HYPERBOLIC GROUPS WITH BOUNDARY AN n-DIMENSIONAL SIERPINSKI SPACE

JEAN-FRANÇOIS LAFONT AND BENA TSHISHIKU

ABSTRACT. For $n \geq 7$, we show that if G is a torsion-free hyperbolic group whose visual boundary $\partial_\infty G \simeq \mathscr{S}^{n-2}$ is an $(n-2)$-dimensional Sierpinski space, then $G = \pi_1(W)$ for some aspherical n-manifold W with nonempty boundary. Concerning the converse, we construct, for each $n \geq 4$, examples of aspherical manifolds with boundary, whose fundamental group G is hyperbolic, but with visual boundary $\partial_\infty G$ *not* homeomorphic to $\mathscr{S}^{n-2}$.

1. Introduction

One of the basic invariants for a hyperbolic group is its boundary at infinity, and a fundamental question is to determine what properties of the group are captured by the topology of the boundary at infinity. For example, the famous *Cannon conjecture* postulates that a hyperbolic group whose boundary at infinity is the 2-sphere S^2 must admit a properly discontinuous, isometric, cocompact action on hyperbolic 3-space $\mathbb{H}^3$.

In [19], Kapovich and Kleiner study groups whose boundary at infinity is a Sierpinski carpet – a boundary version of the Cannon conjecture. In [4], Bartels, Lück, and Weinberger study groups whose boundary at infinity is a sphere S^n of dimension $n \geq 5$ – a high-dimensional version of the Cannon conjecture. In this paper, we consider groups whose boundary at infinity are high-dimensional Sierpinski spaces – thus lying somewhere between the work of Kapovich-Kleiner and that of Bartels-Lück-Weinberger.

The two main theorems are as follows. Let $\mathscr{S}^{n-2}$ denote an $(n-2)$-dimensional *Sierpinski space*. See Section 2 for the definition.

Theorem 1. *Fix $n \geq 7$ and let G be a torsion-free hyperbolic group. If the visual boundary $\partial_\infty G$ is homeomorphic to $\mathscr{S}^{n-2}$, then there exists an n-dimensional compact aspherical topological manifold W with nonempty boundary such that $\pi_1(W) \cong G$. Furthermore, W is unique up to homeomorphism.*

Note that the fundamental group π of a closed aspherical manifold M is an example of a Poincaré duality group. Whether or not all finitely presented Poincaré duality groups arise in this fashion is an open problem that goes back to Wall [16]. So the existence portion of Theorem 1 addresses a relative version of Wall realization problem for a special class of groups. On the other hand, the uniqueness portion of Theorem 1 verifies the Borel conjecture for this same class of groups.

Our second result shows that the converse of Theorem 1 is false – even if one imposes additional strong constraints on the geometry of the aspherical manifold.

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Theorem 2. *For each $n \geq 4$, there exists a compact aspherical manifold M^n with nonempty connected boundary $\partial M^n = N^{n-1}$ such that:*

(1) $G = \pi_1(M)$ is hyperbolic, and $H = \pi_1(N)$ is a proper quasi-convex subgroup in G.
(2) $\partial_\infty(\pi_1(N))$ is homeomorphic to S^{n-2}, but
*(3) $\partial_\infty G \cong \partial_\infty \widetilde{M}$ is **not** homeomorphic to $\mathscr{S}^{n-2}$.*

Moreover, when $n \geq 5$, the manifold M^n supports a locally CAT(-1) *metric with totally geodesic boundary.*

Remark 3. If one just wants a simple counterexample to the converse of Theorem 1, one can proceed as follows: start with a k-dimensional closed hyperbolic manifold K with fundamental group G, where $k < n$. Now embed the hyperbolic k-plane $\mathbb{H}^k$ isometrically inside $\mathbb{H}^n$. Then the G-action on the embedded $\mathbb{H}^k$ extends to an action on the r-neighborhood X of the $\mathbb{H}^k$. Let $M = X/G$, and note that M is aspherical, diffeomorphic to $K \times \mathbb{D}^{n-k}$, with fundamental group G. Clearly $\partial_\infty G$ is homeomorphic to the $(k-1)$-sphere S^{k-1}, and not to Sierpinski $(n-2)$-space $\mathscr{S}^{n-2}$. Of course, in this example, $N = K \times S^{n-k-1}$, so the example fails to have property (1) from Theorem 2.

Remark 4. In Theorem 2 one can construct, in dimensions $n \geq 5$, manifolds satisfying property (1), but failing to have (2). Start with a Davis-Januszkiewicz example of a locally CAT(-1) closed $(n-1)$-manifold N with $\partial_\infty \widetilde{N}$ not homeomorphic to S^{n-2}, chosen so that $N = \partial W^{n+1}$ for some compact manifold W^{n+1}. Then take M to be the relative hyperbolization of W, relative to N (see [15]). Properties of relative hyperbolization readily yield statement (1), while the choice of N ensures that (2) fails. It seems likely that such manifolds M would also have property (3). Indeed, one could visualize the boundary at infinity of $\widetilde{M}$ to be similar to a Sierpinski curve, but instead of having peripheral spheres (see Section 2), it would have peripheral subspaces which are Čech homology spheres instead of genuine spheres (since (2) fails). Such a space is probably not homeomorphic to $\mathscr{S}^{n-2}$. We point out, however, that this approach could not possibly work in dimension $n = 4$, as in this case the boundary would be a closed 3-manifold, which forces (2) to hold.

Structure of paper. In Section 2 we recall the definition of an n-dimensional Sierpinski space. In Sections 3 and 4, we prove Theorems 1 and 2, respectively. In Section 5, we remark on a generalization of Theorem 1 to CAT(0) groups.

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2. n-dimensional Sierpinski space and hyperbolic groups

We use Cannon's definition of n-dimensional Sierpinski space [12] (Cannon uses the term Sierpinski *curve* instead of Sierpinski *space*).

