epsilon/3$ (in fact, the set of such indices i has to lie in the ultrafilter $\mathcal{U}$). Now the problem is that there is no guarantee that this 4-tuple $\{A'_i, B'_i, C'_i, D'_i\}$ is good for the geodesic γ: while it satisfies the orthogonality conditions $\overline{D'_iA'_i} \perp \gamma$ and $\overline{C'_iB'_i} \perp \gamma$, the 4-tuple does not necessarily have the requisite property that $d(D'_i, A'_i) = d(A'_i, B'_i) = d(B'_i, C'_i)$.

So we modify the 4-tuple in the obvious manner, by replacing the points $C'_i, D'_i \in \tilde{M}$ by the points $C''_i \in \overrightarrow{B'_iC'_i}, D''_i \in \overrightarrow{A'_iD'_i}$ chosen so that $d(D''_i, A'_i) = d(A'_i, B'_i) = d(B'_i, C''_i)$. This new sequence of 4-tuples now *does* have the property of being good for γ. So in order to complete the Lemma, we just need to make sure that this new 4-tuple $\{A'_i, B'_i, C''_i, D''_i\}$ has distortion $< \epsilon$. Note that by Lemma 2.2, it is sufficient to show that the distance $d(C''_i, D''_i)$ is not too much larger than $d(A'_i, B'_i)$.

Letting $K := d(A'_i, B'_i)$, we first observe that from the fact that the (unmodified) 4-tuple $\{A'_i, B'_i, C'_i, D'_i\}$ has distortion $< \epsilon/3$ implies that:

$$1 - \epsilon/3 < d(A'_i, D'_i)/K < 1 + \epsilon/3$$

which translates to the estimate:

$$d(D'_i, D''_i) = |d(A'_i, D'_i) - K| < K \cdot \epsilon/3$$

An identical argument gives the estimate $d(C'_i, C''_i) < K \cdot \epsilon/3$. Now the triangle inequality gives us the estimate:

$$|d(C''_i, D''_i) - d(C'_i, D'_i)| \leq d(C'_i, C''_i) + d(D'_i, D''_i) < 2K \cdot \epsilon/3$$

Dividing by K, we obtain:

$$\Big|\frac{d(C''_i, D''_i)}{K} - \frac{d(C'_i, D'_i)}{K}\Big| < 2\epsilon/3$$

But since the original 4-tuple was $\epsilon/3$-undistorted, we have that $d(C'_i, D'_i)/K$ is within $\epsilon/3$ of 1. Hence applying the triangle inequality one last time gives:

$$1 - \epsilon < \frac{d(C''_i, D''_i)}{K} < 1 + \epsilon$$

precisely as desired. This concludes the proof of Lemma 5.1.

□

Now assume that we have a bi-Lipschitz map $\phi : \mathbb{R}^2 \hookrightarrow Cone(\tilde{M})$. We will call the image $\phi(\mathbb{R}^2)$ a *bi-Lipschitz ultraflat*. The primary application of almost undistorted metric squares lies in establishing the following:

Theorem 5.2 (Bi-Lipschitz ultraflat ⇒ flattening sequences.)**.** *Let $\tilde{M}$ be a complete, simply-connected, Riemannian manifold of non-positive sectional curvature, and let $Cone(\tilde{M})$ an asymptotic cone of $\tilde{M}$. For $\gamma \subset \tilde{M}$ a geodesic, let $\gamma_\omega \subset Cone(\tilde{M})$ be the corresponding geodesic in the asymptotic cone of $\tilde{M}$. Assume that:*

- *the sectional curvature of $\tilde{M}$ is bounded below,*
- *there exists $g \in Isom(M)$ which stabilizes γ, and acts cocompactly on g, and*
- *there is a bi-Lipschitz ultraflat $\phi : \mathbb{R}^2 \hookrightarrow Cone(M)$ mapping the x-axis to γ_ω.*

Then γ has a flattening sequence of 4-tuples.

Proof. Our approach is to reduce the problem to one which can be dealt with by methods similar to those in the previous Lemma 5.1. Let $\phi : (\mathbb{R}^2, ||\cdot||) \hookrightarrow (Cone(\tilde{M}), d)$ be the given bi-Lipschitz ultraflat, and let C be the bi-Lipschitz constant. For $r \in \mathbb{R}$, let us denote by $L_r \subset \mathbb{R}^2$ the horizontal line at height r. Note that L_0 coincides with the x-axis in $\mathbb{R}^2$, and hence, by hypothesis, must map to γ_ω under ϕ. Note that to make our various expressions more readable, we are using d to denote distance in $Cone(\tilde{M})$ (as opposed to d_ω), and the norm notation to denote distance inside $\mathbb{R}^2$.

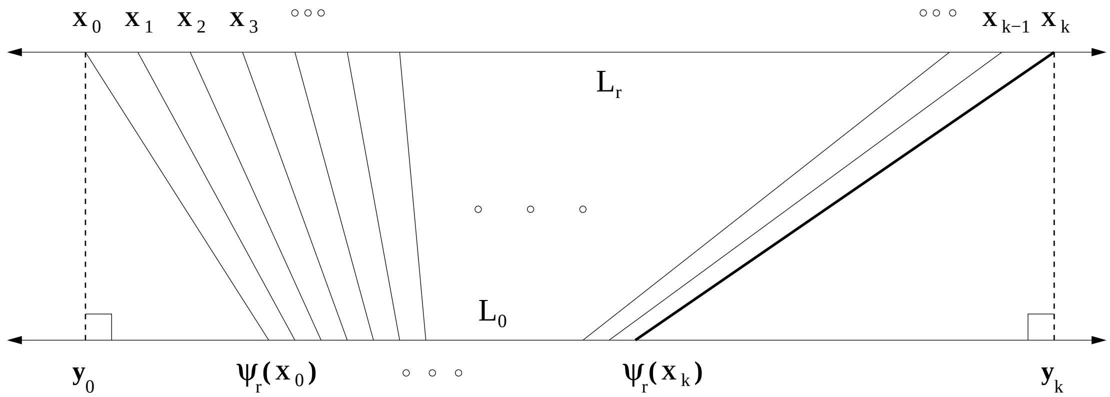

FIGURE 5. Illustration of the proof of our **Assertion**.

