A note on the characteristic classes of non-positively curved manifolds

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Abstract

We give conceptual proofs of some well known results concerning compact non-positively curved locally symmetric spaces. We discuss vanishing and non-vanishing of Pontrjagin numbers and Euler characteristics for these locally symmetric spaces. We also establish vanishing results for Stiefel-Whitney numbers of (finite covers of) the Gromov-Thurston examples of negatively curved manifolds. We mention some geometric corollaries: the MinVol question, a lower bound for degrees of covers having tangential maps to the non-negatively curved duals, and estimates for the complexity of some representations of certain uniform lattices.

Keywords: Dual space, tangential map, locally symmetric spaces, characteristic class, MinVol.

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1. Introduction.

A well known result asserts that closed hyperbolic manifolds have zero Pontrjagin numbers. The standard argument for this consists of observing that closed hyperbolic manifolds are conformally flat, and that conformally flat closed manifolds have zero Pontrjagin numbers by Avez [5]. This note originated in a desire to give a simple, conceptual proof of this basic result, which we do in section 2. Our argument gives an alternate, more geometrical viewpoint that should be contrasted with the classical Hirzebruch proportionality principle. The main advantage of our approach lies in that the characteristic numbers are computed *via an actual map* between the non-positively curved locally symmetric spaces and their non-negatively curved duals.

Now recall that there is another well-known class of negatively curved closed manifolds arising from the Gromov-Thurston construction [14] (see also the older construction of Mostow-Sui [22]). These manifolds are ramified coverings of closed hyperbolic manifolds, where the ramification occurs over a totally-geodesic, codimension two submanifold that is null-homologous. Note that the behavior of characteristic numbers under ramified coverings is unclear (though see the recent result of Izawa [18]). In section 3, we show that the Gromov-Thurston manifolds always have a finite cover that bounds orientably. A byproduct of our argument also gives a very elementary proof of a result of Rohlin [28]: Every orientably closed 3-manifold bounds orientably.

Finally, in section 4, we point out various geometric corollaries of our main results. While many of these are standard, we do include some new results. We conclude in section 5 with some open questions.

2. Characteristic numbers of negatively curved locally symmetric spaces.

Let us start by recalling the construction of the non-negatively curved dual space associated to any non-positively curved closed locally symmetric space. If G is a real Lie group, K it's maximal compact subgroup, we let $G_C = G \otimes \mathbb{C}$ be the complexification of G and G_U the maximal compact subgroup of G_C. The factor spaces G/K and $M_U = G_U/K$ are called dual symmetric spaces [24]. By abuse of language, if Γ is a uniform lattice in G, we will still say that $M := \Gamma\backslash G/K$ and M_U are dual spaces. In [24], Okun showed that if M^n is a closed locally symmetric space, then there is a tangential map from some finite cover $\bar{M}^n$ to the dual symmetric space. We start by showing the following easy Lemma:

Lemma 1: Assume $f : M \longrightarrow N$ is a tangential map between two n-dimensional manifolds. Then

- $p_I(M) = \pm\deg(f) \cdot p_I(N) \in \mathbb{Z}$
- $sw_I(M) = \deg(f) \cdot sw_I(N) \in \mathbb{Z}/2\mathbb{Z}$

where p_I, sw_I denote the Pontrjagin and Stiefel-Whitney numbers associated to a product of Pontrjagin or Stiefel-Whitney classes.

Proof: Since the map is tangential, the pullbacks of Pontrjagin classes (respectively Stiefel-Whitney classes) of N yield the corresponding classes for M. If we denote by $\tau_I(N)$ a product of Pontrjagin classes, we have $f^*(\tau_I(N)) = \tau_I(M)$. Likewise, if $\sigma_I(N)$ denotes a product of Stiefel-Whitney classes, $f^*(\sigma_I(N)) = \sigma_I(M)$. Now we have that:

$$\begin{aligned} p_I(M) &= \langle \tau_I(M), [M] \rangle = \langle f^*(\tau_I(N)), [M] \rangle \\ &= \pm\langle \tau_I(N), f_*([M]) \rangle = \pm\langle (\tau_I(N)), \deg(f) \cdot [N] \rangle \\ &= \pm\deg(f) \cdot \langle (\tau_I(N)), [N] \rangle = \pm\deg(f) \cdot p_I(N). \end{aligned}$$

And the argument for part (b) of the lemma is identical.