Definition. Fix $n \geq 0$. Let $D_1, D_2, \ldots \subset S^{n+1}$ be a sequence of open topological balls such that

(i) $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$,
(ii) $\operatorname{diam}(D_i) \to 0$ with respect to the round metric on S^{n+1}, and
(iii) $\bigcup D_i \subset S^{n+1}$ is dense.

Then $\mathscr{S}^n := S^{n+1} \setminus \bigcup D_i$ is an *n-dimensional Sierpinski space*. The spheres $S^n \cong \partial(\overline{D_i}) \subset \mathscr{S}$ are called *peripheral spheres*.

Example. A 0-dimensional Sierpinski space $\mathscr{S}^0$ is a Cantor set, while the space $\mathscr{S}^1$ is the classical Sierpinski carpet. The Sierpinski space $\mathscr{S}^{n-2}$ arises as the visual boundary of hyperbolic groups (in the sense of Gromov [17]). For example, if W^n is a hyperbolic n-manifold with nonempty totally geodesic boundary, then $\pi_1(W)$ is a hyperbolic group whose visual boundary is a Sierpinski $(n-2)$-space. To see this, observe that the universal cover $\widetilde{W}$ can be embedded as a submanifold of hyperbolic space $\widetilde{W} \hookrightarrow \mathbb{H}^n$. Using the disk model, the visual boundary $\partial_\infty \widetilde{W}$ is a subspace of $\partial_\infty \mathbb{H}^n \cong S^{n-1}$. The boundary components of W lift to countably many disjoint geodesic hyperplanes $\mathbb{H}^{n-1} \subset \mathbb{H}^n$. Each hyperplane has boundary a sphere $\partial_\infty \mathbb{H}^{n-1} \cong S^{n-2}$, which bounds an open ball $\mathbb{D}^{n-1} \subset S^{n-1}$. The visual boundary of $\widetilde{W}$ is obtained by removing this countable collection of open balls, yielding a Sierpinski space $\mathscr{S}^{n-2}$.

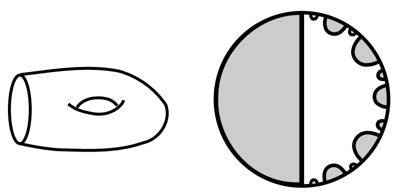

FIGURE 1. A torus with one boundary component, and its universal cover inside the hyperbolic plane.

The simplest example of this is when W is a torus with one boundary component (see Figure 1). More examples are furnished by the following general theorem of Lafont [20].

Theorem 5 (Lafont)**.** *Let M^n be a compact, negatively curved Riemannian manifold with nonempty totally geodesic boundary. Then $\partial_\infty \widetilde{M}$ is homeomorphic to $\mathscr{S}^{n-2}$.*

We remark that the dimension restriction in the statement of [20, Theorem 1.1] is unnecessary thanks to work of Freedman and Quinn (c.f. the MathSciNet review of [25]). As a consequence of this result, the "locally CAT(−1) metric" statement in Theorem 2 cannot be replaced by "negatively curved Riemannian metric".

3. Proof of Theorem 1

Proof. We first prove the existence part of the statement, proceeding in three steps.

Step 1 (Peripheral subgroups and Poincaré duality pairs). Recall that G is a torsion-free hyperbolic group such that $\partial_\infty G \cong \mathscr{S}^{n-2}$. The stabilizer $H \leq G$ of a peripheral sphere $S^{n-2} \subset \mathscr{S}^{n-2}$ is called a *peripheral subgroup*. By Kapovich-Kleiner [19, Theorem 8(1)], there are finitely many peripheral subgroups, up to conjugacy in G. Choose representatives $H_1, \ldots, H_p$ for the conjugacy classes.

In order to show that G is the fundamental group of a manifold with boundary, we first need to establish that G has the same Poincaré duality as a manifold with boundary. To be precise, Kapovich-Kleiner [19, Corollary 12] show that $(G, \{H_i\})$ is a *group PD(n) pair* in the sense of Bieri-Eckmann [7]. This has the following topological consequence (see [18, Theorem 1] and [6, Section 6]): let (X, Y) be the CW-complex pair obtained by taking $Y = \coprod_{i=1}^{p} BH_i$ and defining X to be the mapping cylinder of the map $\coprod BH_i \to BG$. Then (X, Y) is a *CW-complex PD(n) pair* in the sense of Wall [28]. In particular this means that there are isomorphisms $H^i(X; \mathbb{Z}) \cong H_{n-i}(X, Y; \mathbb{Z})$ and $H^{i-1}(Y; \mathbb{Z}) \cong H_{n-i}(Y; \mathbb{Z})$ induced by cap product with $[X] \in H_n(X)$ and $\partial[X] \in H_{n-1}(Y)$, respectively, and that X is a *finitely dominated* CW complex (i.e. there exists a finite CW complex L and maps $X \xrightarrow{i} L \xrightarrow{r} X$ such that $r \circ i = \mathrm{id}_X$).

Step 2 (Preparing for surgery). Let (X, Y) be the pair from Step 0. We now explain why (X, Y) is homotopy equivalent to a pair (K, N) such that

(A) K is a *finite* CW complex, and
(B) N is a manifold.

This will allow us to employ the total surgery obstruction in Step 3.

(A) Wall's finiteness obstruction $\tilde{o}(X) \in \widetilde{K}_0(X)$ vanishes if and only if X is homotopy equivalent to a finite CW complex [27]. Thus to show (A), it suffices to show $\widetilde{K}_0(X) = 0$. This is a corollary of the following powerful result (see [4, Proof of Theorem 1.2] for more information):

Theorem 6 (Bartels-Lück [2], Bartels-Lück-Reich [3])**.** *Let G be a torsion-free hyperbolic group G. Then*

(†) *the (non-connective) K-theory assembly map $H_i(BG; \mathbb{K}_{\mathbb{Z}}) \to K_i(\mathbb{Z}G)$ is an isomorphism for $i \leq 0$ and surjective for $i = 1$;*
(‡) *the (non-connective) L-theory assembly map $H_i\big(BG; {}^w\mathbb{L}_{\mathbb{Z}}^{\langle -\infty \rangle}\big) \to L_i^{\langle -\infty \rangle}(\mathbb{Z}G, w)$ is bijective for every $i \in \mathbb{Z}$ and every orientation homomorphism $w : G \to \{\pm 1\}$.*

The conditions (†) and (‡) are called the Farrell-Jones conjectures in K- and L-theory, respectively. Note that, since G is a torsion-free hyperbolic group, a constructive alternative is to take X a large enough Rips complex (which is automatically a finite simplicial complex). We included the non-constructive proof above, as this "obstruction" point of view will reappear in later arguments.