We now define, for each $r \in [0, \infty)$ a map $\psi_r : L_r \to L_0$ as follows: given $p \in L_r$, we have $\phi(p) \in Cone(\tilde{M})$. Since $\gamma_\omega \subset Cone(\tilde{M})$ is a geodesic inside the CAT(0) space $Cone(\tilde{M})$, there is a well defined, distance non-increasing, projection map $\pi : Cone(\tilde{M}) \to \gamma_\omega$, which sends any given point in $Cone(\tilde{M})$ to the (unique) closest point on γ_ω. Hence given $p \in L_r$, we have the composite map $\pi \circ \phi : L_r \to \gamma_\omega$. But recall that, by hypothesis ϕ maps L_0 homeomorphically to γ_ω. We can now set $\psi_r : L_r \to L_0$ to be the composite map $\phi^{-1} \circ \pi \circ \phi$. We now have the following:

Assertion: For every $r \in [0, \infty)$, there exist a pair of points $p, q \in L_r \subset \mathbb{R}^2$ having the property that $||p - q|| = 1$, and $||\psi_r(p) - \psi_r(q)|| > 1/2$.

Let us explain how to obtain the **Assertion**. We first observe that for arbitrary $x \in L_r$, we have that the distance from x to L_0 is exactly r, and hence from the bi-Lipschitz estimate, we have

$$d(\phi(x), \gamma_\omega) = d(\phi(x), \phi(L_0)) \leq Cr$$

Since π is the nearest point projection onto γ_ω, this implies that $d\big(\phi(x), (\pi \circ \phi)(x)\big) \leq Cr$. Since $(\pi \circ \phi)(x) = \phi(\psi_r(x))$, we can again use the bi-Lipschitz estimate to conclude that:

$$Cr \geq d\big(\phi(x), (\pi \circ \phi)(x)\big) = d(\phi(x), \phi(\psi_r(x)) \geq \frac{1}{C} \cdot ||x - \psi_r(x)||$$

Which gives us the estimate: $||x - \psi_r(x)|| \leq C^2 r$.

Finally, to establish the **Assertion**, let us argue by contradiction (we will ultimately contradict the upper bound on $||x - \psi_r(x)||$ obtained in the previous paragraph). Consider, for integers $k \geq 0$, the point $x_k := (k, r) \in L_r$, and $y_k := (k, 0) \in L_0$. Observe that we clearly have $||x_i - x_{i+1}|| = 1$, and let us assume, by way of contradiction, that *every* pair $\{x_i, x_{i+1}\}$ satisfies $||\psi_r(x_i) - \psi_r(x_{i+1})|| \leq 1/2$. We then observe that we can easily estimate from above the distance between $\psi_r(x_k)$ and the origin y_0:

$$||y_0 - \psi_r(x_k)|| \leq ||y_0 - \psi_r(x_0)|| + \sum_{i=1}^{k} ||\psi_r(x_{i-1}) - \psi_r(x_i)||$$

Note that since the three points x_0, y_0, and $\psi_r(x_0)$ form a right triangle, two of whose sides are controlled, we can estimate from above $||y_0 - \psi_r(x_0)|| \leq r\sqrt{C^4 - 1}$. Combined with our assumption that all the $||\psi_r(x_k) - \psi_r(x_{k+1})|| \leq 1/2$, this yields the estimate:

$$||y_0 - \psi_r(x_k)|| \leq r\sqrt{C^4 - 1} + k/2$$

Since $||y_0 - y_k|| = k$, the estimate above immediately gives us the lower bound:

$$||y_k - \psi_r(x_k)|| \geq k/2 - r\sqrt{C^4 - 1}$$

But now, using the fact that the three points $x_k, y_k, \psi_r(x_k)$ form a right triangle, we obtain the lower bound:

$$||x_k - \psi_r(x_k)|| \geq \sqrt{r^2 + (k/2 - r\sqrt{C^4 - 1})^2}$$

Note that the lower bound above tends to infinity as $k \to \infty$, and hence for k sufficiently large, yields $||x_k - \psi_r(x_k)|| > C^2 r$, which contradicts the previously obtained upper bound $||x_k - \psi_r(x_k)|| \leq C^2 r$. Hence our initial assumption must have been wrong, i.e. there exist a pair $\{x_k, x_{k+1}\}$ satisfying $||\psi_r(x_k) - \psi_r(x_{k+1})|| > 1/2$, completing the proof of the **Assertion**.

For an illustration of this argument, we refer the reader to Figure 5. The parallel lines are L_0 at the bottom, L_r at the top. The points x_i are represented along the line L_r, with pairwise distance $= 1$. The points along L_0 represent the corresponding $\psi_r(x_i)$, with a straight line segment joining each x_i to the corresponding $\psi_r(x_i)$. Our argument is merely making formal the fact that if all the successive distances along L_r are $= 1$, while all the successive distances along L_0 are $< 1/2$, then eventually the bold segment $\overline{x_k \psi_r(x_k)}$ has arbitrary large length (in particular $> C^2 r$, a contradiction).

