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LIMITING PROFILES OF SEMILINEAR ELLIPTIC EQUATIONS WITH LARGE ADVECTION IN POPULATION DYNAMICS II*

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Abstract. Limiting profiles of solutions to a 2×2 Lotka–Volterra competition-diffusion-advection system, when the strength of the advection tends to infinity, are determined. The two species, competing in a heterogeneous environment, are identical except for their dispersal strategies: one is just a random diffusion, while the other is "smarter"—a combination of random diffusion and a directed movement up the environmental gradient. In the previous paper of Lam and Ni [*Discrete Contin. Dyn. Syst.* 28 (2010), pp. 1051–1067], it is proved that in one space dimension, for large advection the "smarter" species concentrates near a selected subset of positive local maximum points of the environment function, establishing a conjecture proposed by Cantrell, Cosner, and Lou. With a different method, we generalize this result to any dimensions with the peaks located under mild hypotheses on the environment function. Moreover, a Liouville-type result which gives the asymptotic profile is proved.

Key words. concentration phenomena, large advection, limiting profile, mathematical ecology, reaction-diffusion equation

AMS subject classifications. 35B40, 35B53, 35B30, 35J57, 92D25

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1. Introduction. In this paper we continue our study in [19, 20] on the shape of coexistence steady states of a reaction-diffusion-advection system from theoretical ecology. We consider the following system proposed in [7],

$$
(1.1)\qquad \begin{cases} U_t = \nabla\cdot(d_1\nabla U - \alpha U\nabla m) + U(m(x) - U - V) & \text{in } \Omega\times(0,\infty),\\ V_t = d_2\Delta V + V(m(x) - U - V) & \text{in } \Omega\times(0,\infty),\\ d_1\dfrac{\partial U}{\partial \nu} - \alpha U\dfrac{\partial m}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,\infty),\\ U(x,0) = U_0(x) \ge 0 \quad\text{and}\quad V(x,0) = V_0(x)\ge 0 & \text{in } \Omega, \end{cases}
$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$ with boundary $\partial\Omega$ and unit outer-normal ν; ∇ is the gradient operator; $\nabla\cdot$ is the divergence operator and $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator; U and V, representing the population densities of two competing species with random dispersal rates d_1, d_2, respectively, are therefore nonnegative; $m(x)$ is a nonconstant function representing the local intrinsic growth rate; $\alpha \ge 0$ is a parameter; and no-flux boundary conditions are imposed on $\partial\Omega$ (see discussions below).

The system (1.1) originates from the diffusive Lotka–Volterra model of two randomly moving competitors in a closed but spatially varying environment. (See [13, 22] and the references therein.) In reality, it is very plausible that besides random dispersal, species could track the local resource gradient and move upward along it. (See, e.g., [2, 5, 6, 10, 24].) The aim of (1.1) is to study the joint effects of random diffusion and directed movement on population dynamics.

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More precisely, we view the local intrinsic growth rate $m(x)$ as describing the quality and quantity of resources available at the point $x \in \Omega$. The two species competing for a common resource are identical except for their dispersal strategies: the species with density V disperses only by random diffusion, while the species with density U disperses by diffusion combined with directed movement up the gradient of m. The dispersal of the two competitors can be understood in terms of the fluxes $J_V = -d_2 \nabla V$ and $J_U = -d_1 \nabla U + \alpha(\nabla m)U$. (See [7] for a derivation of (1.1) and [25] for a discussion of how advection-diffusion equations can be derived in terms of fluxes.) Also, we assume $\alpha \geq 0$ to capture the hypothesis that the species U has a tendency to move up the gradient of m. The no-flux boundary conditions corresponding to the respective dispersal fluxes J_U, J_V reflect the assumption that individuals do not cross the boundary $\partial\Omega$.

To assess whether or not directed movement confers an advantage for either competitor, it suffices to study the existence and stability of steady states, which determines a significant amount of the dynamics of the competition system (1.1) (see [6, 17]). For instance, see Theorem 1.5.

System (1.1) has attracted considerable attention recently. If the diffusion rate d_1 of U is less than the diffusion rate d_2 of V, then for $\alpha \geq 0$ small the slower diffuser U always wipes out its faster-moving competitor V regardless of initial conditions. (See [13, 7].) In other words, $(\tilde{u}, 0)$ is globally asymptotically stable, where $\tilde{u}$ is the unique positive solution to

$$\text{(1.2)} \qquad \begin{cases} \nabla \cdot (d_1 \nabla \tilde{u} - \alpha \tilde{u} \nabla m) + \tilde{u}(m(x) - \tilde{u}) = 0 & \text{in } \Omega, \\ d_1 \frac{\partial \tilde{u}}{\partial \nu} - \alpha \tilde{u} \frac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

As α increases, the species U has a stronger tendency to move towards more favorable regions, and it is expected to continue to win the competition. It is rather surprising that U and V always coexist for α sufficiently large! More precisely, for all α sufficiently large, (1.1) has a *stable* coexistence steady state (U_α, V_α) ($U_\alpha > 0$ and $V_\alpha > 0$). This so-called "advection-mediated coexistence" was discovered in [8] and generalized in [12]. It was further argued that *as α becomes large, the "smarter" competitor moves towards and concentrates in those regions with the most favorable local environments, leaving room in the regions with less resources for the second species to survive*. Furthermore, the above formal argument has been justified mathematically in some special cases. For instance, the following is proved.

THEOREM 1.1 (see [12]). *Assume $\int_\Omega m > 0$. If m has a single critical point x_0 in $\bar{\Omega}$ which is a global maximum point,* $\det D^2 m(x_0) \neq 0$, *and $\frac{\partial m}{\partial \nu} \leq 0$ on $\partial\Omega$, then for any coexistence steady state (U_α, V_α) of (1.1), as $\alpha \to \infty$,*

$$V_\alpha(x) \to \theta_{d_2}(x) \quad \textit{in } C^{1,\beta}(\bar{\Omega}), \textit{ for any } \beta \in (0,1), \textit{ and}$$

$$U_\alpha(x) e^{\alpha[\max_{\Omega} m - m(x)]/d_1} \to 2^{N/2}[m(x_0) - \theta_{d_2}(x_0)] \quad \textit{uniformly in } \Omega,$$

where θ_d is the unique positive solution to

$$\text{(1.3)} \qquad \begin{cases} d\Delta\theta + \theta(m - \theta) = 0 & \textit{in } \Omega, \\ \frac{\partial \theta}{\partial \nu} = 0 & \textit{on } \partial\Omega. \end{cases}$$

(Here the factor $2^{N/2}$ comes from the profile of $U_\alpha \sim U_\alpha(x_0) e^{\alpha[m(x) - \max_\Omega m]/d_1}$ together with the integral constraint $\int_\Omega U_\alpha(m(x) - U_\alpha - V_\alpha)dx = 0$ obtained by integrating the equation over Ω.) In general, we have the following.

Conjecture 1.2 (see [8, 12]). *If $m(x)$ has multiple local maxima, then given any coexistence steady state (U_α, V_α) of* (1.1)*, U_α concentrates at every local maximum point of m as $\alpha \to \infty$.*

Under mild conditions on m, the above conjecture was resolved in [20], when Ω is one-dimensional, and in [19] for higher dimensions under the extra assumption that $m(x)$ has multiple peaks of equal height. It turns out that in both cases, U_α *concentrates precisely at the local maximum points of m where $m - \theta_{d_2}$ is positive.* That is, it does not necessarily concentrate at every local maximum point of m. (See Figure 1.1.) In this paper we are going to resolve the conjecture for all dimensions under mild conditions on m and determine the limiting profile of (U_α, V_α) as $\alpha \to \infty$. In addition, to better understand the different roles played by the advection and reaction terms, we are going to treat the following more general system,

$$(1.4)\qquad \begin{cases} U_t = \nabla \cdot (d_1 \nabla U - \alpha U \nabla m) + U(p - U - V) & \text{in } \Omega \times (0,\infty), \\ V_t = d_2 \Delta V + V(p - U - V) & \text{in } \Omega \times (0,\infty), \\ d_1 \frac{\partial U}{\partial \nu} - \alpha U \frac{\partial m}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

where $m = m(x)$ is not necessarily equal to $p = p(x)$. An important case is $p = \exp m$, which is related to the evolution of optimal dispersal strategies (see [9, 1]). First, we state the assumptions on m and p. Let $\mathfrak{M}$ be the set of all local maximum points of m.

- (H1) $m \in C^2(\bar\Omega)$, and all local maximum points of m are nondegenerate and lie in the interior of Ω (i.e., $\mathfrak{M} \subset \Omega$ and $\det D^2 m \neq 0$ on $\mathfrak{M}$).
- (H2) The set of critical points of m has zero measure.
- (H3) $\frac{\partial m}{\partial \nu} \leq 0$ on $\partial\Omega$ and $\Delta m > 0$ in $\{x \in \bar\Omega : \nabla m = 0\} \setminus \mathfrak{M}$.
- (H4) $p = \chi(m)$ for some strictly increasing function $\chi \in C^\beta(\bar\Omega)$ for some $\beta \in (0,1)$ and $\int_\Omega p\, dx \geq 0$.

Let $\bar\theta_d$ be the unique positive solution to

$$(1.5)\qquad \begin{cases} d\Delta\bar\theta + \bar\theta(p - \bar\theta) = 0 & \text{in } \Omega, \\ \frac{\partial \bar\theta}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

We state our main result concerning the concentration and limiting profile of (U_α, V_α).

Theorem 1.3. *Assume* (H1), (H2), (H3), *and* (H4). *Then, for all α sufficiently large,* (1.4) *has at least one stable coexistence steady state. Moreover, if (U_α, V_α) is any coexistence steady state of* (1.4)*, then as $\alpha \to \infty$,*

- (i) *$V_\alpha(x) \to \bar\theta_{d_2}(x)$ in $C^{1,\beta}(\bar\Omega)$ for any $\beta \in (0,1)$;*
- (ii) *for all $r > 0$, $U_\alpha(x) \to 0$ in $\Omega \setminus [\cup_{x_0 \in \mathfrak{M}} B_r(x_0)]$ uniformly and exponentially;*
- (iii) *for each $x_0 \in \mathfrak{M}$ and each $r > 0$ small,*

$$U_\alpha(x) - 2^{N/2} \max\{p(x_0) - \bar\theta_{d_2}(x_0), 0\} e^{\alpha[m(x) - m(x_0)]/d_1} \to 0 \text{ uniformly in } B_r(x_0).$$

Note that when $p \equiv m$, then $\bar\theta_{d_2} = \theta_{d_2}$, and this establishes Conjecture 1.2.

Remark 1.4.