(B) It remains to see that Y is homotopy equivalent to a closed manifold N^{n-1}. By definition Y is homotopy equivalent to $\coprod_{i=1}^{p} BH_i$. The peripheral subgroups H_i are all hyperbolic groups, and $\partial_\infty H_i$ is identified with the sphere $S^{n-2} \subset \mathscr{S}^{n-2}$ stabilized by H_i (see [19, Theorem 8]). The following result from [4, Theorem A] implies that $Y \simeq \coprod_{i=1}^{p} BH_i$ is homotopy equivalent to a manifold:

Theorem 7 (Bartels-Lück-Weinberger [4])**.** *Fix $n \geq 7$, and let H be a torsion-free hyperbolic group. If $\partial_\infty H \cong S^{n-2}$, then there is a closed aspherical manifold N^{n-1} such that $\pi_1(N) \cong H$.*

Step 3 (The total surgery obstruction). Let (K, N) be the pair from Step 2. The *structure set* $S_\partial^{TOP}(K)$ is defined as the set of equivalence classes of homotopy equivalences $f : (M, \partial M) \to (K, N)$ where $(M, \partial M)$ is a manifold with boundary and $f\big|_{\partial M} : \partial M \to N$ is a homeomorphism (the equivalence relation is h-cobordism rel ∂; see [24, Chapter 18]). Surgery theory provides computable obstructions to determine whether or not (K, N) is homotopy equivalent to a manifold with boundary, i.e. whether or not $S_\partial^{TOP}(K) \neq \emptyset$.

We will follow the algebraic approach detailed in Ranicki [24]. The *total surgery obstruction* $s_\partial(K)$ lives in the *structure group* $\mathbb{S}_n(K)$ and has the property that $s_\partial(K) = 0$ if and only if

(K, N) is homotopy equivalent (rel boundary) to an n-manifold with boundary; see [23, Theorem 1]. The group $\mathbb{S}_n(K)$ fits into the *algebraic surgery exact sequence* [24, Definition 15.19]

$$\cdots \to H_n(K; \mathbb{L}_\bullet) \xrightarrow{A} L_n\big(\pi_1(K)\big) \to \mathbb{S}_n(K) \to H_{n-1}(K; \mathbb{L}_\bullet) \to \cdots$$

where A is the *assembly map* and $\mathbb{L}_\bullet$ is the 1-*connective surgery spectrum* whose 0th space is G/TOP and whose homotopy groups are $\pi_i(\mathbb{L}_\bullet) = L_i(\mathbb{Z})$ for $i \geq 1$.

To show that $S_\partial^{TOP}(K) \neq \emptyset$, we will show that $\mathbb{S}_n(K) = 0$. For this, we need to consider two other versions of the structure groups.

- The *quadratic structure groups* $\mathbb{S}_i(\mathbb{Z}, K)$ are defined in [24, Definition 14.6].
- The group $\overline{\mathbb{S}}_n(K)$ (see [24, Chapter 25]) belongs to the *4-periodic algebraic surgery exact sequence*

$$\cdots \to H_n(K; \overline{\mathbb{L}}_\bullet) \xrightarrow{A} L_n\big(\pi_1(K)\big) \to \overline{\mathbb{S}}_n(K) \to H_{n-1}(K; \overline{\mathbb{L}}_\bullet) \to \cdots$$

  where $\overline{\mathbb{L}}_\bullet$ is the *0-connective surgery spectrum* whose 0th space is $L_0(\mathbb{Z}) \times G/TOP \cong \mathbb{Z} \times G/TOP$ and whose homotopy groups are $\pi_i(\overline{\mathbb{L}}_\bullet) = L_i(\mathbb{Z})$ for $i \geq 0$.

In order to show that $\mathbb{S}_n(K) = 0$, we use the following three facts.

(a) The groups $\overline{\mathbb{S}}_n(K)$ and $\mathbb{S}_n(\mathbb{Z}, K)$ are equal. This follows directly from Ranicki [24, Proposition 15.11(iii)-(iv)]. Here we have used that $\dim K \geq 6$. Note that $L_q(\mathbb{Z}) = 0$ for $q = -1$, and in Ranicki's notation $\mathbb{S}_n\langle 0\rangle(\mathbb{Z}, K) = \overline{\mathbb{S}}_n(K)$ (compare with [24, Page 289]).

(b) The quadratic structure groups $\mathbb{S}_i(\mathbb{Z}, K) \cong \mathbb{S}_i(\mathbb{Z}, BG)$ are 0 for all $i \in \mathbb{Z}$. For the proof, see [4, Proof of Theorem 1.2]. Note that this also uses Theorem 6.

(c) There is an exact sequence

$$H_n\big(K; L_0(\mathbb{Z})\big) \to \mathbb{S}_n(K) \to \overline{\mathbb{S}}_n(K).$$

See Ranicki [24, Theorem 25.3(i)].

From (a) and (b), it follows that $\overline{\mathbb{S}}_n(K) = 0$. Then, by (c), to show $\mathbb{S}_n(K) = 0$ is suffices to show $H_n(K; L_0(\mathbb{Z})) = H_n(K; \mathbb{Z}) = 0$. This can be seen from the long exact sequence in homology of a pair (K, N):

$$H_n(N; \mathbb{Z}) \to H_n(K; \mathbb{Z}) \to H_n(K, N; \mathbb{Z}) \xrightarrow{\partial} H_{n-1}(N; \mathbb{Z}).$$

The group $H_n(N; \mathbb{Z}) = 0$ because N is a PD$(n-1)$ complex. Also $H_n(K, N; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the fundamental class $[K]$, and $\partial[K]$ is a sum of fundamental classes of the components of N. In particular $\partial[K] \neq 0$, so $H_n(K; \mathbb{Z}) = 0$, as desired.

