REVISITING FARRELL'S NONFINITENESS OF NIL

JEAN-FRANÇOIS LAFONT[†], STRATOS PRASSIDIS[*], AND KUN WANG[⋆]

Abstract. We study Farrell Nil-groups associated to a finite order automorphism of a ring R. We show that any such Farrell Nil-group is either trivial, or infinitely generated (as an abelian group). Building on this first result, we then show that any finite group that occurs in such a Farrell Nil-group occurs with infinite multiplicity. If the original finite group is a direct summand, then the countably infinite sum of the finite subgroup also appears as a direct summand. We use this to deduce a structure theorem for countable Farrell Nil-groups with finite exponent. Finally, as an application, we show that if V is any virtually cyclic group, then the associated Farrell or Waldhausen Nil-groups can always be expressed as a countably infinite sum of copies of a finite group, provided they have finite exponent (which is always the case in dimension 0).

1. Introduction

For a ring R and an automorphism $\alpha : R \to R$, one can form the twisted polynomial ring $R_\alpha[t]$, which as an additive group coincides with the polynomial ring $R[t]$, but with product given by $(rt^i)(st^j) = r\alpha^{-i}(s)t^{i+j}$. There is a natural augmentation map $\varepsilon : R_\alpha[t] \to R$ induced by setting $\varepsilon(t) = 0$. For $i \in \mathbb{Z}$, the *Farrell twisted Nil-groups* $NK_i(R, \alpha) := \ker(\varepsilon_*)$ are defined to be the kernels of the induced K-theory map $\varepsilon_* : K_i(R_\alpha[t]) \to K_i(R)$. This induced map is split injective, hence $NK_i(R, \alpha)$ can be viewed as a direct summand in $K_i(R_\alpha[t])$. In the special case where the automorphism α is the identity, the ring $R_\alpha[t]$ is just the ordinary polynomial ring $R[t]$, and the Farrell twisted Nil reduces to the ordinary *Bass Nil-groups*, which we just denote by $NK_i(R)$. We establish the following:

Theorem A. *Let R be a ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. Then $NK_i(R, \alpha)$ is either trivial, or infinitely generated as an abelian group.*

The proof of this result relies heavily on a method developed by Farrell [6], who first showed in 1977 that the lower Bass Nil-groups $NK_*(R)$ with $* \leq 1$, are always either trivial, or infinitely

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generated. This result was subsequently extended to the higher Bass Nil-groups $NK_*(R)$ with $* \geq 1$ by Prasolov [19] (see also van der Kallen [15]). For Farrell's twisted Nils, when the automorphism α has finite order, Grunewald [11] and Ramos [21] independently established the corresponding result for $NK_*(R, \alpha)$ when $* \leq 1$. All these papers used the same basic idea, which we call *Farrell's Lemma*. We exploit the same idea, and establish our own version of Farrell's Lemma (and prove the theorem) in Section 3.

Remark 1.1. Farrell's original proof of his lemma used the transfer map on K-theory. Naïvely, one might want to try to prove **Theorem A** as follows: choose m so that $\alpha^m = \alpha$. Then there is a ring homomorphism from $A = R_\alpha[t]$ to $B = R_\alpha[s]$ sending $t \mapsto s^m$. Call the induced map on K-theory $F_m : K(A) \to K(B)$. Since B is a free (left) A-module of rank m, the transfer map V_m is also defined, and $G_m := V_m \circ F_m = \mu_m$ (multiplication by m). Then follow Farrell's original 1977 argument verbatim to conclude the proof. Unfortunately this approach does not work, for two reasons.

Firstly, the identity $G_m = \mu_m$ does not hold in the twisted case (basically due to the fact that $\oplus_m A$ and B are not isomorphic *as bimodules*). We do not explicitly know what the map G_m does on K-theory, but it is definitely **not** multiplication by an integer. Instead, we have the somewhat more complicated identity given in part (2) of our Lemma 3.1, but which is still sufficient to establish the Theorem.

Secondly, while it is possible to derive the identity in part (2) of Lemma 3.1 using the transfer map as in Farrell's original argument, it is not at all clear how to obtain the analogue of part (3) in higher dimensions by working at the level of K-theory groups. Instead, we have to work at the level of categories, specifically, with the *Nil-category* $NIL(R; \alpha)$ (see Section 2), in order to ensure property (3). The details are in [12].

Next we refine somewhat the information we have on these Farrell Nils, by focusing on the finite subgroups arising as direct summands. In section 4, we establish:

Theorem B. *Let R be a ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. If $H \leq NK_i(R, \alpha)$ is a finite subgroup, then $\bigoplus_\infty H$ also appears as a subgroup of $NK_i(R, \alpha)$. Moreover, if H is a direct summand in $NK_i(R, \alpha)$, then so is $\bigoplus_\infty H$.*

In the statement above, and throughout the paper, $\oplus_\infty H$ denotes the direct sum of countably infinitely many copies of the group H. **Theorem B** together with some group theoretic facts enable us to deduce a structure theorem for certain Farrell Nil-groups. In section 5, we prove:

Theorem C. *Let R be a countable ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. If $NK_i(R, \alpha)$ has finite exponent, then there exists a finite abelian group H, so that $NK_i(R, \alpha) \cong \bigoplus_\infty H$.*

A straightforward corollary of **Theorem C** is the following:

Corollary 1.2. *Let G be a finite group, $\alpha \in Aut(G)$. Then there exists a finite abelian group H, whose exponent divides some power of $|G|$, with the property that $NK_0(\mathbb{Z}G, \alpha) \cong \bigoplus_\infty H$.*

Proof. F. Connolly and S. Prassidis in [2] proved that $NK_0(\mathbb{Z}G, \alpha)$ has finite exponent when G is finite. A. KuKu and G. Tang [16, Theorem 2.2] showed that $NK_i(\mathbb{Z}G, \alpha)$ is $|G|$-primary torsion for all $i \geq 0$. These facts together with **Theorem C** above complete the proof. □

Remark 1.3. It is a natural question whether the above Corollary holds in dimensions other than zero. In negative dimensions $i < 0$, Farrell and Jones showed in [8] that $NK_i(\mathbb{Z}G, \alpha)$ always vanishes when G is finite. In positive dimensions $i > 0$, there are partial results. As mentioned in the proof above, Kuku and Tang [16, Theorem 2.2] showed that $NK_i(\mathbb{Z}G, \alpha)$ is $|G|$-primary torsion. Grunewald [12, Theorem 5.9] then generalized their result to *polycyclic-by-finite* groups in all dimensions. He showed that, for all $i \in \mathbb{Z}$, $NK_i(\mathbb{Z}G, \alpha)$ is mn-primary torsion for every polycyclic-by-finite group G and every group automorphism $\alpha : G \to G$ of finite order, where $n = |\alpha|$ and m is the index of some poly-infinite cyclic subgroup of G (such a subgroup always exists). However, although we have these nice results on the possible orders of torsion elements, it seems there are no known results on the exponent of these Nil-groups. This is clearly a topic for future research.