Note that, from the discussion above, we have associated to any closed locally symmetric space M^n a diagram:

$$M^n \longleftarrow \bar{M}^n \longrightarrow M_U$$

where $\bar{M}^n$ is a finite cover, M_U is the non-negatively curved dual, and the maps in the diagram are tangential. Since a covering map never has zero degree, Lemma 1 tells us that we can solve for the Pontrjagin numbers of M^n:

$$p_I(M^n) = \frac{deg(t)}{deg(f)} \cdot p_I(M_U)$$

Of course, if we are trying to obtain vanishing/non-vanishing of Pontrjagin numbers, it is crucial to know when $deg(t) \neq 0$. Conceivably if $deg(t) = 0$, one could have non-zero Pontrjagin numbers for M_U, but with the corresponding Pontrjagin number for M^n equal to zero. That this does not occur is the content of the next Lemma:

Lemma 2: If t has degree zero, then the Pontrjagin numbers $p_I(M_U)$ are all equal to zero.

Proof: We start by noting that Okun ([24], Corollary 6.5) showed that if G_U and K have equal rank, then t has non-zero degree. Hence if $deg(t) = 0$, we must have $rk(G_U) > rk(K)$. Recall that the *toral rank* of a compact manifold N, denoted by $trk(N)$, is the largest dimension of a torus that has a smooth, rationally-free action on N (see [27]). Now Allday-Halperin [1] have shown that $trk(G_U/K) = rk(G_U) - rk(K)$, hence if $deg(t) = 0$, we have that $trk(M_U) > 0$. But Conner-Raymond [8] have shown that if N is a compact manifold with $trk(N) > 0$, then all the Pontrjagin numbers of N are equal to zero. Applying their result to M_U completes the proof.

For completeness, we point out that by a result of Papadima [27], for the homogenous space $M_U = G_U/K$, we have that the toral rank of M_U is zero if and only if the Euler characteristic of M_U is non-zero. Hence to verify that the map t has non-zero degree, it is sufficient to verify that the Euler characteristic of M_U is non-zero. We refer to Helgason [16] for the classification of the irreducible higher rank non-positively curved closed locally symmetric spaces, as well as for the notation used in our discussion. The results for the classical families can be summarized in the following:

Theorem A: Let M^n be a closed orientable irreducible higher rank locally symmetric space, and assume that M^n is locally modeled on one of the following:

1. $SU(p,q)/S(U_p \times U_q)$
2. $SO_0(p,q)/SO(p) \times SO(q)$ where p and q are not both odd
3. $SO^*(2n)/U(n)$
4. $Sp(n,\mathbb{R})/U(n)$
5. $Sp(p,q)/S(p) \times S(q)$

Then M^n has non-zero Euler characteristic.

Theorem B: Let M^n be a closed orientable irreducible higher rank locally symmetric space, and assume that M^n is locally modelled on one of the following:

1. $SL(n,\mathbb{R})/SO(n)$
2. $SU^*(2n)/Sp(n)$
3. $SO_0(p,q)/SO(p) \times SO(q)$ where p and q are both odd
4. an irreducible globally symmetric spaces of Type IV, see pgs. 515-516 in [16]

Then M^n has all Pontrjagin numbers equal to zero.