Let us now use the **Assertion** to construct almost undistorted metric squares of the type appearing in Lemma 5.1. For each index $j \in \mathbb{N}$, let us take a pair of points $p_j, q_j \in L_j$ whose existence is ensured by the Assertion. Consider now the 4-tuple of points $\{A_j, B_j, Q_j, P_j\}$ in $Cone(\tilde{M})$, defined by $P_j = \phi(p_j)$, $Q_j = \phi(q_j)$, $A_j = (\pi \circ \phi)(p_j)$, and $B_j = (\pi \circ \phi)(q_j)$. An illustration of a few of these 4-tuples is given in Figure 6 above. The slanted surface represents the bi-Lipschitz flat in $Cone(\tilde{M})$, along with the image of the horizontal lines $L_1, L_2 \subset \mathbb{R}^2$ under the map ϕ, and the corresponding 4-tuples of points.

Now for each integer j, we observe that the corresponding 4-tuple of points in $Cone(\tilde{M})$ satisfies the following nice properties:

(1) $d(P_j, Q_j) = d(\phi(p_j), \phi(q_j)) \leq C \cdot ||p_j - q_j|| = C$,

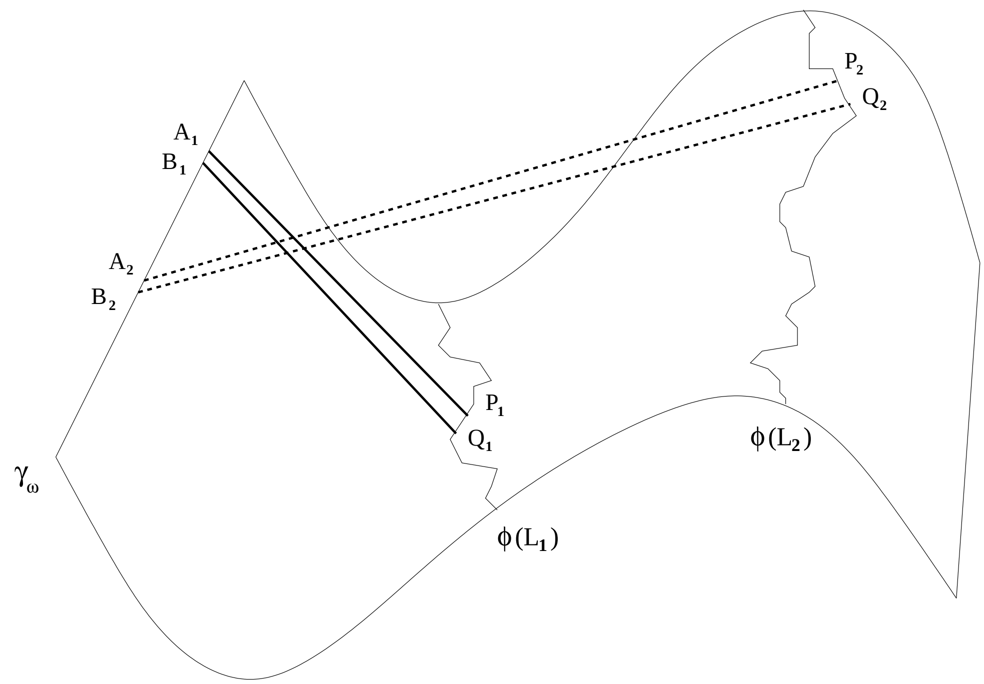

FIGURE 6. Choosing the points $\{A_j, B_j, Q_j, P_j\}$.

(2) $d(A_j, B_j) = d\big(\phi(\psi_j(p_j)), \phi(\psi_j(q_j))\big) \geq \frac{1}{C} \cdot ||\psi_j(p_j) - \psi_j(q_j)|| > 1/2C$,
(3) A_j, B_j are the closest points on γ_ω to P_j, Q_j respectively,
(4) $d(P_j, A_j) = d(P_j, \gamma_\omega) = d(\phi(p_j), \phi(L_0)) \geq j/C$, and similarly for $d(Q_j, B_j)$.

We now proceed to explain how we can use this sequence of 4-tuples to construct a sequence of almost undistorted metric squares along γ_ω. This will be done via a two step process, and the modification at each step is illustrated in Figure 7.

The first step is to replace the original sequence by a new sequence $\{A_j, B_j, P'_j, Q'_j\}$ chosen as follows: if $d(P_j, A_j) \leq d(Q_j, B_j)$, let $P'_j = P_j$, but pick Q'_j to be the unique point on the geodesic segment $\overline{B_jQ_j}$ at distance $d(P_j, A_j)$ from the point B_j (and perform the symmetric procedure if $d(P_j, A_j) \geq d(Q_j, B_j)$). This new sequence of 4-tuples satisfies the same properties and estimates (2)-(4) from above, but of course, the distance $d(P'_j, Q'_j)$ no longer satisfies estimate (1). We now proceed to use the triangle inequality to give a new estimate $(1')$ for the analogous distance for our new 4-tuple. Assuming that we are in the case where $P'_j = P_j$ (the other case is symmetric), we are truncating the segment $\overline{B_jQ_j}$ to have the same length as $\overline{A_jP_j}$; the amount being truncated can be estimated by the triangle inequality:

$$d(Q'_j, Q_j) = d(Q_j, B_j) - d(P_j, A_j) \leq d(P_j, Q_j) + d(A_j, B_j) \leq 2C$$