Remark 1.4. As an example in dimension greater than zero, Weibel [25] showed that $NK_1(\mathbb{Z}D_4) \neq 0$, where D_4 denotes the dihedral group of order 8. He also constructs a surjection $\bigoplus_\infty(\mathbb{Z}_2 \oplus \mathbb{Z}_4) \to NK_1(\mathbb{Z}D_4)$, showing that this group has exponent 2 or 4. It follows from our Corollary that the group $NK_1(\mathbb{Z}D_4)$ is isomorphic to one of the three groups $\bigoplus_\infty(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$, $\bigoplus_\infty \mathbb{Z}_4$, or $\bigoplus_\infty \mathbb{Z}_2$.

For our next application, we recall that there is, for any group G, an assembly map $H_n^G(\underline{E}G; \mathbf{K}_\mathbb{Z}) \to K_n(\mathbb{Z}[G])$, where $H_*^?(-; \mathbf{K}_\mathbb{Z})$ denotes the specific equivariant generalized homology theory appearing in the K-theoretic Farrell-Jones isomorphism conjecture with coefficient in $\mathbb{Z}$, and $\underline{E}G$ is a model for the classifying space for proper G-actions. We refer the reader to Section 5 for a discussion of these notions, as well as for the proof of:

Theorem D. *For any virtually cyclic group V, there exists a finite abelian group H with the property that there is an isomorphism:*

$$\bigoplus_\infty H \cong CoKer\Big(H_0^V(\underline{E}V; \boldsymbol{K}_\mathbb{Z}) \to K_0(\mathbb{Z}[V])\Big)$$

The same result holds in dimension n whenever $CoKer\Big(H_n^V(\underline{E}V; \boldsymbol{K}_\mathbb{Z}) \to K_n(\mathbb{Z}[V])\Big)$ has finite exponent.

We conclude the paper with some general remarks and open questions in Section 6.

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2. Some exact functors

In this section, we define various functors that will be used in our proofs. Let R be an associative ring with unit and $\alpha : R \to R$ be a ring automorphism of finite order, say $|\alpha| = n$. For each integer $i \in \mathbb{Z}$, denote by R_{α^i} the R-bimodule which coincides with R as an abelian group, but with bimodule structure given by $r \cdot x := rx$ and $x \cdot r := x\alpha^i(r)$ (where $x \in R_{\alpha^i}$ and $r \in R$). Note that as left (or as right) R-modules, R_{α^i} and R are isomorphic, but they are in general *not* isomorphic as R-bimodules. For each right R-module M and integer i, define a new right R-module M_{α^i} as follows: as abelian groups, M_{α^i} is the same as M, however the right R-module structure on M_{α^i} is given by $x \cdot r := x\alpha^i(r)$. Clearly $M_{\alpha^n} = M$ and $(M_{\alpha^i})_{\alpha^j} = M_{\alpha^{i+j}}$ as right R-modules. We could have defined $M_{\alpha^i} = M \otimes_R R_{\alpha^i}$, however this has the slight disadvantage that the above equalities would not hold – we would only have natural isomorphisms between the corresponding functors.

Let $\mathbf{P}(R)$ denote the category of finitely generated right projective R-modules. For each $i \in \mathbb{Z}$, there is an exact functor $S_i : \mathbf{P}(R) \to \mathbf{P}(R)$ given by $S_i(P) := P_{\alpha^i}$ on objects and $S_i(\phi) = \phi$ on morphisms. Note that if we forget about the right R-module structures, and just view these as abelian groups and group homomorphisms, then each S_i is just the identity functor. Clearly $S_i \circ S_j = S_j \circ S_i = S_{i+j}$ and $S_n = Id$, so the map $i \mapsto S_i$ defines a functorial $\mathbb{Z}$-action on the category $\mathbf{P}(R)$, which factors through a functorial $\mathbb{Z}_n$-action (recall that n is the order of the ring automorphism α).

We are interested in the *Nil-category* $NIL(R;\alpha)$. Recall that objects of this category are of the form (P, f), where P is an object in $\mathbf{P}(R)$ and $f : P \to P_\alpha = S_1(P)$ is a right R-module homomorphism which is *nilpotent*, in the sense that a high enough composite map of the following form is the zero map:

$$P \xrightarrow{S_{k-1}(f)\circ S_{k-2}(f)\circ\cdots\circ S_1(f)\circ f} P_{\alpha^k}$$

A morphism $\phi : (P, f) \to (Q, g)$ in $NIL(R;\alpha)$ is given by a morphism $\phi : P \to Q$ in $\mathbf{P}(R)$ which makes the obvious diagram commutative, i.e. $S_1(\phi) \circ f = g \circ \phi$. The exact structure on $\mathbf{P}(R)$ induces an exact structure on $NIL(R;\alpha)$, and we have two exact functors

$$F : NIL(R;\alpha) \to \mathbf{P}(R), \quad F(P, f) = P$$

$$G : \mathbf{P}(R) \to NIL(R;\alpha), \quad G(P) = (P, 0)$$

which give rise to a splitting of the K-theory groups $K_i(NIL(R;\alpha)) = K_i(R) \oplus Nil_i(R;\alpha)$, where $Nil_i(R;\alpha) := \mathrm{Ker}\big(K_i(NIL(R;\alpha)) \to K_i(R)\big), i \in \mathbb{N}$ (natural numbers start with 0 in this paper).

Remark 2.1. The Farrell Nil-groups $NK_*(R,\alpha)$ mentioned in the introduction coincide, with a dimension shift, with the groups $Nil_*(R;\alpha^{-1})$ defined above. More precisely, one has for every $i \geq 1$ an isomorphism $NK_i(R,\alpha) \cong Nil_{i-1}(R;\alpha^{-1})$ ([10, Theorem 2.1]).