Proof of Theorems A & B: Let us first note that the duals of the locally symmetric spaces mentioned in theorem A are respectively:

1. $SU(p+q)/S(U_p \times U_q)$
2. $SO(p+q)/SO(p) \times SO(q)$ where p and q are not both odd
3. $SO(2n)/U(n)$
4. $Sp(n)/U(n)$
5. $Sp(p+q)/S(p) \times S(q)$

Since the ranks of the various Lie groups being considered above are $rk(Sp(n)) = n$, $rk(SO(2n)) = rk(SO(2n+1)) = n$, $rk(SU(n)) = n-1$, $rk(U(n)) = n$, and $rk(S(U_p \times U_q)) = p+q-1$, we see that in all the cases of Theorem A, the condition $rk(G_u) = rk(K)$ is satisfied. Since the toral rank of M_u is zero, this implies that the Euler characteristic of M_U is non-zero, giving Theorem A.

Likewise for the cases appearing in Theorem B, we have that the duals are respectively:

1. $SU(n)/SO(n)$
2. $SU(2n)/Sp(n)$

3. $SO(p+q)/SO(p) \times SO(q)$ where p and q are both odd
4. a Lie group

In each of the first three cases, we see that $rk(G_U) > rk(K)$. So by the argument in Lemma 2, we have that all the Pontrjagin numbers of the dual spaces M_U are zero. Hence the Pontrjagin numbers for M^n are likewise zero. For the fourth case, we note that Lie groups are parallelizable, hence have all Pontrjagin numbers zero. This concludes the proof.

Remark: Theorem A above lists the only irreducible higher hank locally symmetric spaces of non-positive curvature which could conceivable have non-vanishing Pontrjagin numbers. Since the procedure for calculating the Pontrjagin numbers of the non-negatively curved duals is well established (see Borel-Hirzebruch [6]), one could in principle find out which of these spaces actually have a non-vanishing Pontrjagin number (note that by Lemma 2, for the spaces in Theorem A, the degree of the tangential map is non-zero). As this procedure is primarily combinatorial in nature, we leave the precise computations to the interested reader, and content ourselves with computing them for the *negatively* curved locally symmetric spaces. In the process, we also discuss the exceptional locally symmetric space $F_{4(-20)}/Spin(9)$ giving rise to Cayley hyperbolic manifolds. We leave to the interested reader the task of deciding for the remaining exceptional cases which of Theorem A or B applies.

Corollary 1: Let M^n be a compact orientable manifold, and assume that one of the following holds:

1. M^n is real hyperbolic
2. M^n is complex hyperbolic, and $n = 4k + 2$
3. M^n is quaternionic hyperbolic, and $n = 8k + 4$

Then M^n has a finite cover that bounds. In the first two cases, there is a finite cover that bounds *orientably* (and hence M^n has all Pontrjagin numbers equal to zero).

Corollary 2: Let M^n be a compact orientable manifold, and assume that one of the following holds:

1. M^n is Cayley hyperbolic (so $n = 16$)
2. M^n is complex hyperbolic, and $n = 4k$
3. M^n is quaternionic hyperbolic of dimension at least 8.

Then M^n has some non-zero Pontrjagin numbers, and hence no finite cover can bound orientably. Furthermore, in the first two cases, we have that *all* Pontrjagin numbers are non-zero.

Since the arguments are closely related, we simultaneously prove both corollaries:

Proof of Corollaries 1 & 2: We note that for the negatively curved symmetric spaces, the duals are easy to compute. Indeed we have that:

- the dual to real hyperbolic space is the sphere,
- the dual to complex hyperbolic space is complex projective space,
- the dual to quaternionic hyperbolic space is quaternionic projective space,
- the dual to Cayley hyperbolic space is the Cayley projective plane.

Since the characteristic classes of the duals are well known, we can apply Lemmas 1 and 2 in each case to obtain information on the negatively curved locally symmetric spaces. $\bar{M}^n$ will always denote the finite cover that supports a tangential map to the positively curved dual. The various cases are:

M^n **is real hyperbolic**: Since the sphere bounds orientably, all its characteristic numbers (both Stiefel-Whitney and Pontrjagin) are zero. Applying Lemma 1, we see that all the characteristic numbers of $\bar{M}^n$ are zero. By a result of Wall [31], this is equivalent to $\bar{M}^n$ bounding orientably, giving (1) of Corollary 1.