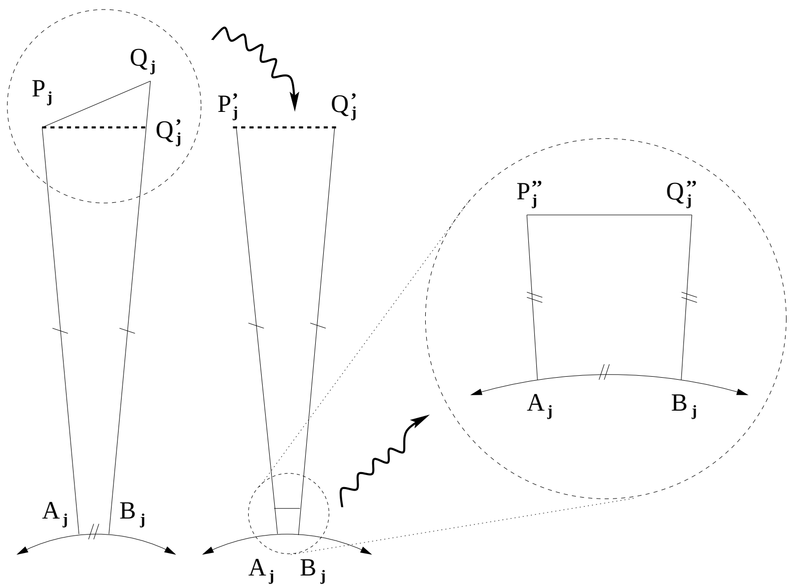

FIGURE 7. Changing $\{A_j, B_j, Q_j, P_j\}$ to an almost undistorted $\Box$.

This in turn allows us to estimate from above the distance:

$$d(P'_j, Q'_j) = d(P_j, Q'_j) \leq d(P_j, Q_j) + d(Q_j, Q'_j) \leq C + 2C = 3C$$

In particular, our new sequence satisfies the following property: ($1'$) for each j, we have the uniform estimate $d(P'_j, Q'_j) \leq 3C$. In addition, our new sequence satisfies the additional property (5) for each j, $d(A_j, P'_j) = d(B_j, Q'_j)$.

Our second step is to further modify the sequence as follows: starting from the index $j \geq C^2$, consider the new sequence of 4-tuples $\{A_j, B_j, Q''_j, P''_j\}$ chosen by picking the points $P''_j \in \overline{A_j P'_j}$ and $Q''_j \in \overline{B_j Q'_j}$ to satisfy the following stronger version of (5):

$$d(A_j, P''_j) = d(A_j, B_j) = d(B_j, Q''_j)$$

Note that this new sequence of 4-tuples still satisfies properties (2) and (3). We now make the:

Claim: The sequence of 4-tuples $\{A_j, B_j, Q''_j, P''_j\}$, as a sequence of maps $\Box \to Cone(\tilde{M})$, is undistorted in the limit.

To establish this, we need to show that the limit (as $j \to \infty$) of the ratios of all distances tend to the corresponding distances in $\Box$ (i.e. tend to 1 or $\sqrt{2}$ according to which ratio of distances is considered). We first observe that, for all $j \geq C^2$, we have by construction the equalities:

$$\frac{d(A_j, P''_j)}{d(A_j, B_j)} = \frac{d(B_j, Q''_j)}{d(A_j, B_j)} = 1$$

which accounts for the relative distances of three of the four sides. Now let us consider the ratio of the fourth side to the first, i.e. the ratio $d(P''_j, Q''_j)/d(A_j, B_j)$. In order to estimate this, we first observe that the points P''_j, Q''_j project to A_j, B_j under the projection map $\pi : Cone(\tilde{M}) \to \gamma_\omega$, and hence since this map is distance non-increasing, we obtain the estimate $d(A_j, B_j) \leq d(P''_j, Q''_j)$. To give an upper bound, we make use of the fact that $Cone(\tilde{M})$ is a CAT(0) space, and hence we have convexity of the distance function. Recall that this tells us that given any two geodesic segments $\alpha, \beta : [0, 1] \to Cone(\tilde{M})$, with parametrization proportional to arclength, and given any $t \in [0, 1]$, we have the estimate:

$$d(\alpha(t), \beta(t)) \leq (1 - t) \cdot d(\alpha(0), \beta(0)) + t \cdot d(\alpha(1), \beta(1)) \tag{8}$$

Let us apply this to the two geodesic segments $\alpha = \overline{A_j P'_j}$ and $\beta = \overline{B_j Q'_j}$. In this situation, we see that $d(\alpha(0), \beta(0)) = d(A_j, B_j)$. Furthermore, we have from properties $(1')$ and (2) the estimate:

$$d(\alpha(1), \beta(1)) = d(P'_j, Q'_j) \leq 3C \leq 6C^2 \cdot d(A_j, B_j)$$

Substituting these estimates into the convexity equation (8), we obtain the following inequality:

$$\frac{d(\alpha(t), \beta(t))}{d(A_j, B_j)} \leq (1 - t) + 6C^2 \cdot t \tag{9}$$

Finally, we recall that $d(A_j, P''_j) = d(A_j, B_j) \leq C$, while from property (4), we have that $d(A_j, P'_j) \geq j/C$. In particular, the parameter t corresponding to the point P''_j is at most C^2/j. Now from property (3), we also know that the function $d(\alpha(t), \beta(t))$ is strictly increasing, giving us the following estimate:

$$\frac{d(P''_j, Q''_j)}{d(A_j, B_j)} \leq \frac{d(\alpha(C^2/j), \beta(C^2/j))}{d(A_j, B_j)} \leq (1 - C^2/j) + 6C^2 \cdot C^2/j \leq 1 + 6C^4/j \tag{10}$$

It is now immediate that this ratio tends to one as $j \to \infty$. Applying Lemma 2.4, we conclude that the sequence of 4-tuples is undistorted in the limit.