We now introduce two exact functors on the category $NIL(R;\alpha)$ which will play an important role in our proofs. On the level of K-theory, one of these yields the twisted analogue of the Verscheibung operators, while the other gives the classical Frobenius operators.

Definition 2.2 (Twisted Verscheibung functors)**.** For each positive integer m, define the twisted Verscheibung functors $V_m : NIL(R;\alpha) \to NIL(R;\alpha)$ as follows. On objects, we set

$$V_m\big((P,f)\big) = (P \oplus P_{\alpha^{-1}} \oplus P_{\alpha^{-2}} \oplus \cdots \oplus P_{\alpha^{-mn}}, \overline{f}) = \Big(\sum_{i=0}^{mn} P_{\alpha^{-i}}, \overline{f}\Big) = \Big(\sum_{i=0}^{mn} S_{-i}(P), \overline{f}\Big)$$

where the morphism

$$\overline{f} : \sum_{i=0}^{mn} P_{\alpha^{-i}} \longrightarrow \Big(\sum_{j=0}^{mn} P_{\alpha^{-j}}\Big)_{\alpha} = \sum_{j=0}^{mn} P_{\alpha^{-j+1}}$$

is defined component-wise by the maps $f_{ij} : P_{\alpha^{-i}} \to P_{\alpha^{-j+1}}$ given via the formula

$$f_{ij} = \begin{cases} id & \text{if } j = i+1, 0 \leq i \leq mn-1 \\ f & \text{if } i = mn, j = 0 \\ 0 & \text{otherwise} \end{cases}$$

In the proof of Lemma 2.5 below, we will see that $\overline{f}$ is nilpotent, so that $V_m\big((P,f)\big)$ does indeed define an object in the category $NIL(R;\alpha)$. If $\phi : (P,f) \to (Q,g)$ is a morphism in the category $NIL(R;\alpha)$, we define the morphism

$$V_m(\phi) : \Big(\sum_{i=0}^{mn} P_{\alpha^{-i}}, \overline{f}\Big) \to \Big(\sum_{i=0}^{mn} Q_{\alpha^{-i}}, \overline{g}\Big)$$

via the formula $V_m(\phi) = \sum_{i=0}^{mn} S_{-i}(\phi)$. One checks that (i) $\overline{g} \circ V_m(\phi) = S_1(V_m(\phi)) \circ \overline{f}$, (ii) $V_m(id) = id$ and (iii) $V_m(\phi \circ \psi) = V_m(\phi) \circ V_m(\psi)$, so that V_m is indeed a functor. Moreover, V_m is exact because each S_{-i} is exact.

Definition 2.3 (Frobenius functors)**.** For each positive integer m, define the Frobenius functors $F_m : NIL(R;\alpha) \to NIL(R;\alpha)$ as follows. On objects, we set $F_m\big((P,f)\big) = (P, \tilde{f})$ where $\tilde{f}$ is the morphism defined by the composition

$$P \xrightarrow{S_{mn}(f) \circ S_{mn-1}(f) \circ \cdots \circ S_1(f) \circ f} P_{\alpha^{mn+1}} = P_{\alpha}$$

(recall that the ring automorphism α has order $|\alpha| = n$). It is immediate that the map $\tilde{f}$ is nilpotent, so that $F_m\big((P,f)\big)$ is indeed an object in $NIL(R;\alpha)$. Now if $\phi : (P,f) \to (Q,g)$ is a morphism in the category $NIL(R;\alpha)$, we define the morphism $F_m(\phi) : (P,\tilde{f}) \to (Q,\tilde{g})$ to coincide with the morphism ϕ. It is obvious that $F_m(id) = id$ and $F_m(\phi \circ \psi) = F_m(\phi) \circ F_m(\psi)$, and one can easily check that $\tilde{g} \circ \phi = S_1(\phi) \circ \tilde{f}$, so that F_m is a genuine functor. Clearly F_m is exact.

Definition 2.4 (α-twisting functors)**.** For each $i \in \mathbb{Z}$, we define the exact functor $T_i : NIL(R;\alpha) \to NIL(R;\alpha)$ as follows. On objects, we set $T_i\big((P,f)\big) = \big(S_{-i}(P), S_{-i}(f)\big)$, and if $\phi : (P,f) \to (Q,g)$ is a morphism, we set $T_i(\phi)$ to be the morphism $S_{-i}(\phi) : S_{-i}(P) \to S_{-i}(Q)$. Observe that, as with the functors S_i on the category $\mathbf{P}(R)$, the functors T_i define a functorial $\mathbb{Z}$-action on the category $NIL(R;\alpha)$, which factors through a functorial $\mathbb{Z}_n$-action.

The relationship between these various functors is described in the following Lemma. We will write G_m for the composite exact functor $G_m = F_m \circ V_m$.

Lemma 2.5. *We have the equality* $G_m = \sum_{i=0}^{mn} T_i$.

Proof. Let (P,f) be an object in $NIL(R;\alpha)$. Then we have $G_m\big((P,f)\big) = \Big(\sum_{i=0}^{mn} S_{-i}(P), \tilde{\overline{f}}\Big)$, where $\tilde{\overline{f}} = S_{mn}(\overline{f}) \circ S_{mn-1}(\overline{f}) \circ \cdots \circ S_1(\overline{f}) \circ \overline{f}$. Note that if we forget the right R-module structures, each S_i is the identity functor on abelian groups. So as a morphism of abelian groups, $\tilde{\overline{f}} = \overline{f}^{mn+1}$. Now recall that $\overline{f}$ is a morphism which cyclicly permutes the $mn+1$ direct summands occuring in its source and target. Using this observation, it is then easy to see that $\tilde{\overline{f}} = \overline{f}^{mn+1}$ is diagonal and equal to $\sum_{i=0}^{mn} S_{-i}(f)$. So on the level of objects, G_m and $\sum_{i=0}^{mn} T_i$ agree. From this, we also see that $\overline{f}$ is nilpotent (as was indicated in Definition 2.2). It is obvious that they agree on morphisms. □

Remark 2.6. It is natural to consider the more general case when $\alpha : R \to R$ has finite order in the *outer automorphism group* of the ring R, i.e. there exists $n \in \mathbb{N}$ and a unit $u \in R$ so that $\alpha^n(r) = uru^{-1}, \forall r \in R$. In this situation, we have for any right R-module M and integer m, an isomorphism $\tau_{m,M} : M_{\alpha^{mn}} \to M$, $\tau_{m,M}(r) := ru^m$ of right R-modules. This gives rise to a natural isomorphism between the functors S_{mn} and $S_0 = Id$. It is then easy to similarly define twisted Verscheibung functors and Frobenius functors, and to verify an analogue of Lemma 2.5. However, in this case, we generally do **not** have that T_n is naturally isomorphic to T_0, unless α fixes u. This key issue prevents our proof of Farrell's Lemma 3.1(2) below (which is the key to the proof of our main theorems) to go through in this more general setting.