This concludes the proof of existence.

Uniqueness. So far we have proven the existence of a compact aspherical manifold W with $\pi_1(W) = G$. To show W is unique, we want to show that $S_\partial^{TOP}(W)$ is a singleton. By [23, Corollary 1 (rel ∂)], it suffices to show that $\mathbb{S}_{n+1}(W) = 0$. By [24, Theorem 25.3(i)], there is an exact sequence

$$0 \to \mathbb{S}_{n+1}(W) \to \overline{\mathbb{S}}_{n+1}(W) \to H_n(W; \mathbb{Z}),$$

and as noted above, $H_n(W; \mathbb{Z}) = 0$. Thus, it suffices to show that $\overline{\mathbb{S}}_{n+1}(W) = 0$. This follows because $\overline{\mathbb{S}}_{n+1}(W) = \mathbb{S}_{n+1}(\mathbb{Z}, W)$ (by the same reason as in Step 3, Fact (a) above), and $\mathbb{S}_{n+1}(\mathbb{Z}, W) = 0$ (see Step 3, Fact (b)). □

4. Proof of Theorem 2

The proof of Theorem 2 is an adaptation of [14, Section (5a), (5c)]. We briefly explain the relative version of [14] and the problem with extending it directly to our case.

The paper [14] uses hyperbolization to construct a closed, locally CAT(-1) manifold M^n with the unusual property that $\partial_\infty \widetilde{M}$ is **not** homomorphic to S^{n-1}. To show this, they establish that $\partial_\infty \widetilde{M} - \{\gamma_+, \gamma_-\}$ is not simply connected, where γ_+, γ_- are the endpoints of a geodesic $\gamma : (-\infty, \infty) \to \widetilde{M}$ whose link is a homology sphere H with $\pi_1(H) \neq 1$. In order to find nontrivial elements of $\pi_1\big(\partial_\infty \widetilde{M} - \{\gamma_+, \gamma_-\}\big)$, [14] studies metric spheres $S_p(r)$ centered at $p = \gamma(0)$. When $s > r$, there are natural *geodesic contraction maps* $\rho^s_r : S_p(s) \to S_p(r)$, which allow one to relate the topology of small spheres to the topology of $\partial_\infty \widetilde{M} = \varprojlim \{S_p(r)\}_{r>0}$. The central property of the maps ρ^s_r that makes the comparison work is that they are *cell-like*.

Following [14], we will construct a triangulated, locally CAT(-1) manifold M with totally geodesic boundary ∂M whose universal cover $\widetilde{M}$ contains a geodesic $\gamma : (-\infty, \infty) \to \widetilde{M}$ whose link is a homology sphere H with $\pi_1(H) \neq 1$. As above, we wish to show $\pi_1(\partial_\infty \widetilde{M} - \{\gamma_+, \gamma_-\}) \neq 1$ (Lemma 8 below then implies that $\partial_\infty \widetilde{M}$ is **not** homeomorphic to $\mathscr{S}^{n-2}$). In this case $\widetilde{M}$ is a manifold with boundary, and the maps $\rho^s_r : S_p(s) \to S_p(r)$ are not surjective for $s >> r$. This prevents us from proceeding directly as in [14]. To bypass this issue, we "cap off" the boundary components of $\widetilde{M}$ to obtain a CAT(-1) manifold $\widehat{M} \supset \widetilde{M}$ to which the arguments of [14] apply; in particular, $\pi_1(\partial_\infty \widehat{M} - \{\gamma_+, \gamma_-\}) \neq 1$. At this point it will be clear from the capping procedure (see specifically Lemma 9 below) that $\pi_1(\partial_\infty \widetilde{M} - \{\gamma_+, \gamma_-\}) \neq 1$.

For the proof of Theorem 2, we need the following elementary fact.

Lemma 8. *For $n \geq 2$, the n-dimensional Sierpinski space $\mathscr{S}^n$ is simply-connected. Moreover, if $F \subset \mathscr{S}^n$ is any finite collection of points in $\mathscr{S}^n$, then $\mathscr{S}^n \setminus F$ is still simply-connected.*

Proof. Model $\mathscr{S}^n$ as the complement, in the standard sphere S^{n+1}, of the interiors of a dense collection of pairwise disjoint round disks D_i whose radii r_i tend to zero. If γ is a curve in $\mathscr{S}^n \subset S^{n+1}$, we can find a bounding disk $\phi : \mathbb{D}^2 \to S^{n+1}$. Perturbing the map a little bit, we can assume that ϕ is transverse to all the D_i. Inductively define $\phi_k : \mathbb{D}^2 \to S^{n+1}$ to have image disjoint from $D_1, \ldots, D_k$, as follows. $\phi^{-1}(\partial D_k)$ is a finite collection of curves in $\mathbb{D}^2$, and each of these curves maps to a curve η_j on $\partial D_k \cong S^n$. Since $n \geq 2$, we can redefine ϕ_{k-1} on the interior of these finitely many curves in $\mathbb{D}^2$, by sending each of these to a bounding disk in ∂D_k for the corresponding η_j. Since the diameter of the D_i shrinks to zero, the maps ϕ_k converge to a map $\phi_\infty : \mathbb{D}^2 \to S^{n+1}$ whose boundary coincides with γ, and whose image is disjoint from the interiors of all the D_i, i.e. the image of ϕ_∞ lies in $\mathscr{S}^n$. A similar argument works even after removing finitely many points in $\mathscr{S}^n$. □

Proof of Theorem 2. We proceed in several steps.

Step 1 (Construction). We construct M using the *strict hyperbolization* construction of Charney-Davis [13]. For simplicity we will focus primarily on the case $n \geq 5$. The case $n = 4$ will be explained at the end of Step 2.