- (i) It is proved in Appendix A of [19] that when d_2 is sufficiently small, if $x_0 \in \mathfrak{M}$, then $p(x_0) - \bar\theta_{d_2}(x_0) > 0$ if and only if $p(x_0) > 0$. When d_2 is large and p has more than one local maximum point, then $p(x_0) - \bar\theta_{d_2}(x_0)$ can sometimes be negative, even when $p(x_0) > 0$. In this case, local maximum points of m can be a “trap” for U there. See Figure 1.1 for a one-dimensional picture.
- (ii) The existence and stability of (U_α, V_α) follow from arguments in [12, 8] and are proved in section 2 (Theorem 2.1).

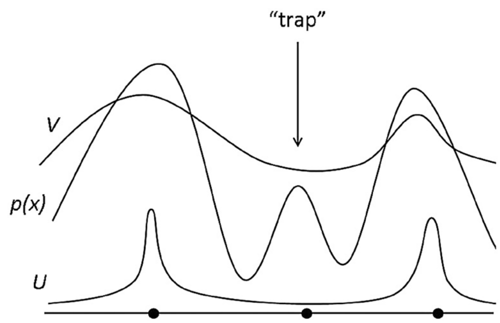

FIG. 1.1. *A trap for U.*

More important, Theorem 1.3 actually describes all possible outcomes of the competition between U and V when α is large: U and V always coexist with the unique limiting population density given by Theorem 1.3. See the end of section 2 for a discussion of the proof.

THEOREM 1.5. *Assume $m \in C^2(\overline{\Omega})$, (H2), and (H4). Then for all α sufficiently large, there exist two coexistence steady states $(\tilde{U}_i, \tilde{V}_i)$, $i = 1, 2$, such that $U_1 \geq U_2$ and $V_1 \leq V_2$, and the set $X = \{(U,V) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : \tilde{U}_1 \geq U \geq \tilde{U}_2$ and $\tilde{V}_1 \leq V \leq \tilde{V}_2\}$ is globally attracting among all nontrivial, nonnegative solutions of (1.4); i.e., given any nontrivial, nonnegative initial condition $(U_0(x), V_0(x))$, the solution $(U(x,t), V(x,t))$ to (1.4) satisfies*

$$\mathrm{dist}_{C(\bar{\Omega}) \times C(\bar{\Omega})}\big((U(\cdot,t), V(\cdot,t)), X\big) \to 0 \quad \text{as } t \to \infty.$$

By way of proving Theorem 1.3, we consider the following closely related single equation:

$$\begin{cases} u_t = \nabla \cdot (d\nabla u - \alpha u \nabla m) + u(p - u) & \text{in } \Omega \times (0,\infty), \\ d\frac{\partial u}{\partial \nu} - \alpha u \frac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.6}$$

Equation (1.6) was proposed in [2] (when $p \equiv m$) to model the population dynamics of a single species with directed movement in a heterogeneous environment. It was proved in [2] that if $\int_\Omega m\,dx > 0$ and $p \equiv m$, then (1.6) has a unique positive steady state u_α for all $\alpha \geq 0$. Moreover, u_α is globally asymptotically stable among nonnegative solutions. Similarly, it was conjectured in [8, 21] that *if $p \equiv m$, then as $\alpha \to \infty$, u_α concentrates precisely on the set of all local maximum points of m.* This conjecture was resolved in [19] under mild conditions.

We shall determine the limiting profile of u_α when Ω is in any dimension and when p is not necessarily equal to m. For the single equation (1.6), we can relax the assumption (H4) on p to the following.

(H5) $p \in C^\beta(\bar{\Omega})$ for some $\beta \in (0,1)$ and $\{x \in \bar{\Omega} : m(x) = \sup_\Omega m\} \subseteq \{x \in \bar{\Omega} : p(x) > 0\}$.

THEOREM 1.6. *Assume (H1), (H2), (H3), and (H5). Then for all α sufficiently large, (1.6) has a unique positive steady state u_α. Moreover, u_α is globally asymptotically stable, and for all small $r > 0$, as $\alpha \to \infty$,*

$$u_\alpha(x) \to 0 \quad \textit{uniformly and exponentially in } \Omega \setminus [\cup_{x_0 \in \mathfrak{M}} B_r(x_0)],$$

while for each $x_0 \in \mathfrak{M}$,

$$u_\alpha(x) - 2^{N/2} \max\{p(x_0), 0\} e^{\alpha[m(x)-m(x_0)]/d} \to 0 \quad \textit{uniformly in } B_r(x_0).$$

Remark 1.7. The existence, uniqueness, and stability of u_α follow from arguments in [2] and are proved in section 2.

The main ingredients in the proof of Theorem 1.6 are the L^∞ estimate in section 3 and the following Liouville-type result, which seems to be new.

PROPOSITION 1.8. *Let B be a symmetric positive definite $N \times N$ matrix, and let $0 < \sigma \in L^\infty_{loc}(\boldsymbol{R}^N)$ such that for some $R_0 > 0$,*

$$\sigma^2 = e^{-x^T Bx} \quad \text{for all } x \in \boldsymbol{R}^N \setminus B_{R_0}(0);$$

then every nonnegative weak solution $w \in W^{1,2}_{loc}(\boldsymbol{R}^N)$ to

$$\nabla \cdot (\sigma^2 \nabla w) = 0 \quad \text{in } \boldsymbol{R}^N \tag{1.7}$$

is a constant.

Proposition 1.8 determines the limiting profiles of u_α and U_α at each $x_0 \in \mathfrak{M}$ and is proved in the appendix.

Remark 1.9. In general, some kind of asymptotic behavior is needed for this kind of result to hold; e.g., it is proved in [4] that for any $0 < \sigma \in L^\infty_{loc}(\mathbf{R}^N)$, a nonnegative weak solution of (1.7) is a constant if there exists $C > 0$ such that $\int_{B_R} \sigma^2 w^2 \le CR^2$ for all large $R > 0$. (See also [14].)

The paper is organized as follows. In section 2, the existence and uniqueness results for positive steady states, and a sketch of the proof of Theorem 1.5, are presented. An L^∞ bound for u_α will be established in section 3. Theorems 1.6 and 1.3 will be proved in sections 4 and 5, respectively. Lastly, some concluding remarks will be given in section 6. The proof of Proposition 1.8 can be found in the appendix. Throughout this paper, C represents some generic constant independent of α.

2. Existence of positive steady-states and proof of Theorem 1.5. In this section we present the existence and stability results for positive steady states of (1.4) and (1.6), as well as the uniqueness result for (1.6). In addition, we shall comment on the proof of Theorem 1.5. The arguments of this section are analogous to those in [2, 8], where the case $p \equiv m$ was treated, and which are presented here for completeness. For later purposes, we study the positive solutions of the following slightly more general equation,

$$\begin{cases} \nabla \cdot (d\nabla u - \alpha u \nabla m) + u(p_\alpha(x) - u) & \text{in } \Omega, \\ d\frac{\partial u}{\partial \nu} - \alpha u \frac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$p_\alpha(x) \in C^\beta(\bar{\Omega}) \text{ for some } \beta \in (0,1), \quad \text{and } \lim_{\alpha \to \infty} p_\alpha = p \text{ in } C^\beta(\bar{\Omega}). \tag{2.2}$$

In particular, (2.1) includes (1.6) as a special case, with p_α being independent of α.

THEOREM 2.1.

(i) *If $m \in C^2(\bar{\Omega})$ and* (H5) *are assumed, then for α sufficiently large, there exists a unique positive solution $u_\alpha \in C^2(\bar{\Omega})$ of* (2.1) *which is globally asymptotically stable.*

(ii) *In addition, if we assume* (H2) *and* (H4), *then for α sufficiently large, there exists at least one stable coexistence steady state $(U_\alpha, V_\alpha) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ of* (1.4).

We also collect some useful facts about u_α.

LEMMA 2.2. *Let u_α be the unique positive solution of* (2.1). *Assume* (2.2) *and* (H1)-(H3). *Then the following statements hold:*

(i) $|u_\alpha|_{L^\infty} \le |p_\alpha|_{L^\infty} + \alpha|\Delta m|_{L^\infty}$.

(ii) $|u_\alpha|_{L^2} \to 0$ *as* $\alpha \to \infty$.

(iii) *For each compact subset K of $\bar{\Omega} \setminus \mathfrak{M}$, there exists $\epsilon' > 0$ and $\alpha_0 > 0$ such that $u_\alpha \le e^{-\epsilon'\alpha}$ in K for all $\alpha \ge \alpha_0$.*

Proof of Theorem 2.1(i). By a transformation $v = e^{-\alpha m/d}u$, (2.1) is equivalent to

$$\begin{cases} \tilde{L}v = \nabla\cdot(de^{\alpha m/d}\nabla v) + e^{\alpha m/d}v(p_\alpha - e^{\alpha m/d}v) = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Fix $\alpha \ge 0$ so that $\int_\Omega e^{\alpha m/d}p_\alpha dx > 0$, which is guaranteed for all large α by (H5). For each such α, we shall construct a pair of upper and lower solutions to show the existence of at least one positive solution for (2.3) (and hence for (2.1)). (See, e.g., [26], and also [3] for a discussion in the time-periodic framework.) First, take $\bar{v} = M$ for some large constant M; then,

$$\begin{cases} \tilde{L}\bar{v} = e^{\alpha m/d}M(p_\alpha - e^{\alpha m/d}M) < 0 & \text{in } \Omega, \\ \frac{\partial \bar{v}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

That means $\bar{v}$ is an upper solution of (2.3). For the lower solution, consider the following eigenvalue problem for μ:

$$\begin{cases} \nabla\cdot(de^{\alpha m/d}\nabla \psi) + e^{\alpha m/d}p_\alpha\psi + \mu e^{\alpha m/d}\psi = 0 & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Now the principal eigenvalue μ_1 of (2.5) is given by

$$\mu_1 = \inf_{\psi\in H^1}\left\{\frac{\int_\Omega e^{\alpha m/d}(d|\nabla\psi|^2 - p_\alpha\psi^2)\,dx}{\int_\Omega e^{\alpha m/d}\psi^2\,dx}\right\}.$$

By considering the test function $\psi \equiv 1$, we have $\mu_1 < 0$ by our choice of α. Denote the eigenfunction corresponding to μ_1 by ψ_1. We may assume $\psi_1 > 0$ and $|\psi_1|_{L^\infty(\Omega)} = 1$. Then $\underline{v} = \epsilon\psi_1$ satisfies, for $\epsilon > 0$ sufficiently small,

$$\begin{cases} \tilde{L}\underline{v} = e^{\alpha m/d}\epsilon\psi_1(-\mu_1(1) - e^{\alpha m/d}\epsilon\psi_1) > 0 & \text{in } \Omega, \\ \frac{\partial \bar{v}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Thus $\underline{v} = \epsilon\psi_1 > 0$ is a lower solution of (2.3). By the method of upper and lower solutions, (2.3) has at least one positive solution v_α. Now by Proposition 3.3 in [6] and the discussion before it, (2.3) has at most one solution, and the unique positive solution thus obtained is globally asymptotically stable among all nonnegative, nontrivial solutions of the corresponding parabolic equation. Therefore, the uniqueness and global asymptotic stability of u_α are proved. By standard elliptic regularity theory, v_α, and hence $u_\alpha = e^{\alpha m/d}v_\alpha$, is in $C^2(\bar{\Omega})$. This proves part (i) of Theorem 2.1. ∎

Having established the existence of u_α, we prove Lemma 2.2.