3. Non-finiteness of Farrell Nils

This section is devoted to proving **Theorem A**.

3.1. **A version of Farrell's Lemma.**

We are now ready to establish our analogue of Farrell's key lemmas from his paper [6].

Lemma 3.1. *The following results hold:*

(1) $\forall j \in \mathbb{N}$, *the induced morphisms* $K_j(V_m), K_j(F_m) : K_j(NIL(R;\alpha)) \to K_j(NIL(R;\alpha))$ *on* K*-theory map the summand* $Nil_j(R;\alpha)$ *to itself;*

(2) $\forall j, m \in \mathbb{N}$, *the identity* $(2+mn)K_j(G_m) - K_j(G_m)^2 = \mu_{1+mn}$ *holds, where the map* μ_{1+mn} *is multiplication by* $1+mn$;

(3) $\forall j \in \mathbb{N}$ *and each* $x \in Nil_j(R;\alpha)$, *there exists a positive integer* $r(x)$, *such that* $K_j(F_m)(x) = 0$ *for all* $m \geq r(x)$.

Proof. (1) Let $H_m := \sum_{i=0}^{mn} S_{-i} : \mathbf{P}(R) \to \mathbf{P}(R)$, one then easily checks $F \circ V_m = H_m \circ F$. We also have $F \circ F_m = F$. Statement (1) follows easily from these.

(2) By the Additivity Theorem for algebraic K-theory, Lemma 2.5 immediately gives us that

$$K_j(G_m) = \sum_{i=0}^{mn} K_j(T_i) = id + m \sum_{i=1}^{n} K_j(T_i)$$

(recall that the functors T_i are n-periodic). Now let us evaluate the square of the map $K_j(G_m)$:

$$\begin{aligned} K_j(G_m)^2 &= \Big(id + m\sum_{i=1}^{n} K_j(T_i)\Big)\Big(id + m\sum_{l=1}^{n} K_j(T_l)\Big) \\ &= id + 2m\sum_{i=1}^{n} K_j(T_i) + m^2 \sum_{i=1}^{n}\sum_{l=1}^{n} K_j(T_{i+l}) \\ &= id + 2m\sum_{i=1}^{n} K_j(T_i) + m^2 \sum_{i=1}^{n}\sum_{l=1}^{n} K_j(T_l) \\ &= id + 2m\sum_{i=1}^{n} K_j(T_i) + m^2 n \sum_{l=1}^{n} K_j(T_l) \\ &= id + (2m + m^2 n)\sum_{i=1}^{n} K_j(T_i) \end{aligned}$$

In the third equality above, we used the fact that the T_i functors are n-periodic, so that shifting the index on the inner sum by i leaves the sum unchanged. Finally, substituting in the expression we have for $K_j(G_m)$ and the expression we derived for $K_j(G_m)^2$, we see that:

$$\begin{aligned} &(2+mn)K_j(G_m) - K_j(G_m)^2 \\ &= (2+mn)\Big(id + m\sum_{i=1}^{n} K_j(T_i)\Big) - \Big(id + (2m+m^2n)\sum_{i=1}^{n} K_j(T_i)\Big) \\ &= (2+mn)id - id = \mu_{(1+mn)} \end{aligned}$$

completing the proof of statement (2).

(3) This result is due to Grunewald [12, Proposition 4.6]. ☐

3.2. **Proof of Theorem A.** The proof of **Theorem A** now follows easily. Let us focus on the case where $i \geq 1$, as the case $i \leq 1$ has already been established by Grunewald [11] and Ramos [21]. So let us assume that the Farrell Nil-group $NK_i(R, \alpha) \cong Nil_{i-1}(R; \alpha^{-1})$ is non-trivial and finitely generated, where $i \geq 1$. Then one can find arbitrarily large positive integers m with the property that the map $\mu_{(1+mn)}$ is an injective map from $Nil_{i-1}(R; \alpha^{-1})$ to itself (for example, one can take m to be any multiple of the order of the torsion subgroup of $Nil_{i-1}(R; \alpha^{-1})$). From Lemma 3.1(2), we can factor the map $\mu_{(1+mn)}$ as a composite

$$\mu_{(1+mn)} = \big(\mu_{(2+mn)} - K_j(G_m)\big) \circ K_j(G_m)$$

and hence there is an arbitrarily large sequence of integers m with the property that the corresponding maps $K_j(G_m) = K_j(F_m) \circ K_j(V_m)$ are injective. This implies that there are infinitely many integers m for which the map $K_j(F_m)$ is non-zero.

On the other hand, let $x_1, \ldots, x_k$ be a finite set of generators for the abelian group $Nil_{i-1}(R; \alpha^{-1})$. Then from Lemma 3.1(3), we have that for any $m \geq \max\{r(x_i)\}$, the map $K_j(F_m)$ is identically zero, a contradiction. This completes the proof of **Theorem A**.

4. Finite subgroups of Farrell Nil-groups

4.1. **A Lemma on splittings.** In order to establish **Theorem B**, we will need an algebraic lemma for recognizing when two direct summands inside an ambient group jointly form a direct summand.