The case $n \geq 5$ is modeled on [14, Section (5c)]. Fix a smooth n-manifold X with non-empty connected boundary Y, equipped with a PL-triangulation. Choose a smooth homology sphere H^{n-2} with non-trivial fundamental group, take a PL-triangulation of H, and consider the double suspension $\Sigma^2 H \cong S^n$, with the obvious induced (no longer PL) triangulation. Take the triangulated connect sum $X \sharp \Sigma^2 H$, obtained by using the interior of a pair of n-simplices in the triangulated X, $\Sigma^2 H$ to take the connect sum (and chosen so that simplex in X does not intersect the boundary of X). Note that, topologically $X \sharp \Sigma^2 H$ is homeomorphic to X, but now has a triangulation that fails to be PL – there is precisely one 4-cycle in the 1-skeleton of the triangulation whose link is H (instead of S^{n-2}). Finally, we let $M^n = h(X \sharp \Sigma^2 H)$, an n-manifold with boundary $N^{n-1} = h(Y)$, and set $G = \pi_1(M)$.

Properties of hyperbolization implies statement (1) in our Theorem, while statement (2) follows from the fact that the triangulation of Y is PL (applying Davis-Januszkiewicz [14, Theorem (3b.2)]). The rest of our proof thus focuses on establishing statement (3) in the theorem – that $\partial_\infty G$ is not homeomorphic to $\mathscr{S}^{n-2}$.

Step 2 (Capping procedure). To show that $\partial_\infty G \neq \mathscr{S}^{n-2}$, first identify $\partial_\infty G \cong \partial_\infty \widetilde{M}$. We use Lemma 8 and show that $\pi_1(\partial_\infty \widetilde{M} \setminus F) \neq 1$, where $F = \{\gamma_+, \gamma_-\}$ consists of two points.

$\widetilde{M}$ is a non-compact CAT(-1) manifold with non-empty boundary, each component of which is isometric to $\widetilde{h(Y)}$. To understand $\partial_\infty \widetilde{M}$, we first define an isometric embedding $\widetilde{M} \hookrightarrow \widehat{M}$ into a CAT(-1) space without boundary. It will be easier to analyze $\widehat{M}$, which is obtained from $\widetilde{M}$ by gluing a certain space Z to each component of $\partial_\infty \widetilde{M}$. Next we define Z and describe its key features.

Let DX be the double of X across Y, with the induced triangulation. We apply a strict hyperbolization of Charney-Davis [13] to obtain a closed n-manifold $h(DX)$ equipped with a locally CAT(-1) metric. The universal cover $\widetilde{h(DX)}$ has boundary at infinity homeomorphic to S^{n-1} (see [14, Theorem (3b.2)]). Take any lift $\widetilde{h(Y)}$ of the separating codimension one submanifold $h(Y) \subset h(DX)$. Then $\widetilde{h(Y)}$ separates $\widetilde{h(DX)}$ into two (isometric) convex subsets. Denote by Z the closure of one of these convex subsets. Then Z is a non-compact locally CAT(-1) n-manifold with totally geodesic boundary $\widetilde{h(Y)}$.

Lemma 9. *The boundary at infinity $\partial_\infty Z$ of Z is homeomorphic to $\mathbb{D}^{n-1}$. The inclusion $\widetilde{h(Y)} = \partial Z$ induces, at the boundary at infinity, an identification $\partial_\infty \widetilde{h(Y)} = S^{n-2} = \partial(\mathbb{D}^{n-1})$.*

Let us momentarily assume Lemma 9 and finish the proof. Form the CAT(-1) space $\widehat{M}$ by gluing a copy of Z to each boundary component of $\partial \widetilde{M}$, by isometrically identifying the copy of $\widetilde{h(Y)}$ inside Z with the boundary component. We have an isometric embedding $\widetilde{M} \hookrightarrow \widehat{M}$, inducing an embedding $\partial_\infty \widetilde{M} \hookrightarrow \partial_\infty \widehat{M}$. Let γ be a lift, in $\widetilde{M} \subset \widehat{M}$ of the singular geodesic in M, i.e. the geodesic whose link is the homology sphere H. The argument in [14, Proof of Theorem 5c.1(iv), pg. 385] applies verbatim to show that $\partial_\infty \widehat{M} - \{\gamma_+, \gamma_-\}$ is not simply-connected. If η denotes a homotopically non-trivial loop in $\partial_\infty \widehat{M} - \{\gamma_+, \gamma_-\}$, then Lemma 9 allows us to use the same argument as in Lemma 8 to homotope η into the subset $\partial_\infty \widetilde{M} = \partial_\infty G$. We conclude that $\partial_\infty G - \{\gamma_+, \gamma_-\}$ fails to be simply connected. From Lemma 8, we conclude that $\partial_\infty G$ is not homeomorphic to $\mathscr{S}^{n-2}$.

The $n = 4$ case proceeds similarly, but is modeled instead on [14, Section (5a)]. Briefly, one lets X be a 4-dimensional simplicial complex whose geometric realization is a homology manifold with non-empty boundary Y, and which contains a singular point in the interior of X (whose link is, for example, the Poincaré homology 3-sphere H). One then looks at the universal cover of the hyperbolization $W = h(X)$. We can "cap off" the boundary components of $\widetilde{W}$ as in the last paragraph to obtain $\widehat{W}$. Then the arguments in [14, Section 3d] shows that $\pi_1(\partial_\infty \widehat{W})$ is non-trivial. Again, using Lemma 9, we can push a homotopically non-trivial loop in $\partial_\infty \widehat{W}$ into the subset $\partial_\infty \widetilde{W} = \partial_\infty G$. From Lemma 8, we conclude that $\partial_\infty G$ is not homeomorphic to $\mathscr{S}^2$. Finally, even though W is not a manifold, it is homotopy equivalent to a manifold: just remove a small neighborhood of the singular cone point, and replace it by a contractible manifold which bounds H. The resulting 4-manifold M has the desired properties.