Proof of Lemma 2.2. (i) follows directly from the maximum principle. (We do not need $\frac{\partial m}{\partial \nu} \le 0$ here since we can transform the equation by $w = e^{-\alpha m/d}u$ as before.)

For (ii) and (iii), we first assume $m(x) > 0$ and $m(x) \geq p_\alpha(x)$ for all $x \in \Omega$ and α. (This assumption will be removed later on.) Let $\tilde{u}$ be the unique positive solution of

$$\begin{cases} \nabla\cdot(d\nabla\tilde{u} - \alpha\tilde{u}\nabla m) + (m-\tilde{u})\tilde{u} = 0 & \text{in } \Omega, \\ d\frac{\partial\tilde{u}}{\partial\nu} - \alpha\tilde{u}\frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then,

$$\begin{cases} \nabla\cdot(d\nabla\tilde{u} - \alpha\tilde{u}\nabla m) + (p_\alpha-\tilde{u})\tilde{u} = (p_\alpha - m)\tilde{u} \leq 0 & \text{in } \Omega, \\ d\frac{\partial\tilde{u}}{\partial\nu} - \alpha\tilde{u}\frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $\tilde{u}$ is an upper solution of (2.3). By the uniqueness of u_α, we deduce by comparison that $u_\alpha \leq \tilde{u}$. Therefore (ii) and (iii) follow from the corresponding properties of $\tilde{u}$ proved in Theorem 1.5(i) of [8] and Theorem 1.5 of [19], respectively.

Finally, to remove the assumption $p_\alpha \leq m$, it suffices to replace m by $m + A$ for some large positive constant A and compare u_α with the unique solution $\tilde{u}$ of

$$\begin{cases} \nabla\cdot[d\nabla\tilde{u} - \alpha\tilde{u}\nabla(m+A)] + [(m+A)-\tilde{u}]\tilde{u} = 0 & \text{in } \Omega, \\ d\frac{\partial\tilde{u}}{\partial\nu} - \alpha\tilde{u}\frac{\partial(m+A)}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$ ∎

Next, we show that (1.4) has at least one coexistence steady state. (Note that p is independent of α in (1.4).)

Proof of Theorem 2.1(ii). By the transformation $W(x) = e^{-\alpha m(x)/d_1}U(x)$, (1.4) becomes

$$(2.7)\qquad \begin{cases} W_t = e^{-\alpha m(x)/d_1}\nabla\cdot(d_1 e^{\alpha m(x)/d_1}\nabla W) \\ \qquad +W(p - e^{\alpha m(x)/d_1}W - V) & \text{in } \Omega\times(0,\infty), \\ V_t = d_2\Delta V + V(p - e^{\alpha m(x)/d_1}W - V) & \text{in } \Omega\times(0,\infty), \\ \frac{\partial W}{\partial\nu} = \frac{\partial V}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases}$$

which is a monotone dynamical system. By the theory of monotone dynamical systems [17, 18, 23, 27], a sufficient condition for the existence of coexistence steady states is the instability of the semitrivial steady states $(e^{-\alpha m(x)/d_1}u_\alpha, 0)$ and $(0, \bar{\theta}_{d_2})$ of (2.7), which is equivalent to the instability of $(u_\alpha, 0)$ and $(0, \bar{\theta}_{d_2})$ of (1.4).

First, we consider the linear instability of $(u_\alpha, 0)$, determined by the following eigenvalue problem:

$$(2.8)\qquad \begin{cases} \nabla\cdot(d_1\nabla\phi - \alpha\phi\nabla m) + (p-2u_\alpha)\phi - u_\alpha\psi + \lambda\phi = 0 & \text{in } \Omega, \\ d_2\Delta\psi + (p-u_\alpha)\psi + \lambda\psi = 0 & \text{in } \Omega, \\ d_1\frac{\partial\phi}{\partial\nu} - \alpha\phi\frac{\partial m}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since (2.8) decouples, it suffices to show that the principal eigenvalue σ_1 of the second equation of (2.8),

$$(2.9)\qquad \begin{cases} d_2\Delta\psi_1 + (p-u_\alpha)\psi_1 + \sigma_1\psi_1 = 0 & \text{in } \Omega, \\ \frac{\partial\psi_1}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

is negative, where ψ_1 is the corresponding positive eigenfunction. But if we divide the equation by ψ_1 and integrate over Ω, we have

$$\sigma_1 = -d_2\int_\Omega \frac{|\psi_1|^2}{\psi_1^2}\,dx - \int_\Omega (p-u_\alpha)\,dx < -\int_\Omega (p-u_\alpha)\,dx \to -\int_\Omega p\,dx < 0$$

as $\alpha \to \infty$, since $\int_\Omega u_\alpha\, dx \to 0$ as $\alpha \to \infty$ by Lemma 2.2(ii). Therefore $(u_\alpha, 0)$ is unstable for α large.

Next we linearize (1.4) at $(0, \bar{\theta}_{d_2})$ and consider the following eigenvalue problem:

$$\begin{cases} \nabla(d_1\nabla\phi - \alpha\phi\nabla m) + (p - \bar{\theta}_{d_2})\phi + \lambda\phi = 0 & \text{in } \Omega, \\ d_2\Delta\psi - \bar{\theta}_{d_2}\phi + (p - 2\bar{\theta}_{d_2})\psi + \lambda\psi = 0 & \text{in } \Omega, \\ d_1\frac{\partial\phi}{\partial\nu} - \alpha\phi\frac{\partial m}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

To show that $(0, \bar{\theta}_{d_2})$ is unstable, again, it suffices to show that the principal eigenvalue ρ_1 of the first equation of (2.10),

$$\begin{cases} \nabla(d_1\nabla\phi - \alpha\phi\nabla m) + (p - \bar{\theta}_{d_2})\phi + \rho\phi = 0 & \text{in } \Omega, \\ d_1\frac{\partial\phi}{\partial\nu} - \alpha\phi\frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

is negative. By the transformation $\varphi = e^{-\alpha m/d_1}\phi$, (2.11) becomes

$$\begin{cases} \nabla(d_1 e^{\alpha m/d_1}\nabla\varphi) + e^{\alpha m/d_1}(p - \bar{\theta}_{d_2})\varphi + \rho e^{\alpha m/d_1}\varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore by variational characterization,

$$\rho_1 = \inf_{\varphi \in H^1(\Omega)} \left\{ \frac{\int_\Omega e^{\alpha m/d_1}[d_1|\nabla\varphi|^2 + (\bar{\theta}_{d_2} - p)\varphi^2]\, dx}{\int_\Omega e^{\alpha m/d_1}\varphi^2\, dx} \right\}.$$

By the maximum principle, we have $\sup_\Omega p > |\bar{\theta}_{d_2}|_{L^\infty(\Omega)}$. Now for $\delta > 0$ small, take a smooth cut-off function φ such that $0 \le \varphi \le 1$,

$$\varphi(x) = \begin{cases} 1 & \text{if } m(x) \ge \sup_\Omega m - 2\delta, \\ 0 & \text{if } m(x) \le \sup_\Omega m - 3\delta, \end{cases}$$

and that $\inf_{\operatorname{supp}\varphi} p > |\bar{\theta}_{d_2}|_{L^\infty(\Omega)}$ (by (H4)). Then $\varphi \not\equiv 0$ and

$$\begin{aligned} \rho_1 &\le \frac{\int_\Omega e^{\alpha m/d_1}[d_1|\nabla\varphi|^2 + (\bar{\theta}_{d_2} - p)\varphi^2]\, dx}{\int_\Omega e^{\alpha m/d_1}\varphi^2\, dx} \\ &\le C\frac{e^{\alpha(m(x_0) - 2\delta)/d_1}}{e^{\alpha(m(x_0) - \delta)/d_1}} + \sup_{\operatorname{supp}\varphi}(\bar{\theta}_{d_2} - p) \\ &\le Ce^{-\delta\alpha/d_1} + |\bar{\theta}_{d_2}|_{L^\infty(\Omega)} - \inf_{\operatorname{supp}\varphi} p \\ &\to |\bar{\theta}_{d_2}|_{L^\infty(\Omega)} - \inf_{\operatorname{supp}\varphi} p < 0 \qquad \text{as } \alpha \to \infty. \end{aligned}$$

Therefore the principal eigenvalue ρ_1 of (2.11) is negative for all α large. Hence $(0, \bar{\theta}_{d_2})$ is unstable for α large. This completes the proof. □

Theorem 1.5 follows from the instability of the two semitrivial steady states $(u_\alpha, 0)$ and $(0, \bar{\theta}_{d_2})$ and a direct application of the theory of monotone dynamical systems. We omit the details of the proof here.

3. L^∞ estimate.

Assume (H1) and (H5). By Theorem 2.1, there exists a unique positive solution u_α of (2.1). The main theorem in this section is as follows.

THEOREM 3.1. *Let u_α be the unique positive solution of* (2.1)*, and assume further* (H2) *and* (H3)*. Then there exists a constant $C > 0$ independent of α such that $|u_\alpha|_{L^\infty(\Omega)} \le C$.*

Remark 3.2. By Lemma 2.2(i), we have only $|u_\alpha|_{L^\infty(\Omega)} \leq O(\alpha)$. The above theorem was first proved in [12] for the special case when m has a unique critical point in $\bar{\Omega}$ via the maximum principle.

Throughout the rest of this paper fix $r_1 > 0$ sufficiently small so that (i) if $x_0, \tilde{x}_0$ are two distinct points in $\mathfrak{M}$, then $B_{r_1}(x_0) \cap B_{r_1}(\tilde{x}_0) = \emptyset$; and (ii) there exist constants $c_1, c_2 > 0$ (by the nondegeneracy of m on $\mathfrak{M}$) such that for all $x_0 \in \mathfrak{M}$, it holds that

$$\begin{cases} c_1|x-x_0| \leq |\nabla m(x)| \leq c_2|x-x_0| & \text{for all } x \in B_{r_1}(x_0), \\ c_1|x-x_0|^2 \leq m(x_0) - m(x) \leq c_2|x-x_0|^2 & \text{for all } x \in B_{r_1}(x_0), \\ c_1|\xi|^2 \leq -\xi^T D^2 m(x_0)\xi \leq c_2|\xi|^2 & \text{for all } \xi \in \mathbf{R}^N. \end{cases} \tag{3.1}$$

DEFINITION 3.3. *For each* $x_0 \in \mathfrak{M}$, $R > 0$, *and* $\alpha > R^2 d/r_1^2$, *define* $\Gamma_{\alpha,R}(x_0) := B_{r_1}(x_0) \setminus \bar{B}_{R\sqrt{d/\alpha}}(x_0)$.