Lemma 4.1. *Let G be an abelian group and $H < G$, $K < G$ be a pair of subgroups. Suppose there are two retractions $\lambda : G \to H$ and $\rho : G \to K$ with the property that $\lambda(K) = \{0\}$. Then there exists a subgroup $L < G$, which is isomorphic to H, and such that $L \oplus K$ is also a direct summand of G.*

Proof. Consider the morphism $(\lambda, \rho) : G \to H \times K$ given by $g \mapsto (\lambda(g), \rho(g))$. It is split by the morphism $\beta : H \times K \to G$ given by $(h, k) \mapsto h - \rho(h) + k$, since λ, ρ are retractions and $\lambda(K) = 0$. Therefore $\beta(H \times K)$ is a direct summand of G. Let $L < G$ be the image of $H \times \{0\}$ under β. By noting $K = \beta(\{0\} \times K)$, we see that $L \oplus K < G$ is a direct summand. □

4.2. **Proof of Theorem B.** We are now ready to prove **Theorem B**. In order to simplify the notation, we will simply write V_m for $K_j(V_m)$, and use a similar convention for F_m and G_m.

Case $i \geq 1$. We first consider the case when $i \geq 1$, and recall that $NK_i(R, \alpha) \cong Nil_{i-1}(R; \alpha^{-1})$. Let $H < Nil_{i-1}(R; \alpha^{-1})$ be a finite subgroup. According to Lemma 3.1(3), since H is finite, there exists an integer $r(H) = \max_{x \in H}\{r(x)\}$, so that $F_m(H) = 0$ for all $m > r(H)$. Let $S \subset \mathbb{N}$ consist of all natural number $m > r(H)$ such that GCD$(1 + mn, |H|)$=1. S contains every multiple of $|H|$ which is greater than $r(H)$, so is an infinite set. Consider the morphisms

$$Nil_{i-1}(R; \alpha^{-1}) \xrightarrow{V_m} Nil_{i-1}(R; \alpha^{-1}) \xrightarrow{F_m} Nil_{i-1}(R; \alpha^{-1})$$

so that the composite is the morphism G_m, and define the subgroup $H_m \leq Nil_{i-1}(R;\alpha^{-1})$ to be $H_m := V_m(H)$. By the defining property of the set S, we have that for $m \in S$, $(\mu_{2+mn} - G_m) \circ G_m = \mu_{1+mn}$ is an isomorphism when restricted to H. Hence G_m is a monomorphism when restricted to H, forcing V_m to also be a monomorphism when restricted to H. So for all $m \in S$, we see that $H_m \cong H$.

We now claim that, for all $m \in S$, $H_m \cap H = \{0\}$. Indeed, since integers in S are larger than $r(H)$, we have $F_m(H) = 0$. But for $m \in S$, the composite map $G_m = F_m \circ V_m$ is an isomorphism from H to $G_m(H) = F_m(H_m)$, so F_m must be injective on H_m. Putting these two statements together, we get that $H_m \cap H = \{0\}$. We conclude that $H \oplus H < Nil_{i-1}(R;\alpha^{-1})$. Applying the same argument to $H \oplus H$ and so on, we conclude $\oplus_\infty H < Nil_{i-1}(R;\alpha^{-1})$.

Next we argue that, if the original subgroup H was a direct summand in $Nil_{i-1}(R;\alpha^{-1})$, then we can find a copy of $H \oplus H$ which is also a direct summand in $Nil_{i-1}(R;\alpha^{-1})$, and which *extends* the original direct summand (i.e. the first copy of H inside the direct summand $H \oplus H$ coincides with the original H).

To see this, let us assume $H < Nil_{i-1}(R;\alpha^{-1})$ is a direct summand, so there exists a retraction $\rho : Nil_{i-1}(R;\alpha^{-1}) \to H$. Let H_m be obtained as above. We first construct a retraction $\lambda : Nil_{i-1}(R;\alpha^{-1}) \to H_m$. Recall that μ_{1+mn} is an isomorphism on H_m, so there exists an integer l so that $\mu_l \circ \mu_{1+mn}$ is the identity on H_m. We define $\lambda : Nil_{i-1}(R;\alpha^{-1}) \to H_m$ to be the composition of the following maps:

$$Nil_{i-1}(R;\alpha^{-1}) \xrightarrow{F_m} Nil_{i-1}(R;\alpha^{-1}) \xrightarrow{\mu_{2+mn}-G_m} Nil_{i-1}(R;\alpha^{-1}) \xrightarrow{\rho} H \xrightarrow{V_m|_H} H_m \xrightarrow{\mu_l} H_m$$

We claim λ is a retraction. Note for $x \in H_m$, there exists $y \in H$ with $V_m(y) = x$. We now evaluate

$$\begin{aligned}
\lambda(x) &= (\mu_l \circ V_m \circ \rho \circ (\mu_{2+mn} - G_m) \circ F_m)(x) \\
&= (\mu_l \circ V_m \circ \rho \circ (\mu_{2+mn} - G_m) \circ F_m)(V_m(y)) \\
&= (\mu_l \circ V_m \circ \rho \circ (\mu_{2+mn} - G_m) \circ G_m)(y) \\
&= (\mu_l \circ V_m \circ \rho \circ ((2+mn)G_m - G_m^2)))(y) \\
&= (\mu_l \circ V_m \circ \rho \circ \mu_{1+mn})(y) \\
&= \big(\mu_l \circ \mu_{1+mn}\big)\big((V_m \circ \rho)(y)\big) \\
&= \big(\mu_l \circ \mu_{1+mn}\big)\big(V_m(y)\big) \\
&= (\mu_l \circ \mu_{1+mn})(x) \\
&= x
\end{aligned}$$

This verifies λ is a retraction. Note also that $\lambda(H) = 0$, since $F_m(H) = 0$ follows from the fact that $m \in S$ (recall that integers in S are larger than $r(H)$). Hence we are in the situation of Lemma 4.1, and we can conclude that $H \oplus H$ also arises as a direct summand of $Nil_{i-1}(R;\alpha^{-1})$. Note that, when applying our Lemma 4.1, we replaced the *second copy* H_m of H by some other (isomorphic) subgroup, but kept the *first copy* of H to be the original H. Hence the direct summand $H \oplus H$ does indeed extend the original summand H. Iterating the process, we obtain that $\oplus_\infty H$ is a direct summand of $Nil_{i-1}(R;\alpha^{-1})$. This completes the proof of **Theorem B** in the case where $i \geq 1$.

Case $i \leq 1$. Next, let us consider the case of the Farrell Nil-groups $NK_i(R,\alpha^{-1})$ where $i \leq 1$. For these, the proof of **Theorem B** proceeds via a (descending) induction on i, with the case $i = 1$ having been established above.