Step 3 (Reducing Lemma 9). To complete the proof of the theorem, we are left with verifying Lemma 9. This is again a minor adaptation of the arguments in [14, Sections 3b, 3c]. Choose a basepoint $x \in \partial Z$, and consider the closed metric r-balls $\overline{B}_Z(r)$, $\overline{B}_{\partial Z}(r)$ in the spaces Z, ∂Z, centered at x, as well as the metric r-spheres $S_Z(r)$ and $S_{\partial Z}(r)$. The proof of Lemma 9 will rely on the following:

<u>Claim 1:</u> For all r, the metric spheres $S_Z(r)$ are manifolds with boundary $S_{\partial Z}(r)$.

<u>Claim 2:</u> For points $p \in S_{\partial Z}(r)$, the complement $\mathrm{Lk}(p) \setminus B_{\mathrm{Lk}(p)}(v; \pi)$ of the metric ball of radius π, centered at $v \in \partial\left(\mathrm{Lk}(p)\right)$ in the link of p, is a cell-like set.

From these two Claims, it is easy to conclude. If one takes a small enough r, then clearly $S_Z(r)$ is homeomorphic to a disk $\mathbb{D}^{n-1}$. In view of Claim 2 and the discussion in [14, pg. 372], there is an $\epsilon > 0$ such that each of the geodesic contraction maps $\rho_r^s : S_Z(s) \to S_Z(r)$ is a cell-like map when $r < s < r + \epsilon$. So by Claim 1, the maps ρ_r^s are cell-like maps between manifolds with boundaries. From the work of Siebenmann [26], Quinn [22], and Armentrout [1] it follows that each ρ_r^s is a *near-homeomorphism* (i.e. can be approximated arbitrarily closely by homeomorphisms), and hence, that all the $S_Z(r)$ are homeomorphic to a disk $\mathbb{D}^{n-1}$, with boundary $\partial S_Z(r) = S_{\partial Z}(r)$.

Finally, we identify the pair $(\partial_\infty Z, \partial_\infty(\partial Z))$ with the inverse limit $\varprojlim \{(S_Z(r), S_{\partial Z}(r))\}_{r>0}$, where the bonding maps are given by the maps ρ_r^s (where $0 < r < s$), which we saw are all near-homeomorphisms. Lemma 9 now follows by applying the main result of Brown [10].

This reduces the proof of Lemma 9 (and hence also of the theorem) to checking Claim 1 and Claim 2 – which are the last two steps of the proof.

Step 4 (Proof of Claim 1). We first argue that the ball $B_Z(r)$ of radius r is a manifold with boundary. It is clear that points $p \in \mathrm{Int}(\widetilde{M})$ at distance $< r$ from the basepoint have manifold neighborhoods. It is also immediate that points $p \in \partial\widetilde{M}$ at distance $< r$ from the basepoint have neighborhoods homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}_+$. Points at distance $= r$ from the basepoint are either in $\mathrm{Int}(\widetilde{M})$ or on $\partial\widetilde{M}$.

For points p in $\mathrm{Int}(\widetilde{M})$, the argument in [14, pg. 372] shows that p has a neighborhood homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}_+$. So the only possible points to worry about are points at distance $= r$, and lying on the subset $\partial\widetilde{M}$. But for such a point p, a similar argument works with no trouble. Let v be the point in $\mathrm{Lk}(p)$ pointing from p to the basepoint x, and consider the closed ball $\overline{B}_{\mathrm{Lk}(p)}(v; \pi/2)$ in the link of p, centered at v, of radius $\pi/2$. For any vector $w \in \overline{B}_{\mathrm{Lk}(p)}(v; \pi/2)$, one can look at the geodesic γ_w emanating from p, in the direction w (γ_w is well-defined close to p). If the direction w is at distance $< \pi/2$ from v, then for a small interval of time $[0, s(w)]$, the geodesic γ_w lies entirely in $B_Z(r)$, with $\gamma_w\left(s(w)\right) \in S_Z(r) \cup B_{\partial Z}(r)$. Note that s varies continuously and $s(w) \to 0$ as $w \to S_{\mathrm{Lk}(p)}(v; \pi/2)$. It follows that p has a neighborhood homeomorphic to the set $\hat{X}$ constructed as follows: take the product $I \times \overline{B}_{\mathrm{Lk}(p)}(v; \pi/2)$, collapse the fibers over the subset $S_{\mathrm{Lk}(p)}(v; \pi/2)$ to 0, and then collapse the subset $\{0\} \times \overline{B}_{\mathrm{Lk}(p)}(v; \pi/2)$ to a single point (which is identified with p) – see Figure 2. By an inductive argument (note

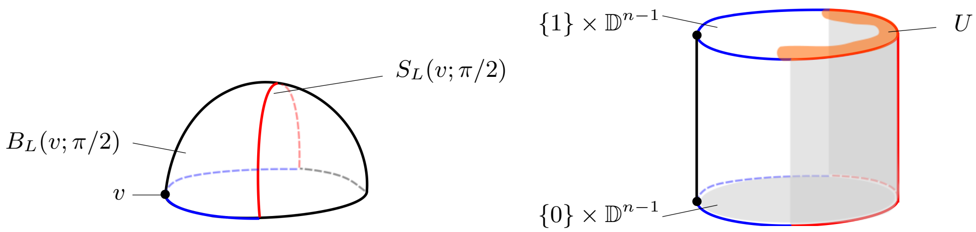

FIGURE 2. Left: The link $L = \mathrm{Lk}(p)$. Right: The space $I \times \overline{B}_{\mathrm{Lk}(p)}(v;\pi/2)$, which is identified with a neighborhood $\hat{X}$ of p after quotienting by the gray region.

that $\dim(\mathrm{Lk}(p)) = \dim(\widetilde{M}) - 1)$ one can assume that $\overline{B}_{\mathrm{Lk}(p)}(v;\pi/2)$ is homeomorphic to a disk $\mathbb{D}^{n-1}$, with the subset $S_{\mathrm{Lk}(p)}(v;\pi/2)$ corresponding to an embedded $\mathbb{D}^{n-2}$ inside $\partial\mathbb{D}^{n-1} \cong S^{n-2}$. Following the construction of $\hat{X}$ given above, we see that $\hat{X}$ is homeomorphic to $\mathbb{D}^n$, with the point corresponding to p lying on $\partial\mathbb{D}^n$. This shows that $B_Z(r)$ is indeed a manifold with boundary, and that the boundary of $B_Z(r)$ naturally decomposes as the union of $S_Z(r) \cup B_{\partial Z}(r)$, where the union is over the common subset $S_{\partial Z}(r)$.