LEMMA 3.4. *For each* $\epsilon \in (0,1)$, *let* $v(x) := e^{-\epsilon\alpha[m(x)-m(x_0)]/d}u_\alpha(x)$. *Then for some* $R_0 = R_0(\epsilon)$, v *satisfies a weak maximum principle in* $\bigcup_{x_0\in\mathfrak{M}} \Gamma_{\alpha,R_0}(x_0)$ *for all* α *large; i.e., for each* $x_0 \in \mathfrak{M}$,

$$\sup_{\Gamma_{\alpha,R_0}(x_0)} v = \sup_{\partial\Gamma_{\alpha,R_0}(x_0)} v \qquad \textit{for all } \alpha \textit{ large.}$$

Proof of Lemma 3.4. v satisfies the equation

$$d\Delta v + (2\epsilon - 1)\alpha\nabla m \cdot \nabla v + v[p_\alpha - u_\alpha + (\epsilon-1)\alpha\Delta m + \epsilon(\epsilon-1)\alpha^2|\nabla m|^2/d] = 0 \text{ in } \Omega.$$

CLAIM 3.5. *There exists* $C_1 > 0$ *such that* $|p_\alpha - u_\alpha + (\epsilon-1)\alpha\Delta m|_{L^\infty(\Omega)} \leq C_1\alpha$ *for all* α *large.*

The claim follows from Lemma 2.2(i) and (2.2).

CLAIM 3.6. *There exists* $R_0 > 0$ *and* α_0 *such that* $\epsilon(\epsilon-1)\alpha^2|\nabla m|^2 \leq -C_1\alpha$ *for all* $x \in \Gamma_{\alpha,R_0}$ *and* $\alpha \geq \alpha_0$, *where* C_1 *is the constant determined in Claim* 3.5.

Claim 3.6 follows from the nondegeneracy of m at x_0. More precisely, let $x \in \Gamma_{\alpha,R}(x_0)$; then by (3.1),

$$\begin{aligned} &\left|\epsilon(\epsilon-1)\alpha^2|\nabla m|^2\right| \\ &\quad \geq |\epsilon(\epsilon-1)\alpha^2|\, c_1^2\, |x-x_0|^2 \\ &\quad \geq |\epsilon(\epsilon-1)\alpha^2|\, c_1^2 \left(R\sqrt{\frac{d}{\alpha}}\right)^2 \\ &\quad = |\epsilon(\epsilon-1)|\, c_1^2\, R^2\alpha d \geq C_1\alpha \qquad \text{if } R \text{ is chosen large.} \end{aligned}$$

By the above claims, $d\Delta v + (2\epsilon-1)\alpha\nabla m \cdot \nabla v \geq 0$ in $\Gamma_{\alpha,R_0}(x_0)$, and the lemma follows from the classical maximum principle. ∎

LEMMA 3.7. *There exists* $\epsilon, C > 0$ *such that for all* α *large,*

$$u_\alpha(x) \leq C(|u_\alpha|_{L^\infty(\Omega)} + 1)e^{\epsilon\alpha[m(x)-m(x_0)]/d} \qquad \textit{in } \Gamma_{\alpha,R_0}(x_0).$$

where R_0 *is defined in Claim* 3.6.

Proof. By the weak maximum principle, for any $x \in \Gamma_{\alpha,R_0}(x_0)$,

$$\begin{aligned}
&u_\alpha(x)e^{-\epsilon\alpha[m(x)-m(x_0)]/d} \\
&\quad\le \sup_{\partial\Gamma_{\alpha,R_0}(x_0)} u_\alpha(x)e^{-\epsilon\alpha[m(x)-m(x_0)]/d} \\
&\quad\le \max\left\{\sup_{\partial B_{r_1}(x_0)} u_\alpha(x)e^{-\epsilon\alpha[m(x)-m(x_0)]/d}, \sup_{\partial B_{R_0\sqrt{d/\alpha}}(x_0)} u_\alpha(x)e^{-\epsilon\alpha[m(x)-m(x_0)]/d}\right\} \\
&\quad:= \max\{I_1, I_2\}.
\end{aligned}$$

By Lemma 2.2(iii) (taking $K = \Omega\setminus[\cup_{x_0\in\mathfrak{M}} B_{r_1}(x_0)]$), if we choose $\epsilon > 0$ such that $\epsilon < d\epsilon'/(2|m|_\infty)$ where ϵ' is as defined in Lemma 2.2(iii), then $I_1 \le 1$, whereas for I_2, by (3.1) again,

$$I_2 = \sup_{\partial B_{R_0\sqrt{d/\alpha}}(x_0)} u_\alpha(x)e^{-\epsilon\alpha[m(x)-m(x_0)]/d} \le |u_\alpha|_{L^\infty(\Omega)} e^{\epsilon c_2 R_0^2}. \qquad \square$$

Proof of Theorem 3.1. We first claim the following.

CLAIM 3.8. *For each $R > 0$ and each $x_0 \in \mathfrak{M}$, there exists $C > 0$ such that*

$$\sup_{B_{R\sqrt{d/\alpha}}(x_0)} u_\alpha \le C \inf_{B_{R\sqrt{d/\alpha}}(x_0)} u_\alpha \quad \textit{for all } \alpha \textit{ large.}$$

The claim follows from the fact that $\tilde{u}(y) := u_\alpha(x_0 + \sqrt{\frac{d}{\alpha}}y)$ satisfies

$$\Delta_y \tilde{u} + P\cdot\nabla_y\tilde{u} + Q\tilde{u} = 0, \tag{3.2}$$

where

$$P(y) = \sqrt{\frac{\alpha}{d}}\nabla_x m\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) \to y^T\cdot D_x^2 m(x_0)$$

and

$$Q(y) = \left[p_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) - u_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) - \alpha\Delta_x m\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right)\right]\Big/\alpha$$

are bounded uniformly for $y \in B_{4R}(0)$ and all α large (by Lemma 2.2(i)). Therefore the claim is a consequence of the following classical Harnack inequality. (See Theorem 8.20 in [15] and a remark after it.)

THEOREM 3.9 (see [15]). *If $w \in W^{1,2}(\Omega)$ satisfies*

$$\begin{cases} D_i(a_{ij}D_j w) + b_i D_i w + cw = 0 & \text{in } \Omega, \\ w \ge 0 & \text{in } \Omega, \end{cases}$$

then for any ball $B_{4R}(y) \in \Omega$, we have

$$\sup_{B_R(y)} w \le C \inf_{B_R(y)} w,$$

where $C \le C_0^{K\log K}$, $C_0 = C_0(N)$, $K = \Lambda/\lambda + \nu R$, $\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2$, and $\nu^2 = (|b|_{L^\infty(B_{4R})}/\lambda)^2 + |c|_{L^\infty(B_{4R})}/\lambda$.

We proceed to show that $|u_\alpha|_{L^\infty(\Omega)}$ is bounded independent of α. Assume for contradiction that $|u_\alpha|_{L^\infty(\Omega)} \to \infty$ as $\alpha \to \infty$. In view of Lemmas 2.2(iii) and 3.7, we must have $|u_\alpha|_{L^\infty(\Omega)} = \sup_{B_{R_1\sqrt{d/\alpha}}(\bar{x}_0)} u_\alpha$ for some $R_1 \geq R_0$ and some $\bar{x}_0 \in \mathfrak{M}$. ($R_1 \geq R_0$ is chosen independent of α, while $\bar{x}_0 \in \mathfrak{M}$ might depend on α.) By Claim 3.8,

$$u_\alpha(x) \geq C|u_\alpha|_{L^\infty(\Omega)} \qquad \text{for all } x \in B_{R_1\sqrt{d/\alpha}}(\bar{x}_0).$$

Therefore,

$$\int_\Omega u_\alpha^2\,dx \geq C\,\alpha^{-N/2}\,|u_\alpha|^2_{L^\infty(\Omega)}. \tag{3.3}$$

On the other hand, by Lemmas 2.2(iii) and 3.7, as well as (2.2),

$$\begin{aligned}
\int_\Omega p_\alpha u_\alpha\,dx &\leq \left\{\int_{\Omega\setminus\bigcup_{\mathfrak{M}} B_{r_1}(x_0)} + \int_{\bigcup_{\mathfrak{M}} \Gamma_{\alpha,R_1}(x_0)} + \int_{\bigcup_{\mathfrak{M}} B_{R_1\sqrt{d/\alpha}}}\right\} p_\alpha u_\alpha\,dx\\
&\leq O(e^{-\epsilon'\alpha}) + C(|u_\alpha|_{L^\infty(\Omega)}+1)\int_{\bigcup_{\mathfrak{M}} \Gamma_{\alpha,R_1}(x_0)} e^{\epsilon\alpha[m(x)-m(x_0)]/d}\,dx\\
&\qquad + C\,|B_{R_1\sqrt{d/\alpha}}||u_\alpha|_{L^\infty(\Omega)}\\
&\leq O(e^{-\epsilon'\alpha}) + C(|u_\alpha|_{L^\infty(\Omega)}+1)\int_{\bigcup_{\mathfrak{M}} B_{r_1}(x_0)} e^{-\epsilon c_1\alpha|x-x_0|^2/d}\,dx\\
&\qquad + C\,|B_{R_1\sqrt{d/\alpha}}||u_\alpha|_{L^\infty(\Omega)}\\
&\leq O(e^{-\epsilon'\alpha}) + C\,\alpha^{-N/2}\,(|u_\alpha|_{L^\infty(\Omega)}+1).
\end{aligned}$$

Now by integrating (2.1) over Ω, we have

$$\int_\Omega u_\alpha^2\,dx = \int_\Omega p_\alpha u_\alpha\,dx. \tag{3.4}$$

Combining, we have

$$C\alpha^{-N/2}|u_\alpha|^2_{L^\infty(\Omega)} \leq C\alpha^{-N/2}(|u_\alpha|_{L^\infty(\Omega)}+1) + O(e^{-\epsilon'\alpha}),$$

$$|u_\alpha|^2_{L^\infty(\Omega)} \leq C|u_\alpha|_{L^\infty(\Omega)} + O(1).$$

But this contradicts the fact that $|u_\alpha|_{L^\infty(\Omega)} \to \infty$, and we arrive at a contradiction. This proves Theorem 3.1. ∎

4. Proof of Theorem 1.6. Now we are in position to prove Theorem 1.6. In fact, for later purposes, we are going to establish the following result for the more general equation (2.1).

THEOREM 4.1. *In addition to the assumptions of Theorem* 1.6, *assume* (2.2). *Then for all α sufficiently large,* (2.1) *has a unique positive solution u_α. u_α is globally asymptotically stable, and for each small $r > 0$, $u_\alpha(x) \to 0$ uniformly and exponentially in $\Omega \setminus \cup_{x_0\in\mathfrak{M}} B_r(x_0)$. Moreover, for each $x_0 \in \mathfrak{M}$,*

$$u_\alpha(x) - 2^{N/2}\max\{p(x_0),0\}e^{\alpha[m(x)-m(x_0)]/d} \to 0 \tag{4.1}$$

uniformly in $B_r(x_0)$.