We remind the reader of the standard technique for extending results known for K_1 to lower K-groups. Take the ring $\Lambda\mathbb{Z}$ consisting of all $\mathbb{N} \times \mathbb{N}$ matrices with entries in $\mathbb{Z}$ which contain only finitely many non-zero entries in each row and each column, and quotient out by the ideal $I \lhd \Lambda\mathbb{Z}$ consisting of all matrices which vanish outside of a finite block. This gives the ring $\Sigma\mathbb{Z} = \Lambda\mathbb{Z}/I$, and we can now define the *suspension functor* on the category of rings by tensoring with the ring $\Sigma\mathbb{Z}$, i.e. sending a ring R to the ring $\Sigma(R) := \Sigma\mathbb{Z} \otimes R$, and a morphism $f : R \to S$ to the morphism $Id \otimes f : \Sigma(R) \to \Sigma(S)$. The functor Σ has the property that there are natural isomorphisms $K_i(R) \cong K_{i+1}(\Sigma(R))$ (for all $i \in \mathbb{Z}$). Moreover, there is a natural isomorphism $\Sigma\big(R_\alpha[t]\big) \cong (\Sigma R)_{Id\otimes\alpha}[t]$, which induces a commutative square

$$\begin{array}{ccc} K_i(R_\alpha[t]) & \longrightarrow & K_i(R) \\ \downarrow \cong & & \downarrow \cong \\ K_{i+1}\big((\Sigma\mathbb{Z}\otimes R)_{Id\otimes\alpha}[t]\big) & \longrightarrow & K_{i+1}(\Sigma\mathbb{Z}\otimes R) \end{array}$$

By induction, for each $m \in \mathbb{N}$, this allows us to identify $NK_{1-m}(R,\alpha)$ with $NK_1(\Sigma^m R, Id^{\otimes m} \otimes \alpha)$, where Σ^m denotes the m-fold application of the functor Σ. Obviously, if the automorphism α has finite order in $\mathrm{Aut}(R)$, the induced automorphism $Id^{\otimes m} \otimes \alpha$ will have finite order in $\mathrm{Aut}\big((\Sigma\mathbb{Z})^{\otimes m} \otimes R\big)$. So for the Farrell Nil-groups $NK_i(R,\alpha)$ with $i \leq 0$, the result immediately follows from the special case of NK_1 considered above. This completes the proof of **Theorem B**.

5. A structure theorem and Nils associated to virtually cyclic groups

In this section, we discuss some applications and establish **Theorem C** and **Theorem D**. For a general ring R, we know by **Theorem A** that a non-trivial Nil-group is an *infinitely generated* abelian group. While finitely generated abelian groups have a very nice structural theory, the picture is much more complicated in the infinitely generated case (the reader can consult [22, Chapter 4] for an overview of the theory). If one restricts to abelian (torsion) groups of *finite exponent*, then it is an old result of Prüfer [20] that any such group is a direct sum of cyclic groups (see [22, item 4.3.5 on pg. 105] for a proof).

5.1. **Proof of Theorem C.** We can now explain how our **Theorem B** allows us to obtain a structure theorem for certain Nil-groups. Let R be a countable ring and $\alpha : R \to R$ be an automorphism of finite order. Then by Proposition 7.1 of the Appendix, we know that $NK_i(R, \alpha)$ is a countable group. If in addition $NK_i(R, \alpha)$ has finite exponent, then by the result of Prüfer mentioned above, it follows that $NK_i(R, \alpha)$ decomposes as a countable direct sum of cyclic groups of prime power order, each of which appears with some multiplicity. In view of our **Theorem B**, any summand which occurs must actually occur infinitely many times. Since the exponent of the Nil-group is finite, there is an upper bound on the prime power orders that can appear, and hence there are only finitely many possible isomorphism types of summands. Let H be the direct sum of a single copy of each cyclic group of prime power order which appear as a summand in $NK_i(R, \alpha)$. It follows immediately that $\bigoplus_\infty H \cong NK_i(R, \alpha)$. This completes the proof of **Theorem C**.

5.2. **Farrell-Jones Isomorphism Conjecture.** In applications to geometric topology, the rings of interest are typically integral group rings $\mathbb{Z}G$. For computations of the K-theory of such groups, the key tool is provided by the (K-theoretic) *Farrell-Jones Isomorphism Conjecture* [7]. Davis and Lück [4] gave a general framework for the formulations of various isomorphism conjectures. In particular, they constructed for any group G, an OrG-spectrum, i.e. a functor $\mathbf{K}_\mathbb{Z} : \text{Or}G \to \mathbf{Sp}$, where Or$G$ is the orbit category of G (objects are cosets $G/H, H < G$ and morphisms are G-maps) and $\mathbf{Sp}$ is the category of spectra. This functor has the property that $\pi_n(\mathbf{K}_\mathbb{Z}(G/H)) = K_n(\mathbb{Z}H)$. As an ordinary spectrum can be used to construct a generalized homology theory, this OrG-spectrum $\mathbf{K}_\mathbb{Z}$ was used to construct a G-equivariant homology theory $H_*^G(-; \mathbf{K}_\mathbb{Z})$. It has the property that $H_n^G(G/H; \mathbf{K}_\mathbb{Z}) = \pi_n(\mathbf{K}_\mathbb{Z}(G/H)) = K_n(\mathbb{Z}H)$ (for all $H < G$ and $n \in \mathbb{Z}$). In particular, on a point, $H_n^G(*; \mathbf{K}_\mathbb{Z}) = H_n^G(G/G; \mathbf{K}_\mathbb{Z}) = K_n(\mathbb{Z}G)$. Applying this homology theory to any G-CW-complex X, the obvious G-map $X \to *$ gives rise to an *assembly map*:

$$H_n^G(X; \mathbf{K}_\mathbb{Z}) \to H_n^G(*; \mathbf{K}_\mathbb{Z}) \cong K_n(\mathbb{Z}G).$$

The Farrell-Jones isomorphism conjecture asserts that, when the space X is a model for the classifying space for G-actions with isotropy in the virtually cyclic subgroups of G, then the above assembly map is an isomorphism. Thus, the conjecture roughly predicts that the K-theory of an integral group ring $\mathbb{Z}G$ is determined by the K-theory of the integral group rings of the virtually cyclic subgroups of G, assembled together in some homological fashion.