Finally, we check that $S_Z(r)$ is an $(n-1)$-manifold with boundary. For points $p \in S_Z(r)$ lying in $\mathrm{Int}(\widetilde{M})$, it follows easily from [14, pg. 372] that these points have neighborhoods homeomorphic to $\mathbb{D}^{n-1}$ with p lying as an interior point. In the case where $p \in S_Z(r)$ lies on $\partial\widetilde{M}$, we look at the neighborhood $\hat{X}$ of p constructed above. Within $\hat{X}$, the subset corresponding to $S_Z(r)$ consists of (the image of) a small neighborhood U of $\{1\} \times S_{\mathrm{Lk}(p)}(v;\pi/2) \cong \mathbb{D}^{n-2}$ inside the slice $\{1\} \times \overline{B}_{\mathrm{Lk}(p)}(v;\pi/2) \cong \mathbb{D}^{n-1}$. Note that the $(n-2)$-disk $S_{\mathrm{Lk}(p)}(v;\pi/2)$ lies in the boundary sphere of the $(n-1)$-disk $\overline{B}_{\mathrm{Lk}(p)}(v;\pi/2)$ (by induction). The image of U thus gives a copy of $\mathbb{D}^{n-1}$, with p lying in the boundary of $\mathbb{D}^{n-1}$. Moreover, the subset of U corresponding to $S_{\partial Z}(r)$ is just a neighborhood of p inside the boundary sphere of $\mathbb{D}^{n-1}$, i.e. is homeomorphic to $\mathbb{D}^{n-2}$. This completes the argument for Claim 1.

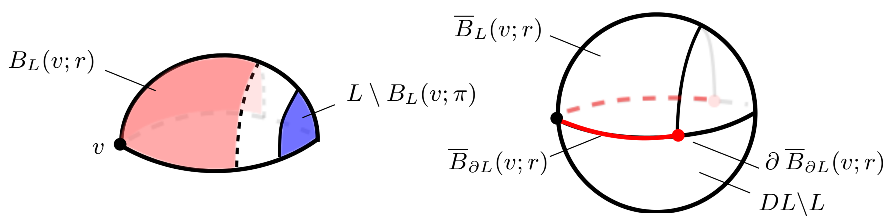

FIGURE 3. The link $L = \mathrm{Lk}(p)$ and its double DL.

Step 5 (Proof of Claim 2). We want to show that the complement $\mathrm{Lk}(p) \setminus B_{\mathrm{Lk}(p)}(v;\pi)$ is cell-like. The set $\mathrm{Lk}(p)$ is homeomorphic to a disk $\mathbb{D}^{n-1}$, so we can think of the set we are interested in as lying within the double $D\,(\mathrm{Lk}(p)) \cong S^{n-1}$. Given an $r \in (0,\pi)$, consider the subset $U_r \subset D\,(\mathrm{Lk}(p)) \cong S^{n-1}$ defined to be the union of $D\,(\mathrm{Lk}(p)) \setminus \mathrm{Lk}(p)$ and the set $B_{\mathrm{Lk}(p)}(v;r)$. See Figure 3. We will show each such U_r is homeomorphic to $\mathbb{R}^{n-1}$. Then by a result of Brown [11] it follows that the union $U_\infty := \bigcup_{r\in(0,\pi)} U_r \subset D\,(\mathrm{Lk}(p)) \cong S^{n-1}$ is also homeomorphic to $\mathbb{R}^{n-1}$. But if a subset of S^{n-1} is homeomorphic to $\mathbb{R}^{n-1}$, its complement is automatically cell-like. Since the complement of U_∞ coincides with $\mathrm{Lk}(p) \setminus B_{\mathrm{Lk}(p)}(v;\pi)$, this would establish Claim 2.

To see that each U_r is homeomorphic to $\mathbb{R}^{n-1}$, we consider their closures $\overline{U}_r$. We have that $U_r = \mathrm{Int}(\overline{U}_r)$, and that $\overline{U}_r$ can be written as the union of a copy of $\mathrm{Lk}(p)$ along with $\overline{B}_{\mathrm{Lk}(p)}(v;r)$, where the union is taken over the common subset $\overline{B}_{\partial\mathrm{Lk}(p)}(v;r)$. Let us analyze the two pieces in this decomposition.

On one of the sides, the subset $\mathrm{Lk}(p)$ is homeomorphic to $\mathbb{D}^{n-1}$, and the common subset $\overline{B}_{\partial\mathrm{Lk}(p)}(v;r)$ is homeomorphic to an embedded $(n-2)$-disk $\mathbb{D}^{n-2}$ inside the boundary sphere $\partial\mathrm{Lk}(p) \cong S^{n-2}$. Note that, by varying the parameter r, we see that

$$S^{n-3} \simeq \partial\overline{B}_{\partial\mathrm{Lk}(p)}(v;r) \subset \partial\mathrm{Lk}(p) \simeq S^{n-2}$$

is bicollared. On the other side, the subset $\overline{B}_{\mathrm{Lk}(p)}(v;r)$ is also homeomorphic to $\mathbb{D}^{n-1}$, and the gluing disk $\mathbb{D}^{n-2} \cong \overline{B}_{\partial\mathrm{Lk}(p)}(v;r)$ inside the boundary sphere $S^{n-2} \cong \partial\overline{B}_{\mathrm{Lk}(p)}(v;r)$ also has complement a disk (by the argument in Claim 1). Thus, we see that $\overline{U}_r$ is obtained by gluing together two closed $(n-1)$-disks, by identifying together two copies of an $(n-2)$-disk, where each copy is nicely embedded in the respective boundary spheres $S^{n-2} \cong \mathbb{D}^{n-1}$. It follows that $\overline{U}_r$ is also homeomorphic to $\mathbb{D}^{n-1}$. This completes the proof of Claim 2 and the proof of the theorem. □