To complete the proof of Theorem 5.2, we would like to apply Lemma 5.1. Looking at the statement of the proposition, we see that we have one more condition we need to ensure, namely we require the sequence of ϵ-undistorted squares $\square \hookrightarrow Cone(\tilde{M})$ to all satisfy $* \in Int(\overline{A_j B_j})$. Note that this is not a priori satisfied by the sequence of 4-tuples we constructed above. In order to ensure this additional condition, we make use of the fact that inside $\tilde{M}$, we assumed that there was a $g \in Isom(\tilde{M})$ acting cocompactly on the geodesic γ. This allows us to make use of Lemma 2.2, which implies that given any pair of points p, q on γ_ω, we have an isometry of $Cone(\tilde{M})$ leaving γ_ω invariant and taking p to q.

To finish, we pick, for each of our previously constructed 4-tuples $\{A_j, B_j, Q''_j, P''_j\}$, a point $p_j \in Int(\overline{A_j B_j}) \subset \gamma_\omega$. Then our Lemma 2.2 ensures the existence of a corresponding isometry Φ_j, leaving γ_ω invariant, and mapping p_j to the distinguished basepoint $* \in Cone(\tilde{M})$. The sequence of image 4-tuples $\Phi_j(\{A_j, B_j, Q''_j, P''_j\})$ now satisfy all the hypotheses of Lemma 5.1. Applying the lemma now completes the proof of Theorem 5.2.

□

Finally, we conclude this section by pointing out that combining Theorem 5.2, Theorem 4.1 (Step 2), and Theorem 3.1 completes the proof of Theorem 1.2 from the introduction.

6. Some applications

Finally, let us discuss some consequences of our main results.

Corollary 6.1 (Constraints on quasi-isometries). *Let $\tilde{M}_1, \tilde{M}_2$ be two simply connected, complete, Riemannian manifolds of non-positive sectional curvature, and assume that $\phi : \tilde{M}_1 \rightarrow \tilde{M}_2$ is a quasi-isometry. Let $\gamma \subset \tilde{M}_1$ be a geodesic, $\gamma_\omega \subset Cone(\tilde{M}_1)$ the corresponding geodesic in the asymptotic cone, and assume that there exists a bi-Lipschitz flat $F \subset Cone(\tilde{M}_1)$ containing the geodesic γ_ω. Then the following dichotomy holds:*

1. *every geodesic η at bounded distance from $\phi(\gamma)$ satisfies $\eta/Stab_G(\eta)$ non-compact, where $G = Isom(\tilde{M}_2)$, or*
2. *every geodesic η at bounded distance from $\phi(\gamma)$ has $rk(\eta) \geq 2$.*

Proof. This follows readily from our Theorem 1.2. Assume that the first possibility does not occur, i.e. there exists a geodesic η at bounded distance from $\phi(\gamma)$ with the property that $Stab_G(\eta) \subset G = Isom(\tilde{M}_2)$ acts cocompactly on η. Then we would like to establish that every geodesic η' at finite distance from $\phi(\gamma)$ has higher rank. We first observe that if there were more than one such geodesic, then the flat strip theorem would imply that any two of them arise as the boundary of a flat strip, and hence that they would all have higher rank.

So we only need to deal with the case where there is a unique such geodesic, i.e. show that the geodesic η has $rk(\eta) \geq 2$. Now recall that the quasi-isometry $\phi : \tilde{M}_1 \rightarrow \tilde{M}_2$ induces a bi-Lipschitz homeomorphism $\phi_\omega : Cone(\tilde{M}_1) \rightarrow Cone(\tilde{M}_2)$. Since $\eta \subset \tilde{M}_2$ was a geodesic at finite distance from $\phi(\gamma)$, we have the containment:

$$\phi_\omega(\gamma_\omega) \subseteq \eta_\omega \subset Cone(\tilde{M}_2).$$

Since $\phi_\omega(\gamma_\omega)$ is a bi-Lipschitz copy of $\mathbb{R}$ inside the geodesic η_ω, we conclude that ϕ maps γ_ω homeomorphically onto η_ω. But recall that we assumed that γ_ω was contained inside a bi-Lipschitz flat $\gamma_\omega \subset F \subset Cone(\tilde{M}_1)$, and hence we see that $\eta_\omega \subset \phi_\omega(F)$ is likewise contained inside a bi-Lipschitz flat.

Furthermore, since $Stab_G(\eta)$ acts cocompactly on η, we see that there exists an element $g \in G = Isom(\tilde{M}_2)$ which stabilizes and acts cocompactly on η. Hence η satisfies the hypotheses of Theorem 1.2, and must have $rk(\eta) \geq 2$, as desired. This concludes the proof of Corollary 6.1.