It is easy to see that Theorem 1.6 is a special case of Theorem 4.1. We first apply Proposition 1.8 to obtain the limiting profile of u_α.

PROPOSITION 4.2. *For each $R > 0$ and each $x_0 \in \mathfrak{M}$,*

$$|u_\alpha(x)e^{\alpha[m(x_0)-m(x)]/d} - u_\alpha(x_0)|_{L^\infty(B_{R\sqrt{d/\alpha}}(x_0))} \to 0 \qquad \text{as } \alpha \to \infty. \tag{4.2}$$

Proof. Since $\alpha[m(x_0 + \sqrt{d/\alpha}y) - m(x_0)]/d \to \frac{1}{2}y^T D^2 m(x_0)y$ uniformly on compact subsets of $\mathbf{R}^N$, it suffice to show that for each $x_0 \in \mathfrak{M}$,

$$u_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right)e^{-\frac{1}{2}y^T D^2 m(x_0)y} - u_\alpha(x_0) \to 0 \quad \text{as } \alpha \to \infty$$

uniformly on every compact subset of $\mathbf{R}^N$. Now let

$$w_\alpha(y) = u_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right)e^{-\frac{1}{2}y^T D^2 m(x_0)y}.$$

Then w_α satisfies the equation

$$\Delta_y w + \tilde{P}\cdot\nabla_y w + \tilde{Q}w = 0 \quad \text{in } \sqrt{\frac{\alpha}{d}}(\Omega - x_0),$$

where

$$\tilde{P} = 2y^T D_x^2 m(x_0) - \sqrt{\frac{\alpha}{d}}\nabla_x m\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right),$$

$$\begin{aligned}\tilde{Q} = \Delta_x m(x_0) &- \Delta_x m\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) + |D_x^2 m(x_0)y|^2 \\ &- y^T D_x^2 m(x_0)\cdot\sqrt{\frac{\alpha}{d}}\nabla_x m\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) \\ &- \left[u_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) - p_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right)\right]\Big/\alpha.\end{aligned}$$

By the L^∞ estimate of u_α (Theorem 3.1) and the fact that $\sqrt{\frac{\alpha}{d}}\nabla_x m(x_0+\sqrt{\frac{d}{\alpha}}y) \to D_x^2 m(x_0)y$ uniformly on compact sets of $\mathbf{R}^N$, we have $\lim_{\alpha\to\infty}\tilde{P} = y^T D_x^2 m(x_0)$ and $\lim_{\alpha\to\infty}\tilde{Q} = 0$ uniformly in compact subsets of $\mathbf{R}^N$ as $\alpha \to \infty$. Hence by elliptic L^p estimates (by Theorem 3.1 again, w_α is bounded in $L^\infty(K)$ uniformly in α for each compact subset K in $\mathbf{R}^N$), after passing to a subsequence if necessary, w_α converges to some limit w_0 uniformly in every compact subset of $\mathbf{R}^N$. Thus w_0 satisfies

$$\nabla\cdot(e^{\frac{1}{2}y^T D_x^2 m(x_0)y}\nabla w_0) = 0 \text{ in } \mathbf{R}^N, \quad w_0(0) < \infty, \quad w_0(y) \geq 0 \text{ in } \mathbf{R}^N,$$

which must be a constant by Proposition 1.8. Now if for some subsequence $\alpha_k \to \infty$, $u_{\alpha_k}(x_0) = w_{\alpha_k}(0)$ converges as $k \to \infty$, then $w_{\alpha_k}(x) - u_{\alpha_k}(x_0) \to 0$ uniformly on compact subsets of $\mathbf{R}^N$. The convergence of the full sequence now follows from the uniqueness of the limit. ∎

To obtain the complete profile of u_α, it suffices to calculate the exact limit of $u_\alpha(x_0)$. We shall prove the following in a series of lemmas.

PROPOSITION 4.3. *For each $x_0 \in \mathfrak{M}$, $\lim_{\alpha\to\infty} u_\alpha(x_0) = \max\{2^{N/2}p(x_0), 0\}$, where $p(x) = \lim_{\alpha\to\infty} p_\alpha(x)$.*

LEMMA 4.4. *Given $\delta > 0$ small,*

$$\left|\sum_{x_0\in\mathfrak{M}} \int_{B_R(0)} u_\alpha^2\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) - p_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) u_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) dy\right| < \delta$$

for all R, α sufficiently large.

Proof. By Lemma 2.2(iii) and (3.4),

$$\sum_{x_0\in\mathfrak{M}} \int_{B_{r_1}(x_0)} u_\alpha^2 - p_\alpha u_\alpha\, dx = O(e^{-\epsilon'\alpha}),$$

$$\left|\sum_{x_0\in\mathfrak{M}} \int_{B_{R\sqrt{d/\alpha}}(x_0)} u_\alpha^2 - p_\alpha u_\alpha\, dx\right| \le \left|\sum_{x_0\in\mathfrak{M}} \int_{\Gamma_{\alpha,R}(x_0)} u_\alpha^2 - p_\alpha u_\alpha\, dx\right| + O(e^{-\epsilon'\alpha}),$$

where ϵ' is as defined in Lemma 2.2(iii). Multiplying by $(\alpha/d)^{N/2}$ and changing coordinates $x = x_0 + \sqrt{\frac{d}{\alpha}}y$, we have, for α sufficiently large,

$$\begin{aligned}
&\left|\sum_{x_0\in\mathfrak{M}} \int_{B_R(0)} u_\alpha^2\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) - p_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) u_\alpha\left(x_0 + \sqrt{\frac{d}{\alpha}}y\right) dy\right| \\
&\quad \le C\alpha^{N/2} \sum_{x_0\in\mathfrak{M}} \int_{\Gamma_{\alpha,R}(x_0)} |u_\alpha^2 - p_\alpha u_\alpha|\, dx + O(\alpha^{N/2}e^{-\epsilon'\alpha}) \\
&\le C\alpha^{N/2} \sum_{x_0\in\mathfrak{M}} \int_{\Gamma_{\alpha,R}(x_0)} e^{\epsilon\alpha[m(x)-m(x_0)]/d} dx + O(\alpha^{N/2}e^{-\epsilon'\alpha}) \\
&\le C \int_{\mathbf{R}^N\setminus B_R(0)} e^{-\epsilon c_1|y|^2} dy + O(\alpha^{N/2}e^{-\epsilon'\alpha}) \\
&< \delta
\end{aligned}$$

if α, R are sufficiently large. The second inequality follows from Lemma 3.7, and the third inequality follows from (3.1). ◻

LEMMA 4.5.

$$\lim_{\alpha\to\infty} \sum_{x_0\in\mathfrak{M}} \int_{\boldsymbol{R}^N} e^{\frac{1}{2}y^T D^2 m(x_0) y} dy \left[u_\alpha^2(x_0) - 2^{N/2}p_\alpha(x_0)u_\alpha(x_0)\right] = 0.$$

Proof. Since $|u|_{L^\infty(\Omega)}$ is uniformly bounded in α, by compactness of bounded sequences in $\mathbf{R}$ it suffices to show that for any $\delta > 0$ and for any sequence $\alpha_k \to \infty$ such that the associated $\lim_{k\to\infty} u_{\alpha_k}(x_0)$ converges at each $x_0 \in \mathfrak{M}$, it holds that (writing $\alpha = \alpha_k$)

$$\left|\sum_{x_0\in\mathfrak{M}} \int_{\mathbf{R}^N} e^{\frac{1}{2}y^T D^2 m(x_0) y} dy [2^{-N/2}u_\alpha^2(x_0) - p_\alpha(x_0)u_\alpha(x_0)]\right| < 4\delta$$

for all $\alpha = \alpha_k$ large. Now,

$$\begin{aligned}
&\left| \sum_{x_0 \in \mathfrak{M}} \int_{\mathbf{R}^N} e^{\frac{1}{2} y^T D^2 m(x_0) y} dy [2^{-N/2} u_\alpha^2(x_0) - p_\alpha(x_0) u_\alpha(x_0)] \right| \\
&\quad = \left| \sum_{x_0 \in \mathfrak{M}} \int_{\mathbf{R}^N} \left[e^{y^T D^2 m(x_0) y} u_\alpha^2(x_0) - e^{\frac{1}{2} y^T D^2 m(x_0) y} p_\alpha(x_0) u_\alpha(x_0) \right] dy \right| \\
&\quad \le \left| \sum_{x_0 \in \mathfrak{M}} \int_{B_R(0)} \left[e^{y^T D^2 m(x_0) y} u_\alpha^2(x_0) - e^{\frac{1}{2} y^T D^2 m(x_0) y} p_\alpha(x_0) u_\alpha(x_0) \right] dy \right| + \delta \\
&\quad \le \sum_{x_0 \in \mathfrak{M}} \left\{ \int_{B_R} \left| e^{y^T D^2 m(x_0) y} u_\alpha^2(x_0) - u_\alpha^2 \left(x_0 + \sqrt{\frac{d}{\alpha}} y \right) \right| dy \right. \\
&\qquad\qquad + \int_{B_R(0)} \left| u_\alpha^2 \left(x_0 + \sqrt{\frac{d}{\alpha}} y \right) - p_\alpha(x_0) u_\alpha \left(x_0 + \sqrt{\frac{d}{\alpha}} y \right) \right| dy \\
&\qquad\qquad \left. + |p_\alpha(x_0)| \int_{B_R} \left| u_\alpha \left(x_0 + \sqrt{\frac{d}{\alpha}} y \right) - e^{\frac{1}{2} y^T D^2 m(x_0) y} u_\alpha(x_0) \right| dy \right\} + \delta \\
&\quad < 4\delta.
\end{aligned}$$

The first inequality holds by fixing $R = R(\delta)$ large. The strict inequality follows from Proposition 4.2 and Lemma 4.4. ∎

LEMMA 4.6. *For each* $x_0 \in \mathfrak{M}$, $\liminf_{\alpha \to \infty} u_\alpha(x_0) \ge 2^{N/2} \max\{p(x_0), 0\}$.