Remark 10. Let us make a small comment on approximating cell-like maps by homeomorphisms, in the case of manifolds with boundary. The attentive reader will probably notice that, in Siebenmann's work [26], there are two cases that require special care. In the 5-dimensional case, he requires the restriction of the map to the boundary to be a homeomorphism (rather than just a cell-like map). This is due to the fact that, at the time [26] was written, it was unclear whether or not cell-like maps of (closed) 4-manifolds could be approximated by homeomorphisms—hence the need of a stronger hypothesis on the boundary map. In view of Quinn's subsequent proof of the 4-dimensional case [22], this stronger hypothesis is no longer needed in the 5-dimensional boundary case. Note that, in our context, the bonding maps, when restricted to the boundary, are always cell-like (but are not homeomorphisms).

The other special case has to do with 3-dimensions. Here there is an added hypothesis that every point pre-image has a neighborhood N which isn't just contractible, but in addition is prime (i.e. if $N = M_1 \# M_2$, then one of the M_i is a standard 3-sphere). The only way this could fail is if one of the M_i were instead a homotopy 3-sphere – but by Perelman's resolution of the Poincaré Conjecture, such a manifold is automatically S^3. So again, in the 3-dimensional case, this additional hypothesis is now unnecessary.

5. Remarks on CAT(0) groups

In this section we remark on generalizing the main result from hyperbolic groups to CAT(0) groups. A proper geodesic space X is called *CAT(0)* if geodesic triangles in X are at least as thin as triangles in Euclidean space [8]. A group G is called *CAT(0)* if there exists a CAT(0) space X on which G acts *geometrically* (that is, isometrically, properly, and compactly).

A CAT(0) space X has a visual boundary $\partial_\infty X$, and if G acts geometrically on X, then G acts on $\partial_\infty X$ by homeomorphisms. In this case $\partial_\infty X$ is called *a* boundary of G. With this terminology we have the following theorem.

Theorem 11. *Let G be a CAT(0) group for which S^{n-1} is a boundary. If $n \geq 6$, then there exists a closed n-dimensional aspherical manifold W such that $\pi_1(W) \simeq G$.*

The proof is almost identical to the proof of Theorem 7 in [4]. We give a short explanation for how to extend that argument to the CAT(0) case.

Proof of Theorem 11. By assumption G acts geometrically on an X with $\partial_\infty X = S^{n-1}$. Denote $\overline{X} = X \cup \partial_\infty X$. We proceed in three steps.

Step 1. BG is homotopy equivalent to a closed aspherical homology n-manifold W such that W has the disjoint disk property. To show this, it suffices to show that G is a PD(n) group and to note that CAT(0) groups satisfy the Farrell-Jones conjectures in K- and L-theory. For then we may use [4, Theorem 1.2], which says that for such a group, BG is homotopy equivalent to a closed aspherical homology n-manifold M with the disjoint disk property.

We explain why G is PD(n) group. First, we know G is of type FP once we know that there exists a finite CW complex $K \simeq BG$ (for then the cellular chain complex of the universal cover $\widetilde{K}$ is a finite length resolution of $\mathbb{Z}$ by finitely generated free G modules). A finite CW complex $K \simeq BG$ for a group G that acts geometrically on a proper CAT(0) space is shown to exist by Lück [21]. Now G is a $PD(n)$ group because

$$H^i(G;\mathbb{Z}G) \cong H^i_c(X) \cong \widetilde{H}^{i-1}(\partial_\infty X) = \widetilde{H}^{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

The first two isomorphisms are described by Bestvina [5]. That this implies G is a PD(n) group is explained in [9, VIII.10.1].

Step 2. The universal cover $\widetilde{W}$ can be compactified $N = \widetilde{W} \cup \partial_\infty X$ such that N is a homology manifold with boundary. To show that N is a homology manifold with boundary it suffices to show that N is a finite-dimensional locally compact ANR and $\partial_\infty X$ is a Z-set in N (see [4, Proposition 2.5]). The pair $(\overline{X}, \partial_\infty X)$ is a Z-structure on G by Bestvina [5, Example 1.2(ii)]. Furthermore, by [5, Lemma 1.4] for any other finite model K for BG, there is a natural Z-structure on $(\overline{K}, \partial_\infty X)$, where $\overline{K} = K \cup \partial_\infty X$. Thus $(N, \partial_\infty X)$ admits a Z-set structure; in particular, N is a Euclidean retract, finite dimensional, and S^{n-1} is a Z-set inside N.

Step 3. $\widetilde{W}$ (and hence also W) is a manifold. This part of the argument is identical to that given in [4, Theorem A]. Quinn's invariant allows one to recognize manifolds among homology manifolds with the disjoint disk property. By the local nature of Quinn's invariant, if $(B, \partial B)$ is a homology manifold with boundary and ∂B is a manifold, then int(B) is a manifold. □

In light of this result and Theorem 1 above, it is natural to ask the following question.

Question. Let G be a CAT(0) group which admits $\mathscr{S}^{n-2}$ as a boundary. Is G the fundamental group of an n-dimensional aspherical manifold with boundary?

Examples of G satisfying the hypothesis of this Question are given by Ruane [25]: every nonuniform lattice $\Gamma \leq \mathrm{SO}(n,1)$ is an example. For these examples, an aspherical manifold with boundary can be obtained by "truncating the cusps" of $\mathbb{H}^n/\Gamma$.

There are some basic problems with answering this Question with the techniques of this paper. For example, it is not obvious that peripheral subgroups of a CAT(0) group with Sierpinski space boundary are CAT(0), or that the double of a CAT(0) group along peripheral subgroups is CAT(0).

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Department of Mathematics, Ohio State University, Columbus, OH 43210
E-mail address: jlafont@math.ohio-state.edu

Department of Mathematics, University of Chicago, Chicago, IL 60615
E-mail address: tshishikub@math.uchicago.edu