□

The statement of our first corollary might seem somewhat complicated. We now proceed to isolate the special case which is of most interest:

Corollary 6.2 (Constraints on perturbations of metrics)**.** *Assume that (M, g_0) is a closed Riemannian manifold of non-positive sectional curvature, and assume that $\gamma_0 \subset M$ is a closed geodesic. Let $\tilde{\gamma}_0 \subset \tilde{M}$ be a lift of γ_0, and assume that $\tilde{\gamma}_0 \subset F$ is contained in a flat F.*

Then if (M, g) is any other Riemannian metric on M with non-positive sectional curvature, and $\gamma \subset M$ is a geodesic (in the g-metric) freely homotopic to γ_0, then the lift $\tilde{\gamma} \subset (\tilde{M}, \tilde{g})$ satisfies $rk(\tilde{\gamma}) \geq 2$.

We can think of Corollary 6.2 as a "non-periodic" version of the Flat Torus theorem. Indeed, in the case where F is $\pi_1(M)$-periodic, the Flat Torus theorem applied to (M, g) implies that $\tilde{\gamma}$ is likewise contained in a periodic flat (and in particular has rank ≥ 2).

Proof. Since M is compact, the identity map provides a quasi-isometry $\phi : (\tilde{M}, \tilde{g}_0) \to (\tilde{M}, \tilde{g})$. The flat F containing $\tilde{\gamma}_0$ gives rise to a flat $F_\omega \subset Cone(\tilde{M}, \tilde{g}_0)$ containing $(\tilde{\gamma}_0)_\omega$. In particular, we can apply the previous Corollary 6.1.

Next note that, since γ_0, γ are freely homotopic to each other, there is a lift $\tilde{\gamma}$ of γ which is at finite distance (in the g-metric) from the given $\tilde{\gamma}_0 \subset (\tilde{M}, \tilde{g})$. Indeed, taking the free homotopy $H : S^1 \times [0,1] \to M$ between $H_0 = \gamma_0$ and $H_1 = \gamma$, we can then take a lift $\tilde{H} : \mathbb{R} \times [0,1] \to \tilde{M}$ satisfying the initial condition $\tilde{H}_0 = \tilde{\gamma}_0$ (the given lift of γ_0). The time one map $\tilde{H}_1 : \mathbb{R} \to \tilde{M}$ will be a lift of $H_1 = \gamma$, hence a geodesic in $(\tilde{M}, \tilde{g})$. Furthermore, the distance (in the g-metric) between $\tilde{\gamma}_0$ and $\tilde{\gamma}$ will clearly be bounded above by the supremum of the g-lengths of the (compact) family of curves $H_p : [0,1] \to (M, g)$, $p \in S^1$, defined by $H_p(t) = H(p, t)$.

Now observe that by construction, the $\tilde{\gamma} \subset (\tilde{M}, \tilde{g})$ from the previous paragraph has $Stab_G(\tilde{\gamma})$ acting cocompactly on $\tilde{\gamma}$, where $G = Isom(\tilde{M}, \tilde{g})$. Hence the first possibility in the conclusion of Corollary 6.1 cannot occur, and we conclude that $\tilde{\gamma}$ has $rk(\tilde{\gamma}) \geq 2$, as desired. This concludes the proof of Corollary 6.2.

□

Next we recall that the classic de Rham theorem [dR] states that any simply connnected, complete Riemannian manifold admits a decomposition as a metric product $\tilde{M} = \mathbb{R}^k \times M_1 \times \ldots \times M_k$, where $\mathbb{R}^k$ is a Euclidean space equipped with the standard metric, and each M_i is metrically irreducible (and not $\mathbb{R}$ or a point). Furthermore, this decomposition is unique up to permutation of the factors. This result was recently generalized by Foertsch-Lytchak to cover finite dimensional geodesic metric spaces [FL]. Our next corollary shows that, in the presence of non-positive Riemannian curvature, there is a strong relationship between splittings of $\tilde{M}$ and splittings of $Cone(\tilde{M})$.

Corollary 6.3 (Asymptotic cones detect splittings)**.** *Let M be a closed Riemannian manifold of non-positive curvature, $\tilde{M}$ the universal cover of M with induced Riemannian metric, and $X = Cone(\tilde{M})$ an arbitrary asymptotic cone of $\tilde{M}$. If $\tilde{M} = \mathbb{R}^k \times M_1 \times \ldots \times M_n$ is the de Rham splitting of $\tilde{M}$ into irreducible factors, and $X = \mathbb{R}^l \times X_1 \times \ldots \times X_m$ is the Foertsch-Lytchak splitting of X into irreducible factors, then $k = l$, $n = m$, and up to a relabeling of the index set, we have that each $X_i = Cone(M_i)$.*

Proof. Let us first assume that $\tilde{M}$ is irreducible (i.e. k=0, n=1), and show that $Cone(\tilde{M})$ is also irreducible (i.e. l=0, m=1). By way of contradiction, let us assume that X splits as a metric product, and observe that this clearly implies that every geodesic $\gamma \subset X$ is contained inside a flat. In particular, from our Theorem 1.1, we see that every geodesic inside $\tilde{M}$ must have higher rank. Applying the Ballmann-Burns-Spatzier rank rigidity result, and recalling that $\tilde{M}$ was irreducible, we conclude that $\tilde{M}$ is in fact an irreducible higher rank symmetric space. But now Kleiner-Leeb have shown that for such spaces, the asymptotic cone is irreducible (see [KlL, Section 6]), giving us the desired contradiction.