M^{2n} **is complex hyperbolic**: Then its dual space is the complex projective space $\mathbb{C}P^n$, which is a 2n-dimensional real manifold. We now have two cases:
(A) If $n = 2k$, then the Pontrjagin numbers are all non-zero ([20] page 185), hence using Lemmas 1 and 2, the same holds for M^{2n}.
(B) If $n = 2k + 1$, then $\mathbb{C}P^n$ bounds orientably ([20] page 186). Arguing as in the real hyperbolic case, we see that $\bar{M}^n$ bounds orientably.
This gives us (2) of Corollaries 1 and 2.

M^{4n} **is quaternionic hyperbolic**: Then its dual space is the quaternionic projective space $\mathbb{O}P^n$, which is a 4n-dimensional real manifold. We again have two cases:
(A) If $n = 2k+1$, then $\mathbb{O}P^n$ bounds, and hence has vanishing Stiefel-Whitney numbers. By Lemma 1, the same holds for $\bar{M}^{2n}$, giving (3) of Corollary 1.
(B) In general, the total Pontrjagin class of $\mathbb{O}P^n$ is given by $(1 + u)^{2n+2}(1 + 4u)^{-1}$, where $u \in H^4(\mathbb{O}P^n)$ is a generator for the truncated polynomial ring $H^*(\mathbb{O}P^n)$. Since the coefficient of u in the power series expansion equals $2n - 2$, we see that the Pontrjagin number $p_1^n(M_U)$ is equal to $(2n - 2)^n$. So provided $n \geq 2$, we can apply Lemmas 1 and 2 to obtain (3) of Corollary 2.

M^{16} **is Cayley hyperbolic**: Then its dual space is the Cayley projective plane $\text{Cay}P^2$. The Cayley plane has two non-vanishing Pontrjagin numbers, namely ${p_2}^2[\text{Cay}P^2] = 36$ and $p_4[\text{Cay}P^2] = 39$ (see Borel-Hirzebruch [6], pages 535-536). Applying Lemma 2, we get that $\bar{M}^{16}$ has some non-vanishing Pontrjagin numbers. This deals with case (1) of Corollary 2, and hence completes the proof of the Corollaries.

Remark: We note that information on the Stiefel-Whitney numbers of the rank one locally symmetric spaces is much harder to obtain. Indeed, anytime the degree of one of the two maps is even, there is a potential loss of information.

Corollary 3: If M^n is a manifold supporting a metric of constant sectional curvature, then all of its Pontrjagin numbers are zero.

Proof: The case of constant negative curvature has been dealt with above. In the remaining two cases, M^n has a finite cover that bounds orientably (either a sphere, or a torus, depending on curvature). The corollary follows.

Remark: Recall that Farrell-Jones have constructed exotic smooth structures on certain closed hyperbolic manifolds, and have shown that these manifolds support Riemannian metrics of negative curvature [10]. There results were subsequently extended to providing exotic smooth structures on a variety of different locally symmetric spaces, see for instance [11], [25], [12], [3], [2]. Observe that while the Pontrjagin classes are smooth invariants, the rational Pontrjagin classes are topological invariants, by a celebrated result of Novikov [23]. Since the Pontrjagin numbers of a manifold only depend on the rational Pontrjagin classes (i.e. the torsion part of the Pontrjagin classes do not influence the Pontrjagin numbers), the discussion in Theorems A & B gives us vanishing (or non-vanishing) results for the Pontrjagin numbers of these exotic manifolds as well.

3. Characteristic numbers of the Gromov-Thurston examples.

Definition: Let X be a oriented differentiable manifold (with or without boundary) on which the cyclic group $\mathbb{Z}_k$ acts semifreely by orientation-preserving diffeomorphisms with fixed set a codimension two submanifold Y. Denote the quotient space by $X' := X/\mathbb{Z}_k$, and the canonical projection map by $\pi : X \to X'$. Let Y be the fixed set of the action on X, and note that $\pi : Y \to Y'$ is a diffeomorphism. Observe that X' is a manifold. We say that the X is an *oriented cyclic ramified cover* of X', of order k, ramified over Y'. If Y' bounds a smooth embedded codimension one submanifold in X', we say that the ramified covering is *nice*.