In view of this conjecture, one can view the K-theory of virtually cyclic groups as the "basic building blocks" for the K-theory of general groups. Focusing on such a virtually cyclic group V, one can consider the portion of the K-theory that comes from the finite subgroups of V. This would be the image of the assembly map:

$$H_n^V(\underline{\text{E}}V; \mathbf{K}_\mathbb{Z}) \to H_n^V(*; \mathbf{K}_\mathbb{Z}) \cong K_n(\mathbb{Z}V)$$

where $\underline{\text{E}}V$ is a model for the classifying space for proper V-actions. While this map is always split injective (see [1]), it is not surjective in general. Thus to understand the K-theory of a virtually cyclic group, we need to understand the K-theory of finite groups, and to understand the cokernels of the above assembly map. The cokernels of the above assembly map can also be interpreted as the obstruction to reduce the family of virtually cyclic groups used in the Farrell-Jones isomorphism conjecture to the family of finite groups - this is the *transitivity principle* (see [7, Theorem A.10]). Our **Theorem D** gives some structure for the cokernel of the assembly map.

5.3. **Proof of Theorem D.** Let V be a virtually cyclic group. Then one has that V is either of the form (i) $V = G \rtimes_\alpha \mathbb{Z}$, where G is a finite group and $\alpha \in \mathrm{Aut}(G)$, or is of the form (ii) $V = G_1 *_H G_2$, where G_i, H are finite groups and H is of index two in both G_i.

Let us first consider case (i). In this case, the integral group ring $\mathbb{Z}[V]$ is isomorphic to the ring $R_{\hat{\alpha}}[t, t^{-1}]$, the $\hat{\alpha}$-twisted ring of Laurent polynomials over the coefficient ring $R = \mathbb{Z}[G]$, where $\hat{\alpha} \in \mathrm{Aut}(\mathbb{Z}[G])$ is the ring automorphism canonically induced by the group automorphism α. Then it is known (see [5, Lemma 3.1]) that the cokernel we are interested in consists of the direct sum of the Farrell Nil-group $NK_n(\mathbb{Z}G, \hat{\alpha})$ and the Farrell Nil-group $NK_n(\mathbb{Z}G, \hat{\alpha}^{-1})$. Applying **Theorem C** and Corollary 1.2 to these two Farrell Nil-groups, we are done.

In case (ii), we note that V has a canonical surjection onto the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, obtained by surjecting each G_i onto $G_i/H \cong \mathbb{Z}_2$. Taking the preimage of the canonical index two subgroup $\mathbb{Z} \leq D_\infty$, we obtain a canonical index two subgroup $W \leq V$. The subgroup W is a virtually cyclic group of type (i), and is of the form $H \rtimes_\alpha \mathbb{Z}$, where $\alpha \in \mathrm{Aut}(H)$. Hence it has associated Farrell Nil-groups $NK_n(\mathbb{Z}H, \hat{\alpha})$.

The cokernel of the relative assembly map for the group V is a *Waldhausen Nil-group* associated to the splitting of V (see [5, Lemma 3.1]). It was recently shown that this Waldhausen Nil-group is always isomorphic to a single copy of the Farrell Nil-group $NK_n(\mathbb{Z}H, \hat{\alpha})$ associated to the canonical index two subgroup $W \leq V$ (see for example [3], [5], or for an earlier result in a similar vein [17]). Again, combining this with our **Theorem C** and Corollary 1.2, we are done, completing the proof of **Theorem D**.

6. Applications and Concluding Remarks

We conclude this short note with some further applications and remarks.

6.1. **Waldhausen's A-theory.** Recall that Waldhausen [24] introduced a notion of algebraic K-theory $A(X)$ of a topological space X. Once the K-theoretic contribution has been split off, one is left with the finitely dominated version of the algebraic K-theory $A^{fd}(X)$. This finitely dominated version satisfies the "fundamental theorem of algebraic K-theory", in that one has a homotopy splitting:

$$A^{fd}(X \times S^1) \simeq A^{fd}(X) \times BA^{fd}(X) \times NA^{fd}_+(X) \times NA^{fd}_-(X) \tag{1}$$

see [14] (the reader should compare this with the corresponding fundamental theorem of algebraic K-theory for rings, see [9]). The Nil-terms appearing in this splitting have been studied by Grunewald, Klein, and Macko [13], who defined Frobenius and Verschiebung operations, F_n, V_n, on the homotopy groups $\pi_*\big(NA^{fd}_{\pm}(X)\big)$. In particular, they show that the composite $V_n \circ F_n$ is multiplication by n [13, Proposition 5.1], and that for any element $x \in \pi_i\big(NA^{fd}_{\pm}(X)\big)$ of finite order, one has $F_n(x) = 0$ for all sufficiently large n (see the discussion in [13, pg. 334, Proof of Theorem 1.1]). Since these two properties are the only ones used in our proofs, an argument identical to the proof of **Theorem B** gives the:

Proposition 6.1. *Let X be an arbitrary space, and let $NA^{fd}_{\pm}(X)$ be the associated Nil-spaces arising in the fundamental theorem of algebraic K-theory of spaces. Then if $H \leq \pi_i\big(NA^{fd}_{\pm}(X)\big)$ is any finite subgroup, then*

$$\bigoplus_{\infty} H \leq \pi_i\big(NA^{fd}_{\pm}(X)\big).$$

Moreover, if H is a direct summand in $\pi_i\big(NA^{fd}_{\pm}(X)\big)$, then so is $\bigoplus_{\infty} H$.

Remark 6.2. An interesting question is whether there exists a "twisted" version of the splitting in equation (1), which applies to bundles $X \to W \to S^1$ over the circle (or more generally, to approximate fibrations over the circle), and provides a homotopy splitting of the corresponding $A^{fd}(W)$ in terms of spaces attached to X and the holonomy map.

6.2. **Cokernels of assembly maps.** For a general group G, one would expect from the Farrell-Jones isomorphism Conjectures that the cokernel of the relative assembly map for G should be "built up", in a homological manner, from the cokernels of the relative assembly maps of the various virtually cyclic subgroups of G (see for example [18] for an instance of this phenomenon). In view of our **Theorem D**, the following question seems relevant:

Question: Can one find a group G, an index $i \in \mathbb{Z}$, and a finite subgroup H, with the property that H embeds in $\mathrm{CoKer}\Big(h_i^G(\underline{E}G) \to K_i(\mathbb{Z}[G])\Big)$, but $\bigoplus_{\infty} H$ does not?