Let us now proceed to the general case: from the metric splitting of $\tilde{M}$, we get a corresponding metric splitting $Cone(\tilde{M}) = \mathbb{R}^k \times Y_1 \times \ldots \times Y_n$, where each $Y_i = Cone(M_i)$. Since each M_i is irreducible, the previous paragraph tells us that each Y_i is likewise irreducible. So we now have two product decompositions of $Cone(\tilde{M})$ into irreducible factors. So assuming that each Y_i is distinct from a point and is not isometric to $\mathbb{R}$, we could appeal to the uniqueness portion of Foertsch-Lytchak [FL, Theorem 1.1] to conclude that, up to relabeling of the index set, each $X_i = Y_i = Cone(M_i)$, and that the Euclidean factors have to have the same dimension $k = l$.

To conclude the proof of our Corollary, we establish that if M is a simply connected, complete, Riemannian manifold of non-positive sectional curvature, and $\dim(M) \geq 2$, then $Cone(M)$ is distinct from a point or $\mathbb{R}$. First, recall that taking an arbitrary geodesic $\gamma \subset M$ (which we may assume passes through the basepoint $* \in M$), we get a corresponding geodesic $\gamma_\omega \subset Cone(M)$, i.e. an isometric embedding of $\mathbb{R}$ into $Cone(M)$. In particular, we see that $\dim(Cone(M)) > 0$. To see that $Cone(M)$ is distinct from $\mathbb{R}$, it is enough to establish the existence of three points $p_1, p_2, p_3 \in$

$Cone(M)$ such that for each index j we have:

$$d_\omega(p_j, p_{j+2}) \neq d_\omega(p_j, p_{j+1}) + d_\omega(p_{j+1}, p_{j+2}) \tag{11}$$

But this is easy to do: take p_1, p_2 to be the two distinct points on the geodesic γ_ω at distance one from the basepoint $* \in Cone(M)$, so that $d_\omega(p_1, p_2) = 2$. Observe that one can represent the points p_1, p_2 via the sequences of points $\{x_i\}$, $\{y_i\}$ along γ having the property that $* \in \overline{x_i y_i}$, and $d(x_i, *) = \lambda_i = d(*, y_i)$, where λ_i is the sequence of scales used in forming the asymptotic cone $Cone(M)$. Now since $\dim(M) \geq 2$, we can find another geodesic η through the basepoint $* \in M$, with the property that $\eta \perp \gamma$. Taking the sequence $\{z_i\}$ to lie on η, and satisfy $d(z_i, *) = \lambda_i$, it is easy to see that this sequence defines a third point $p_3 \in Cone(M)$ satisfying $d_\omega(p_3, *) = 1$. From the triangle inequality, we immediately have that $d_\omega(p_1, p_3) \leq 2$ and $d_\omega(p_2, p_3) \leq 2$. On the other hand, since the Riemannian manifold M has non-positive sectional curvature, we can apply Toponogov's theorem to each of the triangles $\{*, x_i, z_i\}$: since we have a right angle at the vertex $*$, and we have $d(*, x_i) = d(*, z_i) = \lambda_i$, Toponogov tells us that $d(x_i, z_i) \geq \sqrt{2} \cdot \lambda_i$. Passing to the asymptotic cone, this gives the lower bound $d(p_1, p_3) \geq \sqrt{2}$, and an identical argument gives the estimate $d(p_2, p_3) \geq \sqrt{2}$. It is now easy to verify that the three points p_1, p_2, p_3 satisfy (11), and hence $Cone(M) \neq \mathbb{R}$, as desired. This concludes the proof of Corollary 6.3.

□

Before stating our next result, we recall that the celebrated rank rigidity theorem of Ballmann-Burns-Spatzier (see Section 2.3) was motivated by Gromov's well-known rigidity theorem, the proof of which appears in the book [BGS]. Our next corollary shows how in fact Gromov's rigidity theorem can directly be deduced from the rank rigidity theorem. This is our last:

Corollary 6.4 (Gromov's higher rank rigidity [BGS]). *Let M^* be a compact locally symmetric space of $\mathbb{R}$-rank ≥ 2, with universal cover $\tilde{M}^*$ irreducible, and let M be a compact Riemannian manifold with sectional curvature $K \leq 0$. If $\pi_1(M) \cong \pi_1(M^*)$, then M is isometric to M^*, provided $Vol(M) = Vol(M^*)$.*

Proof. Since both M and M^* are compact with isomorphic fundamental groups, the Milnor-Švarc theorem gives us quasi-isometries:

$$\tilde{M}^* \simeq \pi_1(M^*) \simeq \pi_1(M) \simeq \tilde{M}$$

which induce a bi-Lipschitz homeomorphism $\phi : Cone(\tilde{M}^*) \to Cone(\tilde{M})$. Now in order to apply the rank rigidity theorem, we need to establish that every geodesic in $\tilde{M}$ has rank ≥ 2.

We first observe that the proof of Corollary 6.2 extends almost verbatim to the present setting. Indeed, in Corollary 6.2, we used the identity map to induce a bi-Lipschitz homeomorphism between the asymptotic cones, and then appealed to Corollary 6.1. The sole difference in our present context is that, rather than using the identity map, we use the quasi-isometry between $\tilde{M}$ and $\tilde{M}^*$ induced by the isomorphism $\pi_1(M) \cong \pi_1(M^*)$. This in turn induces a bi-Lipschitz homeomorphism between asymptotic cones (see Section 2.1). The reader can easily verify that the rest of the argument in Corollary 6.2 extends to our present setting, establishing that every lift to $\tilde{M}$ of a periodic geodesic in M has rank ≥ 2.