Remark: If a ramified covering is nice, then it is particularly easy to describe it. Indeed, let N be the codimension one embedded submanifold satisfying $\partial N = Y'$. Then the pre-image of N in the ramified cover X will consist of multiple (embedded) copies of N which all coincide along their boundary (which will equal Y). Cutting X open along the pre-images of N will yield k homeomorphic copies of $X' - N$. Now consider the space with boundary the double DN of N, obtained by cutting open X' along N. Then X is obtained by taking k copies of this space, $X_1, \ldots X_k$, and for each space, cyclically gluing ∂X_i^+ to ∂X_{i+1}^-, where $\partial X_i^{\pm}$ denotes the two copies of N in $\partial X_i = DN$.

Proposition: Assume that M^n bounds, and that $p : \bar{M}^n \to M^n$ is an oriented cyclic ramified cover of M^n (ramified over N^{n-2}). If the covering is nice, then $\bar{M}^n$ also bounds. If M^n bounds orientably, then so does $\bar{M}^n$.

Proof: Let $M^n = \partial L^{n+1}$, and note that since the ramified covering is nice, there exists a a smoothly embedded $K_0^{n-1} \subset M^n$ satisfying $\partial K_0^{n-1} = N^{n-2}$. Since M^n is collarable in L^{n+1}, there is a manifold $K^{n-1} \subseteq L^{n+1}$ of dimension $n-1$ with the properties:

- $K^{n-1} \cap \partial L^{n+1} = N^{n-2} = \partial K_0^{n-1}$,
- K^{n-1} and K_0^{n-1} are cobordant in L^{n+1},
- the cobordism W^n is an embedded submanifold satisfying $W^n \cap M^n = K_0^{n-1}$.

Indeed, homotoping K_0^{n-1} (relative $\partial K_0^{n-1} = N^{n-2}$) into a collared neighborhood of M^n in L^{n+1} give both K^{n-1}, and the manifold W^n (the image of the homotopy, which we can assume to have no self-intersections). Now note that $K^{n-1} \subseteq L^{n+1}$ is a codimension two submanifold which bounds W^n. Hence we can take the i-ramified covering of L^{n+1} over K^{n-1} (see the remark preceding this Proposition). But note that on $\partial L^{n+1} = M^n$, this ramified covering yields $\bar{M}^n$. Hence if $\bar{L}^{n+1}$ is the covering, we have $\partial \bar{L}^{n+1} = \bar{M}^n$. Finally, we note that if L^{n+1} is orientable, then so is the ramified covering $\bar{L}^{n+1}$.

Corollary 4: Let M be a closed, orientable, 3-dimensional manifold. Then M bounds orientably. (This result is originally due to Rohlin [28]).

Proof: It is a well known result (due independantly to Hilden [15] and Montesinos [21]) that every closed orientable 3-manifold is ramified covering of the 3-dimensional sphere S^3 along a knot. Since every knot in S^3 bounds a compact embedded surface, this ramified cover is nice. Since S^3 bounds orientably, the proposition gives us the claim.

Remark: The Corollary also follows easily from results of Thom and Wall: the Pontrjagin numbers are automatically zero, since M is 3-dimensional. As for the Stiefel-Whitney numbers, there are only three of them to consider: $s_1^3, s_1^2 s_2$, and s_3. Note that since M is orientable, $s_1 = 0$, so the first two numbers vanish. As for s_3, it is just the mod 2 reduction of the Euler characteristic, which has to be zero as we are in odd dimension. Applying Wall's theorem ([31]), we get that M must bound orientably. The advantage of our approach is that the bounding manifold can be seen *explicitly*, and we avoid appealing to the sophisticated results of Thom and Wall.

Theorem C: Let N be the Gromov-Thurston non-positively curved manifold. Then N has a finite cover that bounds orientably (and hence all Pontrjagin numbers of N are zero).