Proof. Fix $x_0 \in \mathfrak{M}$. If $p(x_0) \le 0$, there is nothing to prove. Now let $p(x_0) > 0$, and let u_α be the unique solution to (2.1). For each α large, u_α is the principal eigenfunction of the following eigenvalue problem with principal eigenvalue $\mu_1 = 0$:

$$(4.3) \qquad \begin{cases} \nabla \cdot (d\nabla\phi - \alpha\phi\nabla m) + (p_\alpha - u_\alpha)\phi + \mu\phi = 0 & \text{in } \Omega, \\ d\frac{\partial\phi}{\partial\nu} - \alpha\phi\frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By the transformation $\phi = e^{\alpha m/d}\psi$, (4.3) is equivalent to the self-adjoint problem

$$(4.4) \qquad \begin{cases} \nabla \cdot (de^{\alpha m/d}\nabla\psi) + (p_\alpha - u_\alpha)e^{\alpha m/d}\psi + \mu e^{\alpha m/d}\psi = 0 & \text{in } \Omega, \\ \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

The variational characterization of problem (4.4) implies

$$0 = \inf_{\psi \in H^1(\Omega)} \left\{ \frac{\int_\Omega e^{\alpha m/d}[d|\nabla\psi|^2 + (u_\alpha - p_\alpha)\psi^2]dx}{\int_\Omega e^{\alpha m/d}\psi^2 dx} \right\}.$$

For any $B_{r_1}(x_0)$ with $r_1 > 0$ small and $0 < \zeta < 1/2$, by the nondegeneracy of $m(x)$ at x_0, we have

$$m(x_0) > \max_{\bar{B}_{r_1}(x_0) \setminus B_{(1-\zeta)r_1}(x_0)} m := M_1.$$

Now take $\zeta > 0$ even smaller such that $M_2 := \min_{\bar{B}_{\zeta r_1}(x_0)} m > M_1$, which is possible by (3.1). Take a smooth test function ψ such that

$$\psi(x) = \begin{cases} 1 \text{ if } x \in B_{(1-\zeta)r_1}(x_0), \\ 0 \text{ if } x \in \mathbf{R}^N \setminus B_{r_1}(x_0), \end{cases} \quad 0 \le \psi(x) \le 1, \; |\nabla\psi(x)| < \frac{2}{\zeta r_1}.$$

Then,

$$\begin{aligned} 0 &\le \frac{\int_\Omega e^{\alpha m/d}[d|\nabla\psi|^2 + (u_\alpha - p_\alpha)\psi^2]dx}{\int_\Omega e^{\alpha m/d}\psi^2 dx} \\ &\le \frac{\int_{B_{r_1}(x_0)} de^{\alpha M_1/d}(\frac{2}{\zeta r_1})^2\,dx}{\int_{B_{\zeta r_1}(x_0)} e^{\alpha M_2/d}\,dx} + \frac{\int_{B_{r_1}(x_0)} e^{\alpha m/d}(u_\alpha - p_\alpha)\psi^2\,dx}{\int_{B_{r_1}(x_0)} e^{\alpha m/d}\psi^2\,dx} \\ &\le \frac{|B_{r_1}|}{|B_{\zeta r_1}|}\frac{4d}{(\zeta r_1)^2}e^{\alpha(M_1-M_2)/d} + \frac{\int_{B_{r_1}(x_0)} e^{\alpha[m-m(x_0)]/d}(u_\alpha - p_\alpha)\,dx}{\int_{B_{(1-\zeta)r_1}(x_0)} e^{\alpha[m-m(x_0)]/d}\,dx}. \end{aligned}$$

This implies that

$$\liminf_{\alpha\to\infty} \frac{\int_{B_{r_1}(x_0)} e^{\alpha[m(x)-m(x_0)]/d}[u_\alpha(x) - p_\alpha(x)]dx}{\int_{B_{(1-\zeta)r_1}(x_0)} e^{\alpha[m(x)-m(x_0)]/d}\,dx} \ge 0. \tag{4.5}$$

By Lebesgue's dominated convergence and (2.2),

$$\lim_{\alpha\to\infty} \frac{\int_{B_{r_1}(x_0)} e^{\alpha[m(x)-m(x_0)]/d}p_\alpha(x)\,dx}{\int_{B_{(1-\zeta)r_1}(x_0)} e^{\alpha[m(x)-m(x_0)]/d}\,dx} = p(x_0).$$

By Proposition 4.2 and Lemma 3.7, for each $R \ge R_0$ and $\eta > 0$, for all α large, we have

$$u_\alpha \le \begin{cases} (1+\eta)u_\alpha(x_0)e^{\alpha[m(x)-m(x_0)]/d} & \text{when } |x-x_0| \le R\sqrt{d/\alpha}, \\ Ce^{\epsilon\alpha[m(x)-m(x_0)]/d} & \text{when } x \in \Gamma_{\alpha,R}(x_0), \end{cases}$$

where ϵ is as given in the statement of Lemma 3.7. Therefore, for any $\eta > 0$ small,

$$\begin{aligned} 0 &\le \liminf_{\alpha\to\infty} \frac{\int_{B_r(x_0)} e^{\frac{\alpha}{d}[m-m(x_0)]}(u_\alpha - p_\alpha)dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\frac{\alpha}{d}[m-m(x_0)]}dx} \\ &\le \liminf_{\alpha\to\infty} \left[(1+\eta)u_\alpha(x_0) \frac{\int_{B_{R\sqrt{d/\alpha}}(x_0)} e^{\frac{2\alpha}{d}[m-m(x_0)]}dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\frac{\alpha}{d}[m-m(x_0)]}dx} \right. \\ &\quad \left. + C\frac{\int_{\Gamma_{\alpha,R}(x_0)} e^{\frac{\epsilon\alpha}{d}[m-m(x_0)]}dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\frac{\alpha}{d}[m-m(x_0)]}dx} \right] - p(x_0) \\ &\le (1+\eta)\liminf_{\alpha\to\infty} \left[u_\alpha(x_0) \frac{\int_{B_{(1-\zeta)r}(x_0)} e^{\frac{2\alpha}{d}[m-m(x_0)]}dx}{\int_{B_{(1-\zeta)r}(x_0)} e^{\frac{\alpha}{d}[m-m(x_0)]}dx} \right] \\ &\quad + C\frac{\int_{\mathbf{R}^N\setminus B_R} e^{-\epsilon c_1|y|^2}dy}{\int_{B_1} e^{-c_2|y|^2}dy} - p(x_0) \end{aligned}$$

where c_1, c_2 are given as in (3.1). Since

$$\lim_{\alpha\to\infty} \frac{\int_{B_{(1-\zeta)r_1}(x_0)} e^{2\alpha[m-m(x_0)]/d}dx}{\int_{B_{(1-\zeta)r_1}(x_0)} e^{\alpha[m-m(x_0)]/d}dx} = 2^{-N/2},$$

taking $R \to \infty$ and $\eta \to 0^+$, we have

$$2^{-N/2} \liminf_{\alpha\to\infty} u_\alpha(x_0) \geq p(x_0). \qquad \square$$

Proposition 4.3 follows from Lemmas 4.5 and 4.6.

Proof of Theorem 4.1. The existence, uniqueness, and global stability are proved in Theorem 2.1. By Lemma 2.2(iii), we have $u_\alpha \to 0$ uniformly and exponentially in any compact subset of $\Omega \setminus \mathfrak{M}$. Finally, (4.1) follows from Lemma 3.7 and Propositions 4.2 and 4.3. $\square$

5. Proof of Theorem 1.3. Assume (H1), (H2), (H3), and (H4). Consider the corresponding system of steady-state equations of (1.4),

$$\begin{cases} \nabla\cdot(d_1\nabla U - \alpha U\nabla m) + U(p-U-V) = 0 & \text{in } \Omega, \\ d_2\Delta V + V(p-U-V) = 0 & \text{in } \Omega, \\ d_1\frac{\partial U}{\partial \nu} - \alpha U\frac{\partial m}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

LEMMA 5.1. *Let* (U_α, V_α) *be a positive solution of* (5.1) *(Theorem* 2.1(ii)*). Then*

$$0 < U_\alpha \leq u_\alpha, \quad 0 < V_\alpha \leq \bar\theta_{d_2},$$

where u_α *is the unique positive steady state of* (1.6) *and* $\bar\theta_{d_2}$ *is the unique positive solution of* (1.5).

Proof. U_α satisfies

$$\begin{cases} \nabla\cdot(d_1\nabla U - \alpha U\nabla m) + U(p-U) = UV_\alpha \geq 0 & \text{in } \Omega, \\ d_1\frac{\partial U}{\partial \nu} - \alpha U\frac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

and is a lower solution of (1.6). Hence $U_\alpha \leq u_\alpha$ by the uniqueness of the steady state u_α. The inequality $V_\alpha \leq \bar\theta_{d_2}$ holds for similar reasons. $\square$

LEMMA 5.2. *Let* u_α *be the unique positive steady state of* (1.6)*; then for any* $p \geq 1$,

$$\lim_{\alpha\to\infty} |u_\alpha|_{L^p(\Omega)} = 0.$$

In particular, by Lemma 5.1, if (U_α, V_α) is any positive solution of (5.1), then for any $p \geq 1$,

$$\lim_{\alpha\to\infty} |U_\alpha|_{L^p(\Omega)} = 0. \tag{5.2}$$

Proof. Let u_α be the unique positive steady state of (1.6). By Lemmas 2.2(iii) and 3.7, for some $R_0 > 0$, we have

$$u_\alpha \leq \begin{cases} e^{-\epsilon'\alpha} & \text{in } \Omega \setminus [\cup_{x_0\in\mathfrak{M}} B_{r_1}(x_0)], \\ C(|u_\alpha|_{L^\infty(\Omega)} + 1)e^{\epsilon\alpha[m(x)-m(x_0)]/d_1} & \text{in } \Gamma_{\alpha,R_0}(x_0),\ x_0 \in \mathfrak{M}, \\ |u_\alpha|_{L^\infty(\Omega)} & \text{in } B_{R_0\sqrt{d_1/\alpha}}(x_0),\ x_0 \in \mathfrak{M}. \end{cases} \tag{5.3}$$

Since $|u_\alpha|_{L^\infty}$ is bounded uniformly in α (Theorem 3.1), $|u_\alpha|_{L^p} \to 0$. Alternatively, one can use interpolation by noting that u_α is bounded uniformly in $L^\infty(\Omega)$ and approaches zero in $L^2(\Omega)$. $\square$

LEMMA 5.3. *Let (U_α, V_α) be a positive solution of* (5.1)*; then for any $\beta \in (0,1)$,*

$$\lim_{\alpha\to\infty} |V_\alpha - \bar\theta_{d_2}|_{C^{1,\beta}(\bar\Omega)} = 0.$$

Proof. Let $\overline{U} \in C^\beta(\bar\Omega)$, and $\alpha_0 > 0$ be fixed such that $0 < U_\alpha \le u_\alpha \le \overline{U}$ for all $\alpha \ge \alpha_0$ and $\int_\Omega (p - \overline{U}) > 0$. (The existence of $\overline{U}$ and α_0 follows from (5.3), the boundedness of $|u_\alpha|_{L^\infty}$, and the fact that $\int_\Omega p > 0$ by (H4).) Let $\underline{V}$ be the unique positive solution to

$$\begin{cases} d_2\Delta V + V(p - \overline{U} - V) = 0 & \text{in } \Omega, \\ \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

The existence is standard. (See, e.g., Lemma 7.1 in [12].) Then

$$d_2\Delta \underline{V} + \underline{V}(p - U_\alpha - \underline{V}) = \underline{V}(\overline{U} - U_\alpha) \ge 0,$$

and so $\underline{V}$ is a lower solution of the single equation

$$\begin{cases} d_2\Delta V + V(p - U_\alpha - V) = 0 & \text{in } \Omega, \\ \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

of which V_α is the unique positive solution. Hence

$$0 < \underline{V} \le V_\alpha. \tag{5.6}$$

Now by Lemma 5.1, $|U_\alpha|_{L^\infty(\Omega)}$ and $|V_\alpha|_{L^\infty(\Omega)}$ are uniformly bounded in α. Therefore, by elliptic L^p estimates applied to the equation $d_2\Delta V_\alpha + V_\alpha(p - U_\alpha - V_\alpha) = 0$, $\{V_\alpha\}_\alpha$ is bounded in $W^{2,p}(\Omega)$ for all $p \ge 1$. Therefore, by possibly passing to a subsequence, we can assume $V_\alpha \to V_0$ in $W^{2,p}(\Omega)$ for some $V_0 > 0$ (by (5.6)) satisfying (1.5). From the fact that $\bar\theta_{d_2}$ is the unique positive solution of (1.5), $V_\alpha \to \bar\theta_{d_2}$ in $C^{1,\beta}(\overline{\Omega})$ as $\alpha \to \infty$. ◻

Proof of Theorem 1.3. The existence and stability of (U_α, V_α) are proved in section 2. (i) is proved by Lemma 5.3, whereas (ii) is a consequence of Theorem 1.6 and Lemma 5.1.