So we now move to the general case, and explain why *every* geodesic in $\tilde{M}$ has higher rank. To see this, assume by way of contradiction that there is a geodesic $\eta \subset \tilde{M}$ with $rk(\eta) = 1$. Note that the geodesic η cannot bound a half-plane. But once we have the existence of such an η, we can appeal to results of Ballmann [Ba1, Theorem 2.13], which imply that η can be approximated (uniformly on compacts) by lifts of periodic geodesics in M; let $\{\tilde{\gamma}_i\} \to \eta$ be such an approximating sequence. Since each $\tilde{\gamma}_i$ has $rk(\tilde{\gamma}_i) \geq 2$, it supports a parallel Jacobi field J_i, which can be taken to satisfy $||J_i|| \equiv 1$ and $\langle J_i, \tilde{\gamma}_i' \rangle \equiv 0$. Now we see that:

- the limiting vector field J defined along η exists, due to the control on $||J_i||$,
- the vector field J along η is a parallel Jacobi field, since both the "parallel" and "Jacobi" condition can be encoded by differential equations with smooth coefficients, solutions to which will vary continuously with respect to initial conditions, and
- J will have unit length and will be orthogonal to η', from the corresponding condition on the J_i.

But this contradicts our assumption that $rk(\eta) = 1$. So we conclude that every geodesic $\eta \subset \tilde{M}$ must satisy $rk(\eta) \geq 2$, as desired.

From the rank rigidity theorem, we can now conclude that $\tilde{M}$ either splits as a product, or is isometric to an irreducible higher rank symmetric space. Since the asymptotic cone of the irreducible higher rank symmetric space is topologically irreducible (see [KlL, Section 6]), and $Cone(\tilde{M})$ is homeomorphic to $Cone(\tilde{M}^*)$, we have that $\tilde{M}$ cannot split as a product. Finally, we see that $\pi_1(M) \cong \pi_1(M^*)$ acts cocompactly, isometrically on two irreducible higher rank symmetric spaces $\tilde{M}$ and $\tilde{M}^*$. By Mostow rigidity [Mo], we have that the quotient spaces are, after suitably rescaling, isometric. This completes our proof of Gromov's higher rank rigidity theorem.

$\square$

Finally, let us conclude our paper with a few comments on this last corollary.

Remarks: (1) The actual statement of Gromov's theorem in [BGS, pg. (i)] does not assume $\tilde{M}^*$ to be irreducible, but rather M^* to be irreducible (i.e. there is no finite cover of M^* that splits isometrically as a product). This leaves the possibility that the universal cover $\tilde{M}^*$ splits isometrically as a product, but no finite cover of M^* splits isometrically as a product. However, in this specific case, the desired result was already proved by Eberlein (see [Eb]). And in fact, in the original proof of Gromov's rigidity theorem, the very first step (see [BGS, pg. 154]) consists of appealing to Eberlein's result to reduce to the case where $\tilde{M}^*$ is irreducible.

(2) In the course of writing this paper, the authors learnt of the existence of another proof of Gromov's rigidity result, which bears some similarity to our reasoning. As the reader has surmised from the proof of Corollary 6.3, the key is to somehow show that M also has to have higher rank. But a sophisticated result of Ballmann-Eberlein [BaEb] establishes that the rank of a non-positively curved Riemannian manifold M can in fact be detected directly from algebraic properties of $\pi_1(M)$, and hence the property of having "higher rank" is in fact algebraic (see also the recent preprint of Bestvina-Fujiwara [BeFu]). The main advantage of our approach is that one can deduce Gromov's rigidity result *directly* from rank rigidity.

(3) We point out that various other mathematicians have obtained results extending Gromov's theorem (and which do not seem tractable using our methods). A variation considered by Davis-Okun-Zheng ([DOZ], requires $\tilde{M}^*$ to be *reducible* and M^* to be an irreducible (the same hypothesis as in Eberlein's rigidity result). However, Davis-Okun-Zheng allow the metric on M to be locally CAT(0) (rather than Riemannian non-positively curved), and are still able to conclude that M is isometric (after rescaling) to M^*. The optimal result in this direction is due to Leeb [L], giving a characterization of certain higher rank symmetric spaces and Euclidean buildings within the broadest possible class of metric spaces, the Hadamard spaces (complete geodesic spaces for which the distance function between pairs of geodesics is always convex). It is worth mentioning that Leeb's result relies heavily on the viewpoint developed in the Kleiner-Leeb paper [KlL].

(4) We note that our method of proof can also be used to establish a *non-compact, finite volume* analogue of the previous corollary. Three of the key ingredients going into our proof were (i) Ballmann's result on the density of periodic geodesics in the tangent bundle, (ii) Ballmann-Burns-Spatzier's rank rigidity theorem, and (iii) Mostow's strong rigidity theorem. A finite volume version of (i) was obtained by Croke-Eberlein-Kleiner (see [CEK, Appendix]), under the assumption that the fundamental group is finitely generated. A finite volume version of (ii) was obtained by Eberlein-Heber (see [EbH]). The finite volume versions of Mostow's strong rigidity were obtained by Prasad in the $\mathbb{Q}$-rank one case [Pr] and Margulis in the $\mathbb{Q}$-rank ≥ 2 case [Ma] (see also [R]). One technicality in the non-compact case is that isomorphisms of fundamental groups no longer induce quasi-isometries of the universal cover. In particular, it is no longer sufficient to just assume $\pi_1(M) \cong \pi_1(M^*)$, but rather one needs a homotopy equivalence $f : M \to M^*$ with the property that f lifts to a quasi-isometry $\tilde{f} : \tilde{M} \to \tilde{M}^*$. We leave the details to the interested reader.

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