Proof of Theorem C: Let M be a real hyperbolic manifold and N be the Gromov-Thurston non-positively curved manifold obtained as a ramified covering of M. From Corollary 1, M has a finite cover $\bar{M}$ that bounds orientably. We claim that there is a space $\bar{N}$ yielding the commutative diagram:

$$\begin{CD} \bar{N} @>{\bar{\psi}}>> \bar{M} \\ @V{\bar{\phi}}VV @VV{\phi}V \\ N @>>{\psi}> M \end{CD}$$

where $\bar{\phi}$ is a covering map and $\bar{\psi}$ is a ramified covering (and ψ is the original ramified covering, ϕ the original covering).

In order to see this, we make the following general observation: assume that X^{n-2} is a smooth embedded codimension two submanifold in Y^n, and let $W \subset Y^n$ be a closed tubular neighborhood of X^{n-2}. Note that W is a $\mathbb{D}^2$-bundle over X^{n-2}, and hence that ∂W is an S^1-bundle over X^{n-2}. Now let $Y' \subset Y^n$ be the manifold with boundary obtained by removing the interior of W from Y^n, and assume that $\bar{Y}' \to Y'$ is a covering map. Then we have:

1. the covering map $f : \bar{Y}' \to Y'$ extends to a covering $\bar{f} : \bar{Y} \to Y$ if and only if, for each fiber F of the bundle $S^1 \to \partial W \to X^{n-2}$, we have that $f^{-1}(F)$ consists of $deg(f)$ disjoint copies of S^1.
2. the covering map $f : \bar{Y}' \to Y'$ extends to a *ramified* covering $\bar{f} : \bar{Y} \to Y$ of degree $deg(f)$ over X^{n-2} if and only if, for each fiber F of the bundle $S^1 \to \partial W \to X^{n-2}$, we have that $f^{-1}(F)$ is connected.

Indeed, one direction of the implications is immediate, since a covering (respectively a ramified covering over X^{n-2}) exhibits precisely the aforementioned behavior on the boundary of a regular neighborhood. Conversely, assume that we have a covering map $f : \bar{Y}' \to Y'$ satisfying one of the above properties. Then note that the pre-image $f^{-1}(\partial W)$ naturally inherits a smooth foliation with S^1 leaves. Now consider the space $\bar{W}$ obtained by smoothly gluing in $\mathbb{D}^2$'s along their boundary to the leaves. Observe that this can be done, since the foliation on $f^{-1}(\partial W)$ is the lift of a fibration, and hence is locally a product. Finally, form the space $\bar{Y}$ by gluing $\bar{Y}'$ with $\bar{W}$ along their common boundary.

Now in case (1) above, we immediately get that the covering map f extends to a covering map $\bar{f}$, by simply extending linearly along each $\mathbb{D}^2$. In case (2), we again extend linearly, but this time also extend the action of $\mathbb{Z}_{deg(f)}$ (by deck transformations) from each S^1 to each $\mathbb{D}^2$. Note that this gives a smooth $\mathbb{Z}_{deg(f)}$ action on $\bar{Y}$, whose fixed point set maps diffeomorphically to the original X^{n-2}.

Now in the setting we have, proceed as follows: if K^{n-2} is the codimension two submanifold of M^n that is being ramified over, then let W be a tubular neighborhood of K, W_0 it's interior. Note that ψ is an actual *covering*, when restricted to the preimage of $M - W_0$ (as we are throwing away a neighborhood of the set where the ramification occurs). Consider the commutative diagram:

$$\begin{CD} M' @>>> \phi^{-1}(M - W_0) \\ @VVV @VV{\phi}V \\ \psi^{-1}(M - W_0) @>>{\psi}> M - W_0 \end{CD}$$

where M' is the pullback of the covering maps. By commutativity of the diagram, we see that the covering $M' \to \phi^{-1}(M - W_0)$ satisfies (2) from our discussion above, while the covering $M' \to \psi^{-1}(M - W_0)$ satisfies (1) from the discussion above. In particular, extending M' as above, we obtain a space $\bar{N}$ which is simultaneously a ramified covering of $\bar{M}$, and an actual covering of N, as desired.