In other words, we are asking whether contributions from the various Nil-groups of the virtually cyclic subgroups of G could *partially cancel out in a cofinite manner*. Note the following special case of this question: is there an example for which this cokernel is a non-trivial finite group?

6.3. **Exotic Farrell Nil-groups.** Our **Theorem C** establish that, for a countable *tame* ring, meaning the associated Farrell Nil-groups has finite exponent, the associated Farrell Nil-groups, while infinitely generated, still remain reasonably well behaved, i.e. are countable direct sums of a fixed finite group. In contrast, for a general ring R (or even, a general integral group ring $\mathbb{Z}G$), all we know about the non-trivial Farrell Nil-groups is that they are infinitely generated abelian groups. Of course, the possibility of having infinite exponent *a priori* allows for many strange possibilities, e.g. the rationals $\mathbb{Q}$, or the Prüfer p-group $\mathbb{Z}(p^\infty)$ consisting of all complex p^i-roots of unity ($i \geq 0$). We can ask:

Question: Can one find a ring R, automorphism $\alpha \in \mathrm{Aut}(R)$, and $i \in \mathbb{Z}$, so that $NK_i(R, \alpha) \cong \mathbb{Q}$? How about $NK_i(R, \alpha) \cong \mathbb{Z}(p^\infty)$? What about if we require the ring to be an integral group ring $R = \mathbb{Z}G$?

Remark 6.3. Grunewald [12, Theorem 5.10] proved that for every group G and every group automorphism α of finite order, $NK_i(\mathbb{Q}G, \alpha)$ is a vector space over the rationals after killing torsion elements for all $i \in \mathbb{Z}$. However this still leaves the possibility that they may vanish.

Or rather, in view of our results, the following question also seems natural:

Question: What conditions on the ring R, automorphism $\alpha \in \mathrm{Aut}(R)$, and $i \in \mathbb{Z}$, are sufficient to ensure $NK_i(R, \alpha)$ is a torsion group of finite exponent? Does $NK_i(\mathbb{Z}G; \alpha)$ have finite exponent for all polycyclic-by-finite groups when α is of finite order?

Finally, while this paper completes our understanding of the finiteness properties of Farrell Nil-groups associated with *finite order* ring automorphisms, nothing seems to be known about the Nil-groups associated with *infinite order* ring automorphisms. This seems like an obvious direction for further research.

7. Appendix

In this appendix, we give a short discussion on the cardinality of Nil-groups. The following proposition is needed in the proof of our **Theorem C** – while presumably well-known to experts, we were unable to find it in the literature.

Proposition 7.1. *Let R be a countable ring and $\alpha : R \to R$ be a ring automorphism. Then the groups $K_i(R)$ and $NK_i(R; \alpha)$ are countable for all $i \in \mathbb{Z}$.*

Proof. Since $NK_i(R; \alpha)$ is a subgroup of $K_i(R_\alpha[t])$ and $R_\alpha[t]$ is countable when R is countable, it is enough to show $K_i(R)$ is countable when R is countable. So let us focus on $K_i(R)$.

We first use Quillen's $+$-construction to treat the case where $i \geq 1$. Consider the infinite general linear group $GL(R)$. Being the countable union of countable groups $GL_n(R)$ ($n \in \mathbb{N}$), we see that $GL(R)$ is countable. By a standard construction, i.e. view $GL(R)$ as a category and take the geometric realization of the nerve of this category, the classifying space $BGL(R)$ can be chosen to be a countable CW-complex. Performing Quillen's $+$-construction to $BGL(R)$, we obtain the algebraic K-theory space $BGL(R)^+$ with $K_i(R) := \pi_i(BGL(R)^+)$, for $i \geq 1$. Note that $BGL(R)^+$ is obtained from $BGL(R)$ by attaching 2-cells and 3-cells indexed by some generating set of the commutator subgroup of $GL(R)$, hence $BGL(R)^+$ is a countable CW-complex. More details of Quillen's $+$-construction can be found, for example, in [23, Theorem 5.2.2].

We now show the homotopy groups of a countable CW-complex is countable. By filtering a countable CW-complex by its countably many finite subcomplexes, it suffices to show homotopy groups of finite CW-complexes are countable. So let us assume X is a finite CW-complex. Since every finite CW-complex has the homotopy type of a finite simplicial complex, we may assume X is a finite simplicial complex. Fix a triangulation Δ_i of S^i. The set of all iterated barycentric subdivisions of Δ_i is countable. Fix a vertex in Δ_i and a vertex in X as base points. By simplicial approximation, any element in $\pi_i(X)$ can be represented by a simplicial map from some iterated barycentric subdivision of Δ_i to X. But the set of such simplicial maps is clearly countable, hence $\pi_i(X)$ is countable. Thus $K_i(R)$ is countable when $i \geq 1$.

Now let us consider the case when $i < 1$. First, we consider $i = 0$. Let $\mathrm{Idem}(R)$ be the set of idempotent matrices in $M(R)$, where $M(R)$ is the union of all $n \times n$ matrices over R, $(n \in \mathbb{N})$. $GL(R)$ acts on $\mathrm{Idem}(R)$ by conjugation, denote the quotient by $\mathrm{Idem}(R)/GL(R)$. This is a semigroup and $K_0(R)$ can be identified with the Grothendieck group associated to this semigroup (see [23, Theorem 1.2.3]). Therefore $K_0(R)$ is countable since $\mathrm{Idem}(R)$ is countable. Now when $i < 0$, the negative K-groups $K_i(R)$ can be inductively defined to be the cokernel of the natural map (see [23, Definition 3.3.1])

$$K_{i+1}(R[t]) \oplus K_{i+1}(R[t^{-1}]) \to K_{i+1}R[t, t^{-1}]$$

Note when R is countable, $R[t], R[t^{-1}]$ and $R[t, t^{-1}]$ are all countable. Hence their K_0-groups are all countable. Thus we inductively have $K_i(R)$ are countable for all $i < 0$. This completes the proof. □

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Department of Mathematics, The Ohio State University, Columbus, OH 43210, U.S.A.
E-mail address: jlafont@math.ohio-state.edu

Department of Mathematics, University of the Aegean, Karlovassi, Samos, 83200 Greece
E-mail address: prasside@aegean.gr

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, U.S.A.
E-mail address: kun.wang@vanderbilt.edu