To finish the proof of Theorem 1.3, it remains to show that for each $x_0 \in \mathfrak{M}$,

$$\left| U_\alpha - 2^{N/2} \max\{p(x_0) - \bar\theta_{d_2}(x_0), 0\} e^{\alpha[m(x)-m(x_0)]/d_1} \right|_{L^\infty(B_r(x_0))} \to 0.$$

But this follows from Theorem 4.1 and Lemma 5.3 by taking p_α to be $p - V_\alpha$. ◻

6. Concluding remarks. We remark that the assumptions (H4) and (H5) serve as sufficient conditions for the existence of u_α, U_α, and V_α for large values of α. In contrast, our methods mainly depend on the nondegeneracy of m at each $x_0 \in \mathfrak{M}$ and apply provided the solutions $u_\alpha, U_\alpha, V_\alpha$ exist for α large.

In our paper, we have made the generalization so that the local growth rate $p(x)$ is not necessarily equal to the $m(x)$ whose gradient determines the direction of directed movement of species U in (1.6) and (1.4). (For instance, $p = \chi(m)$, where χ is an increasing function.) In this way, the different effects of m and p on the solutions can be compared. Roughly speaking, for the single equation (2.1), $m(x)$ (advection effect) determines the shape of the concentrated peaks at local maximum points of $m(x)$, while $p(x)$ (the local resources) determines the heights of those peaks (the population supported at x_0). On the other hand, for the system (1.4), (5.2) tells us that U_α

does not affect V_α in the limit. Therefore, in the limit, the second equation can be solved by setting $U_\alpha = 0$. U_α is then determined by the effective local growth rate $p(x) - V_\alpha(x)$ in the first equation. Loosely stated, if $\mathfrak{M}$ is of measure zero, the system (5.1) decouples as $\alpha \to \infty$ and is "driven" by the second equation.

By Theorems 1.3 and 1.5, for α large, every solution of (1.4) must converge to a common limiting profile regardless of initial conditions as $t \to \infty$. In [8], it is further conjectured that (1.1) has a unique, and hence globally asymptotically stable, coexistence steady state when α is large. Our findings in this paper seem to support this conjecture.

The techniques in this paper seem to be applicable to treating the following two-species competition system introduced in [10, 16] and studied in [5]:

$$\left\{\begin{array}{ll} \nabla\cdot(d_1\nabla U - \alpha U\nabla m) + U(m-U-V) = 0 & \text{in } \Omega, \\ \nabla\cdot(d_2\nabla V - \beta V\nabla m) + V(m-U-V) = 0 & \text{in } \Omega, \\ d_1\frac{\partial U}{\partial\nu} - \alpha U\frac{\partial m}{\partial\nu} = d_2\frac{\partial V}{\partial\nu} - \beta V\frac{\partial m}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{6.1}$$

Among others, the following global stability result is proved in [5], saying that if species V has a fixed large directed movement, then the much more "greedy" species U will always be wiped out regardless of initial conditions.

THEOREM 6.1 (see [5]). *Suppose that $N \le 3$ and all critical points of $m(x)$ are nondegenerate. Then, there exists some $\Lambda_* = \Lambda_*(m,\Omega)$ such that for every $\beta \ge \Lambda_*$, there exists $\Lambda^* = \Lambda^*(\beta, m, \Omega) > 0$ such that $(0, u_{d_2,\beta})$ is globally asymptotically stable for all α sufficiently large, where $u_{d_2,\beta}$ is the unique solution of* (2.1) *with $d = d_2$ and $p \equiv m$.*

By the estimate of $|u_\alpha|_{L^p(\Omega)}$ in Lemma 5.2, we can remove the dimensionality assumption on Ω. The proof is essentially the same as in [5], which we skip here.

THEOREM 6.2. *Suppose that* (H1), (H2), *and* (H3) *hold; then the conclusion of Theorem* 6.1 *holds true in any dimension.*

Recently, the L^∞ bound of u_α independent of α and d was established, which has interesting consequences on the dynamics of (6.1). Please refer to [11] for details.

Appendix. A Liouville-type result. Here we prove Proposition 1.8. By an orthogonal change of coordinates, it suffices to show the following.

PROPOSITION A.1. *Let $0 < \lambda_1 \le \cdots \le \lambda_N$ and $0 < \sigma \in L^\infty_{loc}(\boldsymbol{R}^N)$ such that for some $R_0 > 0$, $\sigma^2 = e^{-\sum_{i=1}^N \lambda_i x_i^2}$ for all $x \in \boldsymbol{R}^N \setminus B_{R_0}(0)$. Then every nonnegative weak solution $w \in W^{1,2}_{loc}(\boldsymbol{R}^N)$ to*

$$\nabla\cdot(\sigma^2\nabla w) = 0 \qquad \text{in } \boldsymbol{R}^N \tag{A.1}$$

is a constant.

Note that (A.1) can be written as

$$\Delta w - \sum_{i=1}^N \lambda_i x_i D_i w = 0 \qquad \text{in } \mathbf{R}^N \setminus B_{R_0}(0). \tag{A.2}$$

First, we note that by local elliptic L^p estimates, w is smooth in $\{x \in \mathbf{R}^N : |x| > R_0\}$ (i.e., when σ is smooth). We shall make use of a general result due to [4].

THEOREM A.2 (see [4]). *If for some positive $\sigma^2 \in L^\infty_{loc}(\boldsymbol{R}^N)$ and constant $C > 0$, $w \in W^{1,2}_{loc}(\boldsymbol{R}^N)$ satisfies*

$$\left\{\begin{array}{l} w\,\nabla\cdot(\sigma^2\nabla w) \ge 0 \text{ in } \boldsymbol{R}^N \text{ locally}, \\ \int_{B_R}\sigma^2 w^2 \le CR^2, \end{array}\right.$$

then w is a constant.

In particular, a sufficient condition for the solution w to (A.1) to be a constant is that $e^{-\sum_{i=1}^N \lambda_i x_i^2} w^2(x)$ be integrable over $\mathbf{R}^N$.

COROLLARY A.3. *Assume* $w \in W^{1,2}_{loc}(\boldsymbol{R}^N)$ *satisfies*

$$\begin{cases} \nabla \cdot (\sigma^2 \nabla w) = 0 & \text{in } \boldsymbol{R}^N \text{ locally,} \\ 0 \le w(x) \le e^{c \sum_{i=1}^N \lambda_i x_i^2} & \text{for some } 0 < c < 1/2, \end{cases}$$

where σ *is as in Proposition* A.1. *Then* w *is a constant.*

We start with some notation concerning the level sets of $e^{-\sum_{i=1}^N \lambda_i x_i^2}$. Define

$$\Sigma_1 := \left\{ y \in \mathbf{R}^N : \sum_{i=1}^N \lambda_i y_i^2 = 1 \right\} \quad \text{and} \quad \Sigma_R := \left\{ x \in \mathbf{R}^N : \sum_{i=1}^N \lambda_i x_i^2 = R^2 \right\}.$$

For each $y \in \Sigma_1$ and $R > 0$, define $\gamma = \gamma(y, R)$ by $\sum_{i=1}^N \lambda_i y_i^2 e^{2\lambda_i \gamma} = R^2$. ($\gamma$ is well-defined since for each $y \in \Sigma_1$, $\gamma \mapsto \sum_{i=1}^N \lambda_i y_i^2 e^{2\lambda_i \gamma}$ is a diffeomorphism from $\mathbf{R}$ to $(0, \infty)$. γ is C^1 by the implicit function theorem.) Next, we define

$$\Phi(R) = \int_{\Sigma_R} \|(\lambda_i x_i)_i\| w(x)\, dS_x. \tag{A.3}$$

Here $(z_i)_i$ is understood as $(z_1, \dots, z_N) \in \mathbf{R}^N$, $\|\cdot\|$ is the usual Euclidean norm in $\mathbf{R}^N$, and dS_y, dS_x are the area elements for the manifolds Σ_1 and Σ_R, respectively.

We are going to prove a differential inequality of Φ that describes the growth of w.

LEMMA A.4.

$$\frac{\sum_{i=1}^N \lambda_i}{\lambda_N R} \Phi(R) \le \Phi'(R) \le \frac{\sum_{i=1}^N \lambda_i}{\lambda_1 R} \Phi(R).$$

Lemma A.4 implies $\frac{d}{dR}\left[R^{-\frac{\sum \lambda_i}{\lambda_1}} \Phi(R)\right] \le 0 \le \frac{d}{dR}\left[R^{-\frac{\sum \lambda_i}{\lambda_N}} \Phi(R)\right]$. In particular,

$$(R/R_0)^{\frac{\sum \lambda_i}{\lambda_N}} \Phi(R_0) \le \Phi(R) \le (R/R_0)^{\frac{\sum \lambda_i}{\lambda_1}} \Phi(R_0) \qquad \text{for all } R \ge R_0. \tag{A.4}$$

Remark A.5. When $\lambda_i = \lambda$ for all i, the equation possesses radial symmetry. In that case, this lemma follows immediately from the observation that the spherical mean of w, which solves an ODE, must be a constant.

Before we prove Lemma A.4, we first express $\Phi(R)$ as an integral over Σ_1.

LEMMA A.6.

$$\Phi(R) = \int_{\Sigma_1} e^{\gamma \sum_i \lambda_i} w((y_i e^{\lambda_i \gamma})_i) \|(\lambda_i y_i)_i\|\, dS_y.$$

Lemma A.6 can be obtained by a change of variables and is a direct consequence of Lemma A.7 below.