Finally, we note that the ramified covering $\bar{\psi}: \bar{N} \to \bar{M}$ is nice. Indeed, in the Gromov-Thurston construction, the ramified covering $\psi: N \to M$ is nice, so we have that $K^{n-2} = \partial L^{n-1}$ for a s mooth, embedded codimension one manifold with boundary. But we have that the map $\bar{\psi}$ is ramified over $\phi^{-1}(K^{n-2})$, which clearly bounds the smooth, embedded codimension one submanifold $\phi^{-1}(L^{n-1})$. This confirms that $\bar{\psi}$ is nice, and since $\bar{M}$ bounds orientably, applying the Proposition, we see that $\bar{N}$ bounds orientably as well. This completes the proof of Theorem C.

Remark: A related (unpublished) result is due to Ardanza-Trevijano Moras [4], and asserts that for the Gromov-Thurston ramified coverings, the individual Pontrjagin classes vanish. We note that while our approach does not give vanishing of individual *classes*, it does give vanishing of the Stiefel-Whitney numbers on a finite cover (which does not follow from the approach in [4]).

4. Geometric applications.

As is well known, characteristic numbers provide obstructions to a wide range of topological problems. To mention but a few, if M^n has a non-zero Pontrjagin number, then

1. no finite cover of M^n bounds orientably.
2. M^n has no orientation reversing self-diffeomorphism.
3. M^n does not support an almost quaternionic structure ([30]).

From our Corollary 2, we immediately get these properties for the rank one locally symmetric manifolds that are either complex hyperbolic (with $n = 4k$), quaternionic hyperbolic or Cayley hyperbolic.

For another application, we note that while our Theorem A does not tell us which of the irreducible, higher-rank, non-positively curved compact manifolds have non-zero Pontrjagin numbers, it does tell us which of these have non-zero Euler characteristic. Again, the Euler characteristic is known to be an obstruction to various topological/geometrical problems, for instance for the spaces discussed in Theorem A, we immediately get that there cannot exist a nowhere vanishing vector field. For a possibly more interesting application, recall that the $MinVol$ of a smooth manifold is defined to be the infimum of the volumes $Vol(M^n, g)$ over all Riemmanian metrics g whose curvatures are bounded between -1 and 1. It is known that if $MinVol(M^n) = 0$, then the Euler characteristic is also zero (see for instance the survey [26]). This gives us:

Corollary 5: Let M^n be a compact orientable manifold which is locally symmetric. If M^n is one of the manifolds allowed in Theorem A, then $MinVol(M^n) > 0$.

Again, this result is known to hold for *all* non-positively curved compact locally symmetric spaces, and follows from the estimates of filling invariants found in Gromov's work [13]. For a more recent (and more general) proof using barycenter map techniques, see Connell-Farb [7].

Corollary 6: Let M^{4n} be a compact orientable manifold which is locally symmetric. Assume that M^{4n} is one of the manifolds allowed in Theorem A. For each partition $I = i_1, i_2, ..., i_r$ of n, let $p_I(M^{4n})$ (respectively $p_I(M_U)$) denote the I-th Pontrjagin number of M^{4n} (respectively of the dual M_U). Note that if $p_I(M_U) \neq 0$, then we also have that $p_I(M^{4n}) \neq 0$ (from Lemma 2). Define

$$\mu(M^{4n}) = LCM_I\{LCM(p_I(M^{4n}), p_I(M_U))/p_I(M^{4n})\}$$

where LCM denotes least common multiple, and the outer LCM is over all partitions I of n for which $p_I(M^{4n}) \neq 0$. If $\bar{M}^{4n} \longrightarrow M^{4n}$ is a degree d cover having a tangential map $\bar{M}^{4n} \longrightarrow M_U$, then $\mu(M^{4n})$ divides d.

Proof: Let r be the degree of the tangential map $\bar{M}^{4n} \longrightarrow M_U$. Then for each I, we have that $d \cdot p_I(M^{4n}) = r \cdot p_I(M_U)$. This implies that $d \cdot p_I(M^{4n})$ is a multiple of $LCM(p_I(M^{4n}), p_I(M_U))$. Hence for each I,we see hat d is a multiple of $\frac{LCM(p_I(M^{4n}),p_I(M_U))}{p_I(M^{4n})}$. This forces d to be a multiple of their least common multiple. Therefore d is a multiple of $\mu(M^{4n})$.

Remark: The argument for the last corollary applies equally well to give an identical estimate for the degree of the tangential map from $\bar{M}^n$ to M_U. Part of our interest in the covering map (rather than the tangential map), stems from the following:

Corollary 7: Let G/K be one of the irreducible globally symmetric spaces allowed as local models in the hypotheses of Theorem A, and assume the dimension of G/K is divisible by 4. Let Γ be a torsion free subgroup of G, and denote by $\Gamma\backslash G/K =: M^{4n}$ the associated locally symmetric space. Consider the flat principal bundle $G/K \times_\Gamma G \longrightarrow M^{4n}$, and extend its structure group to the group G_C. The bundle naturally defines a homomorphism $\rho : \Gamma \longrightarrow G_C \subset GL(k, \mathbb{C})$ (for some suitable k). Let $A \subseteq \mathbb{C}$ be *any* subring of $\mathbb{C}$, finitely generated, with the property that $\rho(\Gamma) \subseteq GL(k, A)$, and let m_1, m_2 be *any* pair of maximal ideals in A with the property that the finite fields A/m_1 and A/m_2 have distinct characteristics. Then $\mu(M^{4n})$ divides the cardinality of the finite group $GL(2k+1, A/m_1) \times GL(2k+1, A/m_2)$.

Proof: Given such a subring and a pair of maximal ideals, Deligne and Sullivan [9] exhibit a finite cover $\bar{M}^{4n}$ of M^{4n} having the property that:
(1) the pullback bundle to $\bar{M}^{4n}$ is trivial,
(2) the degree of the cover divides $|GL(2k+1, A/m_1) \times GL(2k+1, A/m_1)|$.
But Okun shows ([24], proof of theorem 5.1), that there is a tangential map from $\bar{M}^{4n}$ to M_U, hence applying Corollary 6 completes our proof.

Remark: The previous corollary tells us that, in some sense, the *complexity* of the representation $\Gamma \to G_C \subset GL(k, \mathbb{C})$ can be estimated from below in terms of the Pontrjagin numbers of the quotient $\Gamma\backslash G/K$.

5. Some open questions.

There remain a few interesting questions along the line of inquiry we are considering. For starters, Okun has provided sufficient conditions for establishing non-zero degree of the tangential map he constructs. One can ask the:

Question: Are there examples where Okun's tangential map has zero degree? In particular, if one has a locally symmetric space modelled on $SL(n, \mathbb{R})/SO(n)$, does the tangential map to the dual $SU(n)/SO(n)$ have non-zero degree?

Of course, the interest in the special case of $SL(n, \mathbb{R})/SO(n)$ is due to the "universality" of this example: every other locally symmetric space of non-positive curvature isometrically embedds in a space modelled on $SL(n, \mathbb{R})/SO(n)$. Now note that while the relationship between the cohomologies of M^n and M_U (with real coefficients) is well understood (and has been much studied) since the work of Matsushima [19], virtually nothing is known about the relationship between the cohomologies with other coefficients. One can ask:

Question: If $t : M^n \to M_U$ is the tangential map, what can one say about the induced map $t^* : H^*(M_U, \mathbb{Z}_p) \to H^*(M, \mathbb{Z}_p)$?

In particular, the case where $p = 2$ would be of some particular interest, as the Stiefel-Whitney classes lie in these cohomology groups. Finally, we point out that there are other classes of non-positively curved Riemmanian manifolds, arising from Schroeder's cusp closing construction ([29], [17]), doubling constructions, and related techniques.

Question: Compute the characteristic classes for the remaining known examples of non-positively curved manifolds.

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