LEMMA A.7. *Let* $\phi : \Sigma_1 \to \Sigma_R$ *be a diffeomorphism defined by* $(y_1, \dots, y_N) \mapsto (x_1, \dots, x_N) = (y_1 e^{\lambda_1 \gamma}, \dots, y_N e^{\lambda_N \gamma})$, *where* $\gamma = \gamma(y, R)$. *Then the Jacobian* $J\phi(y)$ *is given by*

$$J\phi(y) = \frac{e^{\gamma \sum \lambda_i} \|(\lambda_i y_i)_i\|}{\|(\lambda_i y_i e^{\lambda_i \gamma})_i\|}.$$

The proof of Lemma A.7 is postponed until the end of the section. Also, we have the following.

LEMMA A.8.

$$\frac{d\gamma}{dR}(y_1,\ldots,y_N) = \frac{R}{\|(\lambda_i y_i e^{\lambda_i\gamma})_i\|^2}.$$

Proof. Differentiating $\sum \lambda_i y_i^2 e^{2\lambda_i\gamma} = R^2$ with respect to R, we have

$$2\left(\sum \lambda_i^2 y_i^2 e^{2\lambda_i\gamma}\right)\frac{d\gamma}{dR} = 2R.$$

Hence,

$$\frac{d\gamma}{dR} = \frac{R}{\|(\lambda_i y_i e^{\lambda_i\gamma})_i\|^2} = \frac{R}{\|(\lambda_i x_i)_i\|^2},$$

where $\gamma = \gamma(y,R)$ and $x_i = y_i e^{\lambda_i\gamma}$. ☐

Proof of Lemma A.4. By Lemma A.6, for any $R > R_0$,

$$\begin{aligned}
\Phi(R) &= \int_{\Sigma_1} w(y_i e^{\lambda_i\gamma}) e^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|\,dS_y\\
\Phi'(R) &= \int_{\Sigma_1} [\nabla w(y_i e^{\lambda_i\gamma})\cdot(\lambda_i y_i e^{\lambda_i\gamma})_i]\frac{d\gamma}{dR}e^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|\,dS_y\\
&\quad+\int_{\Sigma_1} w(y_i e^{\lambda_i\gamma})\left(\sum\lambda_i\right)\frac{d\gamma}{dR}e^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|\,dS_y\\
&= \int_{\Sigma_1}\nabla w(y_i e^{\lambda_i\gamma})\cdot\frac{(\lambda_i x_i)_i}{\|(\lambda_i x_i)_i\|}\cdot\frac{Re^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|}{\|(\lambda_i x_i)_i\|}\,dS_y\\
&\quad+\int_{\Sigma_1} w(y_i e^{\lambda_i\gamma})\left(\sum\lambda_i\right)\frac{R}{\|(\lambda_i x_i)_i\|^2}e^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|\,dS_y\\
&= Re^{R^2}\int_{\Sigma_R} e^{-\sum\lambda_i x_i^2}\frac{\partial w}{\partial\nu}\,dS_x\\
&\quad+\int_{\Sigma_1} w(y_i e^{\lambda_i\gamma})\frac{(\sum\lambda_i)Re^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|}{\|(\lambda_i x_i)_i\|^2}\,dS_y\\
&= \int_{\Sigma_1} w(y_i e^{\lambda_i\gamma})\frac{(\sum\lambda_i)Re^{\gamma\sum\lambda_i}\|(\lambda_i y_i)_i\|}{\|(\lambda_i x_i)_i\|^2}\,dS_y
\end{aligned}$$

where we have made use of Lemmas A.7 and A.8 as well as the fact that $e^{-\sum\lambda_i x_i^2} = e^{-R^2}$ on Σ_R for the second-to-last equality. The last equality is a consequence of (A.1). Hence

$$\begin{aligned}
\frac{R\sum\lambda_i}{\max_{\Sigma_R}\|(\lambda_i x_i)_i\|^2}\Phi(R) \le \Phi'(R) &\le \frac{R\sum\lambda_i}{\min_{\Sigma_R}\|(\lambda_i x_i)_i\|^2}\Phi(R)\\
\frac{R\sum\lambda_i}{\max_{\Sigma_R}\sum\lambda_i^2 x_i^2}\Phi(R) \le \Phi'(R) &\le \frac{R\sum\lambda_i}{\min_{\Sigma_R}\sum\lambda_i^2 x_i^2}\Phi(R)\\
\frac{\sum\lambda_i}{R\lambda_N}\Phi(R) \le \Phi'(R) &\le \frac{\sum\lambda_i}{R\lambda_1}\Phi(R).
\end{aligned}$$

The last line is due to $\sum_{i=1}^N \lambda_i x_i^2 = R^2$ on Σ_R, and $0 < \lambda_1 \le \cdots \le \lambda_N$. ☐

By virtue of Corollary A.3, Proposition A.1 is a consequence of the following lemma.

LEMMA A.9. *For all $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that $w(x) \le K(\epsilon) e^{\epsilon \sum \lambda_i x_i^2}$ in $\boldsymbol{R}^N$.*

Proof. Assume to the contrary that there exists $\epsilon_0 > 0$, $R_k \to \infty$, and $z_k = (z_{k,i})_{i=1}^N \in \Sigma_{R_k}$ such that $w(z_k) \ge e^{\epsilon_0 \sum \lambda_i z_{k,i}^2} = e^{\epsilon_0 R_k^2}$. Then apply Theorem 3.9 to $B_4(z_k)$ (with w satisfying (A.2), we have $\Lambda = \lambda = 1$, $\nu = O(R_k)$) to obtain

$$w(x) \ge C_1^{-R_k \log R_k} w(z_k) \ge C_1^{-R_k \log R_k} e^{\epsilon_0 R_k^2} \ge e^{\epsilon_1 R_k^2}$$

whenever $|x - z_k| < 1$, for some $C_1 = C(N)$ and $0 < \epsilon_1 < \epsilon_0$ and for all k large. Then

$$\Phi(R_k) = \int_{\Sigma_{R_k}} w(x) \|(\lambda_i x_i)_i\| dS_x \ge C R_k e^{\epsilon_1 R_k^2}.$$

This contradicts the power-like growth obtained in Lemma A.4. $\square$

Finally, we supply the proof of Lemma A.7.

Proof of Lemma A.7. Fix $R > 0$. Let $\bar{y} = (\bar{y}_i)_{i=1}^N \in \Sigma_1$ and $\phi(y) = (\phi_i(y))_{i=1}^N = (y_i e^{\lambda_i \gamma})_{i=1}^N$. (Here $\gamma = \gamma(y, R)$.) Denote by $T_{\bar{y}}(\Sigma_1)$ the tangent plane of $\Sigma_1 \subset \mathbf{R}^N$ at $\bar{y}$ after translation to the origin. Given $\bar{y}' \in T_{\bar{y}}(\Sigma_1)$, to evaluate $[\nabla \phi(\bar{y})](\bar{y}')$, let $y(t) = (y_i(t))_{i=1}^N$ be a smooth curve on Σ_1 such that $y(0) = \bar{y}$ and $y'(0) = \bar{y}' = (\bar{y}_i')_i$. Then by the definition of a tangent plane,

$$\begin{aligned}
[\nabla \phi(\bar{y})](\bar{y}') &= \frac{d}{dt}\Big|_{t=0} \phi(y(t)) \\
&= \frac{d}{dt}\Big|_{t=0} (y_i(t) e^{\lambda_i \gamma(y(t))})_i \\
&= (\bar{y}_i' e^{\lambda_i \gamma(y(0))})_i + (\lambda_i \bar{y}_i e^{\lambda_i \gamma(y(0))})_i \frac{d\gamma}{dt}(y(0)) \\
&= (\bar{y}_i' e^{\lambda_i \gamma(\bar{y})})_i + (\lambda_i \bar{y}_i e^{\lambda_i \gamma(\bar{y})})_i \frac{d\gamma}{dt}(y(0)) \\
&= P\Big((\bar{y}_i' e^{\lambda_i \gamma(\bar{y})})_i \Big) = P\Big(\Psi\big|_{T_{\bar{y}}(\Sigma_1)}(\bar{y}') \Big),
\end{aligned}$$

where P is the orthogonal projection from $\mathbf{R}^N$ onto $T_{\phi(\bar{y})}(\Sigma_R)$, and $\Psi : \mathbf{R}^N \to \mathbf{R}^N$ is the linear map given by $(y_i)_i \mapsto (y_i e^{\lambda_i \gamma(\bar{y})})_i$ (since $(\lambda_i \bar{y}_i e^{\lambda_i \gamma(\bar{y})})_i \perp T_{\phi(\bar{y})}(\Sigma_R)$ and $[\nabla \phi(\bar{y})](\bar{y}') \in T_{\phi(\bar{y})}(\Sigma_R)$),

$$\Psi((\lambda_i \bar{y}_i)_i) = (\lambda_i \bar{y}_i e^{\lambda_i \gamma(\bar{y})})_i = (\lambda_i \phi_i(\bar{y}))_i. \tag{A.5}$$

That is, the normal to $\bar{y}$ with respect to Σ_1 is mapped under Ψ to the normal to $\phi(\bar{y})$ with respect to Σ_R. Now let $\{e_i\}_{i=1}^N$ and $\{\tilde{e}_i\}_{i=1}^N$ be two orthonormal bases such that

$$\begin{cases} \operatorname{span}\{e_1, e_2, \dots, e_{N-1}\} = T_{\bar{y}}(\Sigma_1), & e_N = \frac{(\lambda_i \bar{y}_i)_i}{\|(\lambda_i \bar{y}_i)_i\|}, \\ \operatorname{span}\{\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{N-1}\} = T_{\phi(\bar{y})}(\Sigma_1), & \tilde{e}_N = \frac{(\lambda_i \phi_i(\bar{y}))_i}{\|(\lambda_i \phi_i(\bar{y}))_i\|}. \end{cases}$$

Then by (A.5), Ψ can be represented by the matrix

$$\Psi = \begin{bmatrix} & & & 0 \\ & P(\Psi|_{T_{\bar{x}_0}(\Sigma_1)}) & & \vdots \\ & & & 0 \\ a_{N,1} & \dots & a_{N,N-1} & a_{N,N} \end{bmatrix},$$

where $a_{N,N} = \frac{\|(\lambda_i \phi_i(\bar{y}))_i\|}{\|(\lambda_i \bar{y}_i)_i\|}$. Hence,

$$\begin{aligned}\det \Psi &= a_{N,N} \cdot \det(P(\Psi|_{T_{\bar{x}_0}(\Sigma_1)})) \\ &= a_{N,N} \cdot J\phi(\bar{y}),\end{aligned}$$

and so

$$J\phi(\bar{y}) = \frac{\det \Psi}{a_{N,N}} = \frac{e^{\gamma \sum \lambda_i} \|(\lambda_i \bar{y}_i)_i\|}{\|(\lambda_i \phi_i(\bar{y}))_i\|}. \qquad \square